Quasiconformal extension of quasisymmetric mappings compatible with a Möbius group

by

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1. Introduction and preliminaries

1A. As is well-known, one can always extend a Möbius transformation of $\bar{\mathbf{R}}^n$ $(=\mathbf{R}^n \cup \{\infty\})$ to a Möbius transformation of the hyperbolic $(n+1)$-space $H^{n+1}=\{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}: x_{n+1}>0\}$. For instance, this can be done as follows. Let $z \in H^{n+1}$. Pick a triple $x=(u, v, w) \in (\bar{\mathbf{R}}^n)^3$ of distinct points such that z is on the hyperbolic line L with endpoints u and v, and such that the hyperbolic ray R with endpoints z and w intersects L orthogonally. Then we write

$$z = p(u, v, w) = p(x). \tag{1.1}$$

If now g is a Möbius transformation of $\bar{\mathbf{R}}^n$, then the extension of g to H^{n+1} is given by

$$g(z) = p(g(u), g(v), g(w)) = pg(x). \tag{1.2}$$

If g is a Möbius transformation, then (1.2) is independent of the choice of the triple satisfying (1.1), but this is not true of non-Möbius g. However, and this observation started this paper, if g is quasiconformal, then (1.2) defines a kind of fuzzy image of z for $z \in H^{n+1}$ which satisfies a certain type of Lipschitz condition. We explain this now in more detail.

First, if two triples $x, x' \in p^{-1}(z)$, then the hyperbolic distance

$$d(pg(x), pg(x')) \leq m, \tag{1.3}$$

where $m \geq 0$ depends only on n and on the dilatation of g (Theorem 3.4). Thus the indeterminacy in the image of z is uniformly bounded for $z \in H^{n+1}$.

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Second, there are constants $M \geq 0$ and $L \geq 1$, depending only on n and on the dilatation of g, such that if $z, z' \in H^{n+1}$ and $d(z, z') \geq M$, then for any $x \in p^{-1}(z)$ and $x' \in p^{-1}(z')$,

$$d(z, z')/L \leq d(pg(x), pg(x')) \leq Ld(z, z'), \tag{1.4}$$

cf. Theorem 3.6.

1B. This paper is an application of these ideas to the following problem. Let G be a group of Möbius transformations of $\bar{\mathbf{R}}^n$ (whose action can be extended to $\bar{H}^{n+1} = H^{n+1} \cup \bar{\mathbf{R}}^n$) and let f be a homeomorphism of $\bar{\mathbf{R}}^n$ which is *G-compatible*; that is, there is a homomorphism $\varphi: G \to G'$ onto another Möbius group G' such that

$$\varphi(g)f(x) = fg(x)$$

whenever $f(x)$ is defined (in which case we say also that f *induces* φ).

We wish to find an extension F of f to $\bar{H}^{n+1}$ which is also G-compatible and preferably also a homeomorphism. Furthermore, if f is quasiconformal, then we wish the extension also to be quasiconformal.

If $n > 1$, then we do not know whether such an extension always exists. But the next theorem is a step in this direction.

THEOREM 1. *Let G be a group of Möbius transformations of $\bar{\mathbf{R}}^n$ and let f be a G-compatible homeomorphism of $\bar{\mathbf{R}}^n$. Then there is a G-compatible and continuous extension F of f to a map $\bar{H}^{n+1} \to \bar{H}^{n+1}$ such that $F(H^{n+1}) \subset H^{n+1}$.*

Furthermore, if $K \geq 1$, then there are $M = M(K, n)$ and $L = L(K, n) \geq 1$ such that if f is K-quasiconformal, if $z, z' \in H^{n+1}$ and if $d(z, z') \geq M$, then

$$d(z, z')/L \leq d(F(z), F(z')) \leq Ld(z, z'). \tag{1.5}$$

This will be proved in Section 3. We give here only the definition of F in H^{n+1}. If $X \subset H^{n+1}$ is non-empty and bounded, there is a well-defined hyperbolic disk of minimal radius containing X. Let $P(X)$ be the center of this disk. We set for $z \in H^{n+1}$

$$F(z) = P(pf(p^{-1}(z))).$$

Note that the set $p^{-1}(z)$ is compact and hence the continuous image $pf(p^{-1}(z))$ of it is bounded. Here we used f also to denote the map $(u, v, w) \mapsto (f(u), f(v), f(w))$.

Then obviously $F(H^{n+1}) \subset H^{n+1}$ and by Theorem 3.1 we get in this manner a continuous and G-compatible map $\bar{H}^{n+1} \to \bar{H}^{n+1}$ extending f which satisfies (1.5) by Theorems 3.4 and 3.6.

A map $F: H^{n+1} \to H^{n+1}$ which satisfies (1.5) is called a *pseduo-isometry*. This notion is due to Mostow [25]; actually, his definition is slightly stronger since he requires that the right-hand inequality of (1.5) is valid for all $z, z' \in H^{n+1}$. One knows that pseudo-isometries of H^{n+1} admit continuous extensions to $\bar{\mathbf{R}}^n$ such that the maps of $\bar{\mathbf{R}}^n$ so obtained are quasiconformal homeomorphisms of $\bar{\mathbf{R}}^n$; see [31] where the proof uses the stronger definition of a pseudo-isometry, but the result is valid also for the weaker one. This theorem is essentially due to Efremovic and Tihomirova [10]. Thus Theorem 1 is also a converse to this result.

1C. *The main theorem.* It is natural to try to deform the pseudo-isometry of Theorem 1 to a quasiconformal extension of f. Indeed, if $G=\{\mathrm{id}\}$, then this can be done since in this case one can find a homeomorphic pseudo-isometric extension [41]. Here we apply the preceding ideas to show that, if $n=1$, quasisymmetric maps allow a quasiconformal G-compatible extension.

We say that a homeomorphism f of $\bar{\mathbf{R}}$ is *quasisymmetric* if $f(\infty)=\infty$ and if for some $k \geq 1$ and all $x, t \in \mathbf{R}$, $t>0$,

$$1/k \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leq k \tag{1.6}$$

in which case we say also that f is k-quasisymmetric. Note that f may be also orientation reversing.

We find it convenient denote the open and closed upper half-planes of $\bar{\mathbf{R}}^2=\bar{\mathbf{C}}$ by U and $\bar{U}$ instead of H^2 and $\bar{H}^2$. We then have

THEOREM 2. *Let $k \geq 1$. Then there is $K=K(k) \geq 1$ such that if G is a group of Möbius transformations of $\bar{\mathbf{R}}$ and if f is a G-compatible k-quasisymmetric map of $\bar{\mathbf{R}}$, then f can be extended to a G-compatible K-quasiconformal homeomorphism of $\bar{U}$.*

We prove this theorem in the course of this paper, and the proof is completed in Section 5E. We indicate here only the main lines of the proof.

Only the case of discrete G is difficult; if G is non-discrete then f is almost always a Möbius transformation. The case in which it need not be is easily reduced to the discrete case and the theorem is shown to be true with the same K, cf. Section 5E.

So assume that G is discrete; i.e., a Fuchsian group. For simplicity we assume that G does not contain parabolic elements. We first show that, given k, there is $M=M(k)$ such that if

$$d(x, g(x)) \geq M \tag{1.7}$$

for all $x \in U$ and $g \in G \setminus \{\mathrm{id}\}$, then there is a $K(k)$-quasiconformal extension of f, cf. Theorem 5.2.

To construct the extension in this special case, we first find a G-invariant triangulation $\mathcal{T}$ of U whose triangles are large, that is, their angles are small, cf. Theorem 4.5. The degree largeness depends on k. We then fix for every vertex a of $\mathcal{T}$ a triple z such that $p(z_a)=a$, p as in (1.1). We do this in a G-invariant manner. If T is a triangle of $\mathcal{T}$ with vertices a_1, a_2, a_3, then there is a non-degenerate hyperbolic triangle T' with vertices $pf(z_{a_i})$ (Theorem 3.8). We define the extension in such a way that $F(T)=T'$, and, indeed, in this manner one obtains a homeomorphic G-compatible extension (Corollary 3.9) which can be made $K(k)$-quasiconformal.

If G is finitely generated and does not contain parabolic elements, then G has a normal subgroup N of finite index which satisfies (1.7), cf. Lemma 5.4. Thus there is a $K(k)$-quasiconformal, N-compatible extension of f. Now Lemma 5.5 implies that then there is also a G-compatible, $K(k)$-quasiconformal extension of f. Lemma 5.5 is based on the existence and uniqueness of Teichmüller's extremal mapping.

If G is not finitely generated, there is a sequence $G_1 \subset G_2 \subset \dots$ of finitely generated subgroups whose union G is. Let F_i be a G_i-compatible, $K(k)$-quasiconformal extension of f. Now a normal family argument gives the G-compatible, $K(k)$-quasiconformal extension of f.

If G contains parabolic elements, then we assume in the first step of the proof that (1.7) is true for all $x \in U$ and $g \in G \setminus H_x$ where $H_x=\{\mathrm{id}\}$ or H_x is a cyclic group generated by a parabolic element of G. Now the triangles of $\mathcal{T}$ may also contain hyperbolic triangles with one vertex in $\bar{\mathbf{R}}$; this vertex is fixed by some parabolic $g \in G$. Otherwise the proof is unchanged.

Finally, we remark that Theorem 2 was proved for finitely generated Fuchsian G by Kra [13] (with K depending on G and f). If f is a G-compatible homeomorphism of $\bar{\mathbf{R}}$, then one knows that f has a G-compatible homeomorphic extension to $\bar{U}$ by [33, Theorem 3] (cf. also [32, pp. 31–33]) if G is discrete and by Section 5 E if G is not.

1 D. *Estimates for K.* Our proof does not give an estimate for K in Theorem 2 (except the estimate (5.6) for k near 1). It does not appear impossible to give such estimates but this would entail further complications in an already complicated proof.

If k is not very large, Lehto [17] showed that one can use the Ahlfors-Weill method to obtain the extension, and then it is easy to estimate K. In fact, we can take

$$K=\frac{2k^{3/2}-1}{2-k^{3/2}} \tag{1.8}$$

if $k<2^{2/3}$, cf. (2.7).

1E. *The case* $n=2$. It seems that our methods can be applied also if $n=2$ to construct a quasiconformal extension of a quasiconformal homeomorphism f of $\bar{\mathbf{R}}^2$ which is compatible with a discrete Möbius group of $\bar{\mathbf{R}}^2$. This is due to the fact that in dimension 2 the complex dilatation allows one to decompose f as $f_k \circ \ldots \circ f_1$ where the dilatation of f_i is near 1 and each f_i is G_i-compatible when $G_i = f_{i-1} G_{i-1} f_{i-1}^{-1}$ and $G_0 = G$. In case of near-conformal f, the construction of Theorem 5.2 can be carried out also for suitable triangulations of the hyperbolic 3-space whose simplexes need not be very large. Thus the idea is the same as in the quasiconformal extension from $\mathbf{R}^2$ to $\mathbf{R}^3$ in Ahlfors [1], only the euclidean geometry is replaced by the non-euclidean, which guarantees G-compatibility.

It is probable that this method works for all discrete G and certainly for all geometrically finite G. In fact, in an earlier version of this paper, we included some results for this case. After this version was written, we were informed of chapter 11 of Thurston [31]. This chapter is as yet incomplete, but in it Thurston intends to prove the same theorem using analytic methods, which should allow a smoother proof than the above combinatorial approach.

1F. *Quasiconformal maps of* $\bar{\mathbf{R}}$. We adopt the following convention regarding quasiconformal maps of $\bar{\mathbf{R}}$. Let f be a homeomorphism of $\bar{\mathbf{R}}$ (not necessarily orientation preserving). It is *quasiconformal* if the following is true for some $K \geq 1$. If $a, b, c, d \in \bar{\mathbf{R}}$ are distinct and follow one another in $\bar{\mathbf{R}}$, let $M(a, b, c, d)$ be the modulus of the quadrilateral with vertices a, b, c, d and interior U, cf. [18, I.2.4] where this was defined for such a, b, c, d which are on positive order on $\partial U = \bar{\mathbf{R}}$; if they are in negative order, we set $M(a, b, c, d) = M(d, c, b, a)$. Then

$$M(a, b, c, d)/K \leq M(f(a), f(b), f(c), f(d)) \leq KM(a, b, c, d) \tag{1.9}$$

for all such quadrilaterals. We say also that f is K-quasiconformal. The smallest number $K \geq 1$ satisfying (1.9) is the *dilatation* $K(f)$ of f. If f fixes ∞, then f is quasisymmetric, that is (1.6) is true for some $k \geq 1$. The smallest number $k \geq 1$ satisfying (1.6) is the *quasisymmetry constant* $Q(f)$ of f. The advantage of the notion of quasiconformality for us is due to the fact that we can now freely compose f with Möbius transformations without changing the dilatation. Also, we can define a metric in the universal Teichmüller space (cf. Section 2B) by means of dilatation since now $K(f \circ g) \leq K(f) K(g)$ and $K(f^{-1}) = K(f)$; these relations are not true if we replace $K(\ )$ by $Q(\ )$.

We note the following relations between $Q(f)$ and $K(f)$. If f is the restriction of a K-quasiconformal map of $\bar{U}$, then $K(f) \leq K$. Since every k-quasisymmetric map can be extended to a k^2-quasiconformal map of $\bar{U}$ (Beurling-Ahlfors [8, Theorem 1]) we have first of the inequalities below, which are valid for all quasisymmetric f,

$$K(f) \leq Q(f)^2 \quad \text{and} \quad Q(f) \leq e^{(K(f)-1)/A} \tag{1.10}$$

where $A=0.2284$. The second inequality follows from [8, p. 131]. Note that, together with (1.10), the Beurling-Ahlfors extension of a quasisymmetric map implies that every K-quasiconformal self-map of $\bar{\mathbf{R}}$ can be extended to a $K'(K)$-quasiconformal self-map of $\bar{U}$.

