Borel selectors for upper semi-continuous set-valued maps

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§ 1. Introduction

A set-valued map F from a topological space X to a topological space Y is said to be *upper semi-continuous*, if the set $\{x: F(x) \cap H \neq \emptyset\}$ is closed in X, whenever H is a closed set in Y. A point-valued function f is said to be a *selector* for such a set-valued map F, if $f(x) \in F(x)$, for all x in X. The function f from X to Y is said to be a Borel measurable function of the *first Borel class* if $f^{-1}(H)$ is a $\mathscr{G}_\delta$-set in X, whenever H is a closed set in Y. Similarly, f is said to be a Borel measureable function of the *second Borel class* if $f^{-1}(H)$ is an $\mathscr{F}_{\sigma\delta}$-set in X, whenever H is a closed set in Y. In [18, Theorems 2 and 3] we prove that, if X and Y are metric spaces and F is an upper semi-continuous set-valued map from X to Y, taking only non-empty values, then F always has a Borel measureable selector of the second Borel class, further, if F only takes non-empty complete values in Y, then F always has a Borel measurable selector of the first Borel class.

Some of the more interesting upper semi-continuous set-valued maps are defined on a subset of a Banach space X and take their values in a Banach space Y with its weak topology, or in a dual Banach space Y^* with its weak-star topology. In [19, Theorem 2] we prove that if F is a weak upper semi-continuous set-valued map, defined on a metric space X, and taking only non-empty weakly compact values, contained in a fixed weakly σ-compact set of a Banach space Y, then F has a weak Borel measurable selector of the first Borel class, which is also a norm Borel measurable selector of the second Borel class. Similarly, see the introduction to [19], if F is a weak-star upper semi-continuous set-valued map, defined on a metric space X, and taking only non-empty, weak-star closed values in the dual space Y^* of a weakly compactly generated

Banach space Y, then F has a weak-star Borel measurable selector of the first Borel class, which is also a norm Borel measurable selector of the second Borel class.

It will be convenient to say that a function f from a metric space X to a metric space Y is of the *first Baire class*, if f is the point-wise limit of a sequence of continuous functions from X to Y. Note that such a function of the first Baire class is automatically a Borel measurable function of the first Borel class, see, for example, [23, p. 386].

Our main aim in this paper is to prove that, if F is a weak (or weak-star) upper semi-continuous set-valued map from a metric space X to a Banach space Y (or to a dual Banach space Y^*) with the Radon-Nikodým property, and F takes only non-empty weakly compact (or weak-star compact) values, then F has a norm Borel measurable selector f of the first Baire class and the set of points of norm discontinuity of f form an $\mathscr{F}_\sigma$-set of the first category in X. The existence of these selectors adds a new facet to the theories of maximal monotone maps, of subdifferential maps, of attainment maps and of metric projections.

Before we introduce our main selection methods we give a brief discussion of the results that can be obtained by use of the well-known nearest point selection, see, for example, [5, pp. 28–29] and [35, p. 209] for previous applications of this method. When the set-valued function F takes non-empty, convex and weakly compact (or weak-star closed) values in a Banach space (or a dual Banach space) with a strictly convex norm, i.e. a norm for which $\|x\|=\|y\|=\|\frac{1}{2}x+\frac{1}{2}y\|$ implies that $x=y$, one can make a selection by taking $f(x)$ to be the unique point $F(x)$ that is nearest to the origin, in norm. In § 2, using this method, we obtain results that have the following consequences.

THEOREM 2. *Let X be a metric space and let Y be a Banach space with an equivalent strictly convex norm. Let F be an upper semi-continuous set-valued map from X to Y with its weak topology. Suppose that F only takes values that are non-empty, convex and weakly compact. Then, using the weak topology on Y, the set-valued map F has a Borel measurable selector f of the first Borel class. Further, the set of points of X, where f is weakly discontinuous, is a set of the first category in X.*

THEOREM 3. *Let X be a metric space and let Y be a Banach space with an equivalent norm whose dual norm on Y^* is strictly convex. Let F be an upper semi-continuous set-valued map from X to Y^* with its weak-star topology. Suppose that F only takes values that are non-empty, convex and weak-star closed. Then, using the weak-star topology on Y, the set-valued map F has a Borel measurable selector f of the first Borel class. Further, the set of points of X, where f is weak-star discontinuous, is a set of the first category in X.*

In subsequent theorems, where we impose rather different conditions on our spaces, we obtain selectors that are norm Borel measurable. However, the conditions in Theorems 2 and 3 are not strong enough to ensure the existence of norm Borel measurable selectors. Theorem 10 below shows that the general existence of norm Borel measurable selectors in a dual Banach space ensures that this dual space has the Radon-Nikodým property. As l_∞ is an example of a dual Banach space, with an equivalent strictly convex dual norm [8], there will be weak-star Borel measurable selectors in l_∞; however, as l_∞ does not enjoy the Radon-Nikodým property, there can, in general, be no norm Borel measurable selectors in this space.

To study the cases, when the values of F are not assumed to be convex, or when there is no equivalent strictly convex norm, we need to use a more sophisticated selection technique. We develop this technique in § 3; it needs, as its starting point, a condition that the space should be "fragmentable" in a sense that we shall define. In the case of a Banach space it will be enough if the space has the point of continuity property. The Banach space Y is said to have the *point of continuity property* if, given any non-empty bounded weakly closed set F in Y there is a point of F at which the restriction to F of the identity map from Y with its weak topology to Y with its norm topology is continuous. A Banach space with the Radon-Nikodým property always has the point of continuity property. The dual Banach space Y^* has the *weak-star point of continuity property* if, given any non-empty bounded weak-star closed set F in Y^*, there is a point of F at which the restriction of F of the identity map from Y^* with its weak-star topology to Y^* with its norm topology is continuous. However, it follows from the work of Namioka and Phelps [25] and Stegall [33] (see also Dulst and Namioka [36]) that Y^* has this weak-star point of continuity property, if, and only if, Y^* has the Radon-Nikodým property.

We prove two main theorems.

THEOREM 7. *Let X be a metric space and let Y be a Banach space with the point of continuity property. Let f be an upper semi-continuous map from X to Y with its weak topology. Suppose that F takes only non-empty, weakly compact values. Then, using the norm topology on Y, the set-valued map F has a Borel measurable selector f of the first Baire class. Further, the set of points of X, where f is norm-discontinuous, is an $\mathcal{F}_\sigma$-set of the first category in X.*

THEOREM 8. *Let X be a metric space and let Y^* be the dual space to a Banach space Y. Suppose that Y^* has the Radon-Nikodým property. Let F be an upper semi-continuous set-valued map from X to Y^* with its weak-star topology. Suppose, further,* *that F takes only values that are non-empty and weak-star closed. Then, using the norm topology on Y^*, the set-valued map F has a Borel measurable selector f of the first Baire class. Further, the set of points of X, where f is norm-discontinuous, is an $\mathcal{F}_\sigma$-set of the first category in X.*

We remark that the conclusion in Theorem 8 holds for all such F only when Y^* has the Radon-Nikodým property, see Theorem 10 below.

A set-valued map F from a Banach space X to its dual space X^* is said to be a *monotone map*, if

$$\langle x_2 - x_1, x_2^* - x_1^* \rangle \geq 0,$$

for all choices of x_1, x_1^*, x_2, x_2^* with

$$x_1^* \in F(x_1) \quad \text{and} \quad x_2^* \in F(x_2).$$

A set-valued map F from X to X^* is said to be a *maximal monotone map*, if it is a monotone map and it has a graph $\bigcup\{\{x\} \times F(x) : x \in X\}$ that is a proper subset of the graph of no monotone map from X to X^*. It follows immediately, by use of Zorn's lemma, that the graph of a monotone map from X to X^* is always contained in the graph of some maximal monotone map from X to X^*. Maximal monotone maps have been extensively studied, see, for example, [5], [6]. We summarize briefly some of the results. For each x in X, the set $F(x)$ is convex and weak-star closed in X^*. Let B^* denote the unit ball in X^* and let F_R, for $R>0$, be the set-valued map defined by

$$F_R(x) = F(x) \cap (RB^*), \quad x \in X.$$

Then, for each $R>0$, F_R is a weak-star upper semi-continuous map from X to X^*. Let D be the domain of F, that is, the set of all x in X for which $F(x)$ is non-empty; let D_0 be the interior of the convex hull of D. Now Rockafellar [31] shows that, if D_0 is non-empty, then $D_0 \subset D$ and D_0 is a convex open set whose closure includes the $\mathcal{F}_\sigma$-set D. He also shows that F is locally bounded at each point of D_0. Kenderov [20] and Robert [29] show that F is weak-star upper semi-continuous at each point of D_0. Using Theorems 3 and 8, we obtain the following result for such maps.

THEOREM 9. *Let X be a Banach space with dual space X^*. Let F be a maximal monotone map from X to X^*. Let D be the domain of F and suppose that the interior D_0 of D is non-empty.*

(a) *If X has an equivalent norm whose dual norm on X^* is strictly convex, then F*

has a weak-star Borel measurable selector f of the first Borel class on D. The set J of points of D_0 where f is weak-star continuous coincides with the set of points of D_0 where F is point-valued. Further J contains a dense $\mathscr{G}_\delta$-subset of D_0.

(b) *If X^* has the Radon-Nikodým property, then F has a norm Borel measurable selector f of the first Baire class on D. The set U of points of D_0, at which f is norm continuous, coincides with the set of points of D_0, at which F is point-valued and norm upper semi-continuous. Further U is a dense $\mathscr{G}_\delta$-subset of D_0.*

Note that the sets J and U will not depend on the choice of the selector f.

Subdifferential maps are special maximal monotone maps, see Rockafellar [30, 32] for a characterization. Consider a lower semi-continuous convex function f defined on a Banach space X and taking values in the extended real line $\mathbf{R} \cup \{+\infty\}$, and taking a finite value for some x in X. The subdifferential $D_x f$ of f at a point x of X is defined to be the set of all elements d^* of X^* satisfying the condition

$$f(x)+\langle d^*, y\rangle \leq f(x+y),$$

for all y in X. Let D be the set of all x for which $f(x)$ is finite. Using the Hahn-Banach theorem it is easy to verify that D is also the set of points x of X for which $D_x f$ is non-empty. As the subdifferential map is necessarily a maximal monotone map, Theorem 9 applies to such maps. In particular, if X has an equivalent norm, such that the corresponding dual norm on X^* is strictly convex, the map $D_x f$ has a weak-star Borel measurable selector d of the first Borel class, the set J of interior points of D where d is weak-star continuous coincides with the set of interior points of D where $D_x f$ is point-valued and further J contains a $\mathscr{G}_\delta$-set dense in D. Now a Banach space X is called a *weak Asplund space* (see [27]) if every continuous convex function on an open convex subset of X is Gâteaux differentiable on a dense $\mathscr{G}_\delta$-set of its domain. Here, the function f is Gâteaux differentiable at the point x, if $D_x f$ reduces to a single point. Thus we have a proof, essentially the proof of Kenderov [38], of Asplund's result that X is a weak Asplund space (he used a different terminology), if X has an equivalent norm, for which the dual norm is strictly convex. The new components of part (a) of Theorem 9 are those concerning the existence, class and continuity properties of the selectors. The proof is quite similar to the work of Kenderov. Indeed, looking at [38] with hindsight, it is remarkable how close Kenderov came to using a selector without actually doing so.

