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Uniqueness of Kähler–Ricci solitons

by

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0. Introduction

In recent years, Ricci solitons have been studied extensively (cf. [Ha], [C2], [T2], etc.). One motivation is that they are very closely related to limiting behavior of solutions of PDE which arise in geometric analysis, such as R. Hamilton's Ricci flow equation and the complex Monge–Ampère equations associated to Kähler–Einstein metrics. Ricci solitons extend naturally Einstein metrics.

Let M be a compact Kähler manifold. A Kähler metric h with its Kähler form ω_h is called a Kähler–Ricci soliton with respect to a holomorphic vector field X if the equation

$$\mathrm{Ric}(\omega_h)-\omega_h=L_X\omega_h \tag{0.1}$$

is satisfied, where $\mathrm{Ric}(\omega_h)$ denotes the Ricci form of ω_h, and L_X is the Lie derivative operator along X (the definition here is slightly stronger than the ordinary one studied, for instance, in [C2]). Since ω_h is d-closed, we may write $L_X\omega_h=\partial\bar\partial\psi$ for some function ψ. It follows that the first Chern class $c_1(M)$ of M is positive and represented by ω_h.

If ω_h is a Kähler–Ricci soliton form with respect to a nontrivial X, then Futaki's invariant with respect to X is

$$F(X)=\int_M |X|^2\omega_h^n\neq 0.$$

By using a result in [F1], we see that there are no Kähler–Einstein metrics on M if M admits a Kähler–Ricci soliton with respect to a nontrivial X. Hence, the existence of Kähler–Ricci solitons is an obstruction to the existence of Kähler–Einstein metrics on compact Kähler manifolds with positive first Chern class. Examples of Kähler–Ricci solitons were found on certain compact Kähler manifolds by E. Calabi [Cal], N. Koiso [Koi] and H. D. Cao [C1], respectively.

The purpose of this paper is to prove the uniqueness of Kähler–Ricci solitons on a given compact Kähler manifold. This extends Bando and Mabuchi's theorem on uniqueness of Kähler–Einstein metrics on Kähler manifolds with positive first Chern class ([BM]). Note that the uniqueness of Kähler–Einstein metrics was proved by E. Calabi in the 1950's for Kähler manifolds with nonpositive first Chern class.

Let $\mathrm{Aut}^{\circ}(M)$ be a connected component containing the identity of the holomorphic transformation group of M. Then there is a semidirect decomposition of $\mathrm{Aut}^{\circ}(M)$ ([FM]),

$$\mathrm{Aut}^{\circ}(M)=\mathrm{Aut}_r(M)\propto R_u,$$

where $\mathrm{Aut}_r(M)\subset\mathrm{Aut}^{\circ}(M)$ is a reductive algebraic subgroup, which is the complexification of a maximal compact subgroup K on M, and R_u is the unipotent radical of $\mathrm{Aut}^{\circ}(M)$.

Our main theorem can be stated as follows.

THEOREM 0.1. *The Kähler–Ricci soliton of M is unique modulo the automorphism subgroup $\mathrm{Aut}_r(M)$; more precisely, if ω_1,ω_2 are two Kähler–Ricci solitons with respect to a holomorphic field X, then there are automorphisms σ in $\mathrm{Aut}^{\circ}(M)$ and τ in $\mathrm{Aut}_r(M)$ such that $\sigma_*^{-1}X\in\eta_r(M)$ and $\sigma^*\omega_2=\tau^*\sigma^*\omega_1$, where $\eta_r(M)$ denotes the Lie algebra of $\mathrm{Aut}_r(M)$. In fact, $\sigma_*^{-1}X$ lies in the center of $\eta_r(M)$.*

We will first reduce the proof of this theorem to solving the complex Monge–Ampére equations

$$\begin{cases}\det(g_{i\bar{j}}+\phi_{i\bar{j}})=\det(g_{i\bar{j}})\exp\{f-t\phi-\theta_X-X(\phi)\},\\ g_{i\bar{j}}+\phi_{i\bar{j}}>0,\end{cases}\tag{0.2}$$

where $t\in[0,1]$, f and θ_X are smooth functions determined by a suitably chosen metric g (cf. §1). The equation (0.2) at $t=1$ is equivalent to (0.1).

We will use the continuity method to solve (0.2). More precisely, if φ is a solution of (0.2) at $t=1$, we need to prove that, modulo automorphisms in $\mathrm{Aut}_r(M)$, there is a smooth family $\{\varphi_s\}$ such that $\varphi_1=\varphi$ and φ_s solves (0.2) at $t=s$. Then the theorem follows from the uniqueness of solutions of (0.2) at $t=0$ proved in [Zh]. There are three main steps in constructing $\{\varphi_s\}$.

Step 1 (cf. §3). We introduce the following functional, which modifies the one used before in [Au], [BM] and [T1]:

$$I(\phi)-J(\phi)=-\int_M\phi e^{\theta_X+X(\phi)}\omega_\phi^n+\int_0^1\int_M\dot{\phi}_s e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n\wedge ds,$$

where $\{\phi_s\}$ is any path from 0 to ϕ. This functional is monotone along any path given by solutions of (0.2). Then by iteration, we can obtain an upper bound of the integrals $\int_M e^{-pt\varphi_t}\omega_g^n$, for some $p>1$, where φ_t are solutions of (0.2) at t.

Step 2 (cf. §4). We introduce the relative extremal function and capacity to derive an a priori C^0-estimate for solutions of the complex Monge–Ampére equations in (0.2). The relative extremal function and capacity were widely used in [BT1], [BT2], [Kol] for the existence and regularity of weak plurisubharmonic solutions of the complex Monge–Ampére equations. Using the L^p-estimate of $e^{-t\phi_t}$ obtained in Step 1, we can prove that ϕ_t are uniformly bounded.

Step 3 (cf. §6). We prove by using the Implicit Function Theorem that (0.2) is solvable for t sufficiently close to 1. Since the corresponding linear operator does have nontrivial kernel in this case, we have difficulties. We will imitate the arguments in [BM]. Using the transformation group $\mathrm{Aut}_r(M)$, we can find a global minimal point of the functional $I-J$ restricted to $\mathrm{Aut}_r(M)$. Then we apply the Implicit Function Theorem to (0.2) at this minimal point.

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1. Reduction to certain complex Monge–Ampére equations

Let (M,g) be an n-dimensional compact Kähler manifold with positive first Chern class $c_1(M)>0$. Denote by $\mathrm{Aut}(M)^\circ$ a connected component containing the identity of the holomorphic transformation group of M. Let K be a maximal compact subgroup of $\mathrm{Aut}^\circ(M)$. Then there is a semidirect decomposition of $\mathrm{Aut}^\circ(M)$ ([FM]),

$$\mathrm{Aut}^\circ(M)=\mathrm{Aut}_r(M)\propto R_u,$$

where $\mathrm{Aut}_r(M)\subset\mathrm{Aut}^\circ(M)$ is a reductive algebraic subgroup and the complexification of K, and R_u is the unipotent radical of $\mathrm{Aut}^\circ(M)$. Let $\eta_r(M)$ be the Lie algebra of $\mathrm{Aut}_r(M)$.

Choose a K-invariant Kähler metric g on M with Kähler form $\omega_g\in 2\pi c_1(M)$. If g is given by $\{g_{i\bar j}\}$ in local coordinates, then

$$\omega_g=\sqrt{-1}\sum g_{i\bar j}\,dz^i\wedge d\bar z^j.$$

Since $\mathrm{Ric}(\omega_g)$ also represents $2\pi c_1(M)$, there is a unique smooth real-valued function f on M such that

$$\begin{cases}\mathrm{Ric}(\omega_g)-\omega_g=\sqrt{-1}\,\partial\bar\partial f,\\ \int_M e^f\omega_g^n=\int_M\omega_g^n=V,\end{cases}\tag{1.1}$$

where $\omega_g^n = \omega_g \wedge \ldots \wedge \omega_g$. Recall that in local coordinates the Ricci curvature is of the form

$$\begin{cases} R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}}), \\ \mathrm{Ric}(\omega_g) = \sqrt{-1}\, R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j. \end{cases}$$

Let ω be a Kähler–Ricci soliton with respect to a holomorphic vector field X on M. Then ω satisfies the equation

$$\mathrm{Ric}(\omega) - \omega = L_X \omega. \tag{1.2}$$

It follows from (1.2) that the $(1,1)$-form $L_X\omega$ is real-valued, i.e. $L_{\mathrm{Im}(X)}\omega = 0$, where $\mathrm{Im}(X)$ denotes the imaginary part of X. Therefore, $\mathrm{Im}(X)$ generates a one-parameter family of isometries of (M, ω). Let K' be the maximal compact subgroup of $\mathrm{Aut}^\circ(M)$ containing such a one-parameter family of isometries. Since K' is conjugate to K, there is an automorphism $\sigma \in \mathrm{Aut}^\circ(M)$ such that $\sigma_*^{-1} X \in \eta_r$, where η_r is the Lie algebra of $\mathrm{Aut}_r(M)$. Furthermore, $\sigma^*\omega$ is a Kähler–Ricci soliton with respect to $\sigma_*^{-1}X$. Thus, it suffices to prove the uniqueness of Kähler–Ricci solitons with respect to any holomorphic vector field $X \in \eta_r$. For simplicity, we may assume that $\sigma = \mathrm{Id}$ and $K = K'$.

Now $\mathrm{Im}(X)$ generates a one-parameter subgroup of isometries of ω_g, or equivalently, $L_X\omega_g$ is a real-valued $(1,1)$-form. So $\bar{\partial} i_X \omega_g = 0$. Since $c_1(M) > 0$, there are no nontrivial harmonic $(0,1)$-forms. By the Hodge Decomposition Theorem, there is a smooth real-valued function θ_X of M such that $i_X \omega_g = \sqrt{-1}\, \bar{\partial}\theta_X$, and consequently,

$$L_X \omega_g = d i_X(\omega_g) = \sqrt{-1}\, \partial\bar{\partial}\theta_X.$$

Moreover, $X(f)$ is real-valued.

Since ω also represents $2\pi c_1(M)$, there is a real-valued function ϕ satisfying

$$\omega = \omega_\phi = \omega_g + \sqrt{-1}\, \partial\bar{\partial}\phi.$$

By adding an appropriate constant to ϕ, we see that (1.2) can be reduced to the complex Monge–Ampére equations

$$\begin{cases} \det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\{f - \phi - \theta_X - X(\phi)\}, \\ g_{i\bar{j}} + \phi_{i\bar{j}} > 0. \end{cases} \tag{1.3}$$

We will normalize θ_X by

$$\int_M e^{\theta_X + X(\phi)} \omega_\phi^n = \int_M \omega_g^n.$$

In order to prove Theorem 0.1, it suffices to show that solutions of (1.3) are unique modulo automorphisms in $\mathrm{Aut}_r(M)$.

Now we consider the following complex Monge–Ampére equations with parameter $t \in [0,1]$:

$$\begin{cases} \det(g_{i\bar{j}}+\phi_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\{f - t\phi - \theta_X - X(\phi)\}, \\ g_{i\bar{j}}+\phi_{i\bar{j}} > 0. \end{cases} \tag{1.4}$$

We observe that the equations in (1.4) are equivalent to the following equations on the Ricci curvature:

$$\mathrm{Ric}(\omega_{\phi_t}) - t\omega_{\phi_t} - (1-t)\omega_g = \sqrt{-1}\,\partial\bar{\partial}(\theta_X + X(\phi_t)) = L_X \omega_{\phi_t}. \tag{1.5}$$

It follows from (1.5) that $X(\phi_t)$ is real-valued whenever ϕ_t solves (1.4) at t.

Set

$$\mathcal{M}_X = \{\phi \in C^\infty(M,R) \mid \omega_\phi = \omega_g + \sqrt{-1}\,\partial\bar{\partial}\phi > 0, \mathrm{Im}(X)(\phi) = 0\}.$$

The following lemma is very important for our further analysis.

LEMMA 1.1 ([Zh]). *There is a uniform constant $C(g,X)$ depending only on the metric g and the field X, such that*

$$\sup_{\mathcal{M}_X} |X(\phi)| \leqslant C.$$

We also have

PROPOSITION 1.1 ([Zh]). *There is a unique solution ϕ_0 of (1.4) at $t=0$ modulo constants.*

In the remaining sections, we shall prove that there is a unique solution of (1.4) at each $t \in (0,1)$.

2. Openness at $t<1$

We adopt the notations in §1. Set

$$\mathcal{W}_X^{k,\alpha} = \{\phi \in C^{k,\alpha}(M) \mid \mathrm{Im}\, X(\phi) = 0\},$$

where k is a nonnegative integer and $\alpha \in (0,1)$.