The proof of our Theorem 2 makes use of (1.10) whose first inequality is obtained by means of the Beurling-Ahlfors extension of a quasisymmetric map. Hence our extension is not independent of the Beurling-Ahlfors extension. However, it is also possible to prove (1.10) directly. Thus the dependence is not essential, and we get a new extension also for the case $G=\{\mathrm{id}\}$. Actually, we only need to know that if $f: \bar{\mathbf{R}} \to \bar{\mathbf{R}}$ is k-quasisymmetric and if g and h are Möbius transformations such that $gfh(\infty)=\infty$, then gfh is k'-quasisymmetric for some $k'=k'(k)$. This follows by Väisälä [43, Section 3].

1G. *Notation and terminology* (in addition to the ones given before). A *Möbius group* G of $\bar{\mathbf{R}}^n$ is a group of Möbius transformations of $\bar{\mathbf{R}}^n$. The action of G extends to $\bar{H}^{n+1}$ and we do not distinguish between these two groups. Note that these groups may contain also orientation reversing elements. A Möbius group has a natural topology, and a *Fuchsian* group is a discrete Möbius group of $\bar{\mathbf{R}}$. If $x \in \bar{H}^{n+1}$ the *stabilizer* of G at x is $G_x=\{g \in G: g(x)=x\}$.

The hyperbolic metric of H^{n+1} is given by $|dx|/x_{n+1}$, $x=(x_1, \ldots, x_{n+1})$. Note that the formulae of hyperbolic trigonometry are valid for this metric. The diameter of a set and the distance between two sets are denoted $d(A)$ and $d(A, B)$. These notations are used also for the euclidean metric of $\mathbf{R}^n$; if confusion is possible we say which metric we mean. The euclidean distance between two points is $|x-y|$. The hyperbolic and euclidean *closed* disks with center x and radius r are denoted $D(x, r)$ and $B(x, r)$, respectively

$$B^n(r)=B^n(0, r), \quad B^n=B^n(1) \quad \text{and} \quad S^n=\{x \in \mathbf{R}^{n+1}: |x|=1\}.$$

The standard basis of $\mathbf{R}^n$ is $e_1, \ldots, e_n$, and a map of $\bar{\mathbf{R}}^n$ is *normalized* if it fixes 0, e_1 and ∞. There is a natural correspondence between homeomorphisms of $\mathbf{R}^n$ and the ones of $\bar{\mathbf{R}}^n$ fixing ∞, and we often do not distinguish between them. We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and $\bar{\mathbf{C}}=\bar{\mathbf{R}}^2=\mathbf{C} \cup \{\infty\}$. The closure $\mathrm{cl}\, A$ and boundary $\mathrm{bd}\, A$ of a set A are taken in $\bar{H}^{n+1}$; bd is also sometimes denoted ∂, and int denotes the interior. We denote by id the map $A \to A$, $x \mapsto x$, for any set A.

2. Some consequences of Theorem 2

Theorem 2 has some interesting consequences for Teichmüller space theory. Before passing to the proof of Theorem 2, we describe them here.

2 A. Let G be a Fuchsian group. Following Lehto [17] we define $Q(G)$ to be the set of G-compatible, univalent maps of the lower half-plane $L=\{z\in \mathbf{C}: \operatorname{im} z<0\}$; $\Delta(G)\subset Q(G)$ is the subset of maps that can be extended to G-compatible quasiconformal maps of the whole plane. We denote $\Delta(1)=\Delta(\{\mathrm{id}\})$. Then $\Delta(G)\subset\Delta(1)\cap Q(G)$.

It is known that if G is finitely generated (and does not contain orientation reversing elements), then

$$\Delta(G) = \Delta(1)\cap Q(G), \tag{2.1}$$

cf. Kra [13] (stated in a slightly different but equivalent manner.) Bers [5, 6, 7] has drawn attention to the question whether (2.1) is in fact valid for all G. It is a consequence of Theorem 2 that this is indeed so.

THEOREM 2.1. *Equation* (2.1) *is valid for every Fuchsian group* G.

Proof. Lehto [17] has shown that (2.1) is a consequence of Theorem 2. Lehto assumes that G is of the first kind and does not contain elliptic nor orientation reversing elements. However, the proof is valid even without these assumptions. The only place where they are used is the footnote on p. 243 to establish that the group G' of Möbius transformations of $\bar{U}$ with $\{g|\mathbf{R}: g\in G'\}=\{fgf^{-1}: g\in G\}$ acts discontinuously in U whenever f is a G-compatible quasisymmetric map. That this is valid for every Fuchsian group G can be seen, for example, from Theorem 2.

2 B. *Metrics on the Teichmüller space.* We define the *Teichmüller space* $T(G)$ of a Fuchsian group G as the set of all normalized, G-compatible and quasisymmetric maps of $\bar{\mathbf{R}}$. *The universal Teichmüller space* $T(1)$ is the Teichmüller space of $G=\{\mathrm{id}\}$. Obviously then $T(G)\subset T(1)$ for all G. One often adds in the definition of $T(G)$ the condition that every $f\in T(G)$ can be extended to a quasiconformal G-compatible map of $\bar{U}$, but we now know by Theorem 2 that this is a consequence of the other conditions. One can identify $T(G)$ with the subset of $\Delta(G)$ consisting of normalized maps, cf. the proof of Theorem 1 in [17].

One can define several natural metrics on $T(G)$. The Teichmüller metric d_G is defined for $f, g \in T(G)$ by

$$d_G(f, g) = \inf \log K(F \circ G^{-1}) \tag{2.2}$$

where the infimum is taken over all quasiconformal G-compatible extensions F of f and G of g to $\bar{U}$. By Theorem 2 such extensions exist. $K(h)$ is the maximal dilatation of h. This metric depends on G, since there may be many groups G such that f and g are G-compatible. In particular, f and g are always G-compatible for $G = \{\mathrm{id}\}$, and in this case we denote

$$d_U = d_{\{\mathrm{id}\}}.$$

Then $d_U(f, g)$ is given by (2.2) where now F and G run over *all* quasiconformal extensions of f and g. Another natural metric d_Q is given by the dilatation of a quasisymmetric map as defined in Section 1 F; thus

$$d_Q(f, g) = \log K(f \circ g^{-1}) \tag{2.3}$$

which is defined for all $f, g \in T(1)$ viewed as quasiconformal self-maps of $\bar{\mathbf{R}}$. Note that if we replace in (2.3) $K(f \circ g^{-1})$ by the quasisymmetry constant $Q(f \circ g^{-1})$ we do not get a metric, cf. 1 F.

If $N \subset G$ is a subgroup of finite index it follows by Lemma 5.5 that

$$d_N | T(G) = d_G \tag{2.4}$$

where we have abbreviated $d|A \times A$ as $d|A$. Bers [6, p. 274] has raised the question whether $d_U|T(G) = d_G$. A result of Strebel [30] shows that this is not true even when $T(G)$ is finite dimensional. The following theorem shows that, however, a Lipschitz condition can still be obtained.

THEOREM 2.2. *Let G be a Fuchsian group and let $A \subset T(G)$. Then A is bounded in one of the metrics d_G, d_U or d_Q if and only if it is bounded in all of them. In addition, if A is bounded, there is a constant $L \geq 1$, which can be determined as soon as one of the numbers $d_G(A)$, $d_U(A)$ or $d_Q(A)$ is given, such that*

$$d_Q|A \leq d_U|A \leq d_G|A \leq L\, d_Q|A.$$

Proof. It is obvious by the definitions of the metrics that $d_Q \leq d_U \leq d_G$. Theorem 2 implies that every d_Q-bounded set is also d_U- and d_G-bounded. Thus it suffices to find L such that $d_G|A \leq L\, d_Q|A$ if $d_Q(A) < \infty$. We can assume that G does not contain orientation reversing elements, otherwise we can pass by (2.4) to the subgroup of orientation preserving elements.

We observe first: If $f \in T(G)$ and $Q(f) < 2^{2/3}$, then there is a G-compatible quasiconformal extension F of f with

$$\frac{K(F)-1}{K(F)+1} \leq 3\frac{Q(f)^{3/2}-1}{Q(f)^{3/2}+1}. \tag{2.5}$$

This is seen as in Lehto [17, § 4], using the Ahlfors-Weill method. For the convenience of the reader, we recapitulate the main points, although Lehto's argument is unchanged; we simply use a result of Lehtinen (instead of Beurling-Ahlfors) to improve the estimate.

We start from the fact that there are conformal mappings f_1 and f_2 of the upper and lower half-planes U and L onto the complementary domains of a Jordan curve such that on $\bar{\mathbf{R}}$

$$f_2 = f_1 \circ f,$$

see [17, p. 242] and [18, II.7.5]. Then one sees as in [17, proof of Theorem 1] that f_2 is G-compatible; that is, $f_2 \in Q(G)$.

Now, Lehtinen [16] implies that if $Q(f) \leq 1.9$ then f has a $Q(f)^{3/2}$-quasiconformal extension ψ to $\bar{U}$. Then $f_1 \circ \psi$ defines a $Q(f)^{3/2}$-quasiconformal extension of f_2 to $\bar{\mathbf{C}}$. Hence the Schwarzian S of f_2 in L satisfies

$$s = \sup_{z=x+iy \in L} 4y^2|S(z)| \leq 6(Q(f)^{3/2}-1)/(Q(f)^{3/2}+1) < 2, \tag{2.6}$$

cf. Kühnau [14, Satz 3*].

In view of (2.6), one can apply the Ahlfors-Weill method [2] to construct a quasiconformal extension w of f_2. Since f_2 is G-compatible, the Schwarzian of f_2 is a holomorphic quadratic differential for G. Hence the complex dilatation of w is a Beltrami differential of G and so w is G-compatible. Then $F = f_1^{-1} \circ w$ defines a G-compatible quasiconformal extension of f to $\bar{U}$. Its dilatation is that of w which equals $s/2$ and (2.5) follows.

Hence, if $Q(f) < 2^{2/3}$, there is by (2.5) a quasiconformal G-compatible extension F of f with dilatation

$$K(F) \leq \frac{2Q(f)^{3/2}-1}{2-Q(f)^{3/2}} = \frac{1+2(Q(f)^{3/2}-1)}{1-(Q(f)^{3/2}-1)}. \tag{2.7}$$

Thus if $Q(f)^{3/2} \leq 1.009$, then

$$\log K(F) \leq 3.01(Q(f)^{3/2}-1) \leq 4.56 \log Q(f). \tag{2.8}$$

By (1.10), $\log Q(f) \leq (K(f)-1)/0.2284 \leq e^{0.001} \log K(f)/0.2284$ if $\log K(f) \leq 0.001$. In view of (2.8), we get now by passing to the metrics

$$d_G(h, g) \leq 20 d_Q(h, g) \tag{2.9}$$

for all $h, g \in T(G)$ such that $d_Q(h, g) \leq 0.001$.

Let now $h, g \in A$ be arbitrary. If $d_Q(h, g) \leq 0.001$, then $d_G(h, g) \leq 20 d_Q(h, g)$ by (2.9). If $d_Q(h, g) \geq 0.001$, then $d_G(h, g) \leq d_G(A) \leq (d_G(A)/0.001)\, d_Q(h, g)$. Thus $d_G|A \leq L\, d_Q|A$ when $L = \max(20, d_G(A)/0.001)$.

Now $d_Q(A) \leq d_U(A) \leq d_G(A) \leq M(d_Q(A))$ where M is an increasing function whose existence follows by Theorem 2. This implies that we can form an estimate for L from above as soon as one of the numbers $d_Q(A)$, $d_U(A)$ or $d_G(A)$ is known, and the theorem is proved.

Remarks. Actually, we could get this theorem also by estimating the maximal dilatation of the extension of f constructed in Theorem 5.2 when $Q(f)$ is near 1 (see (5.6)), but the advantage of the above method is that we now have the explicit estimates (2.7)–(2.9).

If one knows that f has a K-quasiconformal extension to $\bar{U}$ with $K<2$, then one can use this extension for the map ψ above and get that f has a G-compatible quasiconformal extension F with

$$K(F) \leq \frac{2K-1}{2-K}, \tag{2.10}$$

cf. (2.7). This implies that if we consider in (2.9) the metric d_U instead of d_Q, we get that

$$d_G(h, g) \leq c_k\, d_U(h, g) \tag{2.11}$$

if $h, g \in T(G)$ and $d_U(h, g) \leq k < \log 2$ where $c_k \to 3$ as $k \to 0$.

2C. *Nielsen's theorem.* Theorem 2 gives also a new proof of the following theorem which we call Nielsen's theorem since Nielsen [26] proved it in case that G is the cover translation group of a compact surface. For a different proof and references to other proofs see Marden [21].

THEOREM 2.3. *Let G and G' be finitely generated Fuchsian groups of $\bar{U}$ which are of the first kind. Suppose that $\varphi: G \to G'$ is an isomorphism such that φ carries bijectively parabolic elements of G onto parabolic elements of G'. Then there is a quasiconformal homeomorphism F of $\bar{U}$ inducing φ.*

Proof. By Theorem 2, we must only show that there is a quasiconformal homeomorphism f of $\bar{\mathbf{R}}$ inducing φ. The existence of such an f is well-known from Mostow's rigidity theorem. One first constructs a pseudo-isometry $h: U \to U$ inducing φ. Then h is extended to $\bar{U}$, and $f = h|\bar{\mathbf{R}}$ is quasiconformal and induces φ. See for instance [37, Theorem 3.3]. If U/G is compact, then an especially simple proof of the existence of f can be given, see Margulis [22].

The noteworthy feature of this proof, in contrast to other known proofs, is that no special surface topology is involved. In fact, the construction of the pseudo-isometry $h: U \to U$ is quite general and the proof in [37] is valid in all dimensions. Neither does Theorem 2 involve specific surface topology as a perusal of our proof in this paper shows. Note that the proof of the existence and uniqueness of Teichmüller's extremal mapping in [4] does not make use of a specific knowledge of the topology of U/G, once it is known that there is a quasiconformal map inducing φ (which in the present case is constructed using Theorem 5.2). Only some knowledge of the action of G near parabolic cusps is needed.

