THE BLOSSOMING OF SCHRÖDER'S FOURTH PROBLEM

BY

JOHN RIORDAN

The Rockefeller University New York, N. Y. 10021, USA

1. Introduction

Schröder's fourth problem is the last of four enumeration problems associated with various kinds of bracketing of a sum (product) with a fixed number of terms, considered in [12] and published in 1870. In contrast with the earlier (1838) work of E. Catalan, in which the bracketing was restricted to two neighboring terms, Schröder allowed bracketing of any number of terms and in the fourth problem included the effects of term reordering, in a way which will be clearer in the detailed description in the next section.

The identity of this problem with the enumeration of fully labeled (essentially series) series-parallel arrangements has been noticed in [4], a joint paper with L. Carlitz. However the mapping of the bracketings to series-parallel arrangements was not pursued.

More recently, Louis Comtet, [7], has given a mapping of Schroder's bracketings to trees, described as arborescences bifurcante. It is somewhat surprising that these trees, without labels, also appear in A. Cayley's landmark paper [5], of 1857, devoted mainly to the enumeration of unlabeled rooted trees. They appear as a kind of simplification of the main result and are described mainly by the phrase "every branching is at least a bifurcation". The fact that the enumeration is by number of endpoints, rather than total number of points, as in the main result, is not emphasized. In the terminology of Frank Harary and Geert Prins [8, p. 150], these trees are homeomorphically irreducible planted trees, that is, without points of degree two; I prefer the shorter term *series-reduced*.

The object of this paper, in the first place, is to give the mapping of series-parallel arrangements to both the bracketings of Schröder's fourth problem (which Comtet [7] calls schröderiens, for brevity) and to the Cayley–Comtet trees. The mapping is so simple as to arouse the hope that it may remove the stigma from which series-parallel arrangements seem to have suffered in mathematical circles because of their origin in electric circuit theory. Of course, the natural presumption in the electrical setting that element position either in series or parallel is irrelevant, which fits perfectly for Schröder's fourth problem, is disposable for more general uses; indeed its abandonment is required for representation of Catalan bracketing.

Next, series-reduced planted trees are enumerated by number of endpoints other than the root, both without labels and with a fixed number of labels on endpoints. The enumeration is of course by George Pólya's fundamental theorem in enumeration; the fixed number of labels provides a bridge between the unlabeled and fully labeled cases as in [10; p. 129 ff]. However attention is restricted here to the two extreme cases. The procedure supplies two new enumerators, $s_n(y)$ and $S_n(y)$; these are enumerators of trees with n endpoints by number of interior points for the unlabeled and fully labeled cases. It turns out that

$$S_n(y) = \sum_{k=1} b(n-1+k, k)\, y^k \tag{1}$$

with $b(n, k)$ an associated Stirling number of the second kind, in the notation of [10, p. 77]; the sum $S_n(1) \equiv S_n$ is, effectively and apart from notation, Comtet's formula for Schröder's numbers [7, eq. 6].

By way of contrast, a formula is found for series-reduced planted trees, with n labeled points by number of endpoints (root ignored), namely

$$P_n(y) = \sum_{0}^{m} (n)_k\, b(n-1, k)\, y^{n-k}, \quad m = [(n-1)/2] \tag{2}$$

with $(n)_k = n(n-1) \dots (n-k+1)$, the falling factorial.

This is followed by a formal (non-combinatorial) derivation of a polynomial

$$T_n(y) = \sum 2^{n-k} a_{nk} y^k = 2^n Q_n(y/2), \tag{3}$$

which, like $S_n(y)$ has coefficients whose sums are the Schröder numbers. In (3), the coefficients a_{nk} are those appearing in [1], [2], and [3], and $Q_n(y) = \sum a_{nk} y^k$, is defined by $Q_1(y) = y$ and the recurrence

$$Q_n(y) = (2n-1)\, y Q_{n-1}(y) + (y - y^2) Q'_{n-1}(y)$$

with the prime denoting a derivative.

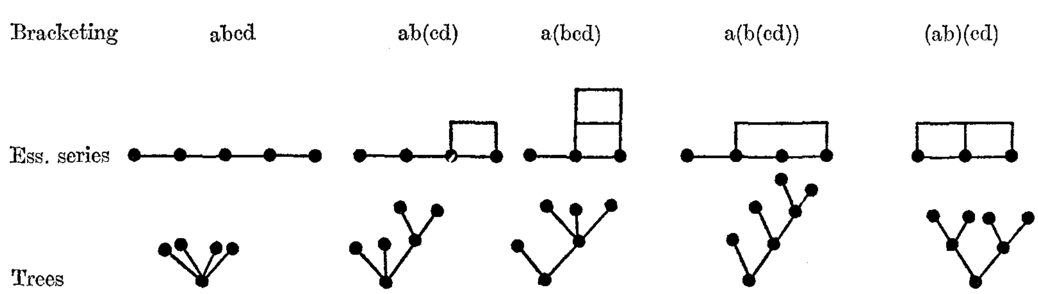
The remaining sections are a small selection of the possible tree enumerations. In section 5, series-reduced rooted trees, derived from their planted correspondents by removal of the planting stem and rerooting, are enumerated by number of lines at the root for both limiting cases. In section 6, the same trees are enumerated by height, again for both limiting cases.