We remark that it follows from Theorem 10, below, that it is not, in general, possible to obtain norm Borel measurable selectors in part (a) of Theorem 9.

In the case when X^* has the Radon-Nikodým property, the map $D_x f$ has a norm

Borel measurable selector of the first Baire class on D, and $D_x f$ is point-valued and is norm upper semi-continuous, relative to the interior D_0 of D, at each point of a $\mathscr{G}_\delta$-set dense in D. A Banach space X is called an *Asplund space* (see [25]) if every continuous convex function on an open convex subset of X is Fréchet differentiable on a set containing a dense $\mathscr{G}_\delta$-set of its domain. Theorem 9 does provide an alternative proof of the result, first proved by Stegall [33], that X is an Asplund space when X^* has the Radon-Nikodým property; however a much simpler proof of Stegall's result has been obtained by Namioka [9, p. 213]. Again the new contributions of part (b) of Theorem 9 are those concerning the existence, class and continuity properties of the selectors (see Asplund [1], Kenderov [21] and Robert [39]).

The next theorem includes a strengthened converse to the result, mentioned in the last paragraph, concerning the existence of a norm Borel measurable selector for a subdifferential map when X^* has the Radon-Nikodým property; it also provides a converse to Theorem 8.

THEOREM 10. *Let Y be a Banach space with dual space Y^*.*

(a) *If the subdifferential map $D_y f$ corresponding to each continuous convex function f defined on Y has a norm Borel measurable selector on a dense $\mathscr{G}_\delta$-set in Y, then Y^* has the Radon-Nikodým property.*

If Y^* has the Radon-Nikodým property, then

(b) *each upper semi-continuous map F, from a metric space X to Y^*, with its weak-star topology, with each value non-empty and weak-star closed, has a norm Borel measurable selector of the first Baire class, and also, more particularly,*

(c) *the subdifferential map $D_y f$ corresponding to each lower semi-continuous convex function defined on Y has a norm Borel measurable selector of the first Baire class on the domain of f.*

Before discussing the next type of upper semi-continuous map, it will be best to recall the relationship between the Radon-Nikodým property and the concept of dentability. Rieffel [28] says that a set D in a Banach space Y is *dentable*, if, for each $\varepsilon>0$, it is possible to find a point y_ε in D that is not in the closed convex hull of

$$D\setminus\{y: \|y-y_\varepsilon\|<\varepsilon\}.$$

It will be convenient to say that D is everywhere dentable if each non-empty bounded subset of D is dentable. Rieffel [28] proves that, if Y is everywhere dentable, then Y has the Radon-Nikodým property. The converse was discovered by Huff [14] and also by

Davis and Phelps [7], using work of Maynard [24]. In our work the Radon-Nikodým property enters *via* this concept of dentability.

Let K be any non-empty, closed, bounded set in a Banach space X. For each x^* in the dual space X^*, let $F(x^*)$ denote the set of x in K, for which

$$\langle x, x^* \rangle = \sup\{\langle k, x^* \rangle : k \in K\}.$$

Note that this definition ensures that $F(0)=K$, and, in general, $F(x^*)$ is the set of points of K at which $\langle x, x^* \rangle$ attains its supremum over K. We call F the attainment map of K. It is easy to verify that F is a monotone map from X^* to X^{**}. However, F will not, in general, be a maximal monotone map, and in some cases $F(x^*)$ will be empty except for x^* in a rather small subset of X^*. Of course, if K is weakly compact, then $F(x^*)$ is non-empty for each x^* in X^*. Conversely, by a theorem of James [16], if K is weakly closed and $F(x^*)$ is non-empty for each x^* in X^*, then K is necessarily weakly compact. When X has the Radon-Nikodým property, and K is convex, a result of Phelps [26] shows that $F(x^*)$ is non-empty, and consists of a single point, for all x^* in a dense $\mathcal{G}_\delta$-set in X^*. A refinement of Phelps' result due to Bourgain [4] obtains the same conclusion on the assumption that K is convex and everywhere dentable. In the particular case when K is compact and convex, the attainment map of K coincides with the subdifferential map of the convex function f defined on X^* by

$$f(x^*) = \sup\{\langle x^*, k \rangle : k \in K\},$$

and so is a maximal monotone map from X^* to X^{**} that happens to take all its values in X.

THEOREM 11. *Let F be the attainment map of a bounded subset K of a Banach space X. Let D^* be the set of points x^* of X^* with $F(x^*) \neq \emptyset$.*

(a) *If K is weakly compact, then $D^*=X^*$, and F has a norm Borel measurable selector of the first Baire class. Further, the set U^* of norm continuity points of f is a $\mathcal{G}_\delta$-set that is dense in X^*.*

(b) *Suppose that K is everywhere dentable and that F is a weakly upper semi-continuous set-valued map that takes only non-empty weakly compact values on a $\mathcal{G}_\delta$-set D contained in and dense in an open set G in X^*. Then F has a norm Borel measurable selector f of the first Baire class. Further, the set U^* of norm continuity points of f, relative to D, is a $\mathcal{G}_\delta$-set that is dense in D.*

In each case, U^* is also the set of points at which F is point-valued and norm upper semi-continuous, relative to D in case (b).

We remark that, when K is convex, the linear functionals x^*, that strongly expose points of K, are just the points of X^* at which F is point-valued, and norm upper semi-continuous.

Let K be a non-empty set in a Banach space X. For each x in X, write

$$\varrho(x) = \inf\{\|k-x\|: k \in K\},$$

and

$$F(x) = \{k: \|k-x\| = \varrho(x) \text{ and } k \in K\}.$$

The set-valued map F is called the nearest point map of K or the metric projection onto K. This map has been much studied. In particular, if K is weakly compact in X, then F is a weak upper semi-continuous set-valued map with non-empty weakly compact values. Further, in various circumstances, there will be a dense $\mathscr{G}_\delta$-set in X on which F is a point-valued and norm continuous (see Kenderov [22]). Our methods enable us to give a simple proof for the following refinement of the known results.

THEOREM 12. *Let X be a Banach space, let K be a set in X that is everywhere dentable, and let F be the metric projection of X onto K. Suppose that the restriction of F, to a $\mathscr{G}_\delta$-set U_0 dense in X, is weakly upper semi-continuous and takes only non-empty weakly compact values. Then F has a norm Borel measurable selector f of the first Baire class on U_0 and f is norm continuous on U_0 at each point of a $\mathscr{G}_\delta$-set U_1 that is contained in U_0 and dense in X. If, in addition, X is strictly convex, then F is point-valued and norm upper semi-continuous with respect to U_0 at each point of U_1.*

We have announced some of these results in [37].

§ 2. Selection in Banach spaces with strictly convex norms

Although our main objective in this section is to prove the existence of Borel measurable selectors for set-valued functions with convex values in a Banach space with a strictly convex norm we first study a more general situation.

Let V be a real vector space. A set K in V is said to be *linearly bounded*, if each half-ray of the form

$$\{\lambda x: \lambda \geqslant 0\},$$

with x a point of V, meets K in a bounded set, regarded as a subset of this real half-line. A set K in V is said to be *absorbent*, if for each point x of V, there is a $\lambda(x)>0$ with $\lambda(x)\,x$

in K. We shall say that a convex set K in V is *linearly strictly convex*, if, whenever h, k are distinct points of K, the translated set $K-\frac{1}{2}(h+k)$ is absorbent in V.

With each linearly bounded absorbent convex set B we associate a real-valued gauge function $\varphi=\varphi_B$ of Minkowski type, by taking

$$\varphi(x)=\inf\{\lambda: \lambda>0 \text{ and } x\in\lambda B\},$$

for each x in X. It is an easy exercise in two-dimensional geometry to verify that the linearly bounded absorbent convex set B is linearly strictly convex, if, and only if, φ_B is strictly convex on V.

Note that all these concepts are real vector space concepts, involving no topology. Note also that, if K is a convex set in a Banach space X, and K is strictly convex, in the usual way, then K is clearly linearly strictly convex.

If a real vector space V contains a linearly bounded, linearly strictly convex and absorbent set B, a natural way to choose a point from a given convex set, is to choose a point at the least possible φ_B-distance from the origin, if there is a unique choice for such a point. The following general selection theorem is based on this idea.

THEOREM 1. *Let X be a metric space and let Z be a locally convex topological vector space. Suppose that Z contains a closed, linearly bounded, linearly strictly convex and absorbent set B. Let F be an upper semi-continuous set-valued map from X to Z with non-empty convex values. Suppose, further, that the set*

$$F(x)\cap\lambda B$$

is compact for each x in X and each $\lambda>0$. Then, for each x in X, $F(x)$ has a unique point $f(x)$ at the least possible φ-distance from 0. Further, f is a Borel measurable selector for F and is of the first Borel class. The real-valued function $\varphi\circ f$ is lower semi-continuous on X. The set of points of X, where f is discontinuous in the locally convex topology, is of the first category in X.

Proof. Write

$$B(0)=\emptyset,\quad B(\varrho)=\varrho B,\quad \text{for } \varrho>0.$$

For all pairs r, n of positive integers, let

$$X(n,r)=\{x: F(x)\cap B(r2^{-n})\neq\emptyset\}\setminus\{x: F(x)\cap B((r-1)2^{-n})\neq\emptyset\}.$$

As F is upper semi-continuous and B is closed convex and absorbent, for each $n\geq 1$, the family

$$\{X(n, r): r \geqslant 1\}$$

is a partition of X into sets that are differences between closed sets. Define set-valued functions $H^{(n)}$ on X, for $n \geqslant 1$, by

$$H^{(n)}(x) = F(x) \cap B(r2^{-n}), \quad \text{for } x \in X(n, r),$$

and take

$$H(x) = \bigcap_{n=1}^{\infty} H^{(n)}(x).$$

Note that, for each $n \geqslant 1$, and for each x in X, the set $H^{(n)}(x)$ is just the first set of the sequence

$$F(x) \cap B(r2^{-n}), \quad r = 1, 2, \ldots,$$

that is non-empty; there is always a first such set as $F(x)$ is non-empty and B is absorbent. If $F(x) \cap B(r2^{-n})$ is the first non-empty set of this sequence, the first non-empty set of the next sequence

$$F(x) \cap B(s2^{-n-1}), \quad s = 1, 2, \ldots,$$

is either

$$F(x) \cap B((2r-1)\,2^{-n-1}),$$

or

$$F(x) \cap B(2r2^{-n-1}).$$

Thus

$$H^{(n)}(x) = F(x) \cap \varrho_n B,$$

with $\varrho_1, \varrho_2, \ldots$ a non-increasing sequence of real numbers, with limit $\varrho(x)$, say. As $H^{(n)}(x)$, $n = 1, 2, \ldots$, is a decreasing sequence of non-empty compact sets, the set $H(x)$ is non-empty and takes the form

$$H(x) = F(x) \cap \varrho(x) B.$$

As B is linearly bounded, we have $H(x) = \{0\}$ when $\varrho(x) = 0$.