The moral is that there is a non-surface-topological proof of Nielsen's theorem. This should be compared with Mostow's rigidity theorem [24, 25] which says that if $\varphi: G \to G'$ is an isomorphism between discrete groups of isometries of hyperbolic n-space, $n \geq 3$, whose orbit spaces have finite volume, then φ is a conjugation by a Möbius transformation. The first step in the proof of this theorem is the same as in the proof of Theorem 2.3: to show that there is a quasiconformal map f of $\bar{\mathbf{R}}^{n-1}$ inducing φ. At this stage there is no essential difference between the cases $n=2$ and $n>2$. Then the regularity of quasiconformal maps (if $n>2$) allows one to show that this quasiconformal map is in fact a Möbius transformation. Here the proof breaks down if $n=2$ since quasisymmetric maps may be very irregular. In fact, f must be then very irregular if it is not a Möbius transformation ([15, 25, 38]): it is a completely singular map which cannot have a non-vanishing, finite derivative at a point x unless x is fixed by a parabolic element of G.

Remarks. (1) By Marden [21], it suffices to assume in Theorem 2.3 that $\varphi(g)$ is parabolic whenever g is. We can recover this result easily as follows. By passing to a subgroup of finite index, we can assume that G is torsionless. Let $P_1, \dots, P_r \subset G$ be the conjugacy classes of maximal parabolic subgroups of G, that is, there is a parabolic $g_i \in P_i$ such that $P_i = \{hg_i^k h^{-1}: h \in G \text{ and } k \in \mathbf{Z},\ k \neq 0\}$ and that g_i is not a power of an element of G. Define similarly $P'_1, \dots, P'_s \subset G'$. Since g^k and hgh^{-1} are parabolic if and only if g is, the assumption that $\varphi(g)$ is parabolic whenever g is, implies that $\varphi(P_i) = P'_j$ for some j. We can assume that $\varphi(P_i) = P'_i$ if $i \leq r$. We must show that $r = s$. If $r < s$, we obtain a contradiction as follows. Let $N \subset G$ and $N' \subset G'$ be the normal subgroups generated by $P_1 \cup \dots \cup P_r$ and $P'_1 \cup \dots \cup P'_r$, respectively. Then G/N is the fundamental group of a compact surface. If $r < s$, G'/N' is the fundamental group of a compact surface with at least one puncture. This readily gives a contradiction.

(2) Suppose that $\varphi: G \to G'$ is an isomorphism between finitely generated Fuchsian groups and that f is a homeomorphism of $\bar{\mathbf{R}}$ which induces φ. If now f is locally quasisymmetric at all points which are not limit points of G, then f is quasisymmetric (cf. Remark 2 in Section 4 B of [37]), and hence one can apply Theorem 2 in this case to obtain a quasiconformal G-compatible extension of f.

3. The triple space

It is a fundamental observation of this paper that it is possible to assign to a triple (x, y, z) of distinct points of $\bar{\mathbf{R}}^n$ a point $w \in H^{n+1}$ by projecting z orthogonally onto the hyperbolic line joining x and y. We now examine this situation in more detail, especially how it is affected by homeomorphisms of $\bar{\mathbf{R}}^n$.

3 A. *The triple space.* Given $n \geq 1$, we define the triple space T^n by

$$T^n = \{(x_1, x_2, x_3) \in (\bar{\mathbf{R}}^n)^3 : x_i \text{ distinct}\}. \tag{3.1}$$

We define a projection $p: T^n \to H^{n+1}$ as follows. If $x \in \bar{H}^{n+1}$ and $y \in \bar{\mathbf{R}}^n$ are distinct, let

$$L(x, y) = \text{the hyperbolic line or ray with endpoints } x \text{ and } y. \tag{3.2}$$

(Note that $y \notin L(x, y)$ and $x \in L(x, y)$ only if $x \in H^{n+1}$.) Now we set for $(x, y, z) \in T^n$,

$$p(x, y, z) = \text{the orthogonal projection of } z \text{ onto } L(x, y); \tag{3.3}$$

that is, $L(x, y)$ and $L(p(x, y, z), z)$ intersect orthogonally at $p(x, y, z)$. We often find it convenient to extend p to a map $T^n \cup \bar{\mathbf{R}}^n \to \bar{H}^{n+1}$ by setting for $x \in \bar{\mathbf{R}}^n$

$$p(x) = x. \tag{3.4}$$

There are two facts about T^n which make it useful for us. The first is that the fiber

$$p^{-1}(w)$$

is compact for all $w \in H^{n+1}$ since it is homeomorphic to the space of 2-frames of $\mathbf{R}^{n+1}$. Thus, in a sense, T^n is not very different from H^{n+1}. The second is that if f is a homeomorphism of $\bar{\mathbf{R}}^n$, then f induces a homeomorphism of T^n, which we denote also by f, by the formula

$$f(x, y, z) = (f(x), f(y), f(z)) \tag{3.5}$$

and furthermore, if f is a Möbius transformation, then it preserves fibers as a map of T^n. That is,

$$p(f(u)) = f(p(u)) \tag{3.6}$$

for all $u \in T^n$ and for all Möbius transformations f of $\bar{\mathbf{R}}^n$.

Consequently, if f is a homeomorphism of $\bar{\mathbf{R}}^n$ which we try to extend to H^{n+1}, we can employ the strategy of first extending f to T^n and then trying to project f back to H^{n+1}. We now give an example of this strategy of which Theorem 5.2 will be a more refined version.

Suppose that the homeomorphism f of $\bar{\mathbf{R}}^n$ is G-compatible for some Möbius group G. That is, there is an isomorphism φ of G onto another Möbius group G' such that

$$f(g(x)) = \varphi(g)(f(x)) \tag{3.7}$$

for all $x \in \bar{\mathbf{R}}^n$. It also remains true if we interprete the maps in it as maps of T^n. This fact allows us to extend f to a G-compatible map $\bar{H}^{n+1} \to \bar{H}^{n+1}$ (which is not in general a homeomorphism) in the following manner.

If $X \subset H^{n+1}$ is non-empty and bounded in the hyperbolic metric, let

$$P(X) = \text{the center of the smallest closed hyperbolic disk containing } X. \tag{3.8}$$

Then a simple calculation shows that $P(X)$ exists and is well-defined (p. 75 of [36], where $n=1$ but the proof is valid for $n>1$ as well). Now $pf(p^{-1}(x)) \subset H^{n+1}$ is compact and hence

$$f(x) = P(pf(p^{-1}(x)) \tag{3.9}$$

defines an extension of f to $\bar{H}^{n+1}$. It is an immediate consequence of (3.7) that the extended f if also G-compatible. In an obvious sense the sets $fp^{-1}(x)$ and $fp^{-1}(y)$ are near each other if x and y are near each other. This implies the continuity of f in H^{n+1}

(see (8) of [36] whose proof is again valid regardless of dimension). We will see that the extension is continuous also in $\bar{\mathbf{R}}^n$ and hence we have

THEOREM 3.1. *Let f be a homeomorphism of $\bar{\mathbf{R}}^n$ which is compatible with a Möbius group G. Then the extension of f to $\bar{H}^{n+1}$ defined by* (3.9) *is continuous and G-compatible.*

Proof. We must only establish the continuity at $\bar{\mathbf{R}}^n$. Let $x\in\bar{\mathbf{R}}^n$ and let W be a closed neighbourhood of $f(x)$ such that $\partial W\cap H^{n+1}$ is a hyperbolic n-plane. The next lemma implies that there is a neighbourhood V of x such that $f(V\cap\bar{\mathbf{R}}^n)\subset W$ and that $pfp^{-1}(y)\subset W$ for all $y\in V\cap H^{n+1}$. Now the center $f(y)$ of the smallest hyperbolic disk containing $pfp^{-1}(y)$ must lie in W since otherwise one could find a smaller hyperbolic disk $D\supset pfp^{-1}(y)$ whose center is the orthogonal projection of $f(y)$ onto ∂W, as a simple geometric argument shows. (See the argument on p. 75 of [36]. Note that we obtain also that $f(y)\in W$ by a limit process from (A) of [36] when $r\to\infty$ in (A).)

LEMMA 3.2. *Let $x\in\mathbf{R}^n$ and let $r>0$. Let $y=(y_1,y_2,y_3)\in T^n$. Then*

(a) *if at least two of the points y_i are in $B^n(x,r)$, then $p(y)\in B^{n+1}(x,(\sqrt{2}+1)r)$,*

(b) *if at most one of the points y_i is in $B^n(x,r)$, then $p(y)\notin B^{n+1}(x,r/(\sqrt{2}+1))$.*

This is true also for $x=\infty$ if we set $B^k(\infty,r)=\bar{\mathbf{R}}^k\setminus B^k(1/r)$.

Proof. If $y_1,y_2\in B^n(x,r)$, then $p(y)\in L(y_1,y_2)\subset B^{n+1}(x,r)$. Suppose then that $y_i,y_3\in B^n(x,r)$, $i\leq 2$. Let z be the orthogonal projection of $p(y)$ onto $L(y_i,y_3)$. Then the hyperbolic triangle with vertices y_i, $p(y)$ and z has a zero angle at y_i, an angle of $\pi/4$ at $p(y)$ and a right angle at z. Then hyperbolic trigonometry implies ([3, 7.9])

$$\cosh d(p(y),z)=\cosh d(p(y),L(y_i,y_3))=1/\sin\pi/4=\sqrt{2} \tag{3.10}$$

Since $L(y_i,y_3)\subset B^{n+1}(x,r)$ and $\operatorname{ar\,cosh}\sqrt{2}=\log(\sqrt{2}+1)=d(\partial B^{n+1}(x,(\sqrt{2}+1)r)\cap H^{n+1},$ $B^{n+1}(x,r))$, case (a) follows.

In case (b), if $y_1,y_2\notin B^n(x,r)$, then $L(y_1,y_2)\cap B^n(x,r)=\emptyset$ and hence $p(y)\notin B^n(x,r)$. Suppose then that $y_i,y_3\notin B^{n+1}(x,r)$. Now as above (3.10) implies that $p(y)\notin B^{n+1}(x,r/(\sqrt{2}+1))$.

Finally, the argument for $x=\infty$ is the same.

3B. *Hyperbolic triangulations and maps of $\bar{\mathbf{R}}^n$.* We now define what one means by a hyperbolic triangulation of H^{n+1}. We first give the definition of a hyperbolic k-simplex.

If $v_0, \ldots, v_k \in \bar{H}^{n+1}$, we denote by $\mathrm{Co}(v_0, \ldots, v_k)$ the smallest subset A of $\bar{H}^{n+1}$ such that $v_i \in A$ and that $A \cap H^{n+1}$ is hyperbolically convex. A (hyperbolic) *k-simplex* of H^{n+1} is a set T which is of the form $T = \mathrm{Co}(v_0, \ldots, v_k) \cap H^{n+1}$ such that the points v_i are not contained in $\mathrm{cl}\,\bar{H}$ for any hyperbolic m-subplane of H^{n+1}, $m<k$. If T is an $(n+1)$-simplex, then an easy induction argument shows that $\mathrm{int}\,T \neq \emptyset$ and that ∂T is a union of n-simplexes. A *face* (or m-face) or T is a simplex T' which is of the form $T' = \mathrm{Co}(u_0, \ldots, u_m) \cap H^{n+1}$ where u_i are distinct and $\{u_0, \ldots, u_m\} \subset \{v_0, \ldots, v_m\}$. *Vertices* of T are the points v_i and it is easy to see that the set T defines uniquely the set of vertices of T.

A (hyperbolic) *triangulation* of H^{n+1} is a collection $\mathcal{T}$ of $(n+1)$-simplexes of $\bar{H}^{n+1}$ such that

(i) *$\mathcal{T}$ is a locally finite cover of H^{n+1}, and*

(ii) *if $T, S \in \mathcal{T}$, then $T \cap S$ is either empty or a common face.*

A *vertex* of $\mathcal{T}$ is a vertex of some simplex of $\mathcal{T}$.

Let f be a homeomorphism of $\bar{\mathbf{R}}^n$ and $\mathcal{T}$ be a triangulation of H^{n+1}. If $F: H^{n+1} \to H^{n+1}$ is a map, we say that F is *compatible* with f and $\mathcal{T}$ if it is true that

(a) *for every vertex v of $\mathcal{T}$, there is $z_v \in T^n \cup \bar{\mathbf{R}}^n$ such that $p(z_v) = v$ and that if $T \in \mathcal{T}$ has vertices $v_0, \ldots, v_{n+1}$, then*

$$F(T) \subset \mathrm{Co}(pf(z_{v_0}), \ldots, pf(z_{v_{n+1}})),$$

(b) *if $T \in \mathcal{T}$, then F and f define together a continuous map* $\mathrm{cl}\,T \to \bar{H}^{n+1}$.

We show that F and f define in fact a continuous map of $\bar{H}^{n+1}$ and give a natural condition when it is a homeomorphism.

THEOREM 3.3. *Let f be a homeomorphism of $\bar{\mathbf{R}}^n$ and $\mathcal{T}$ a triangulation of H^{n+1}. Suppose that $F: H^{n+1} \to H^{n+1}$ is compatible with f and $\mathcal{T}$. Then F and f define a continuous map $\bar{H}^{n+1} \to \bar{H}^{n+1}$.*

Furthermore, suppose that $F|T$ is an embedding for every $T \in \mathcal{T}$ which is always, independently of T, orientation preserving or reversing. Then F and f define a homeomorphism of $\bar{H}^{n+1}$.

Proof. Since $\mathcal{T}$ is locally finite in H^{n+1}, (b) implies that F is continuous at points of H^{n+1}. Hence it suffices to consider the continuity at $\bar{\mathbf{R}}^n$.

Let $x \in \bar{\mathbf{R}}^n$. In view of Lemma 3.2 and the compatibility of F with f and $\mathcal{T}$ (since f is in any case continuous at x), the continuity follows if we can prove:

Let W be a neighbourhood of x in $\bar{H}^{n+1}$. Then there are a smaller neighbourhood V of x and simplexes $T_1, \ldots, T_m$ of $\mathcal{T}$ such that if $T \in \mathcal{T}$, $T \neq \mathcal{T}_i$, and $T \cap V \neq \emptyset$, then

$$T \subset W.$$

If this is not true we can find a sequence $S_1, S_2, \ldots$ of distinct simplexes of $\mathcal{T}$ such that there are points a_i, $b_i \in S_i$ for which

$$\lim_{i\to\infty} a_i = x \quad \text{and} \quad \lim_{i\to\infty} b_i = y \neq x.$$

Let $u \in L(x, y)$. Then every neighbourhood of u intersects with an infinite number of S_i's which contradicts the local finiteness of $\mathcal{T}$.