2. Mappings of Schröder bracketings

It is convenient to focus on bracketings in fixed order. Ignoring the terminal bracket, they are a; ab; abc, a(bc); abcd, ab(cd), a(bcd), a(b(cd)), (ab)(cd) for $n=1,2,3,4$ respectively. As is clear, the terms are in alphabetical order, the order of the bracketing is from right to left, and any number of terms may be bracketed. The numbers of such bracketings are 1,1,2,5 for $n=1(1)4$, which are the numbers of essentially series (or essentially parallel) series-parallel arrangements of like (unlabeled) elements appearing in [10, Table 6 ($m=0$), p. 142].

It may be helpful to remind the reader that a series-parallel arrangement is essentially series when it is a concatenation of parts each of which is either a single element, or a parallel grouping of essentially series arrangements. Each essentially series arrangement has a unique essentially parallel mate or dual, obtained by interchanging the words series and parallel in its verbal description.

The mapping of bracketings to essentially series arrangements and to series-reduced rooted trees seems sufficiently clear for $n=4$, as follows:



In the essentially series representation, the vertical lines are only for the convenience of representing all elements as horizontal lines; alternatively, they may be regarded as signatures of parallel connection. It is evident in the display that a bracket enclosing k like terms is a parallel arrangement of k lines. However the representative of (b(cd)) is the essentially parallel mate of b(cd).

The rules for constructing the trees are as follows: all unbracketed terms are single lines from the root; a bracket of k terms is a single line from the root to a point with k branches; a bracket (b(cd)) is the planted tree corresponding to the rooted tree for b(cd), and similarly for any other double (or multiple) bracket.

It should be noted that the number of terms bracketed is the same as the number of elements in the essentially series arrangement, and the number of endpoints in the series reduced rooted trees. Of course the series-reduced rooted trees are the corresponding planted trees with stem removed and root relocated.

It should also be noticed that series-parallel arrangements may be mapped to expressions with two operators, say addition and multiplication. It is immediate that these expressions divide into two equal classes, essentially additive and essentially multiplicative. The first few essentially additive expressions are: a+b; a+b+c, a+bc; a+b+c+d, a+b+cd, a+bcd, a+b(c+d), (a+b)(c+d).

3. Enumeration of series-reduced planted trees by number of endpoints

As noticed in the introduction, these trees have already been examined by Frank Harary and Geert Prins in [8]. With $p_{m,n}$ the number of such trees with m like (unlabeled) points, n of which are endpoints, and $p(x, y)$ the corresponding generating function: $p(x,y) = \sum x^m y^n p_{m,n}$, one of their results may be written

$$(1+x)p(x, y)+x-xy = x \exp [p(x, y)+ \dots +p(x^n, y^n)/n+ \dots]. \tag{4}$$

The enumerant of interest for current purposes is the number of such trees with n endpoints and i interior points, denoted by $s_{n,i}$. Since $s_{n,i}=p_{n+i,n}$, it follows that

$$s(xy, y) = p(y, x) \tag{5}$$

where $s(x, y) = \Sigma\, x^n y^i s_{n,i}$, and equations (4) and (5) imply

$$(1+y)s(x, y)+y-x = y \exp [s(x, y)+ \dots +s(x^n, y^n)/n+ \dots]. \tag{6}$$

Equation (6) is extended to the case of labeled endpoints by means of the generating function

$$s(x,y,z) = \sum x^n y^i \frac{z^j}{j!} s_{n,i,j}$$

with $s_{n,i,j}$ the number of trees with n endpoints, j of which are labeled with distinct labels, and i interior points; j is a fixed number (which may be null) and of course $j=n$ for $n \leqslant j$. By the version of Polya's theorem given in [10, p. 131] it follows that

$$(1+y)s(x, y, z)+y-x-xz = y \exp [s(x, y, z)+ \dots +s(x^n, y^n)/n+ \dots]. \tag{7}$$