We define an operator F on $\mathcal{M}_X \times [0,1]$ by

$$F(\phi,t) = \log\det(g_{i\bar{j}}+\phi_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - f + t\phi + \theta_X + X(\phi). \tag{2.1}$$

Denote by $L_{(\phi,t)}$ its Fréchet derivative on ϕ. Then we have

$$L_{(\phi,t)}\psi = \Delta'\psi + t\psi + X(\psi), \tag{2.2}$$

where Δ' denotes the Lapalacian operator associated to the Kähler form ω_ϕ.

The following is a simple observation (cf. [Zh]).

LEMMA 2.1. *Let $\phi,\psi\in\mathcal{W}_X^{2,\alpha}$. Then $L_{(\phi,t)}\psi\in\mathcal{W}_X^{0,\alpha}$ and $F(\phi,t)\in\mathcal{W}_X^{0,\alpha}$.*

For any $\phi\in\mathcal{M}_X$, we define an inner product $\langle\cdot,\cdot\rangle$ by

$$\langle f,g\rangle=\int_M fge^{\theta_X+X(\phi)}\omega_\phi^n,$$

where $f,g\in\mathcal{W}_X^{1,\alpha}$. Clearly, one can extend it to an inner product of a Hilbert space containing $\mathcal{M}_X$ as a subspace.

LEMMA 2.2. (i) *Let $\phi\in\mathcal{M}_X$ and $\tilde L_{(\phi,t)}=\Delta'+X(\cdot)$. Then for any two smooth, complex-valued functions f,g,*

$$\int_M g\overline{\tilde L_{(\phi,t)}f}e^{\theta_X+X(\phi)}\omega_\phi^n=\int_M \bar f\tilde L_{(\phi,t)}ge^{\theta_X+X(\phi)}\omega_\phi^n. \tag{2.3}$$

In particular, for any $f,g\in\mathcal{W}_X^{2,\alpha}$,

$$\begin{aligned}\int_M gL_{(\phi,t)}fe^{\theta_X+X(\phi)}\omega_\phi^n&=\int_M fL_{(\phi,t)}ge^{\theta_X+X(\phi)}\omega_\phi^n\\&=-\int_M((\partial f,\partial g)_{\omega_\phi}-tfg)e^{\theta_X+X(\phi)}\omega_\phi^n.\end{aligned}\tag{2.4}$$

This implies that $L_{(\phi,t)}$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$.

(ii) *Suppose that $\phi=\phi_t$ with $t\in(0,1)$ is a smooth solution of (1.4) at $t\in(0,1)$. Then the first eigenvalue of $L_{(\phi,t)}$ is positive.*

(iii) *If $t=1$, the first eigenvalue of $L_\phi=L_{(\phi,1)}$ is zero, and there is a one-to-one correspondence between $\operatorname{Ker}(L_\phi)$ and $\eta_r(M)$, where $\eta_r(M)$ is the Lie algebra of $\operatorname{Aut}_r(M)$. Moreover, we have $[X,Y]=0$ for all $Y\in\eta_r(M)$.*

Proof. (i) Without loss of generality, we may assume that either g or f has compact support in an open subset where there is a local orthonormal coframe $\{\omega_i\}$ such that $\omega_\phi=\sqrt{-1}\,\omega^i\wedge\bar\omega^i$. Then we denote by $X_{\bar i}$ the ith covariant derivative $(\theta_X+X(\phi))_{\bar i}$. Integrating by parts, we deduce

$$\begin{aligned}\int_M g\overline{\tilde L_{(\phi,t)}f}e^{\theta_X+X(\phi)}\omega_\phi^n&=\int_M g(\overline{f_{i\bar i}+X_{\bar i}f_i})e^{\theta_X+X(\phi)}\omega_\phi^n\\&=\int_M(g\bar f_{i\bar i}+gX_i\bar f_{\bar i})e^{\theta_X+X(\phi)}\omega_\phi^n\\&=-\int_M(g_i\bar f_{\bar i}+gX_i\bar f_{\bar i})e^{\theta_X+X(\phi)}\omega_\phi^n+\int_M gX_i\bar f_{\bar i}e^{\theta_X+X(\phi)}\omega_\phi^n\\&=-\int_M g_i\bar f_{\bar i}e^{\theta_X+X(\phi)}\omega_\phi^n=-\int_M(\partial g,\partial f)_{\omega_\phi}e^{\theta_X+X(\phi)}\omega_\phi^n\\&=-\int_M\overline{(\partial f,\partial g)_{\omega_\phi}}e^{\theta_X+X(\phi)}\omega_\phi^n.\end{aligned}\tag{2.5}$$

Similarly, we have

$$\int_M f\overline{\tilde{L}_{(\phi,t)}g}\, e^{\theta_X+X(\phi)}\omega_\phi^n = -\int_M (\partial f, \partial g)_{\omega_\phi} e^{\theta_X+X(\phi)}\omega_\phi^n. \tag{2.6}$$

Then (2.3) follows from (2.5) and (2.6), and so does (2.4).

(ii) Let λ be the first eigenvalue of $L_{(\phi,t)}$ and ψ be one of its eigenfunctions, i.e.

$$\Delta'\psi + t\psi + X(\psi) = -\lambda\psi.$$

Then integrating by parts and using (1.5), we compute

$$\begin{aligned}
&\lambda \int_M \psi_i \psi_{\bar{i}}\, e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\quad= -\int_M (\Delta'\psi + t\psi + X(\psi))_i \psi_{\bar{i}}\, e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\quad= -\int_M (\psi_{j\bar{j}i}\psi_{\bar{i}} + t\psi_i\psi_{\bar{i}})\, e^{\theta_X+X(\phi)}\omega_\phi^n - \int_M (X_{\bar{j}}\psi_{\bar{i}}\psi_{ij} + X_{\bar{j}i}\psi_{\bar{i}}\psi_j)\, e^{\theta_X+X(\phi)}\omega_\phi^n \qquad (2.7)\\
&\quad= \int_M (R_{i\bar{j}} - t\delta_{ij} - X_{\bar{j}i})\psi_{\bar{i}}\psi_j\, e^{\theta_X+X(\phi)}\omega_\phi^n + \int_M \psi_{ij}\psi_{\bar{i}\bar{j}}\, e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\quad= \int_M (1-t) g_{i\bar{j}}\psi_{\bar{i}}\psi_j\, e^{\theta_X+X(\phi)}\omega_\phi^n + \int_M \psi_{\bar{i}\bar{j}}\psi_{ij}\, e^{\theta_X+X(\phi)}\omega_\phi^n.
\end{aligned}$$

Thus we prove $\lambda \geqslant 0$ for any $t \leqslant 1$, and $\lambda > 0$ if $t \in (0,1)$.

(iii) If $t=1$, it follows from (2.7) that $\sum_{i,j} |\psi_{\bar{i}\bar{j}}|^2 = 0$ for any $\psi \in \mathrm{Ker}(L_\phi)$, i.e. $\sum_i \psi_{\bar{i}}\, \partial/\partial z^i \in \eta(M)$, where $\eta(M)$ is the Lie algebra of $\mathrm{Aut}^\circ(M)$. Moreover, by Lemma 2.3.8 in [F2], the imaginary part of $\sum_i \psi_{\bar{i}}\, \partial/\partial z^i$ is a Killing vector field of ω_ϕ. This shows that $\sum_i \psi_{\bar{i}}\, \partial/\partial z^i \in \eta_r(M)$.

Conversely, given any $Y \in \eta_r(M)$, by Theorem A in the Appendix, the imaginary part of Y is a Killing vector field associated with ω_ϕ. By the Hodge theory and using Lemma 2.3.8 in [F2], we can have a smooth, real-valued function ψ such that $i(Y)\omega_\phi = \bar{\partial}\psi$. Thus by (2.4) and (2.7), we have

$$\begin{aligned}
&\int_M (\Delta'\psi + t\psi + X(\psi))(\Delta'\psi + X(\psi))\, e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\qquad= -\int_M (\Delta'\psi + t\psi + X(\psi))_i \psi_{\bar{i}}\, e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\qquad= \int_M |\psi_{ij}|^2 e^{\theta_X+X(\phi)}\omega_\phi^n = 0.
\end{aligned}$$

Since $\lambda \geqslant 0$, we must have $\Delta'\psi + \psi + X(\psi) = 0$, i.e. we have proved $\psi \in \mathrm{Ker}(L_\phi)$.

It follows from $\mathrm{Im}(X)(\psi) = 0$, where ψ is given as above for Y, that $[X, Y] = 0$, since

$$X_j Y_{\bar{j}i} - Y_j X_{\bar{j}i} = (X_j Y_{\bar{j}})_i - (Y_j X_{\bar{j}})_i = (\overline{X(\psi)})_i - (X(\psi))_i = 0. \qquad \square$$

As a direct corollary, we have

COROLLARY 2.1. *Let $\phi=\phi_t$ be a smooth solution of* (1.4) *at t. Then for any $\psi\in\mathcal{W}_X^{1,\alpha}$, we have*

$$\int_M |\bar\partial\psi|^2 e^{\theta_X+X(\phi)}\omega_\phi^n \geqslant t\left(\int_M \psi^2 e^{\theta_X+X(\phi)}\omega_\phi^n - \frac{1}{V}\left(\int_M \psi e^{\theta_X+X(\phi)}\omega_\phi^n\right)^2\right).$$

PROPOSITION 2.1. *Let ϕ_{t_0} be a smooth solution of* (1.4) *at $t=t_0\in(0,1)$. Then there is a small number $\delta>0$ such that there are smooth solutions of equations* (1.4) *for any $s\in(t_0-\delta,t_0]$.*

Proof. Consider the map $F(\phi,t)\colon \mathcal{W}_X^{2,\alpha}\times[0,1]\to\mathcal{W}_X^{0,\alpha}$ defined by (2.1). Let $L_{(\phi_{t_0},t_0)}\colon \mathcal{W}_X^{2,\alpha}\to\mathcal{W}_X^{0,\alpha}$ be the linearization of $F(\phi_{t_0},t_0)$ on the first variable at (ϕ_{t_0},t_0). By Lemma 2.2 and the standard regularity theory for elliptic equations, this linear operator is invertible. Then the Implicit Function Theorem implies that there are $C^{2,\alpha}$-solutions ϕ_s of (1.4) for any $s\in(t_0-\delta,t_0]$, where $\delta>0$ is sufficiently small. Finally, it follows from the standard regularity theory of elliptic equations that ϕ_s are in fact C^∞-smooth. □

3. L^p-estimates for functions $e^{-t\phi}$

First we introduce two functionals on $\mathcal{M}_X$ which are modifications of the corresponding ones in [Au] and [T1]:

$$I(\phi)=\int_M \phi(e^{\theta_X}\omega_g^n-e^{\theta_X+X(\phi)}\omega_\phi^n) \tag{3.1}$$

and

$$J(\phi)=\int_0^1\int_M \dot\phi_s(e^{\theta_X}\omega_g^n-e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n)\wedge ds, \tag{3.2}$$

where ϕ_s is a path from 0 to ϕ in $\mathcal{M}_X$.

As a special case of Lemma 1.2 in [Zh], we have

LEMMA 3.1 ([Zh]). *The functional $J(\phi)$ is well defined, i.e. independent of the path $\{\phi_s\}$.*

Let

$$F(\phi)=J(\phi)-\int_M \phi e^{\theta_X}\omega_g^n=-\int_0^1\int_M \dot\phi_s e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n\wedge ds.$$

By simple computations, one can show that, for any two ω_ϕ and ω_ψ in $\mathcal{M}_X$, the cocycle condition

$$F(\psi)=F(\phi)+F_\phi(\psi) \tag{3.3}$$

is satisfied, where $F_\phi(\psi)=-\int_0^1\int_M \dot\phi_s e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n\wedge ds$, and ϕ_s is a path from ϕ to ψ in $\mathcal{M}_X$.