This proves the first paragraph of the theorem. We now denote also by F the map defined by F and f.

The second paragraph of the theorem follows from the properties of the degree of the map F, cf. [9, 27]. We follow here the exposition of Dold [9].

The map F is a map of the pair $(\bar{H}^{n+1}, \bar{\mathbf{R}}^n)$ onto itself and hence it induces a homomorphism F_* of the homology group $H_{n+1}(\bar{H}^{n+1}, \bar{\mathbf{R}}^n)$ onto itself. There is an integer $\deg F$ such that $F_*(u) = (\deg F)u$ for $u \in H_{n+1}(\bar{H}^{n+1}, \bar{\mathbf{R}}^n)$. This integer is the degree of F. Since $F|\bar{\mathbf{R}}^n$ is a homeomorphism of $\bar{\mathbf{R}}^n$, we have

$$\deg F = \pm 1,$$

cf. [9, IV.4.2].

We conclude the proof by showing that for every $x \in H^{n+1}$, $F^{-1}x$ consists of exactly $|\deg F| = 1$ point. Since $F^{-1}x \subset H^{n+1}$ is compact, $\mathcal{T}$ is locally finite, and $F|T$ is an embedding for every $T \in \mathcal{T}$, in any case $F^{-1}x$ is finite,

$$F^{-1}x = \{x_1, \ldots, x_q\}$$

for some $q \geq 0$. In particular, this means that the local degree of F (regarded as a map of H^{n+1}) over x, denoted $\deg_x F$, can be defined as in [9, IV.5.1]. By [9, IV.5.6],

$$\deg_x F = \deg F = \pm 1.$$

Pick now disjoint open balls $B_i \subset H^{n+1}$ with center x_i. Let $F_i = F|B_i$. Again, $\deg_x F_i$ is defined for all i and [9, IV.5.8] implies that

$$\pm 1 = \deg_x F = \sum_{i=1}^{q} \deg_x F_i.$$

Suppose now that $F^{-1}x\subset\bigcup\{\operatorname{int} T: T\in\mathcal{T}\}$. Then we can choose the balls B_i so small that each F_i is an embedding, which is then by assumption always orientation preserving or reversing independently of i. Hence then either $\deg_x F_i=1$ or $\deg_x F_i=-1$ for all i. Then the above sum formula gives that in this particular case indeed $q=1$ and $F^{-1}x$ is a point.

We get the general case now as follows. Choose a ball B with center x such that $B\cap F(\partial B_i)=\emptyset$ for all i. Then [9, IV.5.12] implies that

$$\deg_x F_i = \deg_y F_i$$

for all $y\in B$ and all i.

Pick now an integer $j\in[1,q]$ and $T\in\mathcal{T}$ such that $x_j\in T$. In view of (a), there is $z\in B_j\cap\operatorname{int} T$ such that, setting $y=F(z)$, $y\in B$ and $F^{-1}y\subset\bigcup\{\operatorname{int} T: T\in\mathcal{T}\}$. As we observed above, then $F^{-1}y$ = a point $=\{z\}$. It follows ([9, IV.5.4 and IV.5.8]) that

$$\deg_y F_i = \deg_x F_i = \begin{cases} 0 & \text{if } i\neq j, \\ \pm 1 & \text{if } i=j. \end{cases}$$

This is a contradiction if $q>1$ since the integer $j\in[1,q]$ can be chosen arbitrarily. Also the case $q<1$ is impossible since then $\deg_x F=0$ by [9, IV.5.4]. This is a contradiction since $\deg_x F=\deg F=\pm 1$ as we observed above. Hence $q=1$ and the theorem is proved.

Remark. Suppose that $f\colon \bar{\mathbf{R}}^n\to\bar{\mathbf{R}}^n$ is a continuous map and $F\colon H^{n+1}\to\bar{H}^{n+1}$ is a map such that F satisfies (a) and (b) with respect to f and a triangulation $\mathcal{T}$ of H^{n+1} when we extend the definition of p for non-distinct $x,y,z\in\bar{\mathbf{R}}^n$ by $p(x,y,z)=w$ if at least two of the points x,y,z equal w. Then one sees as above that F and f define a continuous map $\bar{H}^{n+1}\to\bar{H}^{n+1}$.

3C. *Quasiconformal maps and the triple space.* Our discussion in the preceding sections was valid for all homeomorphisms f of $\bar{\mathbf{R}}^n$. We now consider the situation for quasiconformal f. For this class of mappings we prove results which are generalizations of the facts that Möbius transformations preserve the hyperbolic metric of H^{n+1} and that they induce mappings in T^n preserving the fibers $p^{-1}(x)\subset T^n$ for $x\in H^{n+1}$.

Both of our theorems in this section depend on the fact that the set of normalized K-quasiconformal homeomorphisms of $\bar{\mathbf{R}}^n$ is compact, see [42, 20.5 and 21.1] for $n>1$ and for $n=1$ either [8, Theorem 2] or [40, 3.4–3.7].

THEOREM 3.4. *Let $n\geq 1$ and $K\geq 1$. Then there is $m_K^n\geq 0$ such that if f is a K-quasiconformal homeomorphism of $\bar{\mathbf{R}}^n$ and if $x,y\in T^n$ and $p(x)=p(y)$, then*

$$d(pf(x),pf(y))\leq m_K^n.$$

Proof. If g is a Möbius transformation of $\bar{\mathbf{R}}^n$, then the extension of g to H^{n+1} preserves the hyperbolic metric, the map of T^n induced by g preserves fibers $p^{-1}(u)$ and g commutes with p (see (3.7)). These facts imply that by using auxiliary Möbius transformations we can assume that $p(x)=p(y)=e_{n+1}$ and that f is normalized.

Let $A=\{h(u): u \in p^{-1}(e_{n+1}),\ h$ a normalized K-quasiconformal homeomorphism of $\bar{\mathbf{R}}^n\}$. Now both $p^{-1}(e_{n+1})$ and the set of normalized K-quasiconformal maps of $\bar{\mathbf{R}}^n$ are compact and hence so is A. It follows that the map $A \times A \to \mathbf{R}$, $(u,v) \to d(p(u), p(v))$, attains a maximal value m_K^n for which the theorem is true.

Our next theorem is based on the well-known fact that quasiconformal maps are Hölder continuous ([23, 3.2]; for quasisymmetric maps see [40, 3.10 and 3.14]. Since the argument of [40] is very simple for quasiconformal homeomorphism of $\mathbf{R}^n$, we give the theorem we need here.

LEMMA 3.5. *Let $n \geq 1$ and $K \geq 1$. Then there are $\alpha \geq 1$ and $C \geq 1$ such that if f is a K-quasiconformal homeomorphism of $\mathbf{R}^n$ and if $x, y, z \in \mathbf{R}^n$ are distinct and $|y-x| \leq |z-x|$, then*

$$C^{-1}\left(\frac{|y-x|}{|z-x|}\right)^{\alpha} \leq \frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq C\left(\frac{|y-x|}{|z-x|}\right)^{1/\alpha}. \tag{3.11}$$

Proof. There are $a \in (0,1)$ and $b>0$ such that if h is a normalized K-quasiconformal homeomorphism of $\bar{\mathbf{R}}^n$, then $|u| \leq a$ implies $|h(u)| \leq 1/2$ and $|u| \leq 1$ implies $|h(u)| \leq b$. Using auxiliary similarity maps we get now that, if f, x, y, and z are as in the lemma and $r=|y-x|/|z-x|$ and $r'=|f(y)-f(x)|/f(z)-f(x)|$, then $r \leq a$ implies $r' \leq 1/2$ and $r \leq 1$ implies $r' \leq b$. A recursive argument, using a sequence y_i such that $|y_0-x|/|y-x| \in [a,1]$ and $|y_{i+1}-x|/|y_i-x|=a$, now gives that $r \in [a^{k+1}, a^k]$ implies $r' \leq b2^{-k}$. This implies the right-hand inequality and a similar argument gives the left-hand one.

THEOREM 3.6. *Let $n \geq 1$ and $K \geq 1$. Then there are $L_K^n \geq 1$ and $M_K^n \geq 0$ such that if f is a K-quasiconformal homeomorphism of $\bar{\mathbf{R}}^n$ and if $x_i \in H^{n+1}$ and $u_i \in p^{-1}(x_i)$ for $i=1,2$, then*

$$d(x_1,x_2)/L_K^n \leq d(pf(u_1), pf(u_2)) \leq L_K^n d(x_1,x_2) \tag{3.12}$$

provided that $d(x_1,x_2) \geq M_K^n$.

Proof. As in the proof of the preceding theorem, we can assume that f is normalized, $x_1=e_{n+1}$ and that $x_2=t e_{n+1}$ for some $0<t\leq 1$. If $u_i' \in p^{-1}(x_i)$, then

$$|d(pf(u_1), pf(u_2)) - d(pf(u_1'), pf(u_2'))| \leq 2m_K^n \tag{3.13}$$

by Theorem 2.4.

In view of (3.13), it suffices to estimate the middle term of (3.12) for $u_1=(0,\infty,e_1)$ and $u_2=(0,\infty,te_1)$. Then $f(u_1)=u_1$ and $f(u_2)=(0,\infty,f(te_1))$ and thus

$$d(pf(u_1),pf(u_2))=d(e_{n+1},|f(te_1)|e_{n+1})=|\log f(te_1)|. \tag{3.14}$$

Since $d(x_1,x_2)=d(e_{n+1},te_{n+1})=\log t$, we obtain from this and (3.11), where we set $x=0$, $y=te_1$ and $z=e_1$,

$$-\log C+d(x_1,x_2)/\alpha\leq d(pf(u_1),pf(u_2))\leq\log C+\alpha d(x_1,x_2) \tag{3.15}$$

provided that $|f(te_1)|\leq 1$ which is true if $t\leq C^{-\alpha}$ by (3.11). Combined with (3.13) this implies the theorem.

Remarks. (1) By [23, 3.2 and 3.4], we can choose $\alpha=K^{1/(n-1)}$ in (3.11) if $n>1$. Consequently (3.12) is true with $L_K^n\to K^{1/(n-1)}$ as $d(x_1,x_2)\to\infty$, cf. (3.15).

(2) It is not difficult to define a metric in T^n (resembling the hyperbolic metric of H^{n+1}) in such a way that if a map of T^n is induced by a Möbius transformation of $\bar{\mathbf{R}}^n$, then it is an isometry, and if it is induced by a K-quasiconformal map, then it is an (L,M)-pseudo-isometry with L and M depending only on K and n.

3D. *Transformation of angles.* Next we study how quasiconformal maps of $\bar{\mathbf{R}}^n$ affect angles of hyperbolic triangles, the transformation for triangles being affected via the triple space T^n.

We first prove a result which shows that for large hyperbolic triangles the angle at a vertex and the distance to the opposite side are closely connected.

LEMMA 3.7. *Let T be a hyperbolic triangle of H^{n+1} (possibly with zero angles). Let $v\in H^{n+1}$ be a vertex of T, let $\alpha\in(0,\pi)$ be the angle of T at v and let r be the distance of v to the hyperbolic line containing the side of T opposite to v. Let α_1 and α_2 be the other two angles of T, $\alpha_1\leq\alpha_2$. Then*

$$2\cos\alpha_2/\cosh r<\alpha<2\cos\alpha_1/\sinh r\leq 2/\sinh r. \tag{3.16}$$

If T has a right angle and $\alpha\neq\pi/2$, then

$$\cos\beta/\cosh r<\alpha<\cos\beta/\sinh r\leq 1/\sinh r. \tag{3.17}$$

when β is the other angle of T not equal to $\pi/2$.

Proof. We prove only (3.17) which then obviously implies (3.16). So let the situation be as in (3.17). Then

$$\alpha>\sin\alpha=\cos\beta/\cosh r \tag{3.18}$$

by [3, 7.11.3], and we have the left-most inequality. Using again the equality in (3.18), we get

$$\begin{aligned}\alpha < \sin\alpha/\cos\alpha &= \cos\beta/\cosh r(1-(\cos\beta/\cosh r)^2)^{1/2}\\ &= \cos\beta/(\cosh^2 r-\cos^2\beta)^{1/2}\\ &\leq \cos\beta/\sinh r \leq 1/\sinh r, \end{aligned} \tag{3.19}$$

and we have also the other inequalities of (3.17).

Now we can prove our theorem on distortion of angles which is similar to Theorem 3.6.

THEOREM 3.8. *Let $n\geq 1$ and $K\geq 1$. Then there are $\beta^n_K\in(0,1]$ and $s=s^n_K\geq 1$ with the following property. Let T^1 be a hyperbolic triangle, possibly with zero angles, and let $x^1_i\in\bar{H}^{n+1}$, $i\leq 3$, be the vertices of T^1. Let α^1_i be the angle of T^1 at x^1_i and assume that $0\leq\alpha^1_i\leq\beta^n_K$ for $i\leq 3$. Let f be a K-quasiconformal homeomorphism of $\bar{\mathbf{R}}^n$ and let $z_i\in p^{-1}(x^1_i)$. Set $x^2_i=pf(z_i)$ and let T^2 be the hyperbolic triangle with vertices x^2_i, $i\leq 3$. Then T^2 is a non-degenerate hyperbolic triangle such that if α^2_i is the angle of T^2 at x^2_i, then*

(a) $(\alpha^1_i)^s\leq\alpha^2_i\leq(\alpha^1_i)^{1/s}$.

(b) *Let $n=1$. Let T^i have the orientation induced by the triple (x^i_1,x^i_2,x^i_3). Suppose that T^1 is oriented compatibly with the natural orientation of $U=H^2$. Then this is true of T^2 if and only if f is orientation preserving.*

Proof. Since f^{-1} is also K-quasiconformal, it suffices to prove only the right-hand inequality of (a). We can also assume that $x^1_i\in H^{n+1}$ since this then implies the general case by a limit process, cf. Lemma 3.2.