Suppose that, for some x in X, the set $H(x)$ contains two distinct points h and k, say. Then $\frac{1}{2}(h+k) \in F(x)$, and $\varrho(x) > 0$. As $\varrho(x) B$ is linearly strictly convex, the set

$$\varrho(x) B - \tfrac{1}{2}(h+k)$$

is absorbent. Hence, for some ε with $0<\varepsilon<1$, the point

$$(1+\varepsilon)\tfrac{1}{2}(h+k)$$

belongs to $\varrho(x)B$, and the point $\frac{1}{2}(h+k)$ belongs to

$$F(x)\cap(1+\varepsilon)^{-1}\varrho(x)B.$$

This contradicts the formula

$$\varrho(x)=\inf\{\sigma>0;\ F(x)\cap B(\sigma)\neq\emptyset\}.$$

We conclude that $H(x)$ reduces to a single point, say $h(x)$, for each x in X.

Now let J be any closed set in Z. Then $H(x)\cap J=\emptyset$, if, and only if,

$$J\cap\bigcap_{n=1}^{\infty}H^{(n)}(x)=\emptyset.$$

As $H^{(1)}(x)$, $H^{(2)}(x)$, ... is a decreasing sequence of non-empty compact sets, this holds, if, and only if,

$$J\cap H^{(n)}(x)=\emptyset,$$

for some $n\geq 1$, i.e., if, and only if, for some $n\geq 1$, and some $r\geq 1$,

$$x\in X(n,r)\quad\text{and}\quad F(x)\cap[J\cap B(r2^{-n})]=\emptyset. \tag{1}$$

As F is upper semi-continuous, and J and $B(r2^{-n})$ are closed, the set of x satisfying (1) is a relatively open subset of $X(n,r)$ and so is the difference between two closed sets. Thus the set of x with $H(x)\cap J=\emptyset$ is an $\mathscr{F}_\sigma$-set in X and the set of x with $h(x)$ in J is a $\mathscr{G}_\delta$-set in X. Thus h is the required Borel measurable selector of the first Borel class.

For each x in X, the φ-distance $\varphi(h(x))$ of $h(x)$ from the origin is just the function

$$\varrho(x)=\lim_{n\to\infty}\varrho_n=\inf\{\varrho\colon F(x)\cap\varrho B\neq\emptyset\text{ and }\varrho>0\},$$

introduced above. As $\varrho(x)\geq 0$, for all x, $\varphi(h(x))$ is lower semi-continuous at each point x with $\varphi(h(x))=\varrho(x)=0$. Consider any x_0 with $\varrho(x_0)>0$. Let ε be any real number with $0<\varepsilon<\varrho(x_0)$. Then

$$F(x_0)\cap[\varrho(x_0)-\varepsilon]B=\emptyset.$$

As $[\varrho(x_0)-\varepsilon]B$ is closed, and F is upper semi-continuous, there is a neighbourhood N of x_0 with

$$F(x)\cap[\varrho(x_0)-\varepsilon]B=\emptyset$$

for all x in N. Hence

$$\varphi(h(x)) = \varrho(x) > \varphi(h(x_0)) - \varepsilon,$$

for all x in N. Thus $\varphi \circ h$ is lower semi-continuous at x_0. Hence, by a well-known result (see, for example, [23, p. 394, Theorem 1]), the set of points of X, where $\varphi \circ f$ is discontinuous, is an $\mathscr{F}_\sigma$-set of the first category.

We prove that f is continuous at each point where $\varphi \circ f$ is continous. Let x_0 be a point of X where $\varphi \circ f$ is continuous and suppose that f fails to be continuous at x_0. Then we can choose an open set G containing $f(x_0)$ and a sequence x_n, $n \geq 1$, of points converging to x_0 with

$$f(x_n) \notin G,$$

for $n \geq 1$. By the continuity of $\varphi \circ f$ at x_0, we may make this choice so that

$$\varphi(f(x_n)) \leq \varphi(f(x_0)) + (1/n)$$

for $n \geq 1$. Write $\lambda_0 = 1 + \varphi(f(x_0))$, and consider the compact set

$$K = (F(x_0) \cap \lambda_0 B) \setminus G.$$

If it were possible to find, for each point ζ of K, an open set $U(\zeta)$, containing ζ; but containing $f(x_n)$ for only finitely many n, then, using the compactness of K, it would be possible to find an open set U, containing K, but containing $f(x_n)$ for only finitely many n. Then $F(x_0)$ would be contained in the open set

$$G \cup U \cup \{Z \setminus \lambda_0 B\}.$$

The upper semi-continuity of F would then ensure that

$$f(x_n) \in F(x_n) \subset G \cup U \cup \{Z \setminus \lambda_0 B\},$$

for all sufficiently large n, so that $f(x_n)$ would belong to U for all such n. This contradicts the choice of U. Hence there must be a point ζ in K with the property that all its neighbourhoods contain $f(x_n)$ for infinitely many n. Since ζ does not belong to the neighbourhood G of $f(x_0)$ we have $\zeta \neq f(x_0)$. Since $\zeta \in F(x_0)$ and $f(x_0)$ is the unique φ-nearest point of $F(x_0)$ to the origin, we must have

$$\varphi(\zeta) > \varphi(f(x_0)).$$

Writing $\lambda_1 = \frac{1}{2}\varphi(f(x_0)) + \frac{1}{2}\varphi(\zeta)$, the set $Z \setminus \lambda_1 B$ is an open neighbourhood of the point ζ.

Since all points z of this neighbourhood have

$$\varphi(z) > \lambda_1 > \varphi(f(x_0)),$$

it can contain $f(x_n)$ for only finitely many n. This contradicts the choice of ζ. We conclude that f is continuous at x_0.

We remark that simple examples show that f can be continuous at x_0, without $\varphi \circ f$ being continuous at x_0.

We now use this theorem to prove two theorems, stated in the introduction.

Proof of Theorem 2. Take Z to be the Banach space Y with its weak topology. Then Z is a locally convex topological vector space. Let B be the unit ball of Y corresponding to a strictly convex norm on Y equivalent to the original norm on Y. Then B is a closed, linearly bounded, linearly strictly convex and absorbent set in Z. Now Theorem 1 applies and provides the required selector.

We remark that the method we are using cannot give any general result for the case when we are merely told that F takes only non-empty closed bounded convex values. Suppose that Y is a Banach space and that each non-empty, bounded, weakly closed, convex set in Y contains a point that is nearest to the origin. Let y_0^* be any continuous linear functional on Y, with $\|y_0^*\|=1$. Consider the set C of y in Y with

$$\langle y, y_0^* \rangle \geq 1 \quad \text{and} \quad \|y\| \leq 2.$$

Then C is non-empty, bounded, weakly closed and convex. Let y_0 be the nearest point of C to the origin (or indeed a nearest point). As $\|y_0^*\|=1$, we must have $\|y_0\|=1$, and so $\langle y_0, y_0^* \rangle = 1$. Thus y_0^* attains its norm on the unit ball of Y. By a result of James [15], the space Y must be reflexive. Thus all the bounded weakly closed sets in Y are weakly compact, and we have not escaped from the condition that the values of F should be weakly compact.

Proof of Theorem 3. Take Z to be the Banach space Y^* with its weak-star topology. Then Z is a locally convex topological vector space. Let B be the unit ball of Y^*. Then B is a compact, linearly bounded, linearly strictly convex and absorbent set in Z. Now Theorem 1 applies and provides the required selector f.

We remark that h can be weak-star continuous at x_0, without $\varphi \circ h$ being continuous at x_0. To see this, it suffices to study the behaviour at 0 of the weak-star continuous point-valued map from l_2 to l_2 defined by taking $f(x)=(f_1(x), f_2(x), \ldots)$ with

$$f_n(0) = 0, \quad \text{for } n \geq 1,$$

and, for $x \neq 0$,

$$f_n(x) = \begin{cases} \sqrt{\{\{1/\|x\|\}\}}, & \text{if } \tfrac{1}{2}(n-1) \leq 1/\|x\| \leq \tfrac{1}{2}(n+1) \\ 0, & \text{otherwise} \end{cases}, \quad n \text{ even},$$

$$f_n(x) = \begin{cases} \sqrt{\{\{(1/\|x\|)+\tfrac{1}{2}\}\}}, & \text{if } \tfrac{1}{2}(n-1) \leq 1/\|x\| \leq \tfrac{1}{2}(n+1) \\ 0, & \text{otherwise} \end{cases}, \quad n \text{ odd},$$

where $\{\{t\}\}$ denotes the distance from t to the nearest integer.

§ 3. Selection in completely regular spaces that can be fragmented

As we have explained, in the introduction, a Banach space Y has the Radon-Nikodým property, if, and only if, each non-empty bounded set in Y is dentable. Consider any non-empty dentable set D in the Banach space Y. Then, for each $\varepsilon>0$, it is possible to find a point y_ε in D that is not in the closed convex hull of

$$D \setminus \{y: \|y-y_\varepsilon\| < \varepsilon\}.$$

By the Hahn-Banach theorem, this is equivalent to the condition that, for each $\varepsilon>0$, it is possible to choose y_ε in D, y_ε^* in Y^*, and $\delta_\varepsilon>0$, with $\|y_\varepsilon^*\|=1$ and

$$\operatorname{diam}\{y: y \in D \text{ and } \langle y-y_\varepsilon, y_\varepsilon^*\rangle > -\delta_\varepsilon\} \leq 2\varepsilon.$$

In particular, D has a non-empty relative weakly open subset of diameter at most 2ε.

Now consider a Banach space Y with the point of continuity property, and take D to be a non-empty bounded set in Y. Then the restriction to the weak closure w-cl$D$ of D of the identity map, from Y with its weak topology to Y with its norm topology, has a point of continuity, say d, in w-cl$D$. Thus, for each $\varepsilon>0$, the norm neighbourhood

$$\{y: y \in \text{w-cl}\, D \text{ and } \|y-d\| < \tfrac{1}{2}\varepsilon\}$$

of d in w-cl$D$ contains a relative weak neighbourhood of d in w-cl$D$. Hence D contains a non-empty relative weakly open subset of diameter at most ε.

Finally consider a dual Banach space Y^* having the Radon-Nikodým property. Dulst and Namioka [36] give a direct proof that, if D is a non-empty bounded weak-star closed subset of Y^*, then D contains non-empty relative weak-star open subsets of arbitrarily small diameter. This was previously known as a consequence of the deep results of Asplund and Stegall.

In order to discuss these cases together, it is convenient to introduce a definition.

Definition 1. We say that a Hausdorff space Y can be fragmented using a metric ϱ on the set Y if each non-empty closed subset of Y has non-empty relatively open subsets of arbitrarily small ϱ-diamter.

Although there need be no relation between the topology on Y and the metric ϱ, in the applications the open sets will be ϱ-open as well, and the topology will be completely regular.

By the remarks made in the first three paragraphs of this section we obtain four examples of spaces that can be fragmented; in each case the metric to be used is that provided by a Banach space norm.

