DENUMERABLE MARKOV PROCESSES AND THE ASSOCIATED CONTRACTION SEMIGROUPS ON l

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§ 1. Introduction and summary

1.1. We shall be concerned with the analytical rather than with the probabilistic side of the theory of Markov processes. It will therefore be appropriate to define a *process* as a set $\mathcal{P} \equiv \{p_{ij}\}$ of real-valued functions defined on $\langle 0, \infty \rangle$, where i and j range over some fixed denumerable set E, and

$$\text{I}: \qquad p_{ij}(t) \geq 0 \qquad (t \geq 0);$$

$$\text{II}: \qquad \sum_{\alpha \in E} p_{i\alpha}(t) \leq 1 \qquad (t \geq 0);$$

$$\text{III}: \qquad p_{ij}(u+v) = \sum_{\alpha \in E} p_{i\alpha}(u)\, p_{\alpha j}(v) \qquad (u \geq 0,\ v \geq 0);$$

$$\text{IV}: \qquad p_{ij}(0) = \delta_{ij} = \lim_{t \downarrow 0} p_{ij}(t).$$

The continuity condition IV is designed to exclude excessively irregular behaviour (such as non-measurability) of the p_{ij}; it implies the continuity, uniform for $t \geq 0$, of each p_{ij} (Kendall [16], Th. 3.3).

In the probabilistic theory[1] II is strengthened to

$$\text{II}^*: \qquad \sum_{\alpha \in E} p_{i\alpha}(t) = 1.$$

The $p_{ij}(t)$ are then transition probabilities for a time-homogeneous Markov process with E as its set of states. When the sign of inequality is allowed in II, a probabilistic interpretation is still possible if we suppose that E does not exhaust the set

[1] See Doob [4], [5], Ch. VI.

of states and that the system is such that no return to E is allowed from states outside E. The difference,

$$1-\sum_{\alpha\in E} p_{i\alpha}(t) \qquad (i\in E),$$

is then interpreted as the probability that the system, initially in state i, will be in a state outside E at time t. Such an interpretation is always possible (e.g. by adjoining a single further state, as in § 5.4) and is often useful, and the set $\mathcal{P}\equiv\{p_{ij}\}$ is then sometimes called a *quasi*-process (Jensen [13]). Instead of adopting this practice we shall always call $\mathcal{P}$ a process, but qualify it as "honest" when II* is satisfied.

1.2. One of the aims of the analytical theory is to describe the process completely *in terms of its infinitesimal properties* as $t\downarrow 0$. It is known (Doob [4], Kolmogorov [19], Kendall [16]) that the right-hand derivatives

$$q_{ij}\equiv p'_{ij}(0)=\lim_{t\downarrow 0}\,(p_{ij}(t)-\delta_{ij})/t \tag{1.1}$$

exist,[1] that $0\le q_{ij}<\infty$ when $i\neq j$, and that

$$\sum_{\alpha\neq i} q_{i\alpha}\le -q_{ii}\le\infty. \tag{1.2}$$

All four combinations of $<$ and $=$ envisaged in (1.2) can actually occur for honest processes (Kolmogorov [19], Kendall & Reuter [18], Kendall [17]). However, a knowledge of the coefficients q_{ij} alone does not in general determine the process uniquely (Doob [4]), so that the q_{ij} do not describe the infinitesimal properties of $\mathcal{P}$ in sufficient detail. A better description can be given by introducing an operator P_t on the Banach space l, defined by

$$(P_t x)_j\equiv\sum_{\alpha} x_\alpha\, p_{\alpha j}(t) \qquad (x\in l). \tag{1.3}$$

Properties I–IV of $\mathcal{P}$ imply (see Hille [10], Th. 21.9.2) that $\mathcal{G}\equiv\{P_t: t\ge 0\}$ is a *contraction semigroup*, i.e. that

$$\begin{aligned}
&\mathrm{A}: && P_t x\ge 0 && \text{when } x\ge 0;\\
&\mathrm{B}: && \|P_t x\|\le\|x\| && \text{when } x\ge 0;\\
&\mathrm{C}: && P_0=I,\quad P_{u+v}=P_u P_v;\\
&\mathrm{D}: && \|P_t x-x\|\to 0 \text{ as } t\downarrow 0, \text{ for each } x\in l.
\end{aligned}$$

[1] Unless otherwise stated $p'_{ij}(t)$ will denote the right-hand derivative when $t=0$, and the two-sided derivative when $t>0$.

B can be replaced by

$$\text{B}^*: \quad \|P_t x\| = \|x\| \quad \text{when } x \geq 0,$$

so that $\mathcal{G}$ is a *transition semigroup*, if and only if $\mathcal{P}$ is honest. Conversely every contraction [transition] semigroup $\mathcal{G}$ has a unique representation (1.3) in terms of a process [an honest process] $\mathcal{P}$, so that in analytical contexts we may identify a process $\mathcal{P}$ with the associated semigroup $\mathcal{G}$.

Now $\mathcal{G}$ is uniquely determined by its *infinitesimal generator* Ω. This linear operator is defined by

$$\Omega x \equiv \lim_{t \downarrow 0} (P_t x - x)/t; \tag{1.4}$$

its domain $\mathcal{D}(\Omega)$ consists precisely of those elements for which the (strong) limit in (1.4) exists. Thus $\mathcal{G}$, and therefore $\mathcal{P}$ also, *can* in principle be characterised by its infinitesimal properties, but we are now faced with a new problem: how to recognise and describe in simple terms those operators Ω which generate a contraction or transition semigroup on l.