Let L^1_i be the hyperbolic line containig the side of T^1 opposite to x^1_i and let

$$r^1_i=d(x^1_i,L^1_i). \tag{3.20}$$

In view of (3.16), the right-hand inequality of (a) follows if we can prove:

There are $r=r(n,K)>0$ and $c=c(n,K)>0$ such that if $r^1_i\geq r$ for $i\leq 3$, then T^2 is non-degenerate and

$$cr^1_i\leq d(x^2_i,L^2_i) \tag{3.21}$$

when L^2_i is the hyperbolic line containing the side of T^2 opposite to x^2_i.

We proceed to prove (3.21). Clearly, we can assume that $i=1$ and that the situation is such that f is normalized, $x_1^1\in L(e_{n+1},0)$, and $x_2^1, x_3^1\in L(e_1,-e_1)$. If $r\geq 1$, which we now assume, (3.16) implies that all the angles of T^1 are acute and hence x_2^1 and x_3^1 are in different components of $L(e_1,-e_1)\setminus\{e_{n+1}\}$. Thus we can choose the notation in such a way that

$$x_1^1\in L(e_{n+1},0),\quad x_2^1\in L(e_{n+1},e_1)\quad\text{and}\quad x_3^1\in L(e_{n+1},-e_1). \tag{3.22}$$

We have $d(x_1^1,e_{n+1})=r_1^1\to\infty$ as $r\to\infty$ in (3.21). By (3.16), the angles of $T\to 0$ as $r\to\infty$. By (3.17), then also $d(x_2^1,e_{n+1})\to\infty$ and $d(x_3^1,e_{n+1})\to\infty$ as $r\to\infty$ (see (3.22)). As a consequence we obtain that

$$|x_i^1-x_i|\leq\delta \tag{3.23}$$

where $x_1=0$, $x_2=e_1$ and $x_3=-e_1$ and where $\delta=\delta(r)\to 0$ as $r\to\infty$. Lemma 3.2 implies then that

$$\text{at least two points in the triple } z_i \text{ are in } B^n(x_i,(\sqrt{2}+1)\delta). \tag{3.24}$$

We prove next that, given $\varepsilon>0$, there is $r'=r'(n,K,\varepsilon)$ such that if $r_i^1\geq r'$, then

$$|x_i^2-f(x_i)|\leq\varepsilon. \tag{3.25}$$

Now, the family of normalized K-quasiconformal maps of $\bar{\mathbf{R}}^n$ is equicontinuous (for instance, one can prove this using (3.11).) In the present situation this means that we can find $\delta'=\delta'(n,K,\varepsilon)\in(0,1)$ such that if $u,v\in B^n(2)$ and $|u-v|\leq\delta'$, then $|f(u)-f(v)|\leq\varepsilon/(\sqrt{2}+1)$. Choose now $r'=r'(n,K,\delta')=r'(n,K,\varepsilon)$ so big that (3.23) is true with $\delta=\delta'/(\sqrt{2}+1)$ if $r_i^1\geq r'$. Then by (3.24) and by equicontinuity, at least two points in the triple $f(z_i)$ for each $i\leq 3$ are in $B^n(x_i,\varepsilon/(\sqrt{2}+1))$. This implies (3.25) by Lemma 3.2.

Since f is normalized, $f(x_1)=f(0)=0$ and $f(x_2)=f(e_1)=e_1$. However, f need not fix $x_3=-e_1$, but (3.11) implies that

$$|f(x_3)|\geq C^{-1}\quad\text{and}\quad|f(x_3)-e_1|\geq C^{-1}, \tag{3.26}$$

where $C=C(n,K)\geq 1$.

Fix $r''=r''(n,K)\geq 2$ such that (3.25) is true with $\varepsilon=1/4C$ if $r_i^1\geq r''$. Then by (3.25) and (3.26), the points x_i^2, $i\leq 3$, are distinct and hence the line L_1^2 passing through x_2^2 and x_3^2 is well-defined. Let u and v be its endpoints. Again (3.25) and (3.26) imply that $u, v \notin B^n(1/2C)$ and hence

$$L_1^2 \subset H^{n+1} \setminus B^{n+1}(1/2C) \tag{3.27}$$

if $r_i^1 \geq r''$. By (3.25), $x_1^2 \in B^{n+1}(\varepsilon) = B^{n+1}(1/4C)$ and hence, in particular, T^2 is a non-degenerate triangle.

By (3.22), $x_1^1 = t e_{n+1}$ for some $t \in (0, e^{-r''}) \subset (0, 1/5)$. Thus, by Lemma 3.2, at least two of the points of the triple z_1 are in $B^n((\sqrt{2}+1)t)$. Then (3.11) implies that at least two of the points of the triple $f(z_1)$ are in $B^n(C(\sqrt{2}+1)^{1/\alpha} t^{1/\alpha})$ where $\alpha = \alpha(n, K) \geq 1$. Again by Lemma 3.2, we get, if $r_i^1 \geq r''$,

$$x_1^2 = pf(z_1) \in B^{n+1}(C(\sqrt{2}+1)^{1+1/\alpha} t^{1/\alpha}). \tag{3.28}$$

Now $r_1^1 = d(x_1^1, L(-e_1, e_1)) = d(x_1^1, e_{n+1}) = -\log t$, and then, by (3.27) and (3.28),

$$\begin{aligned} d(x_1^2, L_1^2) &\geq d(\partial B^{n+1}(1/2C) \cap H^{n+1}, \partial B^{n+1}(C(\sqrt{2}+1)^{1+1/\alpha} t^{1/\alpha}) \cap H^{n+1}) \\ &= r_1^1/\alpha + \log 1/2C - \log C(\sqrt{2}+1)^{1+1/\alpha} \end{aligned} \tag{3.29}$$

if $r_i^1 \geq r''$. It is now apparent that (3.21) is true for such c and r as claimed and (a) follows.

Part (b) of Theorem 3.8 is an immediate consequence of (3.25) and (3.26), and it is true as soon as $\varepsilon < 1/2C$ in (3.25).

Combining Theorem 3.8 with Theorem 3.3 we get as a

COROLLARY 3.9. *Let f be a K-quasiconformal homeomorphism of $\bar{\mathbf{R}}$ and let $\mathcal{T}$ be a triangulation of U such that the angles of triangles of $\mathcal{T}$ do not exceed the constant β_K^1 of Theorem* 3.8. *Suppose that $F: U \to U$ is compatible with F and $\mathcal{T}$ (see Section* 3B). *Then $F(T)$ is a non-degenerate triangle for every $T \in \mathcal{T}$, and if $F|T$ is always an embedding, then F and f define a homeomorphism of $\bar{U}$.*

Remark. Note that we cannot generalize Theorem 3.8 for $(n+1)$-simplexes of H^{n+1} if $n>1$. Even if T^1 is a hyperbolic $(n+1)$-simples of H^{n+1} whose vertices are on $\bar{\mathbf{R}}^n$ (and hence the distance of a vertex to the hyperbolic n-plane defined by the opposite face is infinite), then one can always find a quasiconformal map of $\bar{\mathbf{R}}^n$ such that if the hyperbolic $(n+1)$-simplex T^2 is defined as in Theorem 3.8, then T^2 is degenerate.

Similarly, part (b) of Theorem 3.8 and Corollary 3.9 cannot be generalized for $n>1$.

4. Tessellations of the hyperbolic plane

We now describe a method to obtain tessellations of the hyperbolic plane $U=H^2$ whose set of vertices is a given set A which needs to satisfy only a boundedness condition. This tessellation is dual to the familiar tessellation whose 2-cells are

$$F_a=\{x\in U\colon d(x,a)\leq d(x,b) \text{ for } b\in A\}, \tag{4.1}$$

$a\in A$, used to construct fundamental domains for discrete groups of hyperbolic isometries. However, we construct this tessellation directly, without making use of the tessellation $\{F_a\}$.

A set $X\subset U$ is *convex* if it is convex in the hyperbolic metric. In this section we say that a set $X\subset U$ is *closed* if it is closed as a subset of U. We denote the closure and boundary in U by cl_U and by bd_U (or ∂_U); recall that cl and bd (or ∂) are the closure and boundary in $\bar{\mathbf{R}}^2$. We also use this notation if $X\not\subset U$ in which case

$$\partial_U X=\mathrm{bd}\,X\cap U.$$

Here we consider only the 2-dimensional case. Higher-dimensional tessellations have been constructed in [39] using the present method.

4 A. *The tessellation* $\mathcal{T}(A)$. Let $A\subset U$ be a discrete set of points. Given A, we define a subset $V=V(A)$ of $\bar{U}$ as follows.

If $v\in U$, then $v\in V$ if and only if there is $r>0$ such that setting $B_v=D(v,r)$=the closed hyperbolic disk with center v and radius r, then

$$\mathrm{int}\,B_v\cap A=\emptyset \quad\text{and}\quad A_v=B_v\cap A \tag{4.2}$$

contains at least three points.

Note that this is the set of points of U that are at least in three cells F_a, $a\in A$, where F_a is defined by (4.1). That is, $V\cap U$ is the set of vertices of the tessellation $\{F_a\}$.

If $v\in\bar{\mathbf{R}}$, then v is in V if and only if there is a horoball B_v of $\bar{U}$ at v (i.e. B_v is a closed ball in the spherical metric, $B_v\subset U\cup\{v\}$ and ∂B_v is tangent to $\bar{\mathbf{R}}$ at v) such that

$$\mathrm{int}\,B_v\cap A=\emptyset \quad\text{and if}\quad A_v=B_v\cap A, \tag{4.3}$$

then $\partial_U B_v\setminus A_v$ consists of a countable number of intervals both of whose endpoints are in A_v.

Later, when we consider Fuchsian groups of U, the set $V\cap\bar{\mathbf{R}}$ will be the set of parabolic fixed points of the group.

We attach to every $v \in V$ a 2-cell C_v as follows. If $a \in A_v$, there is a unique interval I_a of $\partial B_v \setminus A_v$ such that if I_a is oriented compatibly with the orientation of ∂B_v induced on ∂B_v by the natural orientation of B_v (as a subset of $\bar{\mathbf{R}}^2$), then a is the beginning point of I_a. Let S_a be the hyperbolic line segment with the same endpoints as I_a. Then there is a unique 2-cell $C_v \subset B_v \cap U$ such that

$$\partial_U C_v = \bigcup_{a \in A_v} S_a.$$

Then obviously C_v is a closed and convex subset of U such that ∂C_v is a topological 2-cell. It is compact if and only if $v \in V \cap U$.

A vertex of C_v is a point of A_v and a *side* of C_v is a segment S_v for $v \in A_v$.

We can now define the tessellation $\mathcal{T} = \mathcal{T}(A)$ corresponding to A as

$$\mathcal{T} = \{C_v : v \in V\}.$$

A point is a vertex of $\mathcal{T}$ if it is a vertex of some C_v, and a hyperbolic segment is a side of $\mathcal{T}$ if it is a side of some C_v. We denote the set of vertices of $\mathcal{T}$ by $\mathcal{T}_0$ or $\mathcal{T}_0(A)$, and the set of sides of $\mathcal{T}$ is $\mathcal{T}_1$ or $\mathcal{T}_1(A)$. We can also denote $\mathcal{T}_2 = \mathcal{T}$ which is the set of cells of $\mathcal{T}$.

We now show that the name "tessellation" for $\mathcal{T}$ is justified by

THEOREM 4.1. *Let A be a non-empty discrete subset of U. Then $\mathcal{T} = \mathcal{T}(A)$ is a tessellation in the sense that if $C, C' \in \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ are distinct, then*

$$C \cap C'$$

is either empty, a common vertex or a common side. If $C \in \mathcal{T}_i$, $C' \in \mathcal{T}_{i'}$, and $C \cap C' \in \mathcal{T}_j$, then $j < \max(i, i')$.

Proof. Assume that $C \cap C' \neq \emptyset$. Assume first that both C and C' are in $\mathcal{T}_2$. Then $C = C_v$ and $C' = C_{v'}$ for some distinct $v, v' \in V$. If $\operatorname{int} C \cap \operatorname{int} C' \neq \emptyset$, then there is a vertex a either of C or of C' such that $a \in \operatorname{int} C \cap \operatorname{int} C'$. If, say, a is a vertex of C, then $a \in \operatorname{int} B_{v'}$ which is impossible by (4.2) and (4.3). Hence $\operatorname{int} C \cap \operatorname{int} C' = \emptyset$.

Since both C and C' are convex, it now follows that $C \cap C'$ is contained in a side s of C and in a side s' of C'. These are hyperbolic segments whose endponts lie on ∂B_v and $\partial B_{v'}$, respectively. Hence one sees as above that either $s \cap s'$ is a common vertex or $s = s'$. In the latter case $s = C \cap C'$ which is now a common side.

This proves the theorem if C and C' are 2-cells. Other cases follow easily from this.

Note that in general $\mathcal{T}$ is not a tessellation of U since it need not cover U. For instance, this is always so if A is finite. We now give an additional condition which guarantees that $\mathcal{T}$ covers U.

Let

$$U_A = U \setminus (\cup \{B_v : v \in V \cap \bar{\mathbf{R}}\}) \tag{4.4}$$

which depends only on A and suppose that there is $M>0$ such that

$$U_A \subset \bigcup_{a \in A} D(a, M) \quad \text{and} \quad \partial_U B_v \subset \bigcup_{a \in A_v} D(a, M) \tag{4.5}$$

for every $v \in V$. Under these conditions we have

THEOREM 4.2. *Let A be a discrete subset of U satisfying* (4.5). *Then $\mathcal{T} = \mathcal{T}(A)$ is a locally finite cover of U whose set of vertices is A and*

$$d(F) \leq 4M \tag{4.6}$$

whenever F is a side or a compact cell of $\mathcal{T}$.