(1) A bounded set in a Banach space with the Radon-Nikodým property, taken with its relative weak topology.

(2) A bounded everywhere dentable subset of a Banach space, taken with its relative weak topology.

(3) A bounded subset of a Banach space having the point of continuity property, taken with its relative weak topology.

(4) A bounded subset of a dual Banach space having the Radon-Nikodým property, taken with its relative weak-star topology. Indeed, Stegall [33] (see Dulst and Namioka [36, Proposition 2]) has shown that a weak-star closed convex subset of a dual Banach space is everywhere dentable, if, taken with its relative weak-star topology, it is fragmentable by the norm metric. The converse was obtained earlier by Namioka and Phelps [25, Lemma 3].

We give a fifth example:

(5) Let Y be a weakly compactly generated Banach space and let B^* be the unit ball of the dual space Y^*. Then Y^* with its weak-star topology can be identified with a weakly compact set with its weak topology in a Banach space Z. In the case, B^* with its weak-star topology can be fragmented using the norm on Z.

We remark that the concept of fragmentability is closely related to the point of continuity property. Consider the following definition.

Definition 2. We say that a Hausdorff space Y has the point of continuity property for a metric ϱ on the set Y, if, for each closed subset F of Y, the restriction to F of the identity map from Y with its Hausdorff topology to Y with its ϱ-metric topology has a point of continuity in F.

It is clear that if Y has the point of continuity property for the metric ϱ, then Y can be fragmented by using ϱ. It is easy to verify that, if Y be fragmented using the metric ϱ, then Y has the point of continuity property. It follows, in particular, that a weakly closed bounded subset of a Banach space, taken with its weak topology, has the point of continuity property for the norm, if, and only if, it can be fragmented using the norm.

This point of continuity property for the norm was introduced by Bourgain and Rosenthal [34]. This concept is closely related to the concept of an "épluchable" set introduced earlier by Godefroy [11]. See Edgar and Wheeler [10] for a comparison of these and other related concepts.

Our main objective in this section is to prove a general selection theorem in completely regular spaces that can be fragmented by use of some metric.

THEOREM 4. *Let X be a metric space and let Y be a completely regular Hausdorff space that can be fragmented using a metric ϱ. Let F be an upper semi-continuous set-valued map from X to Y, taking only non-empty compact values. Then F has a selector f that is Borel measurable of the first Borel class both with the completely regular topology and also with the ϱ-topology. Further, the set of points of X where f is ϱ-discontinuous is an $\mathscr{F}_\sigma$-set of the first category in X.*

We first prove two lemmas. The first justifies the use we make of the description 'a space that can be fragmented' for a space satisfying the conditions of Definition 1. These lemmas are the key results of this paper.

LEMMA 1. *Let Y be a completely regular Hausdorff space that can be fragmented. Suppose that $\varepsilon>0$. Then it is possible to choose a disjoint transfinite sequence $\{B_\gamma: 0\le\gamma<\Gamma\}$ of $\mathscr{F}_\sigma$-sets in Y, such that for $0\le\gamma<\Gamma$, we have:*

(a) *the union*

$$\bigcup_{0\le\beta\le\gamma} B_\beta$$

is open in Y;

(b) *B_γ is non-empty with* $\operatorname{diam}\operatorname{cl} B_\gamma<\varepsilon$; *and also*

(c) $\bigcup_{0\le\gamma<\Gamma} B_\gamma = Y.$

Proof. Consider any non-empty closed subset F of Y and any $\varepsilon>0$. Since Y can be fragmented, we can choose an open subset G of Y with $G\cap F$ a non-empty set of diameter less than ε. Choose a point y_0 in $G\cap F$. Since Y is completely regular we can choose a continuous function h on Y with

$$h(y_0)=0$$

and

$$h(y)=1 \quad \text{for all } y \text{ in } Y\setminus G.$$

Now the set F_ε of all points y of F with $h(y)<\frac{1}{2}$ is a non-empty subset of F, with closure of diameter at most ε, with $F\setminus F_\varepsilon$ closed and with F_ε an $\mathscr{F}_\sigma$-set.

We define the sequence $\{B_\gamma: 0\leq\gamma<\Gamma\}$, together with a second sequence $\{R_\gamma: 0\leq\gamma<\Gamma\}$ inductively to satisfy the conditions (a) and (b) and also

(d) $R_\gamma = Y\setminus \bigcup_{0\leq\beta<\gamma} B_\beta$.

We start by taking $R_0=Y$. Using the condition that Y can be fragmented, with $F=Y$, we take B_0 to be a non-empty $\mathscr{F}_\sigma$-set with $\operatorname{diam}\operatorname{cl} B_0<\varepsilon$, with $R_0\setminus B_0$ closed. Then $B_0=Y\setminus(R_0\setminus B_0)$ is open.

Now suppose that for some ordinal $\gamma>0$, the sets B_β, R_β have been defined, for $0\leq\beta<\gamma$, satisfying the conditions (a), (b) and (d). Then

$$\bigcup_{0\leq\beta<\gamma} B_\beta = \bigcup_{0\leq\beta<\gamma}\left\{\bigcup_{0\leq\alpha\leq\beta} B_\alpha\right\}$$

is open. If this union coincides with Y, we take $\Gamma=\gamma$ and the construction terminates. Otherwise the set R_γ defined by

$$R_\gamma = Y\setminus \bigcup_{0\leq\beta<\gamma} B_\beta$$

is closed and non-empty. As Y can be fragmented, we can take B_γ to be a non-empty $\mathscr{F}_\sigma$-set, contained in R_γ, with $\operatorname{diam}\operatorname{cl} B_\gamma<\varepsilon$, and with $R_\gamma\setminus B_\gamma$ closed. Now

$$\bigcup_{0\leq\beta\leq\gamma} B_\beta = \left\{\bigcup_{0\leq\beta<\gamma} B_\beta\right\}\cup B_\gamma = Y\setminus\{R_\gamma\setminus B_\gamma\}$$

is open. Now the conditions (a), (b) and (d) are satisfied for this ordinal γ. As the sequence of sets $\bigcup_{0\leq\beta<\gamma} B_\beta$, $0\leq\gamma$, is strictly increasing, until the whole of Y is covered, this transfinite process terminates with a cover of Y.

Our next lemma is clearly a step towards the proof of Theorem 4. Before we state it we need to explain some further concepts. A family $\{X_\gamma: \gamma\in\Gamma\}$ in a space X is said to be discrete if each point of X has a neighbourhood that meets X_γ for at most one γ in Γ. A family $\{X_\gamma: \gamma\in\Gamma\}$ in X is said to be σ-discrete if one can write $\Gamma=\bigcup_{n=1}^\infty \Gamma(n)$ in a way that ensures that each family $\{X_\gamma: \gamma\in\Gamma(n)\}$, $n\geq1$, is discrete. The family is said to be discretely σ-decomposable if it is possible to write

$$X_\gamma = \bigcup_{n=1}^{\infty} X_\gamma^{(n)}, \quad \text{for } \gamma\in\Gamma,$$

so that the family $\{X_\gamma^{(n)}: \gamma\in\Gamma\}$, is discrete for each $n\geq1$.

LEMMA 2. *Let X be a metric space and let Y be a completely regular Hausdorff space that can be fragmented. Let F be an upper semi-continuous set-valued map from X to Y, taking only non-empty compact values. Then, if $\varepsilon>0$, there is a partition $\{X_\gamma: \gamma\in\Gamma\}$ of X as a discretely σ-decomposable family of sets that are both $\mathscr{F}_\sigma$-sets and also $\mathscr{G}_\delta$-sets and a set-valued map H from X to Y, with:*

(a) *for each x in X, the set $H(x)$ is a non-empty compact subset of $F(x)$ of diameter at most ε;*

(b) *for each γ in Γ, the restriction of H to X_γ is upper semi-continuous; and*

(c) *for each γ in Γ, the diameter of the closure of $H(X_\gamma)$ is at most ε.*

Proof. By Lemma 1, we choose an ordinal Γ and a disjoint transfinite sequence $\{B(\gamma): 0\leq\gamma<\Gamma\}$ of $\mathscr{F}_\sigma$-sets in Y, such that, for $0\leq\gamma<\Gamma$, we have

(α) the union

$$\bigcup_{0\leq\beta\leq\gamma} B(\beta)$$

is open;

(β) $B(\gamma)$ is non-empty with

$$\operatorname{diam}\operatorname{cl} B(\gamma)<\varepsilon,$$

and also

(γ) $\bigcup_{0\leq\beta<\Gamma} B(\gamma)=Y.$

We then define $R(\gamma)$, for $0\leq\gamma<\Gamma$, by

(δ) $R(\gamma)=Y\setminus \bigcup_{0\leq\beta<\gamma} B(\beta)=\bigcup_{\gamma\leq\alpha<\Gamma} B(\alpha).$

Then $R(\gamma)$ is closed for $0\leq\gamma<\Gamma$.

Consider any point x of X. By (γ) and (δ),

$$F(x)\cap\bigcap_{0\leq\gamma<\Gamma} R(\gamma)=\emptyset.$$

As $F(x)\neq\emptyset$, there is a least ordinal $\gamma=\gamma(x)$, with $1\leq\gamma\leq\Gamma$, and

$$F(x)\cap\bigcap_{0\leq\beta<\gamma} R(\beta)=\emptyset.$$

If γ were a limit ordinal, the sequence

$$F(x)\cap\bigcap_{0\leq\beta<\theta} R(\beta),\quad 0\leq\theta<\gamma,$$

would be a decreasing sequence of non-empty compact sets, with a non-empty intersection

$$F(x) \cap \bigcap_{0 \le \beta < \gamma} R(\beta).$$

Thus $\gamma(x)$ is neither a limit ordinal, nor 0, and we can define an ordinal $\eta(x)$ by $\eta(x)+1=\gamma(x)$. This ensures that $0 \le \eta(x) < \Gamma$,

$$F(x) \cap R(\eta(x)) \neq \emptyset,$$

but

$$F(x) \cap R(\gamma) = \emptyset, \quad \text{for } \eta(x) < \gamma < \Gamma.$$

Using (δ), this yields

$$F(x) \cap B(\eta(x)) \neq \emptyset,$$

$$F(x) \cap B(\gamma) = \emptyset, \quad \text{for } \eta(x) < \gamma < \Gamma.$$

Thus we have deferred the choice of $\eta(x)$ until the last possible chance that we have of obtaining a non-empty interesection of $F(x)$ with $B(\eta(x))$.

For each γ with $0 \le \gamma < \Gamma$, we introduce the sets

$$X(\gamma) = \{x : \eta(x) = \gamma\},$$

$$Z(\gamma) = \{x : \eta(x) > \gamma\}.$$

Write

$$T = \bigcup_{x \in X} \{x\} \times F(x),$$

for the graph of F. Now $\eta(x) > \gamma$, if, and only if, $F(x) \cap R(\gamma+1) \neq \emptyset$, so that

$$Z(\gamma) = \text{proj}_X T \cap (X \times R(\gamma+1))\}$$

is a closed set in X, as F is upper semi-continuous. Similarly, $\eta(x)=\gamma$, if, and only if, $F(x) \cap B(\gamma) \neq \emptyset$ and $F(x) \cap R(\gamma+1) = \emptyset$. Hence, using the upper semi-continuity,

$$X(\gamma) = [\text{proj}_X \{T \cap (X \times B(\gamma))\}] \setminus [\text{proj}_X \{T \cap (X \times R(\gamma+1))\}]$$

is an $\mathscr{F}_\sigma$-set in X.