1.3. Summary. We shall mainly deal with processes for which

$$q_i \equiv -q_{ii} < \infty \qquad (i \in E) \tag{1.5}$$

and which satisfy one or other of the differential equations

$$p'_{ij}(t) = \sum_\alpha q_{i\alpha}\, p_{\alpha j}(t), \tag{1.6}$$

$$p'_{ij}(t) = \sum_\alpha p_{i\alpha}(t)\, q_{\alpha j} \tag{1.7}$$

(the celebrated "backward" and "forward" equations of Kolmogorov). The assumption (1.5) has recently been shown (Austin [1]) to imply the existence of continuous derivatives $p'_{ij}(t)$. It will be convenient to use this fact, and we therefore begin by proving a somewhat stronger form of this result in § **2** (Theorem 1). In § **3**, analytical conditions (involving either the q_{ij} or the generator Ω) will be given for the validity of the Kolmogorov equations (1.6) and (1.7). Our remaining results will depend on the existence theorem of Feller [6] for processes with pre-assigned values of the q_{ij}; Feller's proof and some of its corollaries will be discussed in § **4**. In §§ **5-6**, the uniqueness of processes satisfying (1.6) or (1.7) with given q_{ij} will be discussed, and simple uniqueness criteria obtained (Theorems 8 and 10). When there is more than one process with given q_{ij} we are far from having a complete description of *all* such processes; some minor results in this direction will be given in § **7**. Finally § **8** contains several examples, including a complete treatment of uniqueness problems for the general birth-and-death process.

Evidently the problems formulated in § **1.2** will receive only a partial solution in the present paper and several important questions, stated in § **7**, remain open. We should also refer here to two forthcoming papers by Feller [8], [9]; the second of these contains constructions which are considerably more general than those which we use in §§ **5.1** and **6.2**.

1.4. Acknowledgement. I am greatly indebted to David G. Kendall for allowing me to incorporate several of his own results in this paper, and for numerous suggestions and comments during its preparation.

1.5. Notation and terminology

The symbols $p_{ij}(t)$, q_{ij}, $q_i\,(\equiv -q_{ii})$, P_t and Ω have already been defined. The Banach spaces l and m consist respectively of real sequences

$$x \equiv \{x_\alpha : \alpha \in E\} \quad \text{with} \quad \|x\| \equiv \sum_\alpha |x_\alpha| < \infty$$

and real sequences

$$y \equiv \{y_\alpha : \alpha \in E\} \quad \text{with} \quad \|y\| \equiv \sup_\alpha |y_\alpha| < \infty,$$

and we shall write

$$(y, x) \equiv \sum_\alpha y_\alpha x_\alpha \quad \text{when } y \in m \text{ and } x \in l.$$

Also

u^i is the ith "unit vector" in l: $(u^i)_j \equiv \delta_{ij}$;

v^j is the jth "unit vector" in m: $(v^j)_i \equiv \delta_{ij}$;

$e \in m$ is defined by $(e)_i = 1$ (all i).

Observe that

$(v^j, x) = (x)_j$, the jth coordinate of $x \in l$;

$(y, u^i) = (y)_i$, the ith coordinate of $y \in m$;

$p_{ij}(t) = (P_t u^i)_j = (v^j, P_t u^i)$;

$(e, x) = \|x\|$ when $x \geq 0$ $(x \in l)$.[1]

The "dishonesty function" $d_i(t)$ is given by

$$d_i(t) \equiv 1 - \sum_\alpha p_{i\alpha}(t) = 1 - \|P_t u^i\|. \tag{1.8}$$

[1] We write $x \geq 0$ when $x_\alpha \geq 0$ (all $\alpha \in E$), and $x > 0$ when $x \geq 0$ and $x \neq 0$.

It is continuous, and is non-decreasing because

$$\| P_{t+s} u^i \| = \| P_s (P_t u^i) \| \le \| P_t u^i \| \quad \text{for } s \ge 0.$$

The derivative

$$d_i \equiv d_i'(0) = \lim_{t \downarrow 0} d_i(t)/t \tag{1.9}$$

exists and is finite (Kendall [16], Th. 7.1), $d_i \ge 0$, and

$$\sum_{\alpha \neq i} q_{i\alpha} + d_i \le q_i \quad \text{for each } i. \tag{1.10}$$

Finally, we write

$$\psi_{ij}(\lambda) \equiv \int_0^\infty e^{-\lambda t} p_{ij}(t)\, dt \qquad (\lambda > 0) \tag{1.11}$$

for the Laplace transform of p_{ij}, and define the *resolvent operator* Ψ_λ by

$$(\Psi_\lambda x)_j \equiv \sum_\alpha x_\alpha \psi_{\alpha j}(\lambda) \qquad (x \in l). \tag{1.12}$$

Then Ψ_λ is the inverse of $\lambda I - \Omega$:

$$(\lambda I - \Omega) \Psi_\lambda x = x \qquad (x \in l), \tag{1.13}$$

$$\Psi_\lambda (\lambda I - \Omega) x = x \qquad (x \in \mathcal{D}(\Omega)). \tag{1.14}$$

In particular the range $\mathcal{R}(\Psi_\lambda)$ of Ψ_λ coincides with the domain $\mathcal{D}(\Omega)$ of Ω, and $\mathcal{D}(\Omega) = \mathcal{R}(\Psi_\lambda)$ is dense in l. Also $\lambda \Psi_\lambda$ is a contraction operator[1] (for each $\lambda > 0$):

$$\| \lambda \Psi_\lambda x \| \le \| x \| \quad \text{when } x \ge 0. \tag{1.15}$$

If $\{P_t\}$ is a transition semigroup then $\lambda \Psi_\lambda$ is a transition operator:

$$\| \lambda \Psi_\lambda x \| = \| x \| \quad \text{when } x \ge 0; \tag{1.16}$$

conversely if $\lambda \Psi_\lambda$ is a transition operator for *one* $\lambda > 0$ then $\{P_t\}$ is a transition semigroup. We shall also use the facts (see [17], [22]) that

$$(e, \Omega x) \le 0 \quad \text{when } x \ge 0,\ x \in \mathcal{D}(\Omega), \tag{1.17}$$

and that equality holds in (1.17) for all such x if and only if $\{P_t\}$ is a transition semigroup.