Proof. We prove first (4.6). Suppose first that F is a compact cell. Then $F = C_v$ for some $v \in V \cap U$. If $v \in U_A$, then there is $a \in A$ such that $d(v, a) \leq M$. It follows that $d(v, a') \leq M$ for $a' \in A_v$ and hence $d(C_v) \leq 2M < 4M$

If $v \notin U_A$, then $v \in B_u$ for some $u \in V \cap \bar{\mathbf{R}}$. There are consecutive points $a_1, a_2 \in A_u$ such that if L_i is the hyperbolic line with endpoint u and passing through a_i, then v is in the closure of the subdomain D of U bounded by L_1 and L_2. In view of (4.3), a simple geometric argument now shows that

$$A_v \subset \operatorname{cl} D \setminus \operatorname{int} B_u$$

since otherwise there would be $b \in \{a_1, a_2\}$ such that $d(v, b) < d(v, a)$ for $a \in A_v$ which is impossible.

Let $B \supset B_u$ be the horoball of $\bar{U}$ at u such that $d(\partial_U B, \partial_U B_u) = M$. Then also

$$A_v \subset B \setminus \operatorname{int} B_u$$

since otherwise by (4.5) there is $b \in A_u$ such that $d(v, b) < d(v, a)$ for $a \in A_v$. These inclusions imply that

$$d(C_v) = d(A_v) \leq d(D \cap (B \setminus \operatorname{int} B_u)) \leq 4M$$

since two of the four arcs bounding $D \cap (B \setminus \operatorname{int} B_u)$ have diameter M and one has diameter $\leq 2M$.

We have shown that (4.6) is true if F is a compact cell. Thus, if F is a side, then $d(F)\leq 4M$ if F is a side of a compact cell of $\mathcal{T}$. If this is not the case, then F is a side of some C_v where $v\in V\cap\bar{\mathbf{R}}$ and then $d(F)\leq 2M$ by (4.5). Hence (4.6) is true also for sides.

Next we show that $\mathcal{T}$ is locally finite in U. Let $x\in U$ and set

$$V_x=\{v\in V\colon x\notin \operatorname{int} C_v \text{ and } \operatorname{int} D(x,1)\cap C_v\neq\emptyset\}.$$

We show that V_x is finite which implies the local finiteness of $\mathcal{T}$ since $x\in\operatorname{int} C_v$ for at most one $v\in V$ by Theorem 4.1.

If $v\in V_x$, then $\operatorname{int} D(x,1)\cap C_v\neq\emptyset$. Hence there is a side S of C_v such that $D(x,1)\cap S\neq\emptyset$. By (4.6), both endpoints of S are in the set

$$A_x=\{a\in A\colon d(a,x)\leq 4M+1\}$$

which is a finite set. Hence the set of sides S of $\mathcal{T}$ for which $D(x,1)\cap S\neq\emptyset$ is finite. By Theorem 4.1, a side of $\mathcal{T}$ is a side of at most two cells C_v for $v\in V$, and it follows that V_x is finite. We have shown that $\mathcal{T}$ is locally finite.

Since every $T\in\mathcal{T}$ is closed in U, it now follows that $\bigcup\mathcal{T}$ is closed in U. Hence, to show that $\mathcal{T}$ is a cover of U, it suffices to show that (a) $\bigcup\mathcal{T}\neq\emptyset$ and that (b) $\bigcup\mathcal{T}$ is open.

Now $\bigcup\mathcal{T}=\emptyset$ if and only if $V=\emptyset$. We show that $V\neq\emptyset$ by showing that $V\cap\bar{\mathbf{R}}=\emptyset$ implies $V\cap U\neq\emptyset$. Assume now that $V\cap\bar{\mathbf{R}}=\emptyset$ and pick $a\in A$. There is such a by (4.5). Choose then a ray R with endpoint a. Let u be the first point on R from a such that there is $b\in A\setminus\{a\}$ with

$$d(a,u)=d(b,u)\leq d(c,u)$$

for $c\in A$. There is such point by (4.5). Let L be the line

$$L=L_{ab}=\{z\in U\colon d(z,a)=d(z,b)\}. \tag{4.7}$$

Note that $u\in L$ and let $R'\subset L$ be either of the rays with endpoint u. Let $v\in R'$ be the first point from u such that there is $c\in A\setminus\{a,b\}$ with

$$d(a,v)=d(b,v)=d(c,v)\leq d(c',v)$$

for $c'\in A$. Again by (4.5), there is such v and then $v\in V\cap U\neq\emptyset$. It follows that $V\neq\emptyset\neq\bigcup\mathcal{T}$.

Finally, we show that every $x\in\bigcup\mathcal{T}$ has a neighbourhood contained in $\bigcup\mathcal{T}$. If $x\in\operatorname{int} C_v$ for some $v\in V$, this is clear. Suppose then that $x\in S$ for some side S of $\mathcal{T}$ and that x is not an endpoint of S. If S is a side of two cells of $\mathcal{T}$, then x has again a neighbourhood in $\bigcup\mathcal{T}$. So we will prove this.

In any case S is a side of at least one C_v for some $v \in V$. Let a and b be the endpoints of S and define

$$S' = \{z \in U\colon d(z, a) = d(z, b) < d(z, c) \quad \text{for } c \in A \setminus \{a, b\}\}.$$

Then S' is, if not empty, a hyperbolic line, segment, or ray. We claim that

(a) v is an endpoint of S' (and hence $S' \neq \emptyset$), and

(b) if u is the other endpoint of S', then $u \in V$ and S is a side of C_u.

These imply that S is indeed a side of two cells of $\mathcal{T}$, as claimed.

Note first that (a) is clear if $v \in U$. Suppose then that $v \in \bar{\mathbf{R}}$. Let $L = L_{ab} \supset S'$ be as in (4.7). We claim that points of L near v are in S' which implies (a). Let D be the domain of U whose boundary consists of the two hyperbolic lines with endpoint v and passing through a and b, respectively. If $w \in D \setminus B_v$ and $z \in L$, then one sees easily that

$$d(w, z) = d(a, z) + d(w, \partial B_v) + o(w, z) \tag{4.8}$$

where $o(w, z) \to 0$ as $z \to v$ on L uniformly for $w \in D \setminus B_v$.

If $z \in L \cap B_v$, then

$$d(a, z) < d(c, z)$$

for $c \in (A \setminus \{a, b\}) \setminus D$. By (4.8), this is true also for $c \in (A \setminus \{a, b\}) \cap D$ if z is near v. These facts imply (a).

Again, to prove (b), it is clear that $u \in V$ if $u \in U$ and that then $a, b \in U$. If $u \in \bar{\mathbf{R}}$, then, by (4.5), points of S' near u must be in some $B_{u'}$, $u' \in V \cap \bar{\mathbf{R}}$. Hence $u = u' \in V$. Choose now $c \in A_u$. If $a \notin A_u$, then, like in (4.8), one sees that $d(z, c) < d(z, a)$ for $z \in S'$ near u. Hence $a \in A_u$ and, similarly, $b \in A_u$.

Since $S \cap \operatorname{int} C_u = \emptyset$ by Theorem 4.1, a and b must be consecutive points on A_u ($\subset \partial B_u$), and thus S is indeed a side of C_u, and (b) is true.

Finally, we show that if x is a vertex of $\mathcal{T}$, then $\bigcup \mathcal{T}$ contains a neighbourhood of x. Since $\mathcal{T}$ is locally finite, the number of cells of $\mathcal{T}$ with vertex x is finite. Since every side of $\mathcal{T}$ is a side of two cells of $\mathcal{T}$, as we have shown, we can arrange the cells of $\mathcal{T}$ with vertex x in order as $C_1, \ldots, C_p = C_1$ such that $C_i \cap C_{i+1}$ is a common side with vertex x. Then $C_1 \cup \ldots \cup C_{p-1}$ contains a neighbourhood of x. Hence $\bigcup \mathcal{T}$ is open, and consequently $\mathcal{T}$ is a cover of U.

Finally, if $a \in A$, a is in some cell C_v of $\mathcal{T}$ and then (4.2) and (4.3) imply that a is a vertex of C_v. It follows that $A \subset \mathcal{T}_0$ and since the opposite inclusion is trivial, A is indeed the set of vertices of $\mathcal{T}$, and the theorem is proved.

In the actual situation where we use tessellations of this kind, the points of A are not too near each other, and in this case we have

LEMMA 4.3. *Let $A\subset U$ be a set such that $d(a,b)\geq m>0$ if $a,b\in A$ are distinct. Let C_v, $v\in V(A)$, be a cell of the tessellation $\mathcal{T}(A)$. Let $a\in A_v$ and let α be the angle of C_v at a. Then*

$$\alpha\leq 2/\sinh(m/2).$$

Proof. Pick a side S of C_v such that a is an endpoint of S. Let b be the other endpoint of S and let u be the bisector of S. Let T be the hyperbolic triangle with vertices a, u and v. Since $d(v,a)=d(v,b)$ (or, if $v\in\bar{\mathbf{R}}$, a and b are on a horoball tangent to $\bar{\mathbf{R}}$ at v), T has a right angle at u. Hence $d(a,u)$ is the distance of a to the hyperbolic line containing the side of T opposite to a. Then (3.17) implies that the angle of T at a $\leq 1/\sinh d(a,u)=1/\sinh(d(a,b)/2)\leq 1/\sinh m/2$. Since the angle of C_v at a is the sum of two such angles, the lemma follows.

4B. *G-invariant triangulations.* Our aim in this section is to construct certain triangulations invariant under a Fuchsian group. We now fix such a group G and assume that there is an $M>0$ such that if $x\in U$, then

$$d(x,g(x))\geq M \tag{4.9}$$

either for all $g\in G\setminus\{\mathrm{id}\}$ or at least for all $g\in G\setminus H_x$ where H_x is a cyclic group generated by a parabolic element (which depends on x.) (Here H_x is not a stabilizer at x; no $g\in H_x\setminus\{\mathrm{id}\}$ fixes x.) Observe that G cannot contain elliptic elements.

We fix some notation. Let

$$P=\{x\in\bar{\mathbf{R}}\colon x \text{ is fixed by some parabolic } g\in G\},$$

and if $v\in P$, let

$$G_v=\{g\in G\colon g(v)=v\},$$

which is a cyclic group generated by a parabolic element. We fix for every $v\in P$ a horoball B_v of $\bar{U}$ at v such that

$$d(x,g(x))=M/3 \tag{4.10}$$

if $x\in\partial_U B_v$ and g is a generator of G_v. Then (4.9) and (4.10) imply that

$$d(\partial_U B_v,\partial_U B_u)\geq M/3 \tag{4.11}$$

for distinct $v, u \in P$. Finally, we set

$$U' = U \setminus \left(\bigcup_{v \in P} B_v \right). \tag{4.12}$$

We first construct a G-invariant set A which is then used in the construction of a G-invariant tessellation $\mathcal{T}(A)$ as in Section 4 A.

LEMMA 4.4. *There is a G-invariant set $A \subset U$ such that*

(i) $U' \subset \bigcup_{a \in A} D(a, M/3)$,

(ii) $d(a, b) \geqslant M/3$ *for distinct* $a, b \in A$,

(iii) $A \cap B_v = G_v a_v$ *for some* $a_v \in \partial_U B_v$ *when* $v \in P$.

Proof. Pick first a G-invariant set $A' \subset H^{n+1}$ such that

$$A' \cap B_v = G_v a_v$$

for some $a_v \in B_v$. We then construct inductively sets $A_0, A_1, \ldots$ as follows. Pick first a set $X = \{x_1, x_2, \ldots\} \subset U'$ which is dense in U'. Set $A_0 = A'$. If A_i has been defined, we set $A_{i+1} = A_i$ if there is $a \in A_i$ such that $x_{i+1} \in D(a, M/3)$; otherwise we set $A_{i+1} = A_i \cup G x_{i+1}$. An inductive argument, using (4.10) and (4.11) for A_0, now easily shows that $A = A_0 \cup A_1 \cup \ldots$ satisfies the conditions of the lemma.

We then consider the set $V \subset \bar{U}$ defined by (4.2) and (4.3) using the set A constructed in Lemma 4.4. It is obvious that

$$V \cap \bar{\mathbf{R}} = P \tag{4.13}$$

and that the horoballs B_v defined by (4.10) and (4.3) coincide. Hence the set U' of (4.12) is the set U_A of (4.4). Thus conditions (i), (iii) and (4.11) imply that A satisfies (4.5) when one substitutes $M/3$ for M.

It then follows by Theorem 4.2 that the tessellation $\mathcal{T} = \mathcal{T}(A)$ is a locally finite cover of U. Since A is G-invariant, so is $\mathcal{T}$ and thus we can form a G-invariant hyperbolic triangulation of U (see Section 3 B) by dividing the cells of $\mathcal{T}$ into hyperbolic triangles. We do this in the following manner.

If C_v is a compact cell of $\mathcal{T}$ (that is, if $v \in U$), then we choose $u \in A_v$ and let $A_v \setminus \{u\} = \{u_0, \ldots, u_k\}$ where u_i and u_{i+1} are consecutive points on ∂B_b. Let T_i be the triangle with vertices u, u_{i-1} and u_i. Then $\{T_1, \ldots, T_k\}$ is a subdivision of C_v into hyperbolic triangles.

If C_v is non-compact (that is, if $v \in \bar{\mathbf{R}}$), then we enumerate A_v as a_i, $i \in \mathbf{Z}$, where a_i

and a_{i+1} are consecutive points on ∂B_v. We let T_i be the hyperbolic triangle with vertices v, a_i and a_{i+1}. Again, $\{T_i: i \in \mathbf{Z}\}$ is a subdivision of C_v into triangles.

We denote the resulting triangulation of U by $\mathcal{K}$. Obviously, we can do the subdivision in such a way that $\mathcal{K}$ is G-invariant (which has bearing only on subdivision of compact cells). If a triangle T of $\mathcal{K}$ is compact, then T is a *finite* triangle, otherwise T is an *infinite* triangle. A *side* of $\mathcal{K}$ is a side of some triangle of $\mathcal{K}$, again such a side can be finite or infinite according to whether it is compact or non-compact. A *vertex* of $\mathcal{K}$ is a vertex of some triangle of $\mathcal{K}$. Thus also points of $V \cap \bar{\mathbf{R}}$ are vertices of $\mathcal{K}$; again, these are called infinite vertices of $\mathcal{K}$, others are finite vertices. The set of vertices of $\mathcal{K}$ is $A \cup (V \cap \bar{\mathbf{R}}) = A \cup P$.