LEMMA 3.2. *Let ϕ_t be a solution of (1.4) at t. Assume that $\{\phi_t\}$ is a smooth family. Then*

$$\frac{d}{dt}(I(\phi_t)-J(\phi_t))\geqslant 0.$$

In particular, there is a uniform constant C such that

$$I(\phi_t)-J(\phi_t)\leqslant I(\phi_1)-J(\phi_1)\leqslant C.$$

Proof. An easy computation shows

$$\begin{aligned}
-(I(\phi_t)-J(\phi_t))&-\int_M \phi_t e^{\theta_X+X(\phi_t)}\omega^n_{\phi_t}\\
&=J(\phi_t)-\int_M \phi_t e^{\theta_X}\omega_g^n\\
&=J(\phi_1)-\int_M \phi_1 e^{\theta_X}\omega_g^n-\int_M(\phi_t-\phi_1)e^{\theta_X+X(\phi_1)}\omega^n_{\phi_1}\\
&\quad+\int_1^t ds\int_M \dot\phi_s\big(e^{\theta_X+X(\phi_1)}\omega^n_{\phi_1}-e^{\theta_X+X(\phi_s)}\omega^n_{\phi_s}\big).
\end{aligned}\tag{3.4}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}(I(\phi_t)-J(\phi_t))&=-\int_M \phi_t\frac{d}{dt}\big(e^{\theta_X+X(\phi_t)}\omega^n_{\phi_t}\big)\\
&=-\int_M \phi_t(\Delta'\dot\phi_t+X(\dot\phi_t))e^{\theta_X+X(\phi_t)}\omega^n_{\phi_t}.
\end{aligned}\tag{3.5}$$

On the other hand, by differentiating (1.4) with respect to t, we obtain

$$\Delta'\dot\phi_t+X(\dot\phi_t)+t\dot\phi_s=-\phi_t.\tag{3.6}$$

Plugging (3.6) into (3.5) and using part (ii) of Lemma 2.2, we get

$$\frac{d}{dt}(I(\phi_t)-J(\phi_t))=\int_M(\Delta'\dot\phi_t+X(\dot\phi_t)+t\dot\phi_t)(\Delta'\dot\phi_t+X(\dot\phi_t))e^{\theta_X+X(\phi_t)}\omega^n_{\phi_t}\geqslant 0.$$

In particular, there is a uniform constant C such that

$$I(\phi_t)-J(\phi_t)\leqslant I(\phi_1)-J(\phi_1)\leqslant C.$$

LEMMA 3.3. *Let ϕ be in $\mathcal{M}_X$. Then there are uniform constants C_1 and C_2 such that*

$$C_1(I(\phi)-J(\phi)) \leqslant \int_M \phi(\omega_g^n-\omega_\phi^n) \leqslant C_2(I(\phi)-J(\phi)).$$

Proof. Since the proofs of the two inequalities are similar, we prove only the second inequality. Let $\phi_s = s\phi$. Then by using (2.3) in Lemma 2.2 and Lemma 1.1, we have

$$\begin{aligned}\frac{d}{ds}(I(\phi_s)-J(\phi_s)) &= -\int_M \phi_s(\Delta'\dot{\phi}_s + X(\dot{\phi}_s))e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n \\ &= n\sqrt{-1}\int_M e^{\theta_X+X(\phi_s)}\partial\phi_s\wedge\bar{\partial}\dot{\phi}_s\wedge\omega_{\phi_s}^{n-1} \\ &= sn\sqrt{-1}\int_M e^{\theta_X+sX(\phi)}\partial\phi\wedge\bar{\partial}\phi\wedge(s\omega_\phi+(1-s)\omega_g)^{n-1} \\ &\geqslant nC_1'\sqrt{-1}\int_M \partial\phi\wedge\bar{\partial}\phi\wedge\sum_{i=0}^{n-1} s^{i+1}(1-s)^{n-1-i}\omega_\phi^i\wedge\omega_g^{n-1-i}.\end{aligned}\tag{3.7}$$

It follows that

$$\begin{aligned}I(\phi)-J(\phi) &= \int_0^1 \frac{d}{ds}(I(\phi_s)-J(\phi_s))\,ds \\ &\geqslant C_2'\sqrt{-1}\int_M \partial\phi\wedge\bar{\partial}\phi\wedge\sum_{i=0}^{n-1}\omega_\phi^i\wedge\omega_g^{n-1-i}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_M \phi(\omega_g^n-\omega_\phi^n) &= -\sqrt{-1}\int_M \phi\partial\bar{\partial}\phi\wedge\sum_{i=0}^{n-1}\omega_\phi^i\wedge\omega_g^{n-1-i} \\ &= \sqrt{-1}\int_M \partial\phi\wedge\bar{\partial}\phi\wedge\sum_{i=0}^{n-1}\omega_\phi^i\wedge\omega_g^{n-1-i}.\end{aligned}\tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\int_M \phi(\omega_g^n-\omega_\phi^n) \leqslant C_2(I(\phi)-J(\phi)),$$

where C_2 is a uniform constant.

COROLLARY 3.1. *Let $\{\phi_t\}$ be a smooth family of solutions of* (1.4) ($t\in[t_0,1]$). *Then there are two uniform constants C and C', which are independent of t_0, such that for any $t\in[t_0,1]$,*

$$0 \leqslant t\sup_M \phi_t \leqslant C \quad \text{and} \quad \inf_M \phi_t \leqslant 0, \tag{3.9}$$

and

$$-\int_M \phi_t \omega_{\phi_t}^n \leqslant C' \quad \textit{and} \quad -\int_{M_-} t\phi_t \omega_{\phi_t}^n \leqslant C', \tag{3.10}$$

where $M_-=\{x\in M \mid \phi_t\leqslant 0\}$.

Proof. First we shall prove that $\sup_M \phi\geqslant 0$. Notice that by (2.3) in Lemma 2.2,

$$\frac{d}{dt}\int_M e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n=\int_M \tilde{L}_{(\phi,t)}\dot{\phi}_t e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n=0,$$

and hence,

$$\int_M e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n=\int_M e^{\theta_X+X(\phi_1)}\omega_{\phi_1}^n=\int_M e^f\omega_g^n.$$

Integrating (1.4) multiplied by $e^{\theta_X+X(\phi_t)}$, we get

$$\int_M e^{f-t\phi_t}\omega_g^n=\int_M e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n=\int_M e^f\omega_g^n.$$

This shows that

$$\sup_M \phi_t\geqslant 0 \quad \text{and} \quad \inf_M \phi_t\leqslant 0. \tag{3.11}$$

Next by Lemmas 3.2 and 3.3, we see that there is a uniform constant C_1 such that

$$\int_M t\phi_t(\omega_g^n-\omega_{\phi_t}^n)\leqslant tC_1. \tag{3.12}$$

On the other hand, by (1.4), it is easy to see that

$$-\int_M t\phi_t\omega_{\phi_t}^n=-\int_M t\phi_t e^{f-t\phi_t-\theta_X-X(\phi_t)}\omega_g^n\geqslant -C_2\int_{M_+} t\phi_t e^{-t\phi_t}\omega_g^n\geqslant -C_3,$$

where $M_+=\{x\in M\mid \phi_t\geqslant 0\}$. Hence we get

$$t\int_M \phi_t\omega_g^n\leqslant C_4. \tag{3.13}$$

Let $G(x,y)$ be the Green function associated to ω_g satisfying $G(x,\cdot)+C_5\geqslant 0$. Since $\Delta\phi\geqslant -n$ $(\phi=\phi_t)$, by the Green formula, we have

$$\begin{aligned}\sup_M \phi &\leqslant \frac{1}{V}\int_M t\phi\omega_g^n-\max_{x\in M}\left(\frac{1}{V}\int_M (G(x,\cdot)+C_5)\Delta\phi\omega_g^n\right)\\ &\leqslant \frac{1}{V}\int_M \phi\omega_g^n+nC_5.\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we see that there is a uniform constant C such that

$$t\sup_M \phi\leqslant C.$$

By Lemma 3.2 and (3.9), we also have

$$-\frac{1}{V}\int_M \phi\omega_\phi^n \leqslant C_1 - \frac{1}{V}\int_M \phi\omega_g^n \leqslant C_1 + nC_5 - \sup_M \phi \leqslant C_1 + nC_5 = C_6.$$

Moreover,

$$\begin{aligned}-\int_{M_-} t\phi\omega_\phi^n &= -\int_M t\phi\omega_\phi^n + \int_{M_+} t\phi\omega_\phi^n \\ &\leqslant tVC_6 + \int_{M_+} t\phi e^{f-t\phi-\theta_X - X(\phi)}\omega_g^n \leqslant C'.\end{aligned}$$

PROPOSITION 3.1. *Let $\{\phi_t\}$ be a smooth family of solutions of* (1.4) ($t\in[t_0,1]$). *Then there is a uniform $\varepsilon>0$ such that for any $t\in[t_0,1]$,*

$$\int_M \exp\{-(1+\varepsilon)t\phi\}\omega_g^n \leqslant C.$$

Proof. First we choose a uniform constant c such that

$$\bar\phi = \bar\phi_t = \phi_t - \frac{c}{t} \leqslant -1.$$

This is possible since $t\sup_M \phi_t$ is uniformly bounded by Corollary 3.1. Moreover, by (3.10), we have

$$-\int_M t\phi e^{\theta_X + X(\phi)}\omega_\phi^n \leqslant C_1. \tag{3.15}$$

It follows that

$$-\int_M t\bar\phi e^{\theta_X + X(\phi)}\omega_\phi^n \leqslant C_2. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}\int_M (-\bar\phi)^p(\omega_\phi^n - \omega_\phi^{n-1}\wedge\omega_g) &= \sqrt{-1}\int_M (-\bar\phi)^p\partial\bar\partial(\bar\phi)\wedge\omega_\phi^{n-1} \\ &= p\sqrt{-1}\int_M (-\bar\phi)^{p-1}(-\partial\bar\phi)\wedge(-\bar\partial\bar\phi)\wedge\omega_\phi^{n-1} \\ &= \frac{4p}{n(p+1)^2}\int_M |\bar\partial(-\bar\phi)^{(p+1)/2}|^2\omega_\phi^n.\end{aligned}$$

It follows that

$$\int_M |\bar\partial(-\bar\phi)^{(p+1)/2}|^2\omega_\phi^n \leqslant \frac{n(p+1)^2}{4p}\int_M (-\bar\phi)^p\omega_\phi^n. \tag{3.17}$$

Since both θ_X and $X(\phi)$ are uniformly bounded, we can apply the weighted Poincare inequality in Corollary 2.1 and derive from (3.17)

$$\begin{aligned}\int_M(-\bar\phi)^{p+1}e^{\theta_X+X(\phi)}\omega_\phi^n &\leqslant \frac{cp}{t}\int_M(-\bar\phi)^p e^{\theta_X+X(\phi)}\omega_\phi^n+\frac{1}{V}\left(\int_M(-\bar\phi)^{(p+1)/2}e^{\theta_X+X(\phi)}\omega_\phi^n\right)^2\\ &\leqslant \frac{cp}{t}\int_M(-\bar\phi)^p e^{\theta_X+X(\phi)}\omega_\phi^n\\ &\quad+\frac{1}{V}\int_M(-\bar\phi)^p e^{\theta_X+X(\phi)}\omega_\phi^n\cdot\int_M(-\bar\phi)e^{\theta_X+X(\phi)}\omega_\phi^n,\end{aligned}\tag{3.18}$$

where c is a uniform constant. Then by (3.16), we have

$$\int_M(-\bar\phi)^{p+1}e^{\theta_X+X(\phi)}\omega_\phi^n\leqslant\frac{cp}{t}\int_M(-\bar\phi)^p e^{\theta_X+X(\phi)}\omega_\phi^n,\tag{3.19}$$

where c is a uniform constant which may be different from the one in (3.18).

By iterating (3.19) and using (3.16), we have

$$\int_M(-\bar\phi)^{p+1}e^{\theta_X+X(\phi)}\omega_\phi^n\leqslant\frac{c^p(p+1)!}{t^p}\int_M-\bar\phi e^{\theta_X+X(\phi)}\omega_\phi^n\leqslant\frac{c^{p+1}(p+1)!}{t^{p+1}}.\tag{3.20}$$

Now we choose $\varepsilon<1/c$. Then

$$\begin{aligned}\int_M\exp\{-\varepsilon t\bar\phi\}e^{\theta_X+X(\phi)}\omega_\phi^n&=\sum_{p=0}^{+\infty}\frac{(\varepsilon t)^p}{p!}\int_M(-\bar\phi)^p e^{\theta_X+X(\phi)}\omega_\phi^n\\ &\leqslant\sum_{p=0}^{+\infty}(\varepsilon c)^p\leqslant\frac{1}{1-c\varepsilon}.\end{aligned}\tag{3.21}$$

It follows that

$$\begin{aligned}\int_M\exp\{(1+\varepsilon)(-t\phi)\}\omega_g^n&=\int_M\exp\{(1+\varepsilon)(-t\phi)\}e^{-f+t\phi}e^{\theta_X+X(\phi)}\omega_\phi^n\\ &\leqslant C_3\int_M\exp\{-\varepsilon t\bar\phi\}e^{\theta_X+X(\phi)}\omega_\phi^n\leqslant C.\end{aligned}$$

The proposition is proved. □

4. Relative capacity and C^0-estimate

In this section, we establish the a priori C^0-estimate for solutions of (1.4). Such an estimate is a corollary of the results in the last section and the estimates in [Kol] for complex Monge–Ampère equations. For the reader's convenience, we reproduce the C^0-estimate in [Kol]. It uses the relative capacity for plurisubharmonic functions studied in [BT1], [BT2]. All results in this section can be found in either [BT1], [BT2] or [Kol].