[1] See Hille [11], Yosida [24], [25].

§ 2. Differentiability properties

[Throughout § 2 we consider a fixed $i \in E$ for which $q_i < \infty$.]

2.1. In order to study the connexions between differentiability properties of the p_{ij} and properties of the infinitesimal generator Ω, it is useful first to obtain some estimates[1] for the difference quotients

$$\left.\begin{aligned}\Delta_{ij}(t, t+s) &\equiv (p_{ij}(t+s) - p_{ij}(t))/s\\ \Delta_{i}(t, t+s) &\equiv (d_{i}(t+s) - d_{i}(t))/s\end{aligned}\right\}\ (t \geq 0,\ s > 0).$$

Because $\sum_{\alpha} p_{i\alpha}(t) + d_i(t) = 1$, we have

$$\sum_{\alpha} \Delta_{i\alpha}(t, t+s) + \Delta_i(t, t+s) = 0, \tag{2.1}$$

and also

$$\Delta_i(t, t+s) \geq 0, \tag{2.2}$$

because $d_i(t)$ is non-decreasing. Further (cf. Kendall [16], Th. 4.1)

$$(1 - p_{ii}(s))/s \leq q_i \qquad (s > 0). \tag{2.3}$$

Now

$$p_{ij}(t+s) = \sum_{\alpha} p_{i\alpha}(s)\, p_{\alpha j}(t) \geq p_{ii}(s)\, p_{ij}(t)$$

so that by (2.3)

$$\Delta_{ij}(t, t+s) \geq -\frac{1 - p_{ii}(s)}{s}\, p_{ij}(t) \geq -q_i\, p_{ij}(t). \tag{2.4}$$

By summing (2.4) with respect to j, over any set $A \subseteq E$, we obtain

$$\sum_{j \in A} \Delta_{ij}(t, t+s) \geq -q_i \sum_{j \in A} p_{ij}(t) \geq -q_i; \tag{2.5}$$

on the other hand, using (2.1) and (2.2), we find that

$$\sum_{j \in A} \Delta_{ij}(t, t+s) \leq -\sum_{j \notin A} \Delta_{ij}(t, t+s) \leq q_i. \tag{2.6}$$

In particular, taking $A = \{j\}$ in (2.5) and (2.6),

$$|\Delta_{ij}(t, t+s)| \leq q_i; \tag{2.7}$$

also

$$0 \leq \Delta_i(t, t+s) = -\sum_{j} \Delta_{ij}(t, t+s) \leq q_i. \tag{2.8}$$

[1] Due essentially to Austin [1].

From (2.7) and (2.8), p_{ij} and d_i satisfy a Lipschitz condition so that they are absolutely continuous and

$$p_{ij}(t) = \delta_{ij} + \int_0^t p'_{ij}(u)\, du, \tag{2.9}$$

$$d_i(t) = \int_0^t d'_i(u)\, du, \tag{2.10}$$

the derivatives existing almost everywhere. Now take A in (2.5) and (2.6) to be the set of suffixes j for which $\Delta_{ij}(t, t+s) \geq 0$. We obtain

$$\sum_j |\Delta_{ij}| = \sum_{j \in A} \Delta_{ij} + \sum_{j \notin A} (-\Delta_{ij}) \leq 2q_i,$$

from which it follows easily that

$$\sum_j |p'_{ij}(t)| \leq 2q_i \tag{2.11}$$

at any point where all the derivatives exist. Summing (2.9) over $j \in E$,

$$\begin{aligned} \sum_j p_{ij}(t) &= 1 + \sum_j \int_0^t p'_{ij}(u)\, du \\ &= 1 + \int_0^t \Big(\sum_j p'_{ij}(u)\Big)\, du, \end{aligned}$$

the interchange of summation and integration being permissible because of (2.11). On combining this result with (2.10), we obtain

$$0 = \int_0^t \Big(\sum p'_{ij}(u) + d'_i(u)\Big)\, du \qquad (t \geq 0),$$

and therefore the integrand must vanish almost everywhere. This establishes

LEMMA 1. *For almost all $t \geq 0$, the derivatives $p'_{ij}(t)$ and $d'_i(t)$ exist (for all $j \in E$) and satisfy*

$$\sum_j |p'_{ij}(t)| \leq 2q_i, \tag{2.11}$$

$$\sum_j p'_{ij}(t) + d'_i(t) = 0. \tag{2.12}$$

To link differentiability properties with properties of Ω, we now prove

LEMMA 2. *If $\tau \geq 0$ is given, then $P_\tau u^i \in \mathfrak{D}(\Omega)$ if and only if $p'_{ij}(\tau)$ (for all $j \in E$) and $d'_i(\tau)$ exist as right-hand derivatives and satisfy*

$$\sum_j p'_{ij}(\tau) + d'_i(\tau) = 0. \tag{2.13}$$

Proof. If $P_\tau u^i \in \mathcal{D}(\Omega)$, then

$$\Delta_{ij}(\tau, \tau+s) = \left(v^j, \frac{P_s P_\tau u^i - P_\tau u^i}{s}\right)$$

$$\to (v^j, \Omega P_\tau u^i) \quad \text{as } s \downarrow 0,$$

and similarly

$$\Delta_i(\tau, \tau+s) \to -(e, \Omega P_\tau u^i).$$

Hence $p'_{ij}(\tau)$ and $d'_i(\tau)$ exist as right-hand derivatives, and

$$\sum_j p'_{ij}(\tau) = \sum_j (\Omega P_\tau u^i)_j = (e, \Omega P_\tau u^i) = -d'_i(\tau).$$