We introduce the following notation for a triangle T of $\mathcal{K}$, a side S of $\mathcal{K}$, and a vertex v of $\mathcal{K}$:

$$\mathring{T} = \operatorname{int} T,$$

$$\mathring{S} = S \setminus \{\text{endpoints of } S\}, \text{ and}$$

$$\mathring{v} = \{v\} \cap U.$$

We enumerate the properties of $\mathcal{K}$ needed later in

THEOREM 4.5. *Let G be a Fuchsian group acting in U which satisfies* (4.9). *Then there is a hyperbolic triangulation $\mathcal{K}$ of U such that*

(i) *$g(\mathcal{K}) = \mathcal{K}$ for all $g \in G$,*

(ii) *$g(\mathring{T}) \cap \mathring{T} = \emptyset$ if $g \in G \setminus \{\mathrm{id}\}$ and T is a triangle, side or a vertex of $\mathcal{K}$,*

(iii) *if T is a finite side or a finite triangle of $\mathcal{K}$, then*

$$d(T) \in [M/3, 4M/3],$$

(iv) *if α is an angle in a triangle of $\mathcal{T}$, then*

$$\alpha \leq 2/\sinh(M/6),$$

(v) *the set of vertices at infinity of $\mathcal{K}$ is the set P of points of $\bar{\mathbf{R}}$ fixed by some parabolic $g \in G$, and*

(vi) *if a triangle $T \in \mathcal{K}$ has vertex $v \in P$, then the other two vertices u and u' of T are in U and satisfy*

$$d(u, u') = M/3$$

and $u' = g(u)$ for a generator g of G_v.

Proof. It is easy to check that $\mathcal{K}$ is locally finite since $\mathcal{T}$ is, and part (i) is true since $\mathcal{T}(A)$ is G-invariant and we subdivided $\mathcal{T}(A)$ into $\mathcal{K}$ in a G-invariant manner.

Since G acts without fixed points in U, it is obvious that $\mathring{u} \cap g(\mathring{u})=\emptyset$ if u is a vertex of $\mathcal{K}$ and $g \in G\setminus\{\mathrm{id}\}$. For this same reason, $\operatorname{int} C_v \cap g(\operatorname{int} C_v)=\emptyset$ if C_v is a finite cell of $\mathcal{T}(A)$. It follows that $\mathring{T} \cap g(\mathring{T})=\emptyset$ if T is a finite triangle of $\mathcal{K}$ and $g \in G\setminus\{\mathrm{id}\}$. If $v \in P$, then $\operatorname{int} C_v \cap g(\operatorname{int} C_v)=\emptyset$ for $g \in G\setminus G_v$ by (4.11). Considering how the subdivision of C_v was performed, it follows that $\mathring{T} \cap g(\mathring{T})=\emptyset$ if $g \in G\setminus\{\mathrm{id}\}$ and T is an infinite triangle of $\mathcal{K}$.

Finally, if S is a side of $\mathcal{K}$, then the above considerations and the fact that G acts without fixed points in U, imply that $\mathring{S} \cap g(\mathring{S})=\emptyset$ for $g \in G\setminus\{\mathrm{id}\}$. Thus (ii) is true.

Since A satisfies (4.5) with $M/3$ substituted for M, part (iii) is a consequence of (4.6) and part (ii) of Lemma 4.4. Part (iv) is a consequence of Lemma 4.3 and (ii) of Lemma 4.4.

Part (v) follows from (4.13) and (vi) from (iii) of Lemma 4.4 and from (4.10).

5. Quasiconformal extension

In this section we first construct a quasiconformal extension of a quasisymmetric map in a special case (Theorem 5.2) and then prove some auxiliary results which allow to reduce the general case to the special case. Finally, we put all threads together and prove the general extension theorem (Section 5 E).

5 A. *Canonical maps between hyperbolic triangles.* Let $a, b, c \in U$ be three distinct points of the hyperbolic plane not lying on a hyperbolic line. Then there is a non-degenerate hyperbolic triangle with vertices a, b and c which we denote by $T(a, b, c)$.

We now construct a canonical homeomorphism between two such triangles $T_1=T(a_1)$ and $T_2=T(a_2)$ where $a_i=(a_1^i, a_2^i, a_3^i) \in U^3$. We denote this homeomorphism by $f(a_1, a_2)=f$ and it is defined by the following conditions where we have denoted by s_j^i the side of T_i not containing a_j^i.

(1) $f(a_j^1)=a_j^2$ *for* $j=1,2,3$.

(2) $f(s_j^1)=s_j^2$ *and* $f|s_j^1$ *is a linear stretch in the hyperbolic metric,* $j=1,2,3$.

(3) *If* $x \in s_1^i$, *let* s_x^i *be the hyperbolic segment joining* x *and* a_1^i. *Then* $f(s_x^1)=s_{f(x)}^2$ *and* $f|s_x^1$ *is a linear stretch in the hyperbolic metric.*

Obviously in this manner we get a well-defined homeomorphism between triangles not having vertices in $\bar{\mathbf{R}}$. However, in the triangulations we later consider, some triangles may have one vertex in $\bar{\mathbf{R}}$. Therefore we need to define $f(a, b)$ also for such triangles.

We now define such a map. We use the above notation and assume that if a triangle $T=T(a, b, c)$ has vertices a, b, c of which one is in $\bar{\mathbf{R}}$, then $c\in\bar{\mathbf{R}}$ and $a, b\in U$. So assume that $a_i\in U\times U\times\bar{\mathbf{R}}$ and let L_j^i be the hyperbolic line containing s_j^i. Let

$$r=\min_{i,j=1,2} d(a_j^i, L_j^i).$$

Then there are points $\bar{a}_1^i\in s_2^i$ and $\bar{a}_2^i\in s_1^i$, $\bar{a}_j^i\neq a_j^i$, such that, for $i, j=1,2$,

$$d(\bar{a}_j^i, L_j^i)=r/2. \tag{5.1}$$

We now divide T_i into three triangles as follows. Let $T_{i1}=T(\bar{a}_1^i, \bar{a}_2^i, a_3^i)$, $T_{i2}=T(a_1^i, \bar{a}_1^i, a_2^i)$ and $T_{i3}=T(a_2^i, \bar{a}_2^i, \bar{a}_1^i)$. Then (5.1) implies that T_{11} and T_{21} are conformally equivalent: There is a Möbius transformation g of $\bar{U}$ such that $g(a_3^1)=a_3^2$ and that $g(\bar{a}_j^1)=\bar{a}_j^2$ for $j=1,2$. Using this map g and the maps $f(a, b)$ defined above we can now define a homeomorphism $f=f(a_1, a_2)\colon T_1\to T_2$ by setting

$$\begin{aligned} f|T_{11}&=g|T_{11},\\ f|T_{12}&=f((a_1^1, \bar{a}_1^1, a_2^1), (a_1^2, \bar{a}_1^2, a_2^2)),\\ f|T_{13}&=f((a_2^1, \bar{a}_2^1, \bar{a}_1^1), (a_2^2, \bar{a}_2^2, \bar{a}_1^2)). \end{aligned} \tag{5.2}$$

By (1) and (2) and the fact that g preserves hyperbolic metric, this is indeed a well-defined homeomorphism $T_1\to T_2$.

It is important that these homeomorphisms are compatible with Möbius transformations, that is, if g and h are Möbius transformations of $\bar{U}$, then

$$f(g(a), h(b))=h\circ f(a, b)\circ(g^{-1}|T(g(a))). \tag{5.3}$$

Another important property of these maps is that they are quasiconformal. Moreover, the dilatation are uniformly bounded if the triangles vary in a compact set. We give this in

LEMMA 5.1. *Let $c\in(0,1)$. Then there is $K=K(c)\geq 1$ such that the map $f(a_1, a_2)$: $T(a_1)\to T(a_2)$ is K-quasiconformal if the triangles $T(a_i)$ satisfy*

(i) *either both $a_1, a_2\in U^3$ or both $a_1, a_2\in U\times U\times\bar{\mathbf{R}}$,*
(ii) *the angles of $T(a_i)$ do not exceed $\pi-c$, and*
(iii) *the lengths of the finite sides of $T(a_i)$ lie in $[c, 1/c]$.*

Proof. We can normalize the situation by (5.3) in such a way that the first vertex of $T(a_i)$ is e_2. Suppose first that both $T(a_1)$ and $T(a_2)$ are finite triangles. Now, the set of such finite triangles which satisfy (ii) and (iii) is compact in an obvious sense. (Note that (ii) guarantees that it does not contain degenerate triangles.) Since every single $f(a_1, a_2)$ is quasiconformal and since the triangles (and vertices) vary in a compact set, the uniform K-quasiconformality follows; cf. Lemma 4.1 of [35].

Suppose then that both $T(a_1)$ and $T(a_2)$ are infinite triangles. Again by (ii) and (ii), the triangles, as well as the subtriangles T_{ij} in (5.2) into which we have divided $T(a_i)$, vary in a compact set; note that $f(a_1, a_2)$ is conformal in T_{11}.

5 B. *Quasiconformal extension in a special case.* Now we construct a quasiconformal extension to a G-compatible k-quasisymmetric map f, provided that G satisfies a condition depending on k.

THEOREM 5.2. *Let $k \geq 1$. Then there are $M=M(k)>0$ and $K=K(k)\geq 1$ with the following property. Let G be a Fuchsian group such that G and M satisfy condition* (4.9). *Under these conditions any k-quasisymmetric and G-compatible map f admits a K-quasiconformal and G-compatible extension F to a homeomorphism of the closed upper half-plane $\bar{U}$.*

Proof. Let $K'=K'(k)\geq 1$ be a number such that every k-quasisymmetric map is K'-quasiconformal in the sense of Section 1 F. Choose then $M=M(K')=M(k)$ such that

$$2/\sinh(M/6) \leq \beta^1_{K'} \quad \text{and} \quad M \geq M^1_{K'} \tag{5.4}$$

where $\beta^1_{K'} \leq 1$ is as in Theorem 3.8 and $M^1_{K'}$ as in Theorem 3.6.

We show that the theorem is true with this M. We first choose a G-invariant triangulation $\mathcal{T}$ *of U* as in Theorem 4.5. Let A be the set of vertices of $\mathcal{T}$. We represent every triangle $T\in\mathcal{T}$ in the form $T=T(a_T)$ as in the preceding section (i.e., we fix an order for the vertices of T). By (vi) of Theorem 4.5, at most one point of the triple a_T is in $\bar{\mathbf{R}}$, and in accordance with the preceding section, we assume that then the last point of a_T is in $\bar{\mathbf{R}}$. If $T'\in\mathcal{T}$ is another triangle and $T'=g(T)$, $g\in G$, then we assume that $a_{T'}=g(a_T)$.

We next pick for every vertex $a\in A$ a point $z_a\in T^1\cup\bar{\mathbf{R}}$ (cf. Section 3 A) such that $p(z_a)=a$ and do this in a G-invariant manner: $z_{g(a)}=g(z_a)$.

We can now define $F|T$ using the map $f(a, b)$ of the preceding section. If $T=T(a_T)$ and $a_T=(a_{T1}, a_{T2}, a_{T3})$, we set $a'_{Ti}=pf(z_{a_{Ti}})$ and $a'_T=(a'_{T1}, a'_{T2}, a'_{T3})$ and define

$$F|T=f(a_T, a'_T).$$

By (5.4) and Theorem 4.5 (iv), the angles of T do not exceed $\beta^1_{K'} \leq 1$ and hence a'_{Ti} are the vertices of a non-degenerate triangle by Theorem 3.8. Thus $f(a_T, a'_T)$ is indeed defined.

In $\bar{\mathbf{R}}$ we define F by $F|\bar{\mathbf{R}}=f$.

We claim that this gives the required extension. First, we must show that F is well-defined. This is clear if $x \in \bar{\mathbf{R}}$ or if $x \in \operatorname{int} T$ for some $T \in \mathcal{T}$. It is also clear if x is a vertex of $\mathcal{T}$ or if $x \in S$ where S is a finite side of some $T \in \mathcal{T}$, cf. the definition of $f(a, b)$ in Section 5 A. Suppose then that $x \in S$ where S is an infinite side. Let $v \in \bar{\mathbf{R}}$ be the vertex at infinity of S (as we have seen, there is only one such vertex). Let $T_1, T_2 \in \mathcal{T}$ be the triangles of $\mathcal{T}$ such that $S \subset T_i$. Then Theorem 4.5 (vi) implies that there is a generator g of G_v such that $g^{-1}(S)$ and S are sides of T_1 and S and $g(S)$ are sides of T_2. Hence the points $\bar{a}^i_j \in S$ defined by (5.1) are the same regardless of whether we regard S as a side of T_1 or T_2. It follows that F is well-defined also on infinite sides, and hence everywhere.

If f induces $\varphi: G \to G'$, then F also induces φ by (5.3) since $z_{g(a)}=g(z_a)$ and $a_{g(T)}=g(a_T)$ for $g \in G$. Hence F is G-compatible.

Obviously, $F|U$ is compatible with f and $\mathcal{T}$ (Section 3 B), and in addition $F|T$ is an embedding for every $T \in \mathcal{T}$. Then Corollary 3.9 implies that F is a homeomorphism of $\bar{U}$.

Finally, by (5.4) and Theorem 4.5 (iii) and (iv), the angles of the triangles $T \in \mathcal{T}$ do not exceed 1, and the lengths of their finite sides lie in $[M/3, 4M/3]$. Then, in view of Theorem 3.6 and 3.8 the angles of the triangles $F(T)$, $T \in \mathcal{T}$, do not exceed 1, and the lengths of their finite sides lie in $[M/3L^1_{K'}, 4L^1_{K'}M/3]$. Hence the conditions of Lemma 5.1 are satisfied for some $c=c(k) \in (0, 1)$. Thus there is $K=K(c)=K(k) \geq 1$ such that $F|T$ is K-quasiconformal for every $T \in \mathcal{T}$. It follows that F is K-quasiconformal and the theorem is proved.