Let σ be the metric on X, and use

$$N(\xi; i) = \{x : \sigma(x, \xi) < 1/i\}$$

to denote the open ball with centre ξ and radius $1/i$. Let $X(\gamma; i)$ denote the set of all ξ in $X(\gamma)$ for which

$$N(\xi; i) \cap Z(\gamma) = \varnothing,$$

for $0 \leq \gamma < \Gamma$ and $1 \leq i$. Then $X(\gamma; i)$ is the intersection of $X(\gamma)$ with a closed set, and so is an $\mathscr{F}_\sigma$-set.

We show that, for each $i \geq 1$, the family

$$\{X(\gamma; i) \colon 0 \leq \gamma < \Gamma\}$$

is discrete in the completion X^* of X. Let x^* be any point of X^* and suppose that $N(x^*; 2i)$ meets both $X(\theta; i)$ and $X(\varphi; i)$ with $0 \leq \varphi < \Gamma$. Then there are points ξ, ζ of $X(\theta; i)$, $X(\varphi; i)$ with

$$\sigma(\xi, x^*) < 1/(2i), \quad \sigma(\zeta, x^*) < 1/(2i).$$

Thus $\sigma(\xi, \zeta) < 1/i$. As $\xi \in X(\theta; i)$ we have

$$\xi \in X(\theta), \quad \zeta \notin Z(\theta),$$

so that

$$\eta(\xi) = \theta, \quad \eta(\zeta) \leq \theta.$$

As $\zeta \in X(\varphi; i)$ we have

$$\zeta \in X(\varphi), \quad \xi \notin Z(\varphi),$$

so that

$$\eta(\zeta) = \varphi, \quad \eta(\xi) \leq \varphi.$$

Thus

$$\theta = \eta(\xi) \leq \varphi = \eta(\zeta) \leq \theta,$$

and $\theta = \varphi$. Hence, for each $i \geq 1$, the family $\{X(\gamma; i) \colon 0 \leq \gamma < \Gamma\}$ is discrete in X^*, that is, each point of X^* has a neighbourhood that meets at most one member of the family. As

$$X(\gamma) = \bigcup_{i=1}^{\infty} X(\gamma; i), \quad \text{for } 0 \leq \gamma < \Gamma,$$

the family $\{X(\gamma) \colon 0 \leq \gamma < \Gamma\}$ is discretely σ-decomposable in X^*, as each set of the family is a countable union of sets belonging to a countable system of discrete families.

Now $\{X(\gamma): 0\leq\gamma<\Gamma\}$ is a discretely σ-decomposable partition of X into $\mathcal{F}_\sigma$-sets. Hence, the union of any sub-family of this family is an $\mathcal{F}_\sigma$-set, and, in particular, the complement of any set $X(\gamma)$, $0\leq\gamma<\Gamma$, is an $\mathcal{F}_\sigma$-set. So each set $X(\gamma)$, $0\leq\gamma<\Gamma$ is both an $\mathcal{F}_\sigma$-set and a $\mathcal{G}_\delta$-set.

Finally, we define the set-valued function H on X by taking

$$H(x) = F(x) \cap \operatorname{cl} B(\eta(x))$$

for each x in X. This ensures that $H(x)$ is a non-empty compact set of diameter at most ε, for each x in X. Further, the restriction of H to $X(\gamma)$, takes the form

$$H(x) = F(x) \cap \operatorname{cl} B(\gamma), \quad \text{for } x \in X(\gamma),$$

and so is upper semi-continuous on $X(\gamma)$, for $0\leq\gamma<\Gamma$.

By (β) the diameter of the closure of $B(\gamma)$ is less than ε, for each γ in Γ. Hence the closure of $H(X_\gamma)$ is of diameter less than ε, for each γ in Γ. It is easy to check that we have now satisfied all the requirements of the lemma.

Proof of Theorem 4. We define inductively a sequence of ordinals $\Gamma(1), \Gamma(2), \ldots$, a sequence of partitions

$$\{X(\gamma_1, \gamma_2, \ldots, \gamma_n): (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Pi(n)\},$$

$n=1, 2, \ldots$ of X, with

$$\Pi(n) = \Gamma(1)\times\Gamma(2)\times\ldots\times\Gamma(n), \quad n = 1, 2, \ldots,$$

and a sequence $H^{(0)}=F$, $H^{(1)}$, $H^{(2)}, \ldots,$ of set-valued maps from X to Y, satisfying, for each $n\geq1$, the conditions:

(a) $\{X(\gamma_1, \gamma_2, \ldots, \gamma_n): (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Pi(n)\}$ is a discretely σ-decomposable partition of X into sets that are both $\mathcal{F}_\sigma$-sets and also $\mathcal{G}_\delta$-sets;

(b) $H^{(n)}$ takes only non-empty compact values of diameter at most $1/n$, with $H^{(n)}(x)\subset H^{(n-1)}(x)$ for $x \in X$;

(c) $H^{(n)}(X(\gamma_1, \gamma_2, \ldots, \gamma_n))$ has closure with diameter at most $1/n$, for each $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ in $\Pi(n)$; and

(d) for $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ in $\Pi(n)$, with $X(\gamma_1, \gamma_2, \ldots, \gamma_n)\neq\emptyset$, the restriction of $H^{(n)}$ to $X(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is upper semi-continuous.

We obtain the ordinal $\Gamma(1)$, the partition $\{X(\gamma_1):\gamma_1 \in \Pi(1)\}$ and the set-valued map $H^{(1)}$ satisfying the conditions (a), (b), (c) and (d) by a direct application of Lemma 2, with $\varepsilon=1$. Now suppose that, for some $n\geq 1$, the ordinals, the partitions and the set-valued maps satisfying (a), (b), (c) and (d) have been chosen up to and including n. Not of necessity, but for notational convenience, we take the next step using the proof of Lemma 2 as well as its statement. As in the proof of Lemma 2, using Lemma 1 with $\varepsilon=1/(n+1)$ we choose a disjoint transfinite sequence $\{B(\gamma)\colon 0\leq\gamma<\Gamma(n+1)\}$ of $\mathscr{F}_\sigma$-sets in Y and a transfinite sequence $\{R(\gamma)\colon 0\leq\gamma<\Gamma(n+1)\}$ of closed sets satisfying the conditions (α), (β), (γ) and (δ). We replace F in the proof of Lemma 2 by $H^{(n)}$. We proceed just as in the proof of Lemma 2, but operating separately within each set $X(\gamma_1,\gamma_2,\dots,\gamma_n)$ of the partition

$$\{X(\gamma_1,\gamma_2,\dots,\gamma_n)\colon (\gamma_1,\gamma_2,\dots,\gamma_n)\in\Pi(n)\}$$

to obtain a discretely σ-decomposable partition

$$\{X(\gamma_1,\gamma_2,\dots,\gamma_n,\gamma_{n+1})\colon \gamma_{n+1}\in\Gamma(n+1)\}$$

of $X(\gamma_1,\gamma_2,\dots,\gamma_n)$ into sets that are both relative $\mathscr{F}_\sigma$-sets and also relative $\mathscr{G}_\delta$-sets. The only difference from the situation in Lemma 2 is that now some of the sets $X(\gamma_1,\gamma_2,\dots,\gamma_{n+1})$ may be empty. It is clear that the ordinal $\Gamma(n+1)$, the partition

$$\{X(\gamma_1,\gamma_2,\dots,\gamma_{n+1})\colon (\gamma_1,\gamma_2,\dots,\gamma_{n+1})\in\Pi(n+1)\}$$

and the set-valued function $H^{(n+1)}$ obtained in this way satisfy the conditions (a), (b), (c) and (d) with n replaced by $n+1$.

It remains to define f by taking

$$f(x)=\bigcap_{n=0}^{\infty} H^{(n)}(x),\quad \text{for } x\in X,$$

and to verify that f is a well-defined selector for F that is Borel measurable of the first Borel class in the two ways. As the sequence $H^{(0)}(x)=F(x)$, $H^{(1)}(x)$, $H^{(2)}(x),\dots$ is a decreasing sequence of non-empty compact sets with diameters tending to zero, it is clear that $f(x)$ is a well-defined point of $F(x)$ for each x in X. Note that this argument holds, despite the fact that there is no necessary relationship between the topology that we use on Y and the metric on Y, in terms of which the diameters are defined.

To prove that f is of the first Borel class in the Hausdorff topology we consider any closed set J in Y and show that $f^{-1}(J)$ is a $\mathscr{G}_\delta$-set in X. We verify that

$$X\setminus f^{-1}(J)=\bigcup_{n=1}^{\infty}\bigcup\{X(\gamma_1,\gamma_2,\dots,\gamma_n)\setminus(H^{(n)})^{-1}(J)\colon(\gamma_1,\gamma_2,\dots,\gamma_n)\in\Pi(n)\}. \tag{1}$$

Clearly $x \in X \setminus f^{-1}(J)$, if, and only if,

$$f(x) \notin J,$$

or, if, and only if,

$$J \cap \left[\bigcap_{n=1}^{\infty} H^{(n)}(x) \right] = \emptyset.$$

As J is closed, and $H^{(1)}(x)$, $H^{(2)}(x), \ldots$ is a decreasing sequence of compact sets, this holds, if, and only if, for some integer $n \geq 1$, we have

$$J \cap H^{(n)}(x) = \emptyset,$$

or, equivalently,

$$x \in X \setminus (H^{(n)})^{-1}(J).$$

As

$$\{X(\gamma_1, \gamma_2, \ldots, \gamma_n): (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Pi(n)\} \tag{2}$$

is a partition of X, the required formula follows. By the upper semi-continuity of our maps, each set

$$X(\gamma_1, \gamma_2, \ldots, \gamma_n) \setminus (H^{(n)})^{-1}(J)$$

is open relative to $X(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and so is an $\mathscr{F}_\sigma$-set in X. As each family (2) is discretely σ-decomposable, it follows that the union (1) is an $\mathscr{F}_\sigma$-set in X, and $f^{-1}(J)$ is a $\mathscr{G}_\delta$-set, as required.

Now suppose that G is any set in Y that is open in the topology of the metric ϱ. If x is any point of X with $f(x) \in G$, then, for some $n > 0$, the closed ball with centre $f(x)$ and radius $1/n$, defined in terms of ϱ, is contained in G. Hence, using condition (c), for some $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ in $\Pi(n)$, we have

$$x \in X(\gamma_1, \gamma_2, \ldots, \gamma_n)$$

and

$$f(x) \in f(X(\gamma_1, \gamma_2, \ldots, \gamma_n)) \subset H^{(n)}(X(\gamma_1, \gamma_2, \ldots, \gamma_n)) \subset G.$$

Thus

$$f^{-1}(G) = \bigcup_{n=1}^{\infty} \bigcup \{X(\gamma_1, \gamma_2, \ldots, \gamma_n): (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Pi(n) \text{ and } H^{(n)}(X(\gamma_1, \gamma_2, \ldots, \gamma_n)) \subset G\}. \tag{3}$$

As before, it follows that $f^{-1}(G)$ is an $\mathscr{F}_\sigma$-set in X. Hence f is a Borel measurable function of the first Borel class, using the ϱ-topology.