For any compact subset K of a strictly pseudoconvex domain Ω in $\mathbf{C}^n$, its relative capacity in Ω is given by

$$\operatorname{cap}(K,\Omega)=\sup\left\{\int_K(\sqrt{-1}\,\partial\bar\partial u)^n \,\middle|\, u\in \mathrm{PSH}(\Omega),\ -1\leqslant u<0\right\},$$

where $\mathrm{PSH}(\Omega)$ denotes the space of plurisubharmonic functions (abbreviated as psh) in the weak sense. For any open set $U\subset\Omega$, we have

$$\operatorname{cap}(U,\Omega)=\sup\{\operatorname{cap}(K,\Omega)\mid K\subset U,\ K\text{ compact}\}.$$

Recall that the relative extremal function of K with respect to Ω is defined by

$$u_K(z)=\sup\{u(z)\mid u\in \mathrm{PSH}(\Omega)\cap L^\infty(\Omega),\ u<0 \text{ and } u|_K\leqslant -1\}.$$

It is easy to see that $u_K^*(z)=\overline{\lim}_{z'\to z}u_K(z')$ is a psh function, and it is called the upper semicontinuous regularization. A compact set K is said to be regular if $u_K^*=u_K$. Here are the main properties of u_K^* (cf. [BT2], [AT]):

$$u_K^*\in \mathrm{PSH}(\Omega),\quad -1\leqslant u_K^*\leqslant 0,\quad \lim_{z\to\partial\Omega}u_K^*=0,$$
$$(\sqrt{-1}\,\partial\bar\partial u^*)^n=0\quad \text{on } \Omega\backslash K,$$
$$u_K^*=-1\quad \text{on } K, \text{ except on a set of relative capacity zero.}$$

Moreover, we have

$$\operatorname{cap}(K,\Omega)=\int_\Omega(\sqrt{-1}\,\partial\bar\partial u_K^*)^n=\int_K(\sqrt{-1}\,\partial\bar\partial u_K^*)^n. \tag{4.1}$$

LEMMA 4.1. *Let Ω be a strictly pseudoconvex domain in $\mathbf{C}^n$ and u be a smooth solution of the complex Monge–Ampère equation on Ω,*

$$\det(u_{i\bar j})=f.$$

Suppose that u and f satisfy

$$u<0,\quad u(o)>c\ \ (o\in\Omega),\quad \int_K f\,dv\leqslant A\operatorname{cap}(K,\Omega)\frac{\operatorname{cap}(K,\Omega)^{1/n}}{1+\operatorname{cap}(K,\Omega)^{1/n}}, \tag{4.2}$$

for any compact subset K of Ω. If the sets

$$U(s)=\{z\mid u(z)<s\}\cap\Omega''$$

are nonempty and relatively compact in $\Omega''\subset\Omega'\subset\subset\Omega$ for any $s\in[S,S+D]$, where S is some number, then there is a constant S_0, depending only on c,A,D,Ω',Ω, such that

$$\inf_{\Omega''} u\geqslant S_0.$$

Proof. This proof is from [Kol]. Set

$$a(s)=\operatorname{cap}(U(s),\Omega)\quad\text{and}\quad b(s)=\int_{U(s)}\big(\sqrt{-1}\,\partial\bar\partial u\big)^n.$$

First we claim that

$$t^n a(s)\leqslant b(s+t),\quad\text{for all } 0<t<S+D-s. \tag{4.3}$$

Indeed, it suffices to prove that for any compact regular set $K\subset U(s)$,

$$t^n\operatorname{cap}(K,\Omega)\leqslant b(s+t),\quad\text{for all } 0<t<S+D-s.$$

Let u_K be the relative extremal function of K with respect to Ω, and $w=(u-s-t)/t$. Set $V=\{z\mid w(z)<u_K\}\cap\Omega''$. Then one can easily check that $K\subset V\subset U(s+t)$. Moreover, since $U(s+t)$ is relatively compact in Ω'', $w=u_K$ on ∂V. Thus, by the Comparison Principle ([BT2]) and (4.1), we have

$$\begin{aligned}\operatorname{cap}(K,\Omega)&=\int_K\big(\sqrt{-1}\,\partial\bar\partial u_K\big)^n\leqslant\int_V\big(\sqrt{-1}\,\partial\bar\partial u_K\big)^n\\&\leqslant\int_V\big(\sqrt{-1}\,\partial\bar\partial w\big)^n=t^{-n}\int_V\big(\sqrt{-1}\,\partial\bar\partial u\big)^n\\&\leqslant t^{-n}\int_{U(s+t)}\big(\sqrt{-1}\,\partial\bar\partial u\big)^n=t^{-n}b(s+t).\end{aligned}$$

The claim is proved.

Next we define an increasing sequence $s_0,s_1,\dots,s_N$ by setting $s_0=S$ and

$$s_j=\sup\Big\{s\mid a(s)\leqslant\lim_{t\to s_{j-1}^+}ea(t)\Big\}$$

for $j=1,\dots,N$, where N is chosen to be the greatest number satisfying $s_N\leqslant S+D$. Then we have

$$\lim_{t\to s_j^-}a(t)\leqslant\lim_{t\to s_{j-1}^+}ea(t), \tag{4.4}$$

$$a(s_j) \geqslant ea(s_{j-2}) \tag{4.5}$$

and

$$a(S+D) \leqslant \lim_{t\to s_N^+} ea(t). \tag{4.6}$$

By using (4.3), (4.6) and (4.2), for any $t\in(s_N, S+D]$, we obtain

$$(S+D-t)^n a(t) \leqslant b(S+D) \leqslant Aa(S+D)\frac{a(S+D)^{1/n}}{1+a(S+D)^{1/n}} \leqslant Aea(t)a(S+D)^{1/n}.$$

Letting $t\to s_N^+$, it follows that

$$S+D-s_N \leqslant (Ae)^{1/n} a(S+D)^{1/n^2}. \tag{4.7}$$

Now we shall estimate $s_N - s$. Consider two numbers $S<s'<s<S+D$ such that $a(s)\leqslant ea(s')$, and set $t=s-s'$. Then by (4.3), (4.2) and (4.4), we have

$$t^n a(s') \leqslant b(s) \leqslant Aa(s)\frac{a(s)^{1/n}}{1+a(s)^{1/n}} \leqslant Aea(s')a(s)^{1/n}.$$

Letting $s\to s_{j+1}^-$ and $s'\to s_j^+$, it follows that

$$s_{j+1}-s_j = t_j \leqslant (Ae)^{1/n} a(s_{j+1})^{1/n^2}. \tag{4.8}$$

Summing up (4.8) and using (4.5), we get

$$\begin{aligned}\sum_{j=0}^{N-1} t_j &\leqslant (Ae)^{1/n} \sum_{j=0}^{N-1} a(s_{j+1})^{1/n^2}\\ &\leqslant (Ae)^{1/n}\left(\sum_{j=1}^{N-2}\int_{\ln a(s_j)}^{\ln a(s_{j+2})} e^{\tau/n^2}\,d\tau + 2a(S+D)^{1/n^2}\right)\\ &\leqslant 2(Ae)^{1/n}\left(\int_{\ln a(S)}^{\ln a(S+D)} e^{\tau/n^2}\,d\tau + a(S+D)^{1/n^2}\right).\end{aligned} \tag{4.9}$$

Making the change of variable $y=e^{-\tau}$, (4.9) becomes

$$\begin{aligned}S_N - S = \sum_{j=0}^{N-1} t_j &\leqslant 2(Ae)^{1/n}\left(\int_{a(S+D)^{-1}}^{a(S)^{-1}} y^{-(1+1/n^2)}\,dy + a(S+D)^{1/n^2}\right)\\ &\leqslant 2(Ae)^{1/n}(a(S+D)^{1/n^2} + n^2 a(S+D)^{1/n^2}).\end{aligned} \tag{4.10}$$

It was, however, proved in [AT] (also see [Be], or Theorem 1.2.11 in [Kol]) that

$$\operatorname{cap}(\{u<s\}\cap\Omega', \Omega) \leqslant \frac{c'}{|s|},$$

where c' depends only on c and Ω'. It follows that $\operatorname{cap}(\{u<s\}\cap\Omega',\Omega)$ converges uniformly to zero as s tends to $-\infty$, so long as u satisfies $u<0$ and $u(o)>c$. Hence, by (4.7) and (4.10), there is a uniform constant $S_0(c,A,D,\Omega',\Omega)$ such that as $S<S_0$ the following two inequalities hold:

$$S+D-s_N<\tfrac{1}{2}D,\quad s_N-S<\tfrac{1}{2}D.$$

Combining them, we will obtain a contradiction. This shows that $S\geqslant S_0$ and consequently $\inf_{\Omega''}u\geqslant S_0$. □

LEMMA 4.2. *Let Ω be a strictly pseudoconvex domain in $\mathbf{C}^n$, and u be a smooth solution of the complex Monge–Ampére equation on Ω,*

$$\det(u_{i\bar{j}})=f,$$

with $\|f\|_{L^p(\Omega)}\leqslant a$ for some $p>1$. Suppose that u satisfies

$$u<0\quad\textit{and}\quad u(o)>c\ \ (o\in\Omega).$$

Define $U(s)$ as in the last lemma. If $U(s)$ is nonempty and relatively compact in Ω'' for any $s\in[S,S+D]$ for some S, then there is a constant $S_0=S_0(c,p,a,D,\Omega',\Omega)$, such that

$$\inf_{\Omega''}u\geqslant S_0(c,p,a,D,\Omega',\Omega),$$

Proof. Let u_K be the relative extremal function of a regular set K with respect to Ω, and $v=\operatorname{cap}^{-1/n}(K,\Omega)u_K$. Then v is a psh function and satisfies

$$\int_\Omega\left(\sqrt{-1}\,\partial\bar{\partial}v\right)^n=1\quad\text{and}\quad\lim_{z\to\partial\Omega}v=0.$$

By Lemma 2.5.1 in [Kol], we have

$$\lambda(U'(s))\leqslant c\exp\{-2\pi|s|\}$$

for some uniform constant c independent of v, where $\lambda(U'(s))$ is the Lebesgue measure of the set $U'(s)=\{v<s\}$. It follows that for any $q\geqslant1$, there is a uniform constant $c'(q)$ such that $\|v\|_{L^q(\Omega)}\leqslant c'(q)$. Hence,

$$\int_\Omega|v|^n(1+|v|)f\,d\mu\leqslant A(p,\Omega,a).\tag{4.11}$$

On the other hand, we have

$$\begin{aligned}\operatorname{cap}(K,\Omega)^{-1}(1+\operatorname{cap}^{-1/n}(K,\Omega))\int_K f\,d\mu&\leqslant\int_K|v|^n(1+|v|)f\,d\mu\\&\leqslant\int_\Omega|v|^n(1+|v|)f\,d\mu.\end{aligned}\tag{4.12}$$

Combining (4.11) and (4.12), we deduce

$$\int_K f\,dv \leqslant A\,\mathrm{cap}(K,\Omega)\frac{\mathrm{cap}(K,\Omega)^{1/n}}{1+\mathrm{cap}(K,\Omega)^{1/n}}.$$

Then Lemma 4.2 follows from Lemma 4.1. □

The following proposition was proved in [Kol].

PROPOSITION 4.1. *Let* (M,g) *be a compact Kähler manifold and* ϕ *be a smooth solution of the complex Monge–Ampére equations on* M,

$$\begin{cases} \det(g_{i\bar{j}}+\phi_{i\bar{j}})=\det(g_{i\bar{j}})f, \\ \sup_M \phi=0. \end{cases}$$

Then we have

$$\inf_M \phi \geqslant C(M,g,p,\|f\|_{L^p(M)}),$$

where $p>1$ *and* $C=C(M,g,p,\|f\|_{L^p(M)})$.