Remarks. (1) The complex dilatation of the map $f(a, b)$ in Section 5 A depends real analytically on b when a is fixed, cf. Lemma 4.1 of [35], and this is true also if the triangles have one vertex in $\bar{\mathbf{R}}$. If $f \in T(G)$ (cf. Section 2 B), and $x \in \mathbf{R}$, then $f(x)$ depends real analytically on f([34, p. 139]). It follows that the complex dilatation of the map constructed in Theorem 5.2 depends real analytically on $f \in \{g \in T(G): g\ k\text{-quasisymmetric}\}$. Thus there is $c>0$ such that for k near 1 we have the estimate

$$K = 1+c(k-1). \tag{5.6}$$

(2) Actually, one sees as in Lemma 4.1 of [35] that the maps $f(a, b)$ are bilipschitz

maps in the hyperbolic metric and then, as above, a compactness argument shows that the extension of Theorem 5.2 is (in U) bilipschitz with respect to the hyperbolic metric.

However, this property is lost in Theorem 2 since its proof involves passing to extremal quasiconformal mappings and we do not know whether they are bilipschitz maps for the hyperbolic metric. On the other hand, the constant $K(k)$ of Theorem 5.2 is valid for Theorem 2 as well.

5C. *Normal subgroups of Fuchsian groups.* In this section we show that, if a finitely generated Fuchsian group G and $M>0$ are given, then G has a normal subgroup N of finite index such that N and M satisfy condition (4.9). Basically, this is due to the fact that Fuchsian groups are residually finite, i.e., the following lemma is true.

LEMMA 5.3. *Let G be a finitely generated Fuchsian group and let $h_1, \ldots, h_s \in G \setminus \{\mathrm{id}\}$. Then G has a normal subgroup N of finite index not containing the elements h_i.*

Proof. This follows since the lemma is true if G is a finitely generated group of $n \times n$ matrices with entries in a field of characteristic zero, cf. Malcev [20, Theorem VII]. (This can be proved like Lemma 8 of Selberg [29].) And a Fuchsian group is isomorphic to a subgroup of the 3-dimensional ortochroneous Lorentz group, cf. [3, 3.7.7] or [24, Theorem 1.1].

In view of the importance of this lemma in our construction, we give references also to a more direct proof of it. Zieschang–Vogt–Coldewey [44, 4.10.8] proved that G has a normal subgroup N' of finite index not containing torsion elements. Then Hempel [12] showed that N' is residually finite. Since a subgroup of finite index of a finitely generated group contains a normal subgroup of finite index, the lemma follows.

Using Lemma 5.3, we can prove

LEMMA 5.4. *Let G be a finitely generated Fuchsian group and let $M>0$. Then there is a normal subgroup N of G of finite index such that N and M satisfy condition (4.9), i.e. if $x \in U$ and if*

$$A(N, M, x) = \{g \in N\colon d(x, g(x)) < M\},$$

then either $A(N, M, x) = \{\mathrm{id}\}$ or is contained in a subgroup N_x of N generated by a parabolic element.

Proof. Let G_0 be the subgroup of G consisting of orientation preserving elements. If there is $g \in G \setminus G_0$ and if N_0 is a normal subgroup of G_0 such that the lemma is true for G_0 and N_0, then the lemma is true also for G and $N=N_0\cap gN_0g^{-1}$. Hence we can assume that G does not contain orientation reversing elements.

Let then $P\subset\bar{\mathbf{R}}$ be the set of points fixed by a parabolic $g\in G$. Fix for every $v\in P$ an open horoball B_v at v (i.e $B_v\subset U$ is an open 2-ball such that ∂B_v is tangent to $\bar{\mathbf{R}}$ at v). As is well-known, we can do this in such a way that B_v's are disjoint and that $B_{g(v)}=g(B_v)$ for $g\in G$. Let $G_v=\{g\in G: g(v)=v\}$ if $v\in P$. Then

$$\begin{aligned} g(B_v)&=B_v \quad \text{if } g\in G_v \quad \text{and}\\ g(B_v)\cap B_v&=B_{g(v)}\cap B_v=\emptyset \quad \text{if } g\in G\setminus G_v. \end{aligned} \tag{5.7}$$

Define then another horoball $B'_v\subset B_v$ at v by $d(\partial_U B'_v, \partial_U B_v)=M$. Then also $B'_{g(v)}=g(B'_v)$ for $g\in G$ and $v\in P$ and if $x\in B'_v$, then $D(x,M)\subset B_v$. Hence, by (5.7),

$$A(G,M,x)\subset G_v \tag{5.8}$$

if $x\in B'_v$ and $v\in P$ and here G_v is a cyclic group generated by a parabolic element of G.

Let $L(G)\subset\bar{\mathbf{R}}$ be the limit set of G. Let $y\in\bar{\mathbf{R}}\setminus L(G)$. Since G does not contain orientation reversing elements, y has a neighbourhood V in $\bar{U}\setminus L(G)$ such that $A(G,M,x)=\{\mathrm{id}\}$ for $x\in V\cap U$. In the finitely generated case $(\bar{\mathbf{R}}\setminus L(G))/G$ is compact and hence we can find a G-invariant neighbourhood W of $\bar{\mathbf{R}}\setminus L(G)$ in $\bar{U}\setminus L(G)$ such that

$$A(G,M,x)=\{\mathrm{id}\} \tag{5.9}$$

for $x\in W\cap U$.

Since G is finitely generated $(\bar{U}\setminus[L(G)\cup W\cup(\cup_{v\in P}B'_v)])/G$ is compact. Hence there is a compact set $C\subset U$ such that

$$U\subset GC\cup W\cup(\bigcup_{v\in P}B'_v).$$

Let

$$A=\{g\in G\colon d(x,g(x))<M \text{ for some } x\in C\}$$

which is a finite set by compactness of C. Hence, by Lemma 5.3, there is a normal subgroup N of G such that $N\cap A=\{\mathrm{id}\}$.

We claim that this is the sought-for subgroup. Let $x\in U$. If $x\in W$ or $x\in B'_v$ for some $v\in P$, then $A(N,M,x)$ is of the required form by (5.8) and (5.9). If this is not the case, then $x=g(y)$ for some $y\in C$ and $g\in G$. Suppose that there is $h\in N$ such that $d(x,h(x))<M$. Then

$$d(x,h(x))=d(g(y),hg(y))=d(y,g^{-1}hg(y))<M.$$

Since $g^{-1}hg\in N\cap A=\{\mathrm{id}\}$, it follows that $h=\mathrm{id}$, and the lemma is proved.

5D. *Quasiconformal extension and subgroups of finite index.* Macbeath [19] observed that if G is a Fuchsian group with a compact fundamental domain and if $N\subset G$ is a normal subgroup of finite index, then an extremal N-compatible quasiconformal map of U is also G-compatible. We use this idea to show that in the construction of a G-compatible quasiconformal extension of a quasisymmetric map, we can pass to a subgroup of finite index.

LEMMA 5.5. *Let G be a Fuchsian group and f a quasisymmetric G-compatible map. Let N be a subgroup of finite index and assume that there is a K-quasiconformal N-compatible extension F' of f to $\bar{U}$. Then there is a K-quasiconformal G-compatible extension F of f to $\bar{U}$.*

Proof. We can assume that N contains only orientation preserving elements.

We first assume that G is finitely generated. Now G has only finitely many subgroups of the same index as N and then their intersection is a normal subgroup of G of finite index. Hence we can assume that N is a normal subgroup.

Let $\varphi\colon G\to G'$ be the isomorphism induced by f, i.e., $\varphi(g)|\bar{\mathbf{R}}=fgf^{-1}$. Let $L(G)$ be the limit set of G. It is also the limit set of N. If $L(G)\neq\bar{\mathbf{R}}$, choose a set $\{x_1,x_2,\dots\}\subset\bar{\mathbf{R}}\setminus L(G)$ which is dense in $\bar{\mathbf{R}}\setminus L(G)$ and does not contain G-equivalent points. Let, if $n>0$,

$$X_n=\begin{cases}\varnothing & \text{if } L(G)=\bar{\mathbf{R}}\\ G(\{x_1,\dots,x_n\}) & \text{if } L(G)\neq\bar{\mathbf{R}}.\end{cases}$$

Then X_n/N is finite. This and the fact that N is a finitely generated Fuchsian group not containing orientation reversing elements imply that there is a uniquely determined map $F_n\colon\bar{U}\to\bar{U}$, $n\geq 3$, such that

(a) F_n is N-compatible and F induces $\varphi|N$,
(b) $F_n(x)=f(x)$ if $x\in X_n\cup L(G)$, and
(c) F_n is the unique extremal quasiconformal map satisfying conditions (a) and (b).

This is a consequence of the existence and uniqueness of Teichmüller's extremal mapping. (Cf. Bers [4, § 7] where this has been proved in the form we need it.) We show that F_n is G-compatible. Choose $g\in G$. We must show that if $F''=\varphi(g)^{-1}F_n g$, then $F''=F_n$. We show that also F'' satisfies (a) and (b). Then by the uniqueness of the extremal map, we must have $F''=F_n$ since the maximal dilatations of F'' and F_n are equal. Let $h\in N$. Then, if $h'=ghg^{-1}\in N$,

$$\varphi(h)^{-1}F''h=\varphi(h)^{-1}\varphi(g)^{-1}F_n gh=\varphi(g)^{-1}\varphi(h')^{-1}F_n h'g$$
$$=\varphi(g)^{-1}F_n g=F''$$

since F_n induces $\varphi|N$. Thus F'' satisfies (a). If $x\in X_n\cup L(G)$, then $F''(x)=\varphi(g)^{-1}F_n g(x)=\varphi(g)^{-1}fg(x)=f(x)$ since f induces φ and $g(x)\in X_n\cup L(G)$. Thus F'' satisfies also (b), and consequently F_n is G-compatible.

Since we know that there is at least one K-quasiconformal map of $\bar{U}$ satisfying (a) and (b) (i.e. F'), F_n must be also K-quasiconformal.

The normal family properties of quasiconformal mappings imply that there is a sequence $n(1)<n(2)\ldots$ and a K-quasiconformal homeomorphism F of $\bar{U}$ such that $F_{n(i)}\to F$ uniformly in the spherical metric. It is G-compatible since every F_n is, and, by (b), $F(x)=f(x)$ if $x\in L(G)\cup(\bigcup_{n\geq 0}X_n)$ which is dense in $\bar{\mathbf{R}}$. Therefore $F|\bar{\mathbf{R}}=f$ and the lemma is true for finitely generated G.

If G is not finitely generated, there is a sequence $G_1\subset G_2\subset\ldots$ of finitely generated subgroups of G whose union G is. Let $N_i=N\cap G_i$ which is a subgroup of finite index of G_i. Since an N-compatible map is also N_i-compatible, the above proof shows that there is a K-quasiconformal, G_i-compatible extension F_i of f to $\bar{U}$. As above, a normal family argument now shows that there is a G-compatible K-quasiconformal extension F of f to $\bar{U}$.

Remarks. (1) Actually, it would suffice to assume that $N\cap H$ is a subgroup of finite index of H whenever H is a finitely generated subgroup of G.

(2) After the first version of this paper was completed, we got Sakan's paper [28] where he gives a different proof of this theorem.

5 E. *Conclusion of the proof of Theorem* 2. We put now all the pieces together and conclude the proof of Theorem 2. There are two cases.

G is discrete. If G is finitely generated, then Lemma 5.4 implies that G has a normal subgroup N of finite index which satisfies condition (4.9) for $M=M(k)$ of Theorem 5.2. Hence Theorem 5.2 can be applied to construct an N-compatible $K(k)$-quasiconformal extension of f. Then Lemma 5.5 implies that there is also a G-compatible $K(k)$-quasiconformal extension of f.

If G is not finitely generated, pick a sequence $G_1\subset G_2\subset\ldots$ of finitely generated groups whose union G is. Let F_i be a $K(k)$-quasiconformal G_i-compatible extension of f. By passing to a subsequence we obtain that there is a $K(k)$-quasiconformal F such that $F_i\to F$ uniformly on compact sets. Then F is a G-compatible homeomorphism of $\bar{U}$, and Theorem 2 is proved for discrete G.

G is non-discrete. We prove that if f is a G-compatible homeomorphism of $\bar{\mathbf{R}}$, then there is an extension of f to a G-compatible homeomorphism of $\bar{U}$ which is $K(k)$-quasisymmetric if f is k-quasisymmetric, $K(k)$ as for discrete G.

We can assume that G is a closed subgroup of the group of Möbius transformations of $\bar{U}$. Let G_0 be the component subgroup of G containing the identity element. Then (Greenberg [11]) G_0 must be one of the following groups: (a) the group of elliptic transformations fixing a point of U; (b) the group of parabolic, or of parabolic and hyperbolic, transformations fixing a point of $\bar{\mathbf{R}}$; (c) the group of hyperbolic elements fixing a point-pair of $\bar{\mathbf{R}}$; (d) the group of all orientation preserving Möbius transformations of $\bar{U}$.

It is easy to check that then f must be a Möbius transformation except possibly in (c). Suppose now that we have case (c) and let $\{0, \infty\}$ be the point-pair fixed by elements of G_0. Then one sees that f is of the form

$$f(x) = \begin{cases} c|x|^{\alpha}, & x \geqslant 0 \\ c'|x|^{\alpha}, & x \leqslant 0, \end{cases}$$

where $\alpha > 0$ and $cc' > 0$. In particular, f is then always k-quasisymmetric for some $k \geqslant 1$.

It would not be difficult to find directly an extension of f in this case but it can be reduced to the discrete case as follows. Since G_0 is normal in G, one sees that elements of G fix setwise the point-pair fixed pointwise by elements of G_0. Using this fact and the known structure of G_0, one can find discrete subgroups $G_1 \subset G_2 \subset \ldots$ of G whose union is dense in G. Hence there is a $K(k)$-quasiconformal G_i-compatible extension of f to $\bar{U}$ and the proof can be concluded by a normal family argument.

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