We are grateful to R. W. Hansell for drawing our attention to the useful fact that the formula (3) implies that the family $\{f^{-1}(G)$: G ϱ-open$\}$ has a closed σ-discrete base. It follows immediately from (3) that each set $f^{-1}(x)$, with G ϱ-open in Y, is the union of those of the sets of the family

$$\{X(\gamma_1, \gamma_2, \dots, \gamma_n)\colon (\gamma_1, \gamma_2, \dots, \gamma_n) \in \Pi(n) \text{ and } n \geq 1\}$$

that it contains. Thus, introducing a simpler notation, we have a discretely σ-decomposable family $\{\Xi(\theta)\colon \theta \in \Theta\}$ of $\mathscr{F}_\sigma$-sets that form a base for the family $\{f^{-1}(G)$: G ϱ-open$\}$. Now we can write

$$\Xi(\theta) = \bigcup_{n=1}^{\infty} \Xi^{(n)}(\theta), \quad \theta \in \Theta,$$

with each family $\{\Xi^{(n)}(\theta)\colon \theta \in \Theta\}$, $n \geq 1$, a discrete family of closed sets. The family $\{\Xi^{(n)}(\theta)\colon \theta \in \Theta$ and $n \geq 1\}$ is a σ-discrete family of closed sets that is a base for the family $\{f^{-1}(G)$: G ϱ-open$\}$.

It is well known that, if f is a Borel measurable function of the first Borel class from a metric space X to a *separable* metric space Y, then the set of points of X where f is discontinuous is an $\mathscr{F}_\sigma$-set of the first category in X, see [23, p. 394]. Now Hansell [12, Theorem 10] shows that when f satisfies the condition of the last paragraph, then the standard proof can be easily modified to apply without the need for the space Y to be separable. It follows, in our case, that the set of points of X where f is ϱ-discontinuous is an $\mathscr{F}_\sigma$-set of the first category in X.

We can now prove two theorems refining Theorems 7 and 8 stated in the introduction.

THEOREM 5. *Let X be a metric space and let Y_0 be weakly closed subset of a Banach space Y. Suppose that each bounded subset of Y_0 has the point of continuity property for the norm. Let F be an upper semi-continuous set-valued map from X to Y with its weak topology. Suppose that F takes only non-empty weakly compact values contained in Y_0. Then, using the norm topology on Y, the set-valued map F has a Borel measurable selector f of the first Baire class. The set of points of X, where f is norm-discontinuous, is an $\mathscr{F}_\sigma$-set of the first category in X.*

THEOREM 6. *Let X be a metric space and let Y_0 be a weak-star closed subset of the Bananch space Y^* dual to a Banach space Y. Suppose that each bounded subset of*

Y_0 has the weak-star point of continuity property. Let F be an upper semi-continuous set-valued map from X to Y^* with its weak-star topology. Suppose that F takes only non-empty weak-star closed values contained in Y_0. Then, using the norm topology on Y^*, the set-valued map F has a Borel measurable selector f of the first Baire class. Further, the set of points of X where F is norm-discontinuous is an $\mathcal{F}_\sigma$-set of the first category in X.

Note that when $Y_0=Y^*$ the space Y^* has the weak-star point of continuity property, if, and only if, it has the Radon-Nikodým property, see the remarks in the introduction.

Proof of Theorem 5. The conditions ensure that each non-empty bounded weakly closed subset of Y_0, taken with its relative weak topology can be fragmented using the norm.

Write

$$Y_n = \{y \in Y_0 \colon \|y\| \leq n\}, \quad n \geq 1,$$

$$F_n(x) = F(x) \cap Y_n, \quad x \in X,\ n \geq 1,$$

$$X_1 = \{x \colon F(x) \cap Y_1 \neq \emptyset\},$$

and

$$X_n = \{x \colon F(x) \cap Y_{n-1} = \emptyset \text{ but } F(x) \cap Y_n \neq \emptyset\}, \quad n \geq 2.$$

Then, as F is upper semi-continuous, using the weak topology on Y, the family $\{X_n \colon n \geq 1\}$ is a partition of X into sets that are both $\mathcal{F}_\sigma$-sets and also $\mathcal{G}_\delta$-sets. Further, for each $n \geq 1$, the restriction of F_n to X_n is an upper semi-continuous set-valued map into Y_n with its relative weak topology, with non-empty weakly compact values. Thus the set-valued map F_n and the spaces X_n and Y_n satisfy the conditions of Theorem 4, using the norm on Y as the metric ϱ. Hence we can choose a Borel measurable selector f_n of the first Borel class for F_n on X_n, using the norm topology on Y_n, for each $n \geq 1$. Now the function f defined by

$$f(x) = f_n(x), \quad \text{for } x \in X_n,\ n \geq 1,$$

is a Borel measurable selector for F of the first Borel class. Although the continuity properties of f can not be deduced from those of the partial functions f_n on X_n, $n \geq 1$, the argument used in the proof of Theorem 4 applies, and we conclude that the set of points of X where f is norm-discontinuous is an $\mathcal{F}_\sigma$-set of the first category in X.

We prove that f is of the first Baire class on X. Using the norm topology on Y, the family $\{X_n \cap f^{-1}(G)\text{: } G \text{ open}\}$ has a closed σ-discrete base in X_n, for $n \geq 1$. Since each set X_n, $n \geq 1$, is an $\mathscr{F}_\sigma$-set in X, it follows, without difficulty, that the family $\{f^{-1}(G)\text{: } G$ open$\}$ has a closed σ-discrete base in X.

A space Y is said to have the *extension property* with respect to a space X, if every continuous map from a closed subset of X to Y can be extended to a continuous map from X into Y. Now Dugundji and Borsuk, see [3, Theorem 3.1, p. 58], show that a Banach space Y always has the extension property with respect to a metric space X. Further, Hansell, in [13, Theorem 6], shows that if f is a Borel measurable map of the first Borel class from a metric space X to a metric space Y, having the extension property with respect to X, and if the family $\{f^{-1}(G)\text{: } G \text{ open}\}$ has a closed σ-discrete base, then the standard proof for Y the real line (see [23, § 31, Theorem 7]) can be easily modified to show that f is a point-wise limit of a sequence of continuous functions from X to Y. One family $\{\Xi(\theta)\text{: } \theta \in \Theta\}$ of sets in a space X is said to be a *base* for another family $\{X_\gamma\text{: } \gamma \in \Gamma\}$ in X, if each set X_γ, $\gamma \in \Gamma$, is the union of those sets of $\{\Xi(\theta)\text{: } \theta \in \Theta\}$ that it contains.

Fitting these results together, we find that the selector f is of the first Baire class.

Proof of Theorem 6. The result follows from Theorem 4 by the method used to prove Theorem 5.

Theorems 7 and 8, stated in the introduction follow immediately from Theorems 5 and 6.

§ 4. Maximal monotone maps

In this section we obtain selection results for maximal monotone maps. We need three lemmas. The first derives some of the elementary results of the theory in a slightly refined form using a simple proof. The second quotes deeper results due to Rockafellar [31].

LEMMA 3. *Let F be a maximal monotone map from a Banach space X to its dual space X^* and let B^* denote the unit ball of X^*. For $R>0$, let F_R be defined by*

$$F_R(x) = F(x) \cap [RB^*], \quad \textit{for } x \in X.$$

Then, for each $R>0$, the set-valued map F_R is a weak-star upper semi-continuous map with convex weak-star compact values. The domain

$$D = \{x\text{: } F(x) \neq \emptyset\}$$

of F is an $\mathscr{F}_\sigma$-set.

LEMMA 4 (Rockafellar). *Let F be a maximal monotone map from a Banach space X to its dual space X^*. Suppose that the interior D_0 of the convex hull of the domain D of F is non-empty. Then D_0 coincides with the interior of D, is dense in D and is a convex open set. Further, F is weak-star upper semi-continuous and locally bounded on D_0.*

LEMMA 5. *Let F be a monotone map from a Banach space X to its dual space X^*. Suppose that the domain D of F contains x_0 and is dense in a neighbourhood of x_0. Let f be a selector for F on D, and let H be a maximal monotone map whose graph contains the graph of F. If f is weak-star continuous at x_0, then $H(x_0)$ reduces to a single point and H is bounded on some neighbourhood of x_0 and is weak-star upper semi-continuous at x_0. If f is norm continuous at x_0, then H is norm upper semi-continuous at x_0.*

If, in addition, D is a $\mathcal{G}_\delta$-set that is dense in an open set G in X, and f is a norm Borel measurable selector of the first Borel class, then there is a $\mathcal{G}_\delta$-set, contained in D and dense in G that is simultaneously:

(a) *the set of points of D where f is norm continuous;*

(b) *the set of points of D where F is point-valued and norm upper semi-continuous;*

and

(c) *the set of points of D where H is point-valued and norm upper semi-continuous.*

When the selector f is weak-star continuous at x_0 and X is reflexive, it follows immediately from the lemma by use of the results of Rockafellar [31] that x_0 is an interior point of the domain D of H.

We remark that the theory of monotone maps and of maximal monotone maps becomes much simpler when one can work within the interior of their domain. A simple example illustrates the difficulties that occur when the domain has no interior. Consider the set-valued map F from l_2 to l_2 defined by

$$F(x) = \{x\}$$

for those

$$x = (x_1, x_2, x_3, \ldots)$$

in l_2 for which

$$\sum_{n=1}^{\infty} n^2 x_n^2$$

converges, and otherwise by

$$F(x) = \emptyset.$$

It is easy to verify that F is a maximal monotone map and that its domain is dense in l_2, but has empty interior. Further, F is nowhere locally bounded in l_2.

Proof of Lemma 3. We first prove that for each x in X, the set $F(x)$ is convex and weak-star closed. Fix x_0 in X and let x_1^*, x_2^* be points of $F(x_0)$. Then for each choice of ξ, ξ^* with $\xi \in X$, $\xi^* \in F(\xi)$, we have

$$\langle \xi - x_0, \xi^* - x_1^* \rangle \geq 0,$$

$$\langle \xi - x_0, \xi^* - x_2^* \rangle \geq 0,$$

so that

$$\langle \xi - x_0, \xi^* - \{(1-\theta)x_1^* + \theta x_2^*\} \rangle \geq 0,$$

for $0 \leq \theta \leq 1$. As F is a maximal monotone map, we have

$$(1-\theta)x_1^* + \theta x_2^* \in F(x_0)$$

for $0 \leq \theta \leq 1$. Thus $F(x_0)$ is convex.