Conversely, suppose that the right-hand derivatives $p'_{ij}(\tau)$ and $d'_i(\tau)$ exist, and that (2.13) holds. As at (2.11) we have $\sum_j |p'_{ij}(\tau)| \leq 2q_i < \infty$ so that the vector $p'_{i\cdot}(\tau)$ is in l and it will suffice to prove that

$$\sum(s) \equiv \sum_j |\Delta_{ij}(\tau, \tau+s) - p'_{ij}(\tau)| \to 0 \quad \text{as } s \downarrow 0.$$

Given $\varepsilon > 0$, choose the finite set $A \subseteq E$ so that

$$q_i \sum_{j \notin A} p_{ij}(\tau) + \sum_{j \notin A} |p'_{ij}(\tau)| < \varepsilon. \tag{2.14}$$

Then

$$\sum(s) \leq \sum_{j \in A} |\Delta_{ij}(\tau, \tau+s) - p'_{ij}(\tau)| + \sum_{j \notin A} |\Delta_{ij}(\tau, \tau+s)| + \varepsilon,$$

and if $\sum'$ indicates summation over suffixes j such that $\Delta_{ij} < 0$ then[1]

$$\begin{aligned} \sum_{j \notin A} |\Delta_{ij}| &= \sum_{j \notin A} \Delta_{ij} - 2 \sum_{j \notin A}{}' \Delta_{ij} \\ &\leq \sum_{j \notin A} \Delta_{ij} + 2q_i \sum_{j \notin A}{}' p_{ij}(\tau) \quad \text{by (2.5)}, \\ &< \sum_{j \notin A} \Delta_{ij} + 2\varepsilon \qquad\qquad \text{by (2.14)}. \end{aligned}$$

Hence

$$\begin{aligned} \sum(s) &< \sum_{j \in A} |\Delta_{ij} - p'_{ij}(\tau)| + \sum_{j \notin A} \Delta_{ij} + 3\varepsilon \\ &= \sum_{j \in A} |\Delta_{ij} - p'_{ij}(\tau)| - \sum_{j \in A} \Delta_{ij} - \Delta_i + 3\varepsilon \end{aligned}$$

[1] This device is borrowed from AUSTIN [1].

by (2.1), and so

$$\limsup_{s \downarrow 0} \sum(s) \leq 0 - \sum_{j \in A} p'_{ij}(\tau) - d'_i(\tau) + 3\varepsilon$$

$$= \sum_{j \notin A} p'_{ij}(\tau) + 3\varepsilon \quad \text{by (2.13)},$$

$$< 4\varepsilon, \qquad \text{by (2.14)}.$$

Because $\varepsilon > 0$ was arbitrary, it follows that $\sum(s) \to 0$ as required.

It is clear from the proof that $\Omega P_\tau u^i$ is given by

$$(\Omega P_\tau u^i)_j = p'_{ij}(\tau) \tag{2.15}$$

where $p'_{ij}(\tau)$ denotes the right-hand derivative.

2.2. Lemmas 1 and 2, when combined with a simple semigroup argument, give

THEOREM 1. *Suppose that* $q_i < \infty$. *Then* $P_t u^i \in \mathcal{D}(\Omega)$ *for each* $t > 0$, *and* $d_i(\cdot)$, $p_{ij}(\cdot)$ (*for all* $j \in E$) *have continuous derivatives for* $t \geq 0$. *Also*

$$\sum_j |p'_{ij}(t)| \leq 2 q_i \qquad (t > 0), \tag{2.11}$$

$$\sum_j p'_{ij}(t) + d'_i(t) = 0 \qquad (t > 0), \tag{2.12}$$

$$p'_{ij}(t_1 + t_2) = \sum_\alpha p'_{i\alpha}(t_1)(p_{\alpha j}(t_2) \qquad (t_1 > 0,\ t_2 \geq 0). \tag{2.16}$$

Proof. By Lemmas 1 and 2, $P_\tau u^i \in \mathcal{D}(\Omega)$ for almost all $\tau > 0$ and hence for some arbitrarily small τ. Fixing any such τ, it follows that $P_t u^i = P_{t-\tau}(P_\tau u^i)$ has the strongly continuous strong derivative $P_{t-\tau}(\Omega P_\tau u^i)$ for $t > \tau$, so that

$$p'_{ij}(t) = (v^j, P_{t-\tau}\Omega P_\tau u^i)$$

exists and is continuous for $t > \tau$. Similarly $d'_i(t) = -(e, P_{t-\tau}\Omega P_\tau u^i)$ exists and is continuous for $t > \tau$, and

$$d'_i(t) = -\sum_j (P_{t-\tau}\Omega P_\tau u^i)_j = -\sum_j p'_{ij}(t).$$

Because τ can be arbitrarily small, it follows that $p'_{ij}(t)$ and $d'_i(t)$ exist, are continuous, and satisfy (2.12), for all $t > 0$; also (2.11) is already known to hold wherever all the derivatives involved exist. Hence $P_t u^i \in \mathcal{D}(\Omega)$ when $t > 0$, from Lemma 2 (or by observing that $P_t u^i = P_{t-\tau}(P_\tau u^i) \in \mathcal{D}(\Omega)$ if τ is so chosen that $0 \leq \tau < t$ and $P_\tau u^i \in \mathcal{D}(\Omega)$) Also, if $t_1 > 0$ and $t_2 \geq 0$,

$$p'_{ij}(t_1+t_2) = (P_{t_2}\Omega P_{t_1} u^i)_j$$

$$= \sum_\alpha (\Omega P_{t_1} u^i)_\alpha \, p_{\alpha j}(t_2) = \sum_\alpha p'_{i\alpha}(t_1)\, p_{\alpha j}(t_2),$$

which proves (2.16).