Though no use is made of it here, it is worth noting that (6) and (7) imply

$$[(1+y)s(x, y, z)+y-x-xz] \exp s(x, y) = [(1+y)s(x, y)+y-x] \exp s(x, y, z).$$

With all points labeled, $j=n$, the generating function $(s(x, y, z)$ becomes

$$\sum \frac{(xz)^n}{n!} y^i s_{n,i,n} = S(y, w), \quad w = xz.$$

Considering the new variable $w=xz$ as independent of x and taking the limit of (7) for $x\to 0$, it is found that

$$(1+y)S(y,w)+y-w=y\exp S(y,w) \tag{8}$$

with $S(y,w)=\Sigma S_n(y)w^n/n!$, the exponential generating function of $S_n(y)$, and $S_n(y)$ the enumerator of series-reduced planted trees with n labeled endpoints by number of (unlabeled) interior points.

For $y=1$, $s(x,1)\equiv s(x)$, $S(1,x)\equiv S(x)$, (6) and (8) become

$$\begin{gathered} 2s(x)+1-x=\exp[s(x)+\ldots+s(x^n)/n+\ldots], \\ 2S(x)+1-x=\exp S(x), \end{gathered} \tag{9}$$

which apart from notation are, respectively, equations (77a) and (85) of [10, pp. 141, 142]. Note that (85) is transcribed by $A_0(z)+z=2S(z)$.

Note also that the first of (9) is Cayley's result [5], mentioned in the introduction; Cayley's $B(x)$ corresponds to $s(x)$.

Thus, the present enumeration agrees with the series-parallel enumeration so far as the latter goes. However the inclusion of interior points refines the enumeration through the enumerators $s_n(y)$ and $S_n(y)$ of trees with n endpoints, unlabeled and fully labeled, by number of interior points.

To determine $s_n(y)$, note first that $s(x,y)=xs_1(y)+x^2s_2(y)+\ldots$. Hence

$$\sum_1^\infty \frac{1}{n}s(x^n,y^n)=\sum_1^\infty s_n^*(y)\frac{x^n}{n}=\sum_n C_n(s_1^*(y),\ldots,s_n^*(y))\frac{x^n}{n!}$$

with $s_1^*(y)=s_1(y)=y$, $s_2^*(y)=2s_2(y)+s_1(y^2)$,

$$s_n^*(y)=\sum_{d/n} ds_d(y^e),\quad de=n$$

with the sum over all divisors of n (including 1 and n), and with $C_n(t_1,\ldots,t_n)$ the cycle indicator of the symmetric group. For the generating function of C_n see [10, p. 68].

With these results, equation (6) may be rewritten as

$$(1+y)s(x,y)+y-x=y\sum_0 C_n(s_1^*(y),\ldots,s_n^*(y))x^n/n;$$

or equating coefficients of x^n, with δ_{nm} the Kronecker delta

$$(1+y)s_n(y)+y\delta_{n0}-\delta_{n1}=yC_n(s_1^*(y),\ldots,s_n^*(y))/n!. \tag{10}$$

Thus $s_0(y)=0$, $s_1(y)=s_1^*(y)=1$,

$$\begin{gathered} 2(1+y)s_2(y)=yC_2(s_1^*(y),s_2^*(y))=y(2s_2(y)+s_1(y^2)+s_1^2(y)) \\ 2s_2(y)=2y. \end{gathered}$$

Writing $s_n(y) = \Sigma\, s(n, k) y^k$, the coefficients $s(n; k)$ for $n = 2(1)10$, $k = 1(1)n-1$ are as follows

$k \diagdown n$	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7	8
3			2	5	10	16	24	33	44
4				3	12	29	57	99	157
5					6	28	84	192	382
6						11	66	231	615
7							23	157	634
8								46	373
9									98

The sums $s_n(1)$ appear in [13; seq. 558]; the range is 2(1)20. It is interesting to notice (I omit the proof) that $s(n, n-1)$ is the number of completely bifurcative rooted trees with n endpoints associated with the Wedderburn–Etherington bracketing (cf. [6; I, 68]). Comtet's table for these numbers, $n = 1(1)26$ is sequence 298 of [13].

An alternative to (10) may be obtained as follows. First (6) is equivalent to

$$(1+y)\, s(x,y) + y - x = \exp \sum_1 s_n^*(y)\, x^n/n$$

with $s_n^*(y)$ as above. The derivative of this with respect to x is

$$(1+y)\, s_x(x,y) - 1 = \sum_1 x^{n-1} s_n^*(y)\, [(1+y)\, s(x,y) + y - x],$$

which corresponds to

$$n(1+y)\, s_n(y) - \delta_{n1} = (1+y) \sum_1 s_j^*(y)\, s^{n-j}(y) + y s_n^*(y) - s_{n-1}^*(y). \tag{11}$$

All numbers appearing in the table above have been checked by this relation.