Proof. Let $x\in M$ be such that $\phi(x)=\inf_M \phi$ and U be one of its neighborhoods. Without loss of generality, we may assume that there is a smooth, bounded function v such that $\omega_g=\sqrt{-1}\,\partial\bar{\partial}v$ on U, $v\leqslant 0$ and $v(x)\leqslant \inf_{\partial U} v-c_0$ for some positive $c_0>0$. Hence,

$$v(x)+\phi(x)\leqslant \inf_{\partial U}(v+\phi)-c_0.$$

Clearly, if we take $D=c_0-2\varepsilon$, $S=v(x)+\phi(x)+\varepsilon$, $u=v+\phi$, with $\varepsilon\ll 1$, then $U(s)$ as defined in Lemma 4.1 is nonempty and relatively compact in U. On the other hand, since $\Delta\phi\geqslant -n$, by the Green formula, we have

$$0=\sup_M \phi \leqslant \int_M \phi\omega^n+c.$$

Note that in this proof, c, c', etc., always denote uniform constants depending only on M and g. This implies that $\sup_U(v+\phi)\geqslant -c'$. Then by the last lemma, $v(x)+\phi(x)\geqslant -S_0$ for some uniform constant S_0, and so the proposition follows. □

PROPOSITION 4.2. *Let* $\phi=\phi_t$ *be any smooth solution of* (1.4) *at* t. *Then there is a uniform constant* C *such that*

$$\sup_M |\phi_t|\leqslant C.$$

Proof. Let $\bar{\phi}=\phi-\sup_M \phi$. Then by (1.4), $\bar{\phi}$ satisfies the complex Monge–Ampére equations

$$\begin{cases} \det(g_{i\bar{j}}+\bar{\phi}_{i\bar{j}})=\det(g_{i\bar{j}})\exp\{f-t\phi-\theta_X-X(\phi)\}=\det(g_{i\bar{j}})h, \\ \sup_M \bar{\phi}=0. \end{cases} \tag{4.13}$$

By Proposition 3.1, we see that there are a small positive number ε independent of t, and a uniform constant C_1, such that

$$\int_M h^{1+\varepsilon}\omega_g^n \leqslant C_1.$$

Applying Proposition 4.1 to equation (4.13), we get

$$\inf_M \bar{\phi} \geqslant -C_2 \tag{4.14}$$

for some uniform constant C_2. Hence by (3.11), we have proved

$$\sup_M |\phi| \leqslant C_2. \qquad \square$$

5. Higher-order estimates

In this section, we establish a priori C^2- and C^3-estimates for solutions of (1.4). Our proof uses Yau's arguments in [Ya].

PROPOSITION 5.1. *Let $\phi=\phi_t$ be any solution of (1.4) at t. Then there are two uniform constants C and c such that*

$$n+\Delta\phi_t \leqslant C\exp\{c(\phi_t-\inf_M \phi_t)\}\{1+\exp(-t\inf_M \phi_t)\}.$$

Proof. Given a $p\in M$, choose a local coordinate system $(x_1, ..., x_n)$ so that $g_{i\bar{j}}(p)=\delta_{ij}$ and $\phi_{i\bar{j}}(p)=\delta_{ij}\phi_{i\bar{i}}(p)$. Then, following Yau's computations in [Ya], we have

$$\begin{aligned}\Delta'((n+\Delta\phi)\exp\{-c\phi\}) &\\ \geqslant \exp\{-c\phi\}&\big(\Delta(f-t\phi-\theta_X-X(\phi))-n^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\big)-c\exp\{-c\phi\}n(n+\Delta\phi)\\ &+\big(c+\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\big)\exp\{-c\phi\}(n+\Delta\phi)\left(\sum_i \frac{1}{1+\phi_{i\bar{i}}}\right),\end{aligned} \tag{5.1}$$

where Δ' denotes the Lapalacian operator associated with the Kähler form ω_ϕ.

Now assume that p is the maximal point of $(n+\Delta\phi)\exp\{-c\phi\}$. Then at this point, we have $\phi_{l\bar{l}i}=c(n+\Delta\phi)\phi_i$. It follows that

$$\phi_{i\bar{l}l}X_{\bar{i}}=\phi_{l\bar{l}i}X_{\bar{i}}=c(n+\Delta\phi)\phi_i X_{\bar{i}}=c(n+\Delta\phi)X(\phi)\leqslant c(n+\Delta\phi)\sup_M|X(\phi)|. \tag{5.2}$$

Thus by Lemma 1.1, we have

$$
\begin{aligned}
\Delta(-f+t\phi+\theta_X+X(\phi)) &= -\Delta f+t\Delta\phi+(\theta_X+X(\phi))_{i\bar{i}} \\
&= -\Delta f+t\Delta\phi+(X_{\bar{k}}(g_{k\bar{i}}+\phi_{k\bar{i}}))_i \\
&= -\Delta f+t\Delta\phi+X_{\bar{k}}g_{k\bar{i}i}+\phi_{k\bar{i}i}X_{\bar{k}}+X_{\bar{k}i}(g_{k\bar{i}}+\phi_{k\bar{i}}) \\
&\leqslant C_1+(n+\Delta\phi)\big(\sup_k |X_{k\bar{k}}|+t\big)+c(n+\Delta\phi)\sup_M |X(\phi)| \\
&\leqslant C_1+C_2(n+\Delta\phi)
\end{aligned}
\tag{5.3}
$$

for some uniform constants C_1, C_2. Inserting (5.3) into (5.1), we have

$$
\begin{aligned}
&\Delta'((n+\Delta\phi)\exp\{-c\phi\}) \\
&\quad \geqslant \exp\{-c\phi\}\big(-C_1-n^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\big)-\exp\{-c\phi\}(n+\Delta\phi)(n+C_2) \\
&\qquad +\big(c+\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\big)\exp\{-c\phi\}(n+\Delta\phi)\Big(\sum_i \frac{1}{1+\phi_{i\bar{i}}}\Big) \\
&\quad \geqslant -C_3\exp\{-c\phi\}-C_4\exp\{-c\phi\}(n+\Delta\phi) \\
&\qquad +\big(c+\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\big)\exp\{-c\phi\}(n+\Delta\phi)\Big(\sum_i \frac{1}{1+\phi_{i\bar{i}}}\Big).
\end{aligned}
\tag{5.4}
$$

Choose c such that $c+\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\geqslant 1$. Then by (1.4), one can obtain

$$
\begin{aligned}
\Delta'((n+\Delta\phi)\exp\{-c\phi\}) &\geqslant -\exp\{-c\phi\}(C_3+C_4(n+\Delta\phi)) \\
&\quad +C_5\exp\Big\{-c\phi+\frac{t}{n-1}\phi\Big\}(n+\Delta\phi)^{1+1/(n-1)}.
\end{aligned}
\tag{5.5}
$$

Now applying the Maximal Principle to the function $\exp\{-c\phi\}(n+\Delta\phi)$ at p exactly as Yau did in [Ya], we can show that there is a C such that

$$
n+\Delta\phi_t \leqslant C\exp\big\{c\big(\phi_t-\inf_M \phi_t\big)\big\}\big\{1+\exp(-t\inf_M \phi)\big\}.
$$

The proposition is proved. □

Combining Proposition 5.1 with Proposition 4.2, we obtain

COROLLARY 5.1. *Let ϕ_t be any solution of* (1.4) *at t. Then there is a uniform constant C such that $n+\Delta\phi_t\leqslant C$.*

PROPOSITION 5.2. *Let ϕ_t be any solution of* (1.4) *at t. Then there is a uniform constant C such that $\|\phi_t\|_{C^3}\leqslant C$.*

Proof. Let $g'_{i\bar{j}}=g_{i\bar{j}}+\phi_{i\bar{j}}$ and

$$
S=\sum g'^{i\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}}\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}},
$$

where $(g'^{i\bar{j}})$ is the inverse of the matrix $(g'_{i\bar{j}})$.

Using Calabi's computations and Corollary 5.1 as in [Ya], one can show that $S\leqslant C$ for some uniform constant C. Consequently, Proposition 5.2 follows. □

6. Solvability of (1.4) near $t=1$

In this section, we shall prove that there are smooth solutions of (1.4) when t is sufficiently close to 1. This will be done by choosing a special Kähler–Ricci soliton of M with the help of $\mathrm{Aut}_r(M)$. Note that $\mathrm{Aut}_r(M)\subset\mathrm{Aut}(M)$ is a reductive subgroup with Lie subalgebra $\eta_r(M)$ (cf. §1).

First we consider a functional F on $\mathrm{Aut}_r(M)$ by

$$\begin{aligned} F(\varrho) &= I(\varrho^*\omega_\phi)-J(\varrho^*\omega_\phi)=I(\omega_{\phi+\psi})-J(\omega_{\phi+\psi}) \\ &= -\int_M(\phi+\psi)e^{\theta_X+X(\phi+\psi)}\omega^n_{\phi+\psi}+\int_0^1 ds\int_M(\dot{\psi}'_s)e^{\theta_X+X(\psi'_s)}\omega^n_{\psi'_s}, \end{aligned} \tag{6.1}$$

where $\{\psi'_s\}$ is a path in $\mathcal{M}_X$ from 0 to $\phi+\psi$ and $\phi+\psi\in\mathcal{M}_X$ is a smooth function defined by

$$\varrho^*\omega_\phi=\omega_\phi+\sqrt{-1}\,\partial\bar\partial\psi=\omega_g+\sqrt{-1}\,\partial\bar\partial(\phi+\psi).$$

Since $(\varrho^{-1})_*X=X$ by part (iii) of Lemma 2.2, $\varrho^*\omega_\phi$ are all Kähler–Ricci solitons with respect to X. This implies that F is a functional defined on a set of Kähler–Ricci solitons with respect to X. By (1.4), ψ also satisfies the equation

$$\left(\omega_\phi+\sqrt{-1}\,\partial\bar\partial\psi\right)^n=\omega_\phi^n e^{-X(\psi)-\psi}. \tag{6.2}$$

LEMMA 6.1. *The minimal value of F can be attained. In fact, F is proper.*

Proof. By Lemma 3.3, we observe that

$$F(\varrho)=I(\omega_{\phi+\psi})-J(\omega_{\phi+\psi})\geqslant c_1\int_M(\phi+\psi)(\omega_g^n-\omega^n_{\phi+\psi})\geqslant 0. \tag{6.3}$$

Put $G_r=\{\varrho\in\mathrm{Aut}_r(M)\mid F(\varrho)\leqslant r\}$ and $E_r=\{\psi\mid\omega_{\phi+\psi}=\varrho^*\omega_\phi,\,\varrho\in G_r\}$. Then by (6.3), we have

$$\int_M(\phi+\psi)(\omega_g^n-\omega^n_{\phi+\psi})\leqslant\frac{r}{c_1},\quad\text{for all }\psi\in E_r.$$

Hence, there are two uniform constants, c_2 and c_3, such that (by the same arguments as in the proof of Corollary 3.1)

$$\int_M\psi\omega_g^n\leqslant c_2 \tag{6.4}$$

and

$$\int_{M_-}-\psi\omega^n_{\phi+\psi}\leqslant c_3, \tag{6.5}$$

where $M_-=\{x\in M\mid\psi\leqslant 0\}$. By the Green formula associated with ω_g, we deduce from (6.4) that

$$\sup_M\psi\leqslant\int_M\psi\omega_g^n+c_4\leqslant c_2+c_4. \tag{6.6}$$

On the other hand, since ψ satisfies (6.1), using (6.4), (6.5) and the arguments in the proof of Proposition 3.1, we can prove that

$$\int_M e^{-p\psi}\omega_\phi^n \leqslant c_5$$

for some $p>1$. Then Proposition 4.2 implies that

$$\inf_M \psi \geqslant c_6(r), \quad \text{for all } \psi \in E_r. \tag{6.7}$$

Hence, by Corollary 5.1 and Proposition 5.2, we have

$$\|\psi\|_{C^3} \leqslant c_7(r), \quad \text{for all } \psi \in E_r.$$

It implies that E_r is compact in the C^2-topology, and so is G_r. In particular, the minimal value of F can be attained. □

LEMMA 6.2. *Let $\tilde{L}_\phi = \Delta_\phi + X(\cdot)$ and $L_\phi = \tilde{L}_\phi + I$. Then for any $\psi \in \mathcal{W}_X^{3,\alpha}$ and $\phi', \phi'' \in \operatorname{Ker}(L_\phi)$, we have*

(i)

$$\operatorname{Re}(\tilde{L}_\phi(\partial\phi'', \partial\psi)_{\omega_\phi}) = (\partial\bar{\partial}\psi, \partial\bar{\partial}\phi'')_{\omega_\phi} + \operatorname{Re}(\partial(\tilde{L}_\phi\psi), \partial\phi'')_{\omega_\phi}, \tag{6.8}$$

and, in particular, if $\psi = \phi' \in \operatorname{Ker}(L_\phi)$,

$$L_\phi(\operatorname{Re}(\partial\phi'', \partial\psi)_{\omega_\phi}) = (\partial\bar{\partial}\phi', \partial\bar{\partial}\phi'')_{\omega_\phi}; \tag{6.9}$$