We have now proved all the assertions of the theorem, except for the existence[1] and continuity of $p'_{ij}(t)$ and $d'_i(t)$ *at* $t=0$: thus we must still prove that $p'_{ij}(t)$ and $d'_i(t)$ have finite limits as $t \downarrow 0$. Let $0<\tau<t$ and $s>0$. Then

$$\Delta_{ij}(t, t+s) = \sum_\alpha \Delta_{i\alpha}(\tau, \tau+s)\, p_{\alpha j}(t-\tau)$$

$$\geq \Delta_{ij}(\tau, \tau+s)\, p_{jj}(t-\tau) - q_i \sum_{\alpha \neq j} p_{i\alpha}(\tau)\, p_{\alpha j}(t-\tau)$$

$$= \Delta_{ij}(\tau, \tau+s)\, p_{jj}(t-\tau) - q_i\,[p_{ij}(t) - p_{ij}(\tau)\, p_{jj}(t-\tau)].$$

If we let $s \downarrow 0$ and then, keeping t fixed, let $\tau \downarrow 0$, we obtain

$$p'_{ij}(t) \geq (\limsup_{\tau \downarrow 0} p'_{ij}(\tau))\, p_{jj}(t) - q_i\,[p_{ij}(t) - \delta_{ij}\, p_{jj}(t)],$$

and because $p_{jj}(t)>0$ (Kendall [16], Th. 3.2) the limit superior is finite. We now let $t \downarrow 0$ and obtain $\liminf_{t\downarrow 0} p'_{ij}(t) \geq \limsup_{\tau \downarrow 0} p'_{ij}(\tau)$, so that $p'_{ij}(t)$ has a (finite) limit as $t \downarrow 0$. A similar argument, starting from

$$\Delta_i(t, t+s) = \Delta_i(\tau, \tau+s) + \sum_\alpha \Delta_{i\alpha}(\tau, \tau+s)\, d_\alpha(t-\tau)$$

$$\geq \Delta_i(\tau, \tau+s) - q_i \sum_\alpha p_{i\alpha}(\tau)\, d_\alpha(t-\tau)$$

$$= \Delta_i(\tau, \tau+s) - q_i\,[d_i(t) - d_i(\tau)],$$

shows that $d'_i(t)$ has a finite limit, and this concludes the proof of Theorem 1.

Remarks. (1) The existence and continuity of $p'_{ij}(t)$ for $t>0$, when $q_i<\infty$, was first proved for honest processes by Austin [1]; his argument as it stands does not appear to prove continuity at $t=0$. A second proof by Chung [2] establishes continuity for $t \geq 0$.

(2) The new part of Theorem 1, that $P_t u^i \in \mathcal{D}(\Omega)$ for $t>0$, will be needed in the proof of Theorem 4 and is also of independent interest.

[1] The existence of $p'_{ij}(0)$ and $d'_i(0)$ is of course known already (Kolmogorov [19], Kendall [16]), but it will follow independently from our proof in the present special case.

(3) There exist honest processes for which $q_i < \infty$ (all i) but $\sum_\alpha q_{i\alpha} < 0$ for some i (Kolmogorov [19], Kendall & Reuter [18]). This shows that (2.12) may fail at $t=0$, and also that u^i need not belong to $\mathcal{D}(\Omega)$ (by Lemma 2 it does so if and only if $\sum q_{i\alpha} + d_i = 0$). Thus Theorem 1 is in a sense best possible. On the other hand, there also exist processes for which $q_i = \infty$ for some i and yet $P_t u^i \in \mathcal{D}(\Omega)$ for all $t > 0$ (Kendall & Reuter [18]).

§ 3. The Kolmogorov differential equations

3.1. It is easy to show, by letting $s \downarrow 0$ in the identities

$$\begin{aligned}\Delta_{ij}(t, t+s) &= \sum_\alpha \Delta_{i\alpha}(0, s)\, p_{\alpha j}(t) \\ &= \sum_\alpha p_{i\alpha}(t)\, \Delta_{\alpha j}(0, s),\end{aligned}$$

that the *inequalities*

$$p'_{ij}(t) \geq \sum_\alpha q_{i\alpha}\, p_{\alpha j}(t), \tag{3.1}$$

$$p'_{ij}(t) \geq \sum_\alpha p_{i\alpha}(t)\, q_{\alpha j} \tag{3.2}$$

hold whenever $p'_{ij}(t)$ exists (see Doob [4]). In general, these can be strict inequalities,[1] but it is desirable to find conditions under which the differential *equations*, already quoted in (1.6) and (1.7)

$$(B_{ij}): \quad p'_{ij}(t) = \sum_\alpha q_{i\alpha}\, p_{\alpha j}(t), \tag{3.3}$$

$$(F_{ij}): \quad p'_{ij}(t) = \sum_\alpha p_{i\alpha}(t)\, q_{\alpha j}, \tag{3.4}$$

will hold. Conditions of a probabilistic nature, involving the continuity properties of sample functions of the Markov process, have been given by Doob [4]. We shall give some different conditions, of an analytical nature, involving the infinitesimal generator Ω and its relation to the coefficients q_{ij}.

It will be convenient from now on to view the constants q_{ij} in a slightly different light, and they will no longer be *defined* by $q_{ij} \equiv p'_{ij}(0)$. We now start with a *given* set $Q \equiv \{q_{ij}\}$ of (finite) real numbers q_{ij} such that

$$q_{ij} \geq 0 \quad (i \neq j), \qquad \sum_\alpha q_{i\alpha} \leq 0 \quad (\text{all } i), \tag{3.5}$$

[1] See the examples in §§ **8.1–8.3**.

and then consider processes for which either (3.3) or (3.4) holds. It will also be necessary to interpret (3.3) [or (3.4)] either in the *strict* sense that p'_{ij} is continuous and (3.3) [or (3.4)] holds for all $t \geq 0$, or in the *wide* sense that p_{ij} is absolutely continuous for $t \geq 0$, and (3.3) [or (3.4)] holds for almost all $t \geq 0$. It will, however, be shown that the two interpretations are equivalent; this implies that even when (3.3) [or (3.4)] is merely known to hold in the wide sense, then $p'_{ij}(0) = q_{ij}$. Thus there will be no conflict with the definition $q_{ij} \equiv p'_{ij}(0)$ which has been used so far.