Now consider any point x_0^* in the weak-star closure of $F(x_0)$. For each choice of ξ, ξ^* with $\xi \in X$, $\xi^* \in F(\xi)$, and each $n \geq 1$, the set of points x^* with

$$\langle \xi - x_0, x^* - x_0^* \rangle \geq -1/n,$$

is a weak-star neighbourhood of x_0^*, and so contains a point, x_n^* say, of $F(x_0)$. Hence

$$\langle \xi - x_0, x_n^* - x_0^* \rangle \geq -1/n,$$

and, as $x_n^* \in F(x_0)$,

$$\langle \xi - x_0, \xi^* - x_n^* \rangle \geq 0.$$

Thus

$$\langle \xi - x_0, \xi^* - x_0^* \rangle \geq -1/n,$$

for $n \geqslant 1$, so that

$$\langle \xi - x_0, \; \xi^* - x_0^* \rangle \geqslant 0.$$

As F is maximal, this implies that $x_0^* \in F(x_0)$. Thus $F(x_0)$ is weak-star closed.

We use $X^*_{(*)}$ to denote X^* with its weak-star topology. We prove that the graph

$$M_R = \{(x, x^*) : x \in X \text{ and } x^* \in F_R(x)\}$$

of F_R is closed in $X \times X^*_{(*)}$. Let (ξ, ξ^*) be a point of the closure of M_R in $X \times X^*_{(*)}$. Consider a general point (x_0, x_0^*) of the graph M of F, with $x_0 \neq \xi$. Let $\varepsilon > 0$ be given and write

$$\delta = \frac{\frac{1}{2}\varepsilon}{\|x_0^*\| + R}.$$

The set of points (x, x^*) with

$$\|x - \xi\| < \delta,$$

$$\langle x_0 - \xi, x_0^* - x^* \rangle < \langle x_0 - \xi, x_0^* - \xi^* \rangle + \tfrac{1}{2}\varepsilon,$$

is a neighbourhood of (ξ, ξ^*) in $X \times X^*_{(*)}$. So we may choose a point, (x_1, x_1^*) say, of M_R in this neighbourhood of (ξ, ξ^*). Then $\|x_1^*\| \leqslant R$. Further, as F is a monotone map,

$$\langle x_0 - x_1, \; x_0^* - x_1^* \rangle \geqslant 0.$$

Thus

$$\begin{aligned}\langle x_0 - \xi, x_0^* - \xi^* \rangle &> \langle x_0 - \xi, x_0^* - x_1^* \rangle - \tfrac{1}{2}\varepsilon \\ &= \langle x_0 - x_1, x_0^* - x_1^* \rangle + \langle x_1 - \xi, x_0^* - x_1^* \rangle - \tfrac{1}{2}\varepsilon \\ &\geqslant -\|x_1 - \xi\| \cdot \|x_0^* - x_1^*\| - \tfrac{1}{2}\varepsilon \\ &\geqslant -\|x_1 - \xi\| \, \{\|x_0^*\| + R\} - \tfrac{1}{2}\varepsilon \geqslant -\varepsilon.\end{aligned}$$

Hence

$$\langle x_0 - \xi, x_0^* - \xi^* \rangle \geqslant 0,$$

for all x_0, x_0^* with $x_0^* \in F(x_0)$. As F is a maximal monotone map this ensures that $\xi^* \in F(x^*)$. As the ball $\|x^*\| \leqslant R$ is weak-star closed, we also have $\|\xi^*\| \leqslant R$. Hence $(\xi, \xi^*) \in M_R$ as required. So M_R is closed in $X \times X^*_{(*)}$.

Consider any weak-star closed set J in X^*. As

$$F_R^{-1}(J) = \text{proj}_X \{M_R \cap \{X \times J\}\} = \text{proj}_X \{M_R \cap \{X \times (J \cap RB^*)\}\}$$

with B^* the unit ball of X^*, this set $F_R^{-1}(J)$ is the projection of the closed set

$$M_R \cap \{X \times (J \cap RB^*)\}$$

of $X \times X^*_{(*)}$ onto X through the weak-star compact set RB^*. Hence $F_R^{-1}(J)$ is closed in X. Thus F_R is weak-star upper semi-continuous. By the first paragraph, F_R takes convex weak-star compact values.

As

$$D = \bigcup_{R=1}^{\infty} F_R^{-1}(X^*),$$

the set D is an $\mathscr{F}_\sigma$-set.

Proof of Lemma 4. This follows immediately from Theorem 1 and Corollary 1.2 of Rockafellar [31]. Rockafellar does not state his results in quite this way, but his results do imply those stated here.

Proof of Lemma 5. Let H be a maximal monotone map from X to X^* whose graph contains that of F. Then the domain E of H contains the domain D of F.

We may suppose that D is dense in the neighbourhood

$$B(x_0; \varepsilon) = \{x: \|x - x_0\| < \varepsilon\},$$

of x_0, with $\varepsilon > 0$.

Now suppose that the selector f, defined on D, is weak-star continuous at x_0. It follows that f must be bounded on some relatively open subset of D containing x_0. Changing the value of $\varepsilon > 0$, if necessary, we may suppose that, for some $M > 0$,

$$\|f(x)\| \leqslant M,$$

for all x in $D \cap B(x_0; \varepsilon)$.

Our first aim is to prove that H is bounded in some relatively open subset of E containing x_0. Suppose that this is not the case. Then we can choose sequences (ξ_n) and (ξ_n^*) with

$$\|\xi_n - x_0\| \leqslant \varepsilon/(4n),$$

$$\|\xi_n^*\| \geqslant n,$$

$$\xi_n^* \in H(\xi_n),$$

for each $n \geq 1$. Now the set of points x satisfying

$$\|x-\xi_n\| < \tfrac{1}{2}\varepsilon,$$

$$\langle x-\xi_n, -\xi_n^*/\|\xi_n^*\|\rangle < -\tfrac{1}{4}\varepsilon,$$

is a non-empty open subset of $B(x_0; \varepsilon)$. Since D is dense in $B(x_0; \varepsilon)$, we can choose d_n in D satisfying

$$\|d_n-\xi_n\| < \tfrac{1}{2}\varepsilon,$$

$$\langle d_n-\xi_n, -\xi_n^*/\|\xi_n^*\|\rangle < -\tfrac{1}{4}\varepsilon,$$

for each $n \geq 1$. Since H is a monotone map we have

$$\begin{aligned} 0 &\leq \langle d_n-\xi_n, f(d_n)-\xi_n^*\rangle \\ &= \langle d_n-\xi_n, -\xi_n^*\rangle + \langle d_n-\xi_n, f(d_n)\rangle \\ &\leq \|\xi_n^*\| \langle d_n-\xi_n, -\xi_n^*/\|\xi_n^*\|\rangle + \tfrac{1}{2}\varepsilon M \\ &< -\tfrac{1}{4}n\varepsilon + \tfrac{1}{2}\varepsilon M \to -\infty, \end{aligned}$$

as $n \to \infty$. This contradiction shows that H is bounded on some neighbourhood of x_0. Changing the value of ε, if necessary, and choosing a new M, we may suppose that

$$\|h\| \leq M \text{ for all } h \text{ in } H(x) \text{ with } x \text{ in } E \cap B(x_0, \varepsilon).$$

The next step is to show that $H(x_0)$ must reduce to a single point. Suppose that $H(x_0)$ contains a point x_0^* other than the point $f(x_0)$ of $F(x_0) \subset H(x_0)$ selected by f. Choose ξ in X with

$$\langle \xi, f(x_0)-x_0^*\rangle = -2.$$

Then $x_0+n^{-1}\xi \in B(x_0; \varepsilon)$ for all sufficiently large n, say for $n \geq n_0$. Since D is dense in $B(x_0; \varepsilon)$, we can choose η_n with

$$x_0+n^{-1}\eta_n \in D,$$

$$\|\eta_n\| \leq 2\|\xi\|,$$

$$\|\eta_n-\xi\| < 1/(2M),$$

for $n \geq n_0$. Since H is a monotone map,

$$
\begin{aligned}
0 &\leqslant n\langle x_0+n^{-1}\eta_n-x_0, f(x_0+n^{-1}\eta_n)-x_0^*\rangle \\
&= \langle \eta_n-\xi, f(x_0+n^{-1}\eta_n)-x_0^*\rangle \\
&\quad + \langle \xi, f(x_0+n^{-1}\eta_n)-f(x_0)\rangle \\
&\quad + \langle \xi, f(x_0)-x_0^*\rangle \\
&\leqslant \|\eta_n-\xi\|\, 2M-2 \\
&\quad + \langle \xi, f(x_0+n^{-1}\eta_n)-f(x_0)\rangle \\
&\leqslant -1+\langle \xi, f(x_0+n^{-1}\eta_n)-f(x_0)\rangle \\
&\to -1,
\end{aligned}
$$

as $n\to\infty$, using the weak-star continuity of f at x_0 relative to D. This contradiction shows that $H(x_0)=F(x_0)=f(x_0)$.

Using the boundedness of H in the neighbourhood $B(x_0;\varepsilon)$ and Lemma 3, we see that H is necessarily weak-star upper semi-continuous in $B(x_0;\varepsilon)$ and so, in particular, at x_0.

We now suppose that f is norm continuous at x_0 and show that H is norm upper semi-continuous at x_0. Since a norm continuous function is certainly weak-star continuous, we may use the results of the previous paragraphs. Suppose, for sake of argument, that H is not norm upper semi-continuous at x_0. Then we can choose $\delta>0$, and sequences ξ_n, ξ_n^* with

$$
\begin{gathered}
\|\xi_n-x_0\|<\varepsilon/(2n), \\
\xi_n^*\in H(\xi_n), \\
\|\xi_n^*-f(x_0)\|\geqslant\delta,
\end{gathered}
$$

for each $n\geqslant 1$.

For each $n\geqslant 1$, we can choose η_n in X with $\|\eta_n\|=2$ and

$$\langle \eta_n, \xi_n^*-f(x_0)\rangle \geqslant \delta.$$

Then, for $n\geqslant 1$,

$$\|\xi_n+(\varepsilon/4n)\eta_n-x_0\|<\varepsilon/n\leqslant\varepsilon.$$

By the density of D in $B(x_0;\varepsilon)$ we can choose ζ_n with

$$\|\eta_n-\zeta_n\|<1/n,$$

and with

$$\xi_n+(\varepsilon/4n)\zeta_n \quad \text{in } D.$$

Since $\xi_n+(\varepsilon/4n)\zeta_n$ converges in norm to x_0 through points of D, the image points $f(\xi_n+(\varepsilon/4n)\zeta_n)$ converge in norm to $f(x_0)$ as $n\to\infty$.

Using the fact that H is a monotone map, we deduce that

$$\begin{aligned}
0 &\leq 4n\langle \xi_n+(\varepsilon/4n)\zeta_n-\xi_n, f(\xi_n+(\varepsilon/4n)\zeta_n)-\xi_n^*\rangle \\
&= \varepsilon\langle \eta_n, f(x_0)-\xi_n^*\rangle \\
&\quad +\varepsilon\langle \eta_n, f(\xi_n+(\varepsilon/4n)\zeta_n)-f(x_0)\rangle \\
&\quad +\varepsilon\langle \zeta_n-\eta_n, f(\xi_n+(\varepsilon/4n)\zeta_n)-\xi_n^*\rangle \\
&\leq -\varepsilon\delta+2\varepsilon\|f(\xi_n+(\varepsilon/4n)\zeta_n)-f(x_0)\|+2\varepsilon M\|\zeta_n-\eta_n\| \\
&\to -\varepsilon\delta,
\end{aligned}$$

as $n\to\infty$. This contradiction shows that H is norm upper semi-continuous at x_0.