For the enumerators $S_n(y)$, things are a little simpler. Since

$$\exp \left(y_1 x + y_2 \frac{x^2}{2!} + \dots + y_n \frac{x^n}{n!} + \dots \right) = \sum_0 Y_n(y_1, \dots, y_n) \frac{x^n}{n!}$$

is the equation of definition for Bell multivariable polynomials, it follows from (8) that

$$(1+y)\, S_n(y) + y\delta_{n0} - \delta_{n1} = y\, Y_n(S_1(y), \dots, S_n(y)). \tag{12}$$

Thus $S_0(y) = 0$, $S_1(y) = 1$, $S_2(y) = y$, $S_3(y) = y + 3y^2$, and it may be guessed that

$$S_n(y) = \sum_0^{n-1} b(n-1+k, k)\, y^k, \quad n = 1, 2, \dots \tag{13}$$

with $b(n, k)$ the associated Stirling number of the second kind, defined in [10, p. 77]. This is proved as follows. Differentiating (8), with partial derivatives indicated by suffixes, it is found that

$$yS_y(y, w)\,D(y, w) = S(y, w) - w,$$
$$S_w(y, w)\,D(y, w) = 1,$$

with $D(y, w) = 1 + w - (1+y)\,S(y, w)$. The first of these is equivalent to

$$y(1+y)\,S_y(y, w)\,D(y, w) = 1 - w - D(y, w).$$

Hence

$$(1-xy)\,S_w(y, w) - y(1+y)\,S_y(y, w) = 1,$$

or, with a prime denoting a derivative,

$$S_{n+1}(y) - nyS_n(y) - y(1+y)\,S'_n(y) = \delta_{n0}. \tag{14}$$

Writing $S_n(y) = \Sigma\, \sigma(n, k)y^k$ it follows from (14) that

$$\sigma(n+1, k) = k\sigma(n, k) + (n-1+k)\sigma(n, k-1),$$

which is the recurrence [10, p. 78]

$$b(n+1, k) = kb(n, k) + nb(n-1, k-1)$$

with n replaced by $n-1+k$, so that $\sigma(n, k) = b(n+k-1, k)$.

It should also be noticed (I owe this remark to Neil Sloane) that

$$S_n(y) = Z_n(y, \ldots, y)$$

with $Z_n(z_1, \ldots, z_n)$ the multivariable polynomial for derivatives of inverse functions, in the notation of [11, p. 180]; (the substantiation of this statement by table 5.2 [11, p. 181] needs the following corrections: the coefficient of $z_4 z_1^2$ is 378, not 278, a term $2100 z_3 z_2^2$ is missing, and $15400 z_3^2 z_1$ should be $15\,400 z_2^3 z_1$).

I do not take space to derive the curious identity

$$(1+x)Z_n(xS_2(y), \ldots, xS_{n+1}(y)) = xS_{n+1}(y(1+x)),$$

which for $x = 1$ becomes

$$2Z_n(S_2(y), \ldots, S_{n+1}(y)) = S_{n+1}(2y).$$

In contrast with this result for fully labeled endpoints, consider the case of all points labeled. As above, $p(x, y, z)=\Sigma\, p_{m,n,j}x^m y^n z^j/j!$, with $p_{m,n,j}$ the number of trees with m points, j of which are labeled, and n endpoints. Then

$$(1+x+xz)p(x, y, z)+x(1-y)(1+z) = x(1+z) \exp [p(x, y, z)+ \dots + p(x^n, y^n)/n + \dots]. \quad (15)$$

Omitting details, this leads to

$$(1+w)P(y, w)+w-wy = w \exp P(y, w) \quad (16)$$

with $P(y, w)=\Sigma\, P_n(y)w^n/n!$ and $P_n(y)$ the enumerator of series-reduced planted trees with n labeled points by number of endpoints (other than the root). I do not take space to prove that

$$P_n(y)=\sum_0^N (n)_k b(n-1, k)\, y^{n-k}, N=[(n-1/2)], \quad (17)$$

with $(n)_k=n(n-1)\dots(n-k+1)$, the falling factorial. The first few of these polynomials are

n	1	2	3	4	5	6	7
$P_n(y)$	y	0	$3y^2$	$4y^3$	$60y^3+5y^4$	$360y^4+6y^5$	$3\,150y^4+1\,050y^4+7y^6$

Note that $P_n(y)\equiv 0 \pmod n$; it may also be shown that

$$P_{n+p}(1) \equiv nP_n(1)+n(n-1)P_{n-1}(1), \pmod p$$

with p an odd prime.