To formulate our conditions for the validity of (3.3) or (3.4) we define operators Q and Q_0 as follows: $\mathcal{D}(Q)$ is the set of $x \in l$ such that

(i) $\sum_{\alpha} x_{\alpha} q_{\alpha j}$ converges absolutely for each j, and

(ii) $\sum_{j} \left| \sum_{\alpha} x_{\alpha} q_{\alpha j} \right| < \infty$.

Q is given by

$$(Qx)_j \equiv \sum_{\alpha} x_{\alpha} q_{\alpha i} \qquad (x \in \mathcal{D}(Q)). \tag{3.6}$$

$\mathcal{D}(Q_0) \equiv \mathcal{D}_0$, the set of "finite" vectors in l (vectors $x \in l$ with only finitely many non-zero components); note that $\mathcal{D}_0 \subseteq \mathcal{D}(Q)$. Q_0 is the restriction of Q to $\mathcal{D}(Q_0)$.

Both Q_0 and Q have domains dense in l, because $\mathcal{D}_0$ is dense.

3.2. The backward equations. The strict and wide interpretations of

$$(B_{ij}): \qquad p'_{ij}(t) = \sum_{\alpha} q_{i\alpha} p_{\alpha j}(t)$$

are easily seen to be equivalent. If (B_{ij}) holds in the wide sense, then

$$p_{ij}(t) = \delta_{ij} + \int_0^t \left(\sum_{\alpha} q_{i\alpha} p_{\alpha j}(u)\right) du \qquad (t \geq 0).$$

Here the integrand is continuous, the series defining it being uniformly convergent, and by differentiating we see that (B_{ij}) holds in the strict sense. We may therefore always use the strict interpretation; putting $t=0$, we see that $p'_{ij}(0) = q_{ij}$. We now prove

LEMMA 3. *Let* $Q = \{q_{ij}\}$ *satisfy* (3.5) *and suppose that* $\mathcal{P} \equiv \{p_{ij}(t)\}$ *is a process with generator* Ω. *For any given* $i \in E$, (B_{ij}) *will hold for all* $j \in E$ *if and only if* $u^i \in \mathcal{D}(\Omega)$ *and* $(\Omega u^i)_j = q_{ij}$.

Proof. If $u^i \in \mathcal{D}(\Omega)$ and $(\Omega u^i) = q_{ij}$ then $P_t u^i$ has the strong derivative $P_t \Omega u^i$ for $t \geq 0$. Hence $p_{ij}(t) = (v^j, P_t u^i)$ has a continuous derivative for $t \geq 0$, given by

$$p'_{ij}(t) = (v^j, P_t \Omega u^i) = \sum_\alpha (\Omega u^i)_\alpha p_{\alpha j}(t) = \sum_\alpha q_{i\alpha} p_{\alpha j}(t),$$

so that (B_{ij}) holds for all $j \in E$.

Conversely if (B_{ij}) holds for all $j \in E$, then by taking Laplace transforms on both sides we obtain

$$\lambda \psi_{ij}(\lambda) = \delta_{ij} + \sum_\alpha q_{i\alpha} \psi_{\alpha j}(\lambda) \qquad (\lambda > 0), \tag{3.7}$$

where ψ_{ij} denotes the Laplace transform of p_{ij} (cf. (1.11)). In terms of the resolvent operator Ψ_λ (cf. (1.12)), (3.7) can be written as

$$\Psi_\lambda (\lambda u^i - \xi) = u^i, \tag{3.8}$$

where $(\xi)_\alpha \equiv q_{i\alpha}$. But Ψ_λ is $1-1$ and inverse to $\lambda I - \Omega$, whence (3.8) implies that $u^i \in \mathcal{D}(\Omega)$ and $\lambda u^i - \xi = \lambda u^i - \Omega u^i$, i.e. $(\Omega u^i)_j = (\xi)_j = q_{ij}$.

THEOREM 2. *Let $\mathcal{Q} \equiv \{q_{ij}\}$ satisfy* (3.5) *and let $\mathcal{P} \equiv \{p_{ij}(t)\}$ be a process with generator Ω. Then (B_{ij}) holds for all i, j in E if and only if Ω is an extension of Q_0.*

Proof. This follows at once from Lemma 3, because $u^i \in \mathcal{D}(Q_0)$ and $(Q_0 u^i)_j = q_{ij}$.

THEOREM 3. *Let $\mathcal{P} \equiv \{p_{ij}(t)\}$ be any process, and define $q_{ij} \equiv p'_{ij}(0)$, $d_i \equiv d'_i(0)$. Then for any one $i \in E$, (B_{ij}) holds for all $j \in E$ if and only if q_{ii} is finite and*

$$\sum_\alpha q_{i\alpha} + d_i = 0. \tag{3.9}$$

Proof. If q_{ii} is finite and (3.9) holds, Lemma 2 (applied at $\tau = 0$) shows that $u^i \in \mathcal{D}(\Omega)$ and that $(\Omega u^i)_j = q_{ij}$ (cf. (2.15)). Hence (B_{ij}) holds by Lemma 3.

Conversely, if (B_{ij}) holds,[1] then $u^i \in \mathcal{D}(\Omega)$ and $(\Omega u^i)_j = q_{ij}$ by Lemma 3. Lemma 2 now shows that (3.9) must hold.