(ii)

$$\begin{aligned} -\int_M \phi(\partial\bar{\partial}\psi, \partial\bar{\partial}\phi'')_{\omega_\phi} e^{\theta_X + X(\phi)}\omega_\phi^n \\ = \int_M (\phi'\phi'' - \operatorname{Re}(\partial\phi', \partial\phi'')_{\omega_\phi}) L_\phi\psi e^{\theta_X + X(\phi)}\omega_\phi^n. \end{aligned} \tag{6.10}$$

Proof. (i) Direct computations show that

$$X(\partial\phi'', \partial\psi) = X_{\bar{i}}(\phi''_j\psi_{\bar{j}})_i = X_{\bar{i}}\phi''_j\psi_{\bar{j}i}$$

and

$$(\partial(X(\psi)), \partial\phi'') = (X_i\psi_{\bar{i}})_j\phi''_{\bar{j}} = X_i\phi''_{\bar{j}}\psi_{\bar{i}j}.$$

Hence,

$$\operatorname{Re}(X(\partial\phi'', \partial\psi)) = \operatorname{Re}(\partial(X(\psi)), \partial\phi''). \tag{6.11}$$

On the other hand, we have

$$\Delta_\phi(\partial\psi, \partial\phi'') = (\psi_i\phi''_{\bar{i}})_{j\bar{j}} = (\partial\bar{\partial}\psi, \partial\bar{\partial}\phi'') + (\partial(\Delta_\phi\psi), \partial\phi''),$$

and it follows that

$$\operatorname{Re}(\Delta_\phi(\partial\phi'',\partial\psi))=(\partial\bar\partial\psi,\partial\bar\partial\phi'')+\operatorname{Re}((\partial(\Delta_\phi\psi),\partial\phi'')). \tag{6.12}$$

Combining (6.11) and (6.12), we obtain

$$\operatorname{Re}(\tilde L_\phi(\partial\phi'',\partial\psi))=(\partial\bar\partial\psi,\partial\bar\partial\phi'')+\operatorname{Re}(\partial(\tilde L_\phi\psi),\partial\phi'').$$

Now assume that $\psi=\phi'\in\operatorname{Ker}(L_\phi)$, i.e. $L_\phi\phi'=0$. Then

$$X(\partial\phi',\partial\phi'')=X_{\bar j}(\phi'_i\phi''_{\bar i})_j=(X(\phi''))_{\bar i}\phi'_i=(X_j\phi''_{\bar j})_{\bar i}\phi'_i=X_{j\bar i}\phi''_{\bar j}\phi'_i,$$

and similarly, we also have

$$X(\partial\phi'',\partial\phi')=X_{j\bar i}\phi'_{\bar j}\phi''_i.$$

We can deduce from these equations that

$$\begin{aligned}X(\operatorname{Re}(\partial\phi',\partial\phi''))&=\tfrac12X((\partial\phi',\partial\phi'')+(\partial\phi'',\partial\phi'))\\&=\tfrac12(X(\partial\phi',\partial\phi'')+X(\partial\phi'',\partial\phi'))\\&=\operatorname{Re}(X(\partial\phi',\partial\phi'')).\end{aligned} \tag{6.13}$$

Since $L_\phi\psi=0$ for $\psi=\phi'$, (6.8) becomes

$$\operatorname{Re}(L_\phi(\partial\phi',\partial\phi''))=(\partial\bar\partial\phi'',\partial\bar\partial\phi')=\operatorname{Re}(L_\phi(\partial\phi'',\partial\phi')). \tag{6.14}$$

Putting (6.13) and (6.14) together, we have

$$\begin{aligned}L_\phi(\operatorname{Re}(\partial\phi',\partial\phi''))&=\operatorname{Re}(\Delta_\phi(\partial\phi',\partial\phi''))+(\partial\phi',\partial\phi'')+X(\operatorname{Re}(\partial\phi',\partial\phi''))\\&=\operatorname{Re}(L_\phi(\partial\phi',\partial\phi''))=(\partial\bar\partial\phi',\partial\bar\partial\phi'').\end{aligned}$$

(ii) Let $L_\phi\psi=\xi$. Note that $\xi\in\mathcal W^{1,\alpha}_X$. Since $\tilde L_\phi\phi''=-\phi''$, we can derive

$$\begin{aligned}&\int_M(\phi'\phi''-\operatorname{Re}(\partial\phi',\partial\phi''))\xi e^{\theta_X+X(\phi)}\omega_\phi^n\\&\quad=-\int_M(\phi'\tilde L_\phi\phi''+\operatorname{Re}(\partial\phi',\partial\phi''))\xi e^{\theta_X+X(\phi)}\omega_\phi^n\\&\quad=-\int_M(\phi'\xi\tilde L_\phi\phi''+\operatorname{Re}(\partial(\phi'\xi),\partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n\\&\qquad+\int_M\phi'\operatorname{Re}((\partial\xi,\partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n\\&\quad=\int_M\phi'\operatorname{Re}((\partial\xi,\partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n\quad\text{(by (2.4))}\\&\quad=\int_M\phi'\operatorname{Re}((\partial\psi,\partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n\\&\qquad+\int_M\phi'\operatorname{Re}((\partial(\tilde L_\phi\psi),\partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n.\end{aligned}$$

Applying (6.8) in part (i), we have

$$
\begin{aligned}
&\int_M (\phi'\phi'' - \mathrm{Re}(\partial\phi', \partial\phi''))L_\phi\psi e^{\theta_X + X(\phi)}\omega_\phi^n \\
&\quad = \int_M \phi' \,\mathrm{Re}((\partial\psi, \partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n + \int_M \phi' \,\mathrm{Re}(\tilde{L}_\phi(\partial\phi'', \partial\psi))e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\qquad - \int_M \phi'(\partial\bar{\partial}\psi, \partial\bar{\partial}\phi'')e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\quad = \int_M \phi' \,\mathrm{Re}((\partial\psi, \partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n + \int_M \tilde{L}_\phi\phi' \,\mathrm{Re}((\partial\psi, \partial\phi''))e^{\theta_X+X(\phi)}\omega_\phi^n \\
&\qquad - \int_M \phi'(\partial\bar{\partial}\psi', \partial\bar{\partial}\phi'')e^{\theta_X+X(\phi)}\omega_\phi^n \quad \text{(by (2.3) in Lemma 2.2)} \\
&\quad = -\int_M \phi'(\partial\bar{\partial}\psi, \partial\bar{\partial}\phi'')e^{\theta_X+X(\phi)}\omega_\phi^n \quad (\text{by } L_\phi\phi' = 0). \qquad \square
\end{aligned}
$$

LEMMA 6.3. *Let ω_ϕ be a minimal point of F. Then*
(i) *for any $\psi \in \mathrm{Ker}(L_\phi)$,*

$$\int_M \phi\psi e^{\theta_X+X(\phi)}\omega_\phi^n = 0;$$

(ii) *the second variation formula is*

$$D^2F_{\omega_\phi}(\psi, \psi') = \int_M \left(1 + \tfrac{1}{2}\tilde{L}_\phi\phi\right)\psi\psi' e^{\theta_X+X(\phi)}\omega_\phi^n. \tag{6.15}$$

Proof. Let ϱ_s be the one-parameter subgroup generated by the real part of (the holomorphic vector field associated to) $\bar{\partial}\psi$, and write

$$\omega_{\phi_s} = \varrho_s^*\omega_\phi = \omega_\phi + \sqrt{-1}\,\partial\bar{\partial}\phi_s = \omega_g + \sqrt{-1}\,\partial\bar{\partial}(\phi + \phi_s).$$

Then one can easily see that $\dot{\phi}_s|_{s=0} = (d\phi_s/ds)|_{s=0} = \psi$ modulo constants. Hence, by (6.1), we have

$$
\begin{aligned}
0 &= \frac{d}{ds}F(\varrho_s)\Big|_{s=0} \\
&= -\frac{d}{ds}\int_M (\phi+\phi_s)e^{\theta_X+X(\phi+\phi_s)}\omega_{\phi_s}^n\Big|_{s=0} + \int_M \dot{\phi}_s|_{s=0}e^{\theta_X+X(\phi)}\omega_\phi^n \\
&= -\int_M \phi(\tilde{L}_{\omega_{\phi_s}}\dot{\phi}_s)|_{s=0}e^{\theta_X+X(\phi)}\omega_\phi^n \\
&= -\int_M \phi\tilde{L}_\phi\psi e^{\theta_X+X(\phi)}\omega_\phi^n.
\end{aligned} \tag{6.16}
$$

Notice that ω_{ϕ_s} satisfies (6.2). Differentiating (6.2) with respect to s at $s=0$, we obtain

$$L_\phi \psi = \Delta_\phi \psi + X(\psi) + \psi = 0. \tag{6.17}$$

Inserting (6.17) into (6.16), we get

$$\frac{d}{ds} F(\varrho_s)\Big|_{s=0} = \int_M \phi \psi e^{\theta_X + X(\phi)} \omega_\phi^n = 0.$$

(ii) Let $\psi, \psi' \in \operatorname{Ker}(L_\phi)$, and let ϱ_s and ϱ_t be two one-parameter subgroups generated by the real parts of (holomorphic vector fields associated to) $\bar\partial\psi$ and $\bar\partial\psi'$, respectively. Set

$$\omega_{s,t} = (\varrho_s \varrho_t)^* \omega_\phi = \omega_\phi + \sqrt{-1}\,\partial\bar\partial \phi_{s,t} = \omega_g + \sqrt{-1}\,\partial\bar\partial(\phi + \phi_{s,t}).$$

Clearly,

$$\frac{d}{ds} \phi_{s,t}\Big|_{s,t=0} = \psi$$

and

$$\frac{d}{dt} \phi_{s,t}\Big|_{s,t=0} = \psi'.$$

Since $\omega_{s,t}$ satisfies (6.2), by differentiating (6.2) with respect to s, we obtain

$$L_{\omega_{s,t}} \frac{\partial}{\partial s} \phi_{s,t} = 0. \tag{6.18}$$

Differentiating (6.18) with respect to t yields

$$L_{\omega_{s,t}} \frac{\partial^2}{\partial s \partial t} \phi_{s,t} = \left(\partial\bar\partial \frac{\partial}{\partial s} \phi_{s,t}, \partial\bar\partial \frac{\partial}{\partial t} \phi_{s,t} \right)_{\omega_{s,t}}.$$

Evaluating this at $(s,t)=(0,0)$, we get

$$L_{\omega_\phi} \frac{\partial^2}{\partial s \partial t} \phi_{s,t}\Big|_{(0,0)} = \left(\partial\bar\partial \frac{\partial}{\partial s} \psi, \partial\bar\partial \frac{\partial}{\partial t} \psi' \right)_{\omega_\phi}. \tag{6.19}$$

Combining (6.19) with (6.9) in Lemma 6.2, we obtain

$$\frac{\partial^2}{\partial s \partial t} \phi_{s,t}\Big|_{(0,0)} = \operatorname{Re}(\partial\psi, \partial\psi')_{\omega_\phi} \quad \text{modulo } \operatorname{Ker}(L_\phi). \tag{6.20}$$

Now we compute

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}F(\varrho_s\varrho_t)\Big|_{(0,0)} &= -\frac{\partial}{\partial s}\int_M(\phi+\phi_{s,t})\tilde{L}_{\omega_{s,t}}\Big(\frac{\partial}{\partial t}\phi_{s,t}\Big)e^{\theta_X+X(\phi_{s,t})}\omega^n_{s,t}\Big|_{(0,0)}\\
&= \frac{\partial}{\partial s}\int_M(\phi+\phi_{s,t})\frac{\partial}{\partial t}\phi_{s,t}e^{\theta_X+X(\phi_{s,t})}\omega^n_{\phi_{s,t}}\Big|_{(0,0)} \quad \text{(by (6.18))}\\
&= \int_M\phi\frac{\partial^2}{\partial s\partial t}\phi_{s,t}\Big|_{(0,0)}e^{\theta_X+X(\phi)}\omega^n_\phi+\int_M\psi\psi' e^{\theta_X+X(\phi)}\omega^n_\phi\\
&\quad+\int_M\phi\psi'\tilde{L}_\phi\psi e^{\theta_X+X(\phi)}\omega^n_\phi\\
&= \int_M\phi(\mathrm{Re}(\partial\psi,\partial\psi')+\psi'\tilde{L}_\phi\psi)e^{\theta_X+X(\phi)}\omega^n_\phi\\
&\quad+\int_M\psi\psi' e^{\theta_X+X(\phi)}\omega^n_\phi \quad \text{(by (6.20))}\\
&= \frac12\int_M\phi((\partial\psi',\partial\psi)+(\partial\psi,\partial\psi')+\psi'\tilde{L}_\phi\psi+\psi\tilde{L}_\phi\psi')e^{\theta_X+X(\phi)}\omega^n_\phi\\
&\quad+\int_M\psi\psi' e^{\theta_X+X(\phi)}\omega^n_\phi \quad \text{(by } L_\phi\psi=0,\ L_\phi\psi'=0)\\
&= \frac12\int_M\phi\tilde{L}_\phi(\psi\psi')e^{\theta_X+X(\phi)}\omega^n_\phi+\int_M\psi\psi' e^{\theta_X+X(\phi)}\omega^n_\phi\\
&= \int_M\big(\psi\psi'+\tfrac12\psi\psi'\tilde{L}_\phi\phi\big)e^{\theta_X+X(\phi)}\omega^n_\phi \quad \text{(by (2.4) in Lemma 2.2).} \qquad \square
\end{aligned}$$