4. Polynomials $T_n(x)$

These are polynomials arising from the following formula given, apart from notation, by Schröder in [12]

$$S_{n+1} = \sum_0 S(n+r, r) \sum_0^{n-r} (-1)^s \binom{2n+1}{s} 2^{n-r-s} \quad (18)$$

with $S(n, k)$ the Stirling number of the second kind. The sum on the right is a convolution, which suggests the two generating functions

$$S_n^*(x) = \sum_0 S(n+r, r)\, x^r$$

$$C_n(x) = \sum_0 x^r \sum_0^r (-1)^s \binom{2n+1}{s} 2^{r-s}.$$

The evaluation $C_n(x) = (1-x)^{2n+1}(1-2x)^{-1}$ follows from interchange of the order of summation, and it is shown in [1, p. 280] that

$$S_n^*(x) = (1-x)^{-2n-1}Q_n(x) \tag{19}$$

with $Q_0(x) = 1$, $Q_1(x) = x$, and, with a prime denoting a derivative,

$$Q_n(x) = (2n-1)xQ_{n-1}(x) + (x-x^2)Q'_{n-1}(x), \quad n = 1, 2, \ldots.$$

The coefficients of $Q_n(x)$ are denoted by a_{nk} in [1], and $a_{n0} = \delta_{n0}$, $a_{nk} = ka_{n-1,k} + (2n-k)a_{n-1,k-1}$. It is worth noting here that $Q_n(x)$ is the enumerator of trapezoidal words with n elements by number of distinct elements. Trapezoidal words are such that the ith element takes the values $1, 2, \ldots, 2i-1$; for $n=3$, the words, classified by number of distinct elements, are 111; 112, 113, 114, 115, 121, 122, 131, 133; 123, 124, 125, 132, 134, 135. Thus the enumerator is $Q_3(x) = x + 8x^2 + 6x^3$.

Since the product $S_n^*(x)C_n(x)$ is $Q_n(x)(1-2x)^{-1}$, it is apparent that

$$S_{n+1} = \sum_1^n a_{nk}2^{n-k}$$

and if $T_0(x) = 1$,

$$T_n(x) = \sum_1^n a_{nk}2^{n-k}x^k = 2^n Q_n(x/2), \quad n = 1, 2, \ldots \tag{20}$$

then $T_n(1) = S_{n+1}$.

The following short table of coefficients T_{nk} is included for concreteness

$k \diagdown n$	1	2	3	4	5
1	1	2	4	8	16
2		2	16	88	416
3			6	116	1 312
4				24	888
5					120

Note that by (20) and the recurrence for $Q_n(x)$

$$T_n(x) = (2n-1)xT_{n-1}(x) + (2x-x^2)T'_{n-1}(x)$$

and, if $T_n(x) = xt_n(x)$,

$$2t_n(1+x) = P_{n+1}(x), \quad n = 1, 2, \ldots$$

with $P_n(x)$ the polynomial associated with fully labeled series-parallel networks, appearing in [10. p. 143].

5. Series-reduced rooted trees by lines at the root

As mentioned before, the rooted trees in question are obtained from the corresponding planted trees by removing the planting stem and rerooting. Their enumerator for trees with n like endpoints by number of lines at the root is denoted $l_n(y)$; the corresponding enumerator for n labeled endpoints is $L_n(y)$.

The first few values of $l_n(y)$ may be found by inspection of the trees; thus $l_2(y)=y^2$, $l_3(y)=y^2+y^3$, $l_4(y)=3y^2+y^3+y^4$. By definition of series-reduced trees, it is apparent that there are at least two and at most n lines at the root of a tree with n endpoints. Moreover, each line at the root is the stem of a planted tree with j endpoints, for which there are s_j choices ($s_j=s_j(1)$), but if there are k lines each with j endpoints, the number of choices is $\binom{s_j+k-1}{k}$, the number of combinations with repetition of s_j things, k at a time. Thus

$$l_4(y)=\left(s_3 s_1+\binom{s_2+1}{2}\right)y^2+s_2\binom{s_1+1}{2}y^3+\binom{s_1+3}{2}y^4.$$

From the identity

$$\exp\sum_1 s(x^k)\,y^k/k=(1-xy)^{-s_1}(1-x^2y)^{-s_2}\ldots=1+x(s_1y)+x^2\left[s_2y+\binom{s_1+1}{2}y^2\right]+\ldots$$

it follows, writing $\lambda_1(y)=s_1y=y$, $\lambda_n(y)=s_ny+l_n(y)$, $n=2, 3, \ldots$, and $\lambda(x, y)=\Sigma_1\, x^n\lambda_n(y)$, that

$$1+\lambda(x,y)=\exp\sum_1 s(x^k)\,y^k/k=\exp\sum_1 \sigma_n(y)\,x^n/n \tag{21}$$

with $\sigma_n(y)=\Sigma_{d|n}\, ds_d y^e$, $de=n$.