COROLLARY. *(B_{ij}) always holds (for all $j \in E$) when $\sum_\alpha q_{i\alpha} = 0$.*

Remarks. Theorem 3 shows in particular that for *honest* processes (B_{ij}) holds for all i and j if and only if q_{ii} is finite and

$$\sum_\alpha q_{i\alpha} = 0 \quad \text{for all } i. \tag{3.10}$$

[1] Of course, this is understood to imply that q_{ii} is finite.

This result is due to Doob [4]. The first example of an honest process for which (B_{ij}) does not hold was given by Kolmogorov [19]; see also Kendall & Reuter [18].

In future we shall call the set $Q \equiv \{q_{ij}\}$ *conservative* if it satisfies (3.10).

3.3. The forward equations. These read

$$(F_{ij}): \qquad p'_{ij}(t) = \sum_{\alpha} p_{i\alpha}(t)\, q_{\alpha j}.$$

They will require more delicate handling than (B_{ij}), since we can no longer assert *a priori* that the series on the right converges uniformly. We shall prove a result (Theorem 4) analogous to Theorem 2, but there will be no counterpart to Theorem 3. This is hardly surprising because the forward equations (F_{ij}) may fail even for honest processes for which $Q \equiv \{p'_{ij}(0)\}$ is conservative (Doob [4]; see also the remarks at the end of § 5.2).

THEOREM 4. *Let $Q \equiv \{q_{ij}\}$ satisfy (3.5) and let $\mathcal{P} \equiv \{p_{ij}(t)\}$ be generated by Ω. Then the following three statements are equivalent:*

(1) *(F_{ij}) holds in the wide sense for all i, j in E;*

(2) *(F_{ij}) holds in the strict sense for all i, j in E;*

(3) *Ω is a restriction of Q.*

Any one of these statements implies that $p'_{ij}(0) = q_{ij}$.

Proof. Suppose that (1) holds. Because p_{ij} is absolutely continuous and bounded, the Laplace transform of p'_{ij} exists for $\lambda > 0$ and equals $\lambda\psi_{ij}(\lambda) - \delta_{ij}$, so that from (F_{ij}) we obtain

$$\lambda\psi_{ij}(\lambda) = \delta_{ij} + \sum_{\alpha} \psi_{i\alpha}(\lambda)\, q_{\alpha j} \qquad (i, j \in E;\ \lambda > 0); \tag{3.11}$$

the termwise integration required for this is justified because the terms of the series $\sum p_{i\alpha}(t)\, q_{\alpha j}$ are non-negative except perhaps when $\alpha = j$. We shall deduce from (3.11) that $\Psi_\lambda x \in \mathcal{D}(Q)$ and

$$(\lambda I - Q)\Psi_\lambda x = x \qquad (\lambda > 0) \tag{3.12}$$

for all $x \in l$. By linearity it suffices to prove this when $x \geq 0$. If we write $y \equiv \Psi_\lambda x$ then

$$\begin{aligned}\sum_{\alpha \neq j} y_\alpha q_{\alpha j} &= \sum_{\alpha \neq j} \Big(\sum_{\beta} x_\beta \psi_{\beta\alpha}(\lambda)\Big) q_{\alpha j} \\ &= \sum_{\beta} x_\beta (\lambda\psi_{\beta j}(\lambda) - \delta_{\beta j} - \psi_{\beta j}(\lambda)\, q_{jj}), \quad \text{by (3.11)}, \\ &= (\lambda y - x)_j - y_j q_{jj},\end{aligned}$$

the inversion of summations being justified because $x_\beta \psi_{\beta\alpha}(\lambda) q_{\alpha j} \geq 0$. Hence $\Sigma y_\alpha q_{\alpha j}$ is absolutely convergent, and its sum is $(\lambda y - x)_j$. This shows that $y \in \mathcal{D}(Q)$ and $Qy = \lambda y - x$, i.e. that $\Psi_\lambda x \in \mathcal{D}(Q)$ and that (3.12) holds. Now Ψ_λ is $1-1$ and inverse to $\lambda I - \Omega$, so that (3.12) implies that Ω is a restriction of Q, and we have therefore proved that (1) implies (3).

Now assume that (3) holds. Then (3.12) will hold because $\Omega \subseteq Q$; if we put $x = u^i$ and take the jth components in (3.12), we obtain (3.11). From (3.11) it follows that the Laplace transform of $\sum_\alpha p_{i\alpha} q_{\alpha j}$ exists for $\lambda > 0$, so that $\sum_\alpha p_{i\alpha} q_{\alpha j}$ is summable over any finite interval $\langle 0, T \rangle$.

Dividing both sides of (3.11) by λ, we now have

$$\psi_{ij}(\lambda) = \lambda^{-1} \delta_{ij} + \lambda^{-1} \sum_\alpha \psi_{i\alpha}(\lambda) q_{\alpha j}. \tag{3.13}$$

The two sides of (3.13) are the Laplace transforms of the continuous functions $p_{ij}(t)$ and

$$\delta_{ij} + \int_0^t (\sum_\alpha p_{i\alpha}(u) q_{\alpha j}) \, du,$$

so that by Lerch's theorem these two functions coincide for $t \geq 0$. This shows (by differentiating) that (F_{ij}) holds in the wide sense, and we have therefore proved that (3) implies (1).

(1) and (3) are now known to be equivalent, and we shall show next that they imply that $p'_{ij}(0) = q_{ij}$.