PROPOSITION 6.1. *Let ω_ϕ be a minimum point of F. Then there is a $\delta>0$ such that* (1.4) *is solvable for any $t\in(1-\delta,1]$.*

Proof. Let $\psi=\phi_t-\phi$. Then (1.4) is equivalent to

$$\begin{cases} \big(\omega_\phi+\sqrt{-1}\,\partial\bar\partial\psi\big)^n=\omega^n_\phi\exp\{-t\psi+(1-t)\phi-X(\psi)\},\\ \omega_\phi+\sqrt{-1}\,\partial\bar\partial\psi>0.\end{cases} \tag{6.21}$$

Decompose ψ into $\theta+\hat\psi$ with $\theta\in\mathrm{Ker}(L_\phi)$ and $\hat\psi\in\mathrm{Ker}^\perp(L_\phi)$. Let P_0 be the projection from $\mathcal{M}_X$ to $\mathrm{Ker}(L_\phi)$. Consider

$$(I-P_0)\bigg(\log\frac{(\omega_\phi+\sqrt{-1}\,\partial\bar\partial(\theta+\hat\psi))^n}{\omega^n_\phi}+X(\theta+\hat\psi)\bigg)=-t\hat\psi+(1-t)\phi. \tag{6.22}$$

Since the linearization $(I-P_0)L_\phi$ along $\hat\phi$ at $(t,\theta,\hat\psi)=(1,0,0)$ is invertible in $\mathrm{Ker}^\perp(L_\phi)$, by the Implicit Function Theorem, for any small θ and t close to 1, there is a unique solution $\hat{\psi}_{t,\theta}$ such that $\theta+\hat{\psi}_{t,\theta}$ solves (6.22). Then (6.21) is reduced to the equation $\Gamma(t,\theta)=0$ on θ, where

$$\Gamma(t,\theta)=\frac{1}{1-t}P_0\biggl(\log\frac{(\omega_\phi+\sqrt{-1}\,\partial\bar{\partial}(\theta+\hat{\psi}_{t,\theta}))^n}{(\omega_\phi+\sqrt{-1}\,\partial\bar{\partial}(\theta+\hat{\psi}_{1,\theta}))^n}+X(\hat{\psi}_{t,\theta}-\hat{\psi}_{1,\theta})\biggr)-\theta. \tag{6.23}$$

Moreover, we can compute

$$\Gamma(1,\theta)=P_0(\Delta_\theta\xi_{1,\theta}+X(\xi_{1,\theta}))-\theta,$$

where $\xi_{1,\theta}=(d\psi_{t,\theta}/dt)|_{(1,\theta)}$, and Δ_θ is the Lapalacian operators associated to the Kähler forms $\omega_\phi+\sqrt{-1}\,\partial\bar{\partial}(\theta+\hat{\psi}_{1,\theta})$.

On the other hand, since $\xi_{1,\theta}\in\operatorname{Ker}^\perp(L_\phi)$, one can prove

$$\tilde{L}_\phi\xi_{1,\theta}\in\operatorname{Ker}^\perp(L_\phi). \tag{6.24}$$

It follows that

$$\tilde{L}_\phi(D_\theta\xi_{1,\theta}(\theta'))\in\operatorname{Ker}^\perp(L_\phi), \tag{6.25}$$

where $D_\theta\xi_{1,\theta}$ denotes the Fréchet derivative of $\xi_{1,\theta}$ on θ. Moreover, by differentiating (6.22) at $(t,\theta)=(1,0)$, we have

$$L_\phi\xi_{1,\theta}|_{(1,0)}=-\phi\in\operatorname{Ker}^\perp(\omega_\phi). \tag{6.26}$$

Since

$$D_\theta\hat{\psi}_{1,0}(\theta')=0,$$

by (6.25), we get the Fréchet derivative of $\Gamma(1,\theta)$ with respect to θ at $(1,0)$ as

$$D_2\Gamma(1,0)\theta'=D_\theta\Gamma(1,\theta)|_{(1,0)}(\theta')=P_0((\partial\bar{\partial}\theta',\partial\bar{\partial}\xi_{1,0}))-\theta'. \tag{6.27}$$

Next we shall compute the Hessian of $D_2\Gamma(1,0)$. By (6.27), for any $\theta',\theta''\in\operatorname{Ker}(L_\phi)$, we have

$$\begin{aligned}
&(D_2\Gamma(1,0)\theta',\theta'')\\
&=-\int_M\theta'\theta''e^{\theta_X+X(\phi)}\omega_\phi^n+\int_M\theta''(\partial\bar{\partial}\theta',\partial\bar{\partial}\xi_{1,0})e^{\theta_X+X(\phi)}\omega_\phi^n\\
&=-\int_M\theta'\theta''e^{\theta_X+X(\phi)}\omega_\phi^n\\
&\qquad-\int_M(\theta'\theta''-\operatorname{Re}(\partial\theta',\partial\theta''))L_\phi\xi_{1,0}e^{\theta_X+X(\phi)}\omega_\phi^n\quad\text{(by (6.10))}
\end{aligned}$$

$$\begin{aligned}
&= -\int_M \theta'\theta'' e^{\theta_X + X(\phi)}\omega_\phi^n \\
&\quad + \int_M (\theta'\theta'' - \mathrm{Re}(\partial\theta', \partial\theta''))\phi e^{\theta_X + X(\phi)}\omega_\phi^n \quad \text{(by (6.26))} \qquad (6.28)\\
&= -\int_M \theta'\theta'' e^{\theta_X + X(\phi)}\omega_\phi^n \\
&\quad -\frac{1}{2}\int_M \phi((\partial\theta'', \partial\theta') + (\partial\theta', \partial\theta'') + \theta'\tilde{L}_\phi\theta' + \theta'\tilde{L}_\phi\theta'') e^{\theta_X + X(\phi)}\omega_\phi^n \\
&= -\int_M \theta'\theta'' e^{\theta_X + X(\phi)}\omega_\phi^n - \frac{1}{2}\int_M \phi\tilde{L}_\phi(\theta'\theta'') e^{\theta_X + X(\phi)}\omega_\phi^n \\
&= -\int_M \theta'\theta''\left(1 + \tfrac{1}{2}\tilde{L}_\phi\phi\right) e^{\theta_X + X(\phi)}\omega_\phi^n \leqslant 0.
\end{aligned}$$

So if the derivative $D_2\Gamma(1,\sigma)$ is invertible, then the equations (6.23) are solvable for any t close to 1. This proves that there is a small number δ such that (1.4) is solvable for any $t\in(1-\delta, 1]$.

In general, we can use a trick in [BM]. For any small ε, define $\omega_\varepsilon = (1-\varepsilon)\omega + \varepsilon\omega_\phi$. Then $(1-\varepsilon)\phi$ is a solution of (1.4) at $t=1$ with ω_g replaced by ω_ε. Moreover, analogously to (6.27), we have

$$D_2\Gamma_\varepsilon(1,0) = -(1-\varepsilon)\tilde{L}_\phi - \mathrm{Id} = (1-\varepsilon)D_2\Gamma(1,0) - \varepsilon\,\mathrm{Id} < 0.$$

This implies that $D_2\Gamma_\varepsilon(1,0)$ is invertible in $\Lambda_1(\omega_\phi)$ for any $\varepsilon\neq 0$. Therefore, there are solutions ϕ_t^ε of (1.4) for t sufficiently close to 1 and with ω_ε in place of ω. Now it suffices to prove that $\phi_t^\varepsilon \to \phi_t$ as $\varepsilon\to 0$.

Notice that

$$\frac{d}{dt}(I_{\omega_\varepsilon}(\phi_t^\varepsilon) - J_{\omega_\varepsilon}(\phi_t^\varepsilon)) \geqslant 0.$$

It follows from this and Lemma 3.3 that

$$\begin{aligned}
\int_M \phi_t^\varepsilon(\omega_\varepsilon^n - \omega_{\phi_t^\varepsilon}^n) &\leqslant c_1(I_{\omega_\varepsilon}(\phi_t^\varepsilon) - J_{\omega_\varepsilon}(\phi_t^\varepsilon)) \\
&\leqslant c_1(I_{\omega_\varepsilon}((1-\varepsilon)\phi) - J_{\omega_\varepsilon}((1-\varepsilon)\phi)) \\
&\leqslant c_2\int_M (1-\varepsilon)\phi(\omega_\varepsilon^n - \omega_\phi^n) \leqslant c_3\|\phi\|_{C^0}.
\end{aligned}$$

Hence, by the same arguments as in the proof of Corollary 3.1, we can prove that

$$0 \leqslant t\sup_M \phi_t^\varepsilon \leqslant c_4 \quad \text{and} \quad \inf_M \phi_t^\varepsilon \leqslant 0, \tag{6.29}$$

and

$$\int_{M_-} -t\phi_t^\varepsilon \omega_{\phi_t^\varepsilon}^n \leqslant c_5, \tag{6.30}$$

where $M_- = \{x \in M \mid \phi_t^\varepsilon \leqslant 0\}$. Then arguing as in the proof of Propositions 3.1, 4.2, 5.2 and Corollary 5.1, we can deduce that

$$\|\phi_t^\varepsilon\|_{C^3} \leqslant c_6.$$

This shows that there is a sequence $\{\varepsilon_i\} \to 0$ and a $C^{2,\alpha}$-function ϕ_t ($\alpha \in (0,1)$) such that

$$\|\phi_t^{\varepsilon_i} - \phi_t\|_{C^{2,\alpha}} \to 0 \quad \text{as } \varepsilon_i \to 0.$$

By the standard elliptic regularity theory, ϕ_t is in fact a smooth solution of (1.4) at t. The proposition is proved. □

LEMMA 6.4. *There is a small number $\delta > 0$ such that there is a unique solution of* (1.4) *for each $t \in (0, \delta)$.*

Proof. Define a functional on $\mathcal{W}_X^{2,\alpha}$ by

$$a(\phi) = \int_0^1 \int_M \dot{\phi}_s e^{\theta_X + X(\phi_s)} \omega_{\phi_s} \wedge ds,$$

where ϕ_s is any path from 0 to ϕ on $\mathcal{W}_X^{2,\alpha}$. We define a normalized map

$$F(\phi, t)\colon \mathcal{W}_X^{2,\alpha} \times [0,1] \to \mathcal{W}_X^{0,\alpha}$$

by

$$F(\phi, t) = \log\det(g_{i\bar{j}} + \phi_{i\bar{j}}) - \log\det(g_{i\bar{j}}) - f + t\phi + \theta_X + X(\phi) - a(\phi).$$

Then by Proposition 1.1, we see that there is a unique smooth function ϕ_0 such that

$$F(\phi_0, 0) = 0 \quad \text{and} \quad a(\phi_0) = 0.$$

Furthermore, the linearization of $F(\phi, t)$ on the first variable at $(\phi_0, 0)$ is given by

$$L_{(\phi_0,0)}\psi = \Delta'_{\omega_{\phi_0}}\psi + X(\psi) - \int_M \psi e^{\theta_X + X(\phi_0)} \omega_{\phi_0}^n.$$

Since this linear operator is invertible, by the Implicit Function Theorem and the standard regularity theory for elliptic equations, there is a small number $\delta > 0$ such that there is a unique smooth function ϕ_t satisfying $F(\phi_t, t) = 0$ for any $0 \leqslant t < \delta$. Let

$$\varphi_t = \phi_t + \frac{a(\phi_t)}{t}.$$

Then φ_t is the unique solution of (1.4) for any $0 < t < \delta$. □

Proof of Theorem 0.1. Suppose that ω_1, ω_2 are Kähler–Ricci solitons with respect to X. As showed in §1, we may assume that $X \in \eta_r$, and that $\mathrm{Aut}_r(M)$ is the complexification of the isometry groups of ω_1, ω_2. Choosing $\omega_g = \omega_1$, $\omega_2 = \omega_g + \sqrt{-1}\,\partial\bar\partial\phi$, we can easily see that (1.4) becomes

$$\begin{cases} \det(g_{i\bar j} + \phi_{i\bar j}) = e^{-t\phi - X(\phi)} \det(g_{i\bar j}), \\ g_{i\bar j} + \phi_{i\bar j} > 0. \end{cases} \tag{6.31}$$

Define

$$S(\phi) = \{t \mid \text{there is a smooth path } \{\phi_s\}_{t \leqslant s \leqslant 1} \text{ such that } \phi_s \text{ solves (6.31) at } s \text{ and } \phi_1 = \phi\}.$$

Clearly, $1 \in S(\phi)$. By Propositions 4.2, 5.2 and Corollary 5.1, we have that $S(\phi)$ is a closed set in $[0,1]$.