Equation (21) implies the two results

$$\begin{aligned}\lambda_n(y) &= S_n(\sigma_1(y), \ldots, \sigma_n(y)),\\ n\lambda_n(y) &= \sigma_n(y)+\sum_{j=1}\sigma_j(y)\,\lambda_{n-j}(y),\end{aligned} \tag{22}$$

with $S_n(t_1, \ldots, t_n)$ the cycle index of the symmetric group.

With $l_n(x)=\Sigma\, l_{nk}x^k$, the coefficients for $n=5(1)9$ are:

$k \backslash n$	2	3	4	5	6	7	8	9
5	7	3	1	1				
6	20	8	3	1	1			
7	55	22	8	3	1	1		
8	162	63	23	8	3	1	1	
9	477	188	65	23	8	3	1	1

It is clear that $l_{nn} = l_{n,n-1} = 1$, $l_{n,n-2} = s_3 + 1 = 3$, $n = 4, 5, \ldots$; for any $k \geqslant 2$, $l_{n,n-k}$ is monotone increasing for $n = k + 2(1)2k$ and $l_{2k+j,k+j} = l_{2k,k}$, $j = 1, 2, \ldots$. Thus the table above may be truncated by the following table for $l_{n,n-k}$

$k \backslash n-k-2$	0	1	2	3	4	5
2	7	8				
3	20	22	23			
4	55	63	65	66		
5	162	188	196	195	199	
6	477	564	590	598	600	601

Turn now to the enumerator $L_n(y)$ (fully labeled endpoints). The identity corresponding to (21) is

$$1 + L(x, y) = \exp\,(yS(x)) - yS(x) \tag{23}$$

with $L(x, y) = \Sigma\, L_n(y)x^n/n!$, $S(x) = \Sigma\, S_n x^n/n!$. Thus $L_0(y) = 0$. It follows easily from (23) that

$$L_n(y) + yS_n = Y_n(yS_1, \ldots, yS_n), \quad n = 1, 2, \ldots, \tag{24}$$

$$L_{n+1}(y) = y \sum_0 \binom{n}{j} S_{j+1} L_{n-j}(y) + y^2 \sum_0 \binom{n}{j} S_{j+1} S_{n-j}, \quad n = 0, 1, \ldots.$$

In the first of (24), Y_n is a Bell polynomial; in the second the last term is $y^2 L(n+1, 2)$ or as will appear below, $y^2(S_{n+1} + nS_n)/2$. From either of equations (24), it may be found that: $L_2(y) = y^2$, $L_3(y) = 3y^2 + y^3$, $L_4(y) = 19y^2 + 6y^2 + y^3$.

Following Comtet [6; I, p. 144], write

$$Y_n(yS_1, \ldots, yS_n) = \sum_1^n y^k B_{n,k}(S_1, S_2, \ldots),$$

so that

$$L(n, k) = B_{n,k}(S_1, S_2, \ldots), \quad k = 2(1)n.$$

Then first, by a known formula [10, p. 48]

$$L(n, 2) = \sum_0 \binom{n-1}{j} S_j S_{n-j},$$

and by differentiation of the second of equations (9),

$$(1 + x)S'(x) = 1 + 2S(x)S'(x) \tag{25}$$

(the prime denotes a derivative); hence

$$S_{n+1} + nS_n = \delta_{n0} + 2 \sum \binom{n}{j} S_j S_{n+1-j} = \delta_{n0} + 2L(n+1, 2).$$

Thus $L(1,2) = 0$ and

$$2L(n+1,2) = S_{n+1} + nS_n, \quad n = 1, 2, \ldots.$$

Multiplying (25) through by $S^k(x)$ to get

$$(1+x)S^k(x)S'(x) = S^k(x) + 2S^{k+1}(x)S'(x),$$

and noting that $DS^k(x) = kS^{k-1}(x)S'(x)$, $D = d/dx$, it follows that

$$(1+x)DS^{k+1}(x)/(k+1) = S^k(x) + 2S^{k+2}(x)/(k+2). \tag{26}$$

Using the identity [11, p. 180]

$$S^k(x) = k! \sum_{n=k} B_{n,k}(S_1, S_2, \ldots)x^n/n! = k! \sum_{n=k} L(n,k)x^n/n!, \quad k = 2(1)n,$$