We now consider the case when D is a $\mathscr{G}_\delta$-set that is dense in an open set G in X, and f is a Borel measurable selector of the first Borel class. The set D is a completely metrizable space and f is a Borel measurable function of the first Borel class from D to X^* with its norm topology. By a result, going back to Baire's thesis, as recently extended to the non-separable case by Hansell [13], the points of norm continuity of f form a $\mathscr{G}_\delta$-set U that is dense in D and so also dense in G. This set U can now be immediately identified with the sets (a), (b) and (c) in the statement of the lemma.

We now prove the main result of this section.

Proof of Theorem 9. First consider the case when X has a equivalent norm whose dual norm on X^* is strictly convex. Let φ denote this strictly convex norm on X^*, and let Φ^* denote the corresponding unit ball

$$\Phi^* = \{x^*: \varphi(x^*)\leq 1\}.$$

By Lemma 3, for each integer $r>0$, the set-valued function

$$F_r(x) = F(x)\cap[r\Phi^*], \quad \text{for } x\in X,$$

is weak-star upper semi-continuous with weak-star compact values. Despite the fact that $F(x)$ itself may not be weak-star upper semi-continuous, the arguments in Theorems 1 and 3 apply and the nearest point selector f for $F(x)$ is a weak-star Borel measurable selector of the first Borel class on D. Further, the set of weak-star discontinuities of f is contained in an $\mathscr{F}_\sigma$-set of the first category in D. Since we have taken the interior D_0 of D to be non-empty, the set J of points of D_0 where f is weak-star continuous, contains a $\mathscr{G}_\delta$-set that is dense in D_0. Applying Lemma 5 to the monotone map obtained by restricting F to D_0, we can identify J with the set of points of D_0 where F is point-valued.

Now consider the case when X^* has the Radon-Nikodým property. By Lemma 3, for each integer $r>0$, the set-valued function

$$F_r(x) = F(x) \cap [rB^*], \quad \text{for } x \in X,$$

is weak-star upper semi-continuous with convex weak-star compact values. Using Theorem 8, we can choose a norm Borel measurable selector f_r of the first Borel class for F_r on its domain D_r, which is a closed set in X (see the proof of Lemma 3). Define f on D by taking

$$f(x) = f_1(x), \quad \text{if } x \in D_1,$$

and

$$f(x) = f_r(x), \quad \text{if } x \in D_r \setminus D_{r-1},\ r \geq 2.$$

It is easy to verify that f is a norm Borel measurable selector of the first Borel class for F on D, and that $\{f^{-1}(G)\colon G \text{ norm open in } X^*\}$ has a closed σ-discrete base. It follows, as in the proof of Theorem 5, that f is of the first Baire class on D.

Restricting F and f to the open set D_0 we obtain a monotone map (no longer maximal, in general) with a norm Borel measurable selector of the first Baire class. It follows immediately, from Lemma 5, that there will be a $\mathscr{G}_\delta$-set contained in D_0, dense in D_0, and so also in D, that is both the set of all points of D_0 at which f is norm continuous with respect to D_0 and also the set of points of D_0 at which F is point-valued and norm upper semi-continuous with respect to D_0.

§ 5. Subdifferentials and the Radon-Nikodým property

As we have explained in the introduction, the subdifferential of a lower semi-continuous convex function f defined on a Banach space X, taking values in the extended real line, and taking finite values on some convex set with a non-empty interior, is a maximal monotone map from X to X^*. When X^* has the Radon-Nikodým property, Theorem 9 applies, and we find, in particular, that the subdifferential has a norm Borel measurable selector of the first Baire class on its domain of definition. In this section, we first prove the strengthened converse to this result, stated in the introduction.

Proof of Theorem 10. Let F be a subdifferential map with domain of definition D. Let U be a $\mathcal{G}_\delta$-set that is dense in the interior D_0 of D, and suppose that f is a norm Borel measurable selector for F on U. Then U is a completely metrizable space. By an extension due to Hansell [13] of a classical result [23, p. 400], there is a $\mathcal{G}_\delta$-subset U_1 of U, dense in U, with the restriction of f to U_1 norm continuous at each point of U_1.

We can now apply Lemma 5. We find that F is point-valued and norm upper semi-continuous at each point of U_1 which is a $\mathcal{G}_\delta$-set in X that is dense in D_0. Thus the convex function giving rise to the subdifferential is Fréchet differentiable on a dense $\mathcal{G}_\delta$-subset of the set where it is finite. Thus X is an Asplund space (see the introduction for the definition), and, by a result of Namioka and Phelps [25], the space X^* has the Radon-Nikodým property.

§ 6. Attainment maps

Throughout this section, K will be a non-empty weakly closed set in a Banach space X. The attainment map F of K is the set-valued map defined on the dual space X^* of X by the formula

$$F(x^*) = \{x : x \in X \text{ and } \langle x, x^* \rangle = \sup \{\langle k, x^* \rangle : k \in K\}\},$$

so that $F(x^*)$ is the set of points of K at which the linear functional x^* attains its supremum over K. We prove Theorem 11, stated in the introduction.

Proof of Theorem 11. We first verify that F is a monotone map from X^* to X^{**}. Let x_0^* and x_1^* be distinct points of X^* and suppose that $x_0 \in F(x_0^*)$ and $x_1 \in F(x_1^*)$. Then

$$\langle x_0, x_0^* \rangle = \sup\{\langle k, x_0^* \rangle : k \in K\} \geq \langle x_1, x_0^* \rangle,$$

and similarly

$$\langle x_1, x_1^* \rangle \geq \langle x_0, x_1^* \rangle,$$

so that

$$\langle x_1 - x_0, x_1^* - x_0^* \rangle = \langle x_1, x_1^* \rangle - \langle x_0, x_1^* \rangle + \langle x_0, x_0^* \rangle - \langle x_1, x_0^* \rangle \geq 0.$$

Thus F is monotone.

We next consider case (a), when K is weakly compact. In this case all linear functionals attain their suprema on K, so that $D^*=X^*$. We verify that F is weakly upper semi-continuous. Consider any point x_0^* in X^* and any weakly open set G in X with

$$F(x_0^*) \subset G.$$

Then $K \setminus G$ is weakly compact and $\langle x, x_0^* \rangle$ obtains its supremum, q say, over $K \setminus G$ at some point h of $K \setminus G$. As h is not in $F(x_0^*)$, this supremum q must be strictly less than the value, p say, taken by $\langle x, x_0^* \rangle$ on $F(x_0^*)$. So

$$\langle x, x_0^* \rangle \leq q \quad \text{for } x \text{ in } K \setminus G,$$

$$\langle x, x_0^* \rangle = p \quad \text{for } x \text{ in } F(x_0^*).$$

As $K \setminus G$ and $F(x_0^*)$ are bounded, we can choose $\varepsilon > 0$, so that

$$\langle x, x_1^* \rangle \leq \tfrac{2}{3}q + \tfrac{1}{3}p \quad \text{for } x \text{ in } K \setminus G,$$

$$\langle x, x_1^* \rangle \geq \tfrac{1}{3}q + \tfrac{2}{3}p \quad \text{for } x \text{ in } F(x_0^*),$$

for all x_1^* with $\|x_1^* - x_0^*\| < \varepsilon$. It follows that $F(x_1^*) \subset G$ for all such x_1^*. Thus F is weakly upper semi-continuous.

As K is weakly compact, it is dentable (see [8, p. 204]). As F takes only non-empty weakly compact values contained in K, Theorem 5 applies and F has a norm Borel measurable selector f of the first Baire class on X^*. As F is a monotone map from X^* to X^{**}, Lemma 5 applies and there is a $\mathscr{G}_\delta$-set U^* dense in X^* that is both the set of norm continuity points of f and also the set of points where F is point valued and norm upper semi-continuous.

The conditions in (b) have been chosen to ensure that the same proof applies with only the obviously necessary modifications.

§ 7. Metric projection

In this section we prove Theorem 12 stated and discussed in the introduction.

Proof of Theorem 12. Under our hypotheses, Theorem 5 applies and F has a norm Borel measurable selector f of the first Baire class on U_0. By the result of Hansell [13], used in the proof of Lemma 5, there is a $\mathscr{G}_\delta$-set U_1 dense in U_0, and so also dense in X, with f norm continuous on U_0 at each point of U_1.

Now suppose that X is strictly convex. To prove that F is point-valued and norm upper semi-continuous, with respect to U_0 at a point x_1 of U_1 it is enough to show that, for each $\varepsilon>0$, there is an open neighbourhood N of x_1, with $\operatorname{diam} F(N\cap U_0)\leqslant\varepsilon$. Suppose, on the contrary, that $x_1\in U_1$, but that, for some $\varepsilon>0$, we have

$$\operatorname{diam} F(N\cap U_0)>\varepsilon,$$

for all open neighbourhoods N of x_1. As f is continuous on U_0 at x_1, we can take N to be an open neighbourhood of x_1 with

$$\|f(x)-f(x_1)\|<\tfrac{1}{8}\varepsilon,$$

for all x in $N\cap U_0$. As $f(x_1)\in F(N\cap U_0)$ and $\operatorname{diam} F(N\cap U_0)\geqslant\varepsilon$, we can choose x_2 in $N\cap U_0$ and h in $F(x_2)$ with $\|h-f(x_1)\|>\frac{1}{2}$. As N is open in X, we can choose x_3 to be a point of N in the relative interior of the line segment $[x_2, h]$. As the unit ball of X is strictly convex, every point of the ball, defined by

$$\|x-x_3\|\leqslant\|h-x_3\|,$$

other that h, is in the interior of the ball defined by

$$\|x-x_2\|\leqslant\|h-x_3\|.$$

As $h\in F(x_2)\subset K$, and $\|h-x_2\|=\varrho(x_2)$, it follows that $F(x_3)=\{h\}$. As $\|h-f(x_1)\|>\frac{1}{2}\varepsilon$, we can choose ξ^* in X^*, with

$$\langle h-f(x_1), \xi^*\rangle>\tfrac{1}{4}.$$

so the half-space

$$\langle x, \xi^*\rangle>\tfrac{1}{8}+\langle f(x_1), \xi^*\rangle$$

is a weak open set containing $F(x_3)$. By the weak upper semi-continuity of F, we can choose an open neighbourhood M of x_3 contained in N with $F(M\cap U_0)$ contained in this open set. As U_0 is dense in X, we can choose x_4 in $M\cap U_0\subset N\cap U_0$. Then

$$\langle f(x_4)-f(x_1), \xi^*\rangle>\tfrac{1}{8},$$

but

$$\|f(x_4)-f(x_1)\|<\tfrac{1}{8}.$$

This is impossible as $\|\xi^*\|=1$. The required result follows.

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