By Lemma 6.2, there is a $\tau \in \mathrm{Aut}_r(M)$ such that the functional $I - J$ attains its minimum at τ. For simplicity, we may assume that $\tau = \mathrm{Id}$. Then by Propositions 2.1 and 6.1, we see that $S(\phi)$ is open. Hence, $S(\phi) = [0,1]$, and consequently, there is a smooth family $\{\phi_t\}$ such that ϕ_t solves (6.31) at t and $\phi_1 = \phi$. By Lemma 6.4, however, there is a small number $\delta > 0$ such that equation (6.31) has a unique solution 0, for any $0 < t < \delta$. Then it follows from Proposition 2.1 that $\phi_t = 0$ for all $0 < t < 1$, and consequently $\phi = \phi_1 = 0$. The theorem is proved. □

7. Appendix

In this Appendix, we prove a structure theorem for the automorphism group $\mathrm{Aut}^\circ(M)$ of any compact Kähler manifold M admitting Kähler–Ricci solitons. This structure theorem was used in the proof of Lemma 2.2 and in §6.

Let ω_ϕ be a Kähler–Ricci soliton of M. We will adopt the notations in §2. As before, we have the following Hermitian inner product on the space $C^\infty(M, \mathbf{C})$ of complex-valued functions:

$$(f, g) = \int_M f\bar g e^{\theta_X + X(\phi)} \omega_\phi^n, \quad \text{for } f, g \in C^\infty(M, \mathbf{C}).$$

Let L and $\bar L$ be two linear complex operators defined by

$$Lf = \Delta f + f + (\bar\partial f, \bar\partial(\theta_X + X(\phi)))_{\omega_\phi}, \quad \text{for } f \in C^\infty(M, \mathbf{C}),$$

and

$$\bar L f = \Delta f + f + (\bar\partial(\theta_X + X(\phi)), \bar\partial(\bar f))_{\omega_\phi}, \quad \text{for } f \in C^\infty(M, \mathbf{C}),$$

where Δ is the Laplacian of ω_ϕ. One can check easily that

$$\bar{L}f = \overline{L\bar{f}}, \quad \text{for any } f \in C^\infty(M, \mathbf{C}).$$

Using the same arguments as in the proof of Lemma 2.2, we can show (cf. Theorem 2.4.3 in [F2])

LEMMA A.1. (1) *Both L and $\bar{L}$ are self-adjoint elliptic operators with respect to the inner product* $(\cdot,\cdot)$.

(2) $\mathrm{Ker}(L)$ *is isomorphic to $\eta(M)$ through the linear map $\Phi: f \to \uparrow\bar{\partial}f$, where $\eta(M)$ denotes the Lie algebra of holomorphic vector fields, and $\uparrow\partial f$ is the vector field of type* $(1,0)$ *defined by $\omega(\uparrow\bar{\partial}f, Y) = \bar{\partial}f(\bar{Y})$ for any vector field Y of type* $(1,0)$.

Let E_λ be the eigenspace of $\bar{L}$ with eigenvalue λ. Then we define subspaces of $\mathrm{Ker}(L)$ as follows:

$$E_0' = \{f \in \mathrm{Ker}(L) \cap \mathrm{Ker}(\bar{L}) \mid f \text{ is a real-valued function}\},$$
$$E_0'' = \{f \in \mathrm{Ker}(L) \cap \mathrm{Ker}(\bar{L}) \mid f \text{ is a purely imaginary-valued function}\},$$
$$E_\lambda = \{f \mid f \in \mathrm{Ker}(L) \cap \Lambda_\lambda\}.$$

Put

$$\eta_0' = \{Y = \uparrow\bar{\partial}f \mid f \in E_0'\},$$
$$\eta_0'' = \{Y = \uparrow\bar{\partial}f \mid f \in E_0''\},$$
$$\eta_\lambda = \{Y = \uparrow\bar{\partial}f \mid f \in E_\lambda\}.$$

Then by Lemma A.1, the above sets are all subspaces of $\eta(M)$. Moreover, we have the following Cartan decomposition of $\eta(M)$.

LEMMA A.2. (1) $\eta(M) = \eta_0' + \eta_0'' + \sum_{\lambda>0} \eta_\lambda$.

(2) *For each* $Y \in \eta_\lambda$, $[X, Y] = \lambda Y$.

(3) $[\eta_0'', \eta_0''] \subset \eta_0''$, $[\eta_0', \eta_0'] \subset \eta_0''$ *and* $[\eta_0', \eta_0''] \subset \eta_0'$.

(4) $[\eta_\lambda, \eta_\mu] \subset \eta_{\lambda+\mu}$.

Proof. Let $Y \in \eta(M)$. Then by Lemma A.1, there is a unique smooth complex-valued function $f \in \mathrm{Ker}(L)$ such that

$$i_Y \omega_\phi = \sqrt{-1}\,\bar{\partial}f.$$

Moreover,

$$\bar{L}f = (\bar{L} - L)f = (\theta_X + X(\phi))_{\bar{j}} f_j - f_{\bar{j}}(\theta_X + X(\phi))_j.$$

Using the identities

$$(\theta_X + X(\phi))_{\bar{i}\bar{j}} = 0 \quad \text{and} \quad f_{\bar{i}\bar{j}} = 0,$$

we deduce

$$\begin{aligned}(\bar{L}f)_{\bar{i}} &= ((\theta_X + X(\phi))_{\bar{j}} f_j)_{\bar{i}} - (f_{\bar{j}}(\theta_X + X(\phi))_j)_{\bar{i}} \\ &= (\theta_X + X(\phi))_{\bar{j}} f_{j\bar{i}} - f_{\bar{j}}(\theta_X + X(\phi))_{j\bar{i}} = [X, Y]_{,\bar{i}}.\end{aligned} \tag{A.1}$$

This shows that

$$\uparrow\bar{\partial}(\bar{L}f) = [X, Y]$$

is a holomorphic vector field. By Lemma A.1, we obtain

$$\bar{L}f \in \mathrm{Ker}(L), \quad \text{for all } f \in \mathrm{Ker}(L),$$

i.e. $\mathrm{Ker}(L)$ is an invariant subspace of the operator $\bar{L}$. So again by Lemma A.1, we see that the restriction of $\bar{L}$ on $\mathrm{Ker}(L)$ has only finite, nonnegative eigenvalues λ such that

$$\mathrm{Ker}(L) = E_0 \oplus \sum_{\lambda>0} E_\lambda,$$

and consequently,

$$\eta(M) = \eta_0 + \sum_{\lambda>0} \eta_\lambda. \tag{A.2}$$

Let $Y \in \eta_0$. Then there is a smooth complex-valued function $f \in \mathrm{Ker}(L) \cap \mathrm{Ker}(\bar{L})$, i.e. f satisfies

$$Lf = \bar{L}f = 0.$$

It follows that

$$L\bar{f} = \overline{\bar{L}f} = 0,$$

and consequently,

$$L(\mathrm{Re}(f)) = L(\mathrm{Im}(f)) = 0.$$

This shows that E_0 can be decomposed into $E_0 = E_0' \oplus E_0''$, and so we have

$$\eta_0 = \eta_0' + \eta_0''. \tag{A.3}$$

Combining (A.2) and (A.3), we get

$$\eta(M) = \eta_0' + \eta_0'' + \sum_{\lambda>0} \eta_\lambda.$$

(2) Let $Y \in \eta(M)$. Then by part (1), there is a unique smooth complex-valued function $f \in E_\lambda$ such that $Y = \uparrow\bar{\partial} f$. Thus by (A.1), we have

$$\lambda Y_{,\bar{i}} = \lambda f_{\bar{i}} = (\bar{L}f)_{\bar{i}} = [X, Y]_{,\bar{i}}.$$

This shows that

$$[X,Y]=\lambda Y, \quad \text{for all } Y\in\eta_\lambda.$$

(3) can be proved as follows. Let $Y\in\eta_0''$. Then there is a smooth purely imaginary function f such that $Y=\uparrow\bar{\partial}f$. It follows that

$$L_Y\omega_\phi=\partial i_Y\omega_\phi=\sqrt{-1}\,\partial\bar{\partial}f.$$

This shows that $L_Y\omega_\phi$ is a purely imaginary two-form, and consequently, $\mathrm{Re}(Y)$ is a Killing vector field. Thus, for any $Y_1,Y_2\in\eta_0''$, $[\mathrm{Re}(Y_1),\mathrm{Re}(Y_2)]$ is still a Killing vector field. This proves that $[Y_1,Y_2]\in\eta_0''$. Similarly, we can prove that

$$[\eta_0',\eta_0']\subset\eta_0'' \quad \text{and} \quad [\eta_0',\eta_0'']\subset\eta_0'.$$

(4) is a direct corollary of the Jacobi identity and part (2). □

THEOREM A. *If M is a compact Kähler manifold having a Kähler–Ricci soliton, then the maximal compact, connected subgroup of* $\mathrm{Aut}^\circ(M)$ *is conjugate to the identity component of the isometry group of M.*

Proof. We use the same arguments as in the proof of Theorem 3 in [Cal]. First by parts (1) and (3) of Lemma A.2, η_0'' is the Lie algebra of the holomorphic isometry group $H(M,\omega_\phi)$ of ω_ϕ. Let G be a maximal compact, connected subgroup of $\mathrm{Aut}^\circ(M)$ with Lie algebra $\tilde{\eta}$. Suppose that $G\supsetneq H_0(M,\omega_0)$, where $H_0(M,\omega_\phi)$ denotes the identity component of $H(M,\omega_\phi)$. Then there is a holomorphic vector field $Y\in\tilde{\eta}$ such that $Y\notin\eta_0''$. Let

$$Y=Y_0'+Y_0''+\sum_{\lambda>0}Y_\lambda,$$

where $Y_0'\in\eta_0'$, $Y_0''\in\eta_0''$ and $Y_\lambda\in\eta_\lambda$. Since $Z_0=\sqrt{-1}\,X\in\eta_0''$, Z_0 generates a one-parameter group of isometries of M. It follows that

$$\mathrm{Ad}(\exp\{tZ_0\})(Y)=Y_0'+Y_0''+\sum_{\lambda>0}e^{t\lambda\sqrt{-1}}Y_\lambda\in\tilde{\eta}.$$

By taking appropriate linear combinations, we have

$$Y_0'+Y_0''\in\tilde{\eta}$$

and

$$Y_\lambda\in\tilde{\eta}, \quad \text{for any } \lambda>0.$$

If $\sum_{\lambda>0} Y_\lambda \neq 0$, however, then by part (2) of Lemma A.2, Z_0 and Y_λ would generate a solvable, nonabelian Lie subalgebra of $\tilde{\eta}$, contradicting that $\tilde{\eta}$ generates a compact group. So we have

$$\sum_{\lambda>0} Y_\lambda = 0 \quad \text{and} \quad Y = Y_0' + Y_0''.$$

On the other hand, by the assumption $Y \notin \eta_0''$, Y_0' must be a nontrivial element of $\tilde{\eta}$. Let u be a smooth real-valued function of M such that $Y_0' = \uparrow\bar{\partial}u$. Then

$$\frac{d}{dt}(u(\exp\{tY_0'\})) = |\bar{\partial}u|^2(\exp\{tY_0'\}) = |Y_0'|^2.$$

Therefore, the one-parameter subgroup $\{\exp\{tY_0'\}\}$ cannot be contained in any compact group, which contradicts the fact that G is compact. The contradiction shows that $H_0(M, \omega_g) = G$. □

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