and its immediate consequence

$$DS^k(x) = k! \sum_{n=k} L(n+1,k)x^n/n!, \quad k = 2(1)n$$

in (26) leads to the recurrence

$$L(n+1, k+1) + nL(n, k+1) = L(n, k) + 2(k+1)L(n+1, k+2), \tag{27}$$

which holds for $k = 1(1)n$ if $L(n, 1)$ is taken as S_n. Note that the case $k = 1$ then gives

$$4L(n+1, 3) = L(n+1, 2) + nL(k, 2) - S_n$$

or, since $2L(n+1, 2) = S_{n+1} + nS_n$,

$$8L(n+1, 3) = S_{n+1} + 2(n-1)S_n + n(n-1)S_{n-1}.$$

Similarly it is found that

$$48L(n+1, 3) = S_{n+1} + 3(n-2)S_n + (3n^2 - 9n + 4)S_{n-1} + n(n-1)/n-2)S_{n-2}.$$

Working from the upper end, it is apparent that $L(n+1, n+1) = L(n, n) = L(2, 2) = 1$. Then

$$L(n+1, n) = L(n, n-1) + 2nL(n+1, n+1) - L(n, n) = L(n, n-1) + n = \binom{n+1}{2}$$

the last step by the boundary condition $L(3, 2) = 3$.

Omitting details, two further results are

$$L(n+1, n-1) = \binom{n+1}{3} + 3\binom{n+2}{4}$$

$$L(n+1, n-2) = \binom{n+1}{4} + 10\binom{n+2}{5} + 15\binom{n+3}{6}$$

in which the numerical coefficients are instances of the familiar numbers $b(k+j, j)$. Thus it may be guessed that

$$L(n, n-k) = \sum_0^k \binom{n+j-1}{j+k} b(k+j, j), \tag{28}$$

which is readily verified by (27) and the recurrence for $b(n, k)$ quoted above. Note that the sum starts with zero only to include $L(n, n) = b(0, 0) = 1$; for $k>0$, $b(k, 0) = 0$. Of course, (28) is equivalent to

$$L(n, k) = \sum_0^{n-k} \binom{n+j-1}{k-1} b(n-k+j, j), \tag{28a}$$

and

$$L_n(y) = \sum_{k=2}^{n} y^k \sum_{j=0}^{n-k} \binom{n+j-1}{k-1} b(n-k+j, j).$$

Additional values of $L(n, k)$ are as follows

$n \backslash k$	2	3	4	5	6	7	8
5	170	55	10	1			
6	1 966	645	125	15	1		
7	27 860	9 226	1 855	245	21	1	
8	467 244	155 764	32 081	4 480	434	28	1

6. Series-reduced rooted trees by height

The height of any tree, issueing from the root, is the number of lines in the longest path from the root to an endpoint. With several lines at the root, the height of the tree is the maximum of the heights of the several trees. If the enumerators by height are denoted $h_n(y)$ and $H_n(y)$ (for the unlabeled, and labeled cases, resp.), the following table may be found by inspection:

n	2	3	4	5
$h_n(y)$	y	$y+y^2$	$y+3y^2+y^3$	$y+5y^2+5y^3+y^4$
$H_n(y)$	y	$y+3y^2$	$y+13y^2+12y^3$	$y+50y^2+125y^3+60y^4$

Of course $h_n(1) = s_n$, $H_n(1) = S_n$.

These results may be verified by classifying the trees by their *specification*, that is, by the partition $1^{p_1}, 2^{p_2}, \ldots, n^{p_n}$ of n, with p_i the number of planted trees with i endpoints issueing from the root; note that $p_1 + p_2 + \ldots + p_n \geq 2$, $p_1 + 2p_2 + \ldots + np_n = n$. For $n = 2, 3, 4$ the specifications are 1^2; $21, 1^3$; $31, 22$; 21^2; 1^4. Writing $h(1^{p_1} \ldots n^{p_n}, y)$ for the height enumerator of (unlabeled) trees with specification $1^{p_1} \ldots n^{p_n}$ it is clear that

$$h(1^{p_1}, y) = y, \quad h(1^{p_1} 2^{p_2}, y) = y^2, \quad p_2 = 1, 2, \ldots$$

Note that in the second all p_1 (with one endpoint) have height one, all p_2 trees have height 2, that is, height enumerator $y^2 = yh_2(y)$. Similarly the height enumerator of a (planted) tree with j endpoints, issueing from the root, is $yh_j(y)$ and $h(1^{p_1}2^{p_2}j, y) = yh_j(y)$. Thus

$$\begin{aligned} h_4(y) &= h(13, y) + h(22, y) + h(211, y) + h(1\,111, y) \\ &= yh_3(y) + 2y^2 + y = y + 3y^2 + y^3, \end{aligned}$$

$$\begin{aligned} h_5(y) &= h(14, y) + h(23, y) + h(1^23, y) + h(12^2, y) + h(1^32, y) + h(1^5, y) \\ &= yh_4(y) + 2yh_3(y) + 2y^2 + y = y + 5y^2 + 5y^3 + y^4. \end{aligned}$$

For $n = 6$, the term $h(3^2, y)$ requires special consideration, since $h(3^{p_3}, 1) = \binom{s_3 + p_3 - 1}{p_3} = p_3 + 1$. But it is easy to see that

$$h(3^{p_3}, y) = y^2 + p_3 y^3, \quad p_3 = 1, 2, \dots,$$

since there is only one way of choosing p_3 trees of height 2. It is a little harder to show that

$$h(4^{p_4}, y) = y^2 - y^3 + \binom{p_4 + 3}{3} y^3 + \binom{p_4 + 3}{4} y^4, \quad p_4 = 1, 2, \dots,$$

and

$$h(5^j, y) = y^2 - y^3 + \binom{j+5}{j}(y^3 - y^4) + \binom{j+10}{j} y^4 + \binom{j+10}{j-1} y^5, \quad j = 1, 2, \dots,$$

$$h(6^j, y) = y^2 - y^3 + \binom{j+9}{j}(y^3 - y^4) + \binom{j+24}{j}(y^4 - y^5) + \binom{j+31}{j} y^5 + \binom{j+31}{j-1} y^6.$$

Since $h(6, y) = yh_6(y) = y^2 + 9y^3 + 15y^4 + 7y^5 + y^6$, the numbers in the binomial coefficients appearing in $h(6^7, y)$ are recognized as 9, $9 + 15$, $9 + 15 + 7$. Since a similar remark applies to its predecessors, it is easy to guess the structure of $h(n^j, y)$, $n = 1, 2, \dots$, which I have not taken the time to prove.

The dominance relation implicit in height enumeration is accommodated by an operator $\circ$, such that

$$y^{j_1} \circ y^{j_2} \dots \circ y^{j_k} = y^J$$

with $J = \max(j_1, \dots, j_k)$. Thus a tree of specification 43 is evaluated by

$$yh_4(y) \circ yh_3(y) = (y^2 + 3y^3 + y^4) \circ (y^2 + y^3) = y^2 + 7y^3 + 2y^4.$$

The coefficients $h(n, k)$ of $h_n(y)$ for $n = 6(1)10$ are:

$n \backslash k$	1	2	3	4	5	6	7	8	9
6	1	9	15	7	1				
7	1	13	37	29	9	1			
8	1	20	84	97	47	11	1		
9	1	28	175	286	193	69	13	1	
10	1	40	354	788	690	333	95	15	1

It is apparent that $h(n, n-1)=1$, $h(n, n-2)=2n-5$, $n=2, 3, \ldots$, and it may be verified that

$$h(n, n-3) = 2(n-1)(n-5)+5, \quad n=5, 6, \ldots,$$
$$h(n, n-4) = 8\binom{n-2}{3} - 4\binom{n-2}{2} - 3, \quad n = 8, 9, 10.$$

The sequences $h(n, j)$, $j=2, 3, 4, 5$ do not appear in [13].

For the labeled case, the formula similar to the first of (24) is

$$H_n(y) = yY_n(H_1(y), \ldots, H_n(y)) - yH_n(y)$$

with the products in the expansion of Y_n evaluated by the operator $\circ$; *ex gratia*

$$H_1(y)\circ H_n(y) = H_2(y)\circ H_n(y) = H_n(y),$$
$$H_3(y)\circ H_3(y) = (y+3y^2)\circ(y+3y^2) = y+15y^2.$$

The coefficients $H(n, k)$ for $n=6(1)9$ are

$n \backslash k$	1	2	3	4	5	6	7	8
6	1	201	1 080	1 110	360			
7	1	875	9 352	11 170	10 290	2 520		
8	1	4 138	84 917	229 936	218 400	102 480	20 160	
9	1	21 145	820 521	3 370 941	4 372 704	2 948 400	1 103 760	181 440

It is evident that $H(n, n-1)=n!/2$, and that

$$H(n, n-2) = nH(n-1, n-3)+nH(n-1, n-2), \quad n=5, 6, \ldots,$$
$$H(n, n-3) = nH(n-1, n-4)+2\binom{n}{2}H(n-1, n-3)+8\binom{n}{3}H(n-1, n-2), \quad n=7, 8, \ldots.$$

Also $H(n, 1)=1$, and $H(n, 2)=B_n-2$, $n=2, 3, \ldots$, with $B_n=Y_n(1, \ldots, 1)$, a Bell number.

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