First (1) gives

$$\begin{aligned} p_{ij}(t) &= \delta_{ij} + \int_0^t (\sum_\alpha p_{i\alpha}(u) q_{\alpha j}) \, du \\ &\geq \delta_{ij} + q_{ij} \int_0^t p_{ii}(u) \, du + (1 - \delta_{ij}) q_{jj} \int_0^t p_{ij}(u) \, du, \end{aligned}$$

and hence

$$\bar{q}_{ij} \equiv p'_{ij}(0) \geq q_{ij} + (1 - \delta_{ij}) q_{jj} \delta_{ij} = q_{ij}; \tag{3.14}$$

this shows that $\bar{q}_{ii} \geq q_{ii} > -\infty$, so that all $\bar{q}_{ij}$ are finite. Next (cf. (3.2))

$$p'_{ij}(t) \geq \sum_\alpha p_{i\alpha}(t) \bar{q}_{\alpha j}$$

whenever $p'_{ij}(t)$ exists, and by combining this with (F_{ij}) we find that

$$\sum_\alpha p_{i\alpha}(t) (\bar{q}_{\alpha j} - q_{\alpha j}) \leq 0 \qquad (\text{all } i, j \text{ in } E)$$

for almost all $t \geq 0$. But here each term in the series is non-negative, so that each term is zero for almost all t and in particular $p_{ii}(t)(\bar{q}_{ij} - q_{ij}) = 0$ *p.p.* This gives $\bar{q}_{ij} = q_{ij}$ (because $p_{ii}(t) > 0$ for all $t \geq 0$), i.e. $p'_{ij}(0) = q_{ij}$ as asserted.

It remains only to show that (1) (or (3)) implies (2). Since we now know that $p'_{ij}(0) = q_{ij}$ when (1) holds, it follows that $p'_{ii}(0)$ is finite. By Theorem 1, $p_{ij}(t)$ has a continuous derivative for $t \geq 0$, and because $P_t u^i \in \mathfrak{D}(\Omega)$ for $t > 0$ we have

$$\begin{aligned} p'_{ij}(t) &= (v^j, \Omega P_t u^i) \\ &= (v^j, Q P_t u^i) \qquad \text{because } \Omega \subseteq Q \quad \text{(by (3))}, \\ &= \sum_\alpha (P_t u^i)_\alpha q_{\alpha j} = \sum_\alpha p_{i\alpha}(t) q_{\alpha j}; \end{aligned}$$

this also holds at $t = 0$ because $p'_{ij}(0) = q_{ij}$ and therefore (F_{ij}) holds in the strict sense. This completes the proof of Theorem 4.

Remark. Theorem 4 shows that the sum of the series $\sum\limits_\alpha p_{i\alpha}(t) q_{\alpha j}$ is continuous whenever (F_{ij}) is known to hold in the wide sense. It then follows from Dini's theorem that the series converges uniformly in any finite interval $0 \leq t \leq T$.

3.4. Theorems 2 and 4 indicate a close connexion between Ω and the q_{ij} when either the backward or the forward equations hold. No equally simple connexion seems to be known when neither set of equations holds (for a non-trivial case of this phenomenon, see § **8.3**). However, it is always possible, in principle, to calculate the q_{ij} when Ω is known (even when the assumption, that $q_i < \infty$ for all i, is dropped). The calculation is based on a simple Abelian argument (Kendall [17]): because $p_{ij}(t) - \delta_{ij} \sim q_{ij} t$ for small t, it follows that

$$\begin{aligned} q_{ij} &= \lim_{\lambda \to \infty} \lambda^2 \int_0^\infty (p_{ij}(t) - \delta_{ij}) e^{-\lambda t} \, dt \\ &= \lim_{\lambda \to \infty} (\lambda^2 \psi_{ij}(\lambda) - \lambda \delta_{ij}) \\ &= \lim_{\lambda \to \infty} \lambda (v^j, (\lambda \Psi_\lambda - I) u^i). \end{aligned} \tag{3.15}$$

Similarly we find that

$$d_i = \lim_{\lambda \to \infty} \lambda (e, (I - \lambda \Psi_\lambda) u^i). \tag{3.16}$$

Because Ψ_λ is the inverse of $\lambda I - \Omega$, it may be regarded as known when Ω is given, and thus (3.15) and (3.16) will enable us to calculate q_{ij} and d_i when Ω is known.

§ 4. Feller's existence theorem

4.1. Feller's construction. The infinitesimal properties of a process may, as we have seen, be specified roughly by means of the q_{ij} $(\equiv p'_{ij}(0))$ and precisely by means of the infinitesimal generator Ω. In any attempt to construct all possible processes one may therefore well use the q_{ij} for a preliminary classification. If this is done, the first problem is: given a set $Q \equiv \{q_{ij}\}$ of finite constants such that

$$q_{ij} \geq 0 \quad (i \neq j), \qquad \sum_{\alpha} q_{i\alpha} \leq 0 \qquad (\text{all } i), \tag{4.1}$$

do there exist processes such that $p'_{ij}(0) = q_{ij}$? This question, formulated for a much more general class of processes, was answered in the fundamental paper of Feller [6], who constructed *one* such process. Feller's result will be repeatedly used, and we shall also need to refer explicitly to his construction,[1] which we now describe.

Define the functions f^n_{ij} recursively by

$$f^0_{ij}(t) \equiv 0,$$

$$f^{n+1}_{ij}(t) \equiv \delta_{ij} e^{-q_i t} + e^{-q_i t} \int_0^t \Big(\sum_{\alpha \neq i} q_{i\alpha} f^n_{\alpha j}(u) \Big) e^{q_i u} \, du, \tag{4.2}$$

where as usual $q_i \equiv -q_{ii}$ (≥ 0). The f^n_{ij} increase with n and tend to finite limits

$$f_{ij}(t) \equiv \lim_{n \to \infty} f^n_{ij}(t),$$

and they also satisfy the recurrence relations

$$f^{n+1}_{ij}(t) = \delta_{ij} e^{-q_j t} + e^{-q_j t} \int_0^t \Big(\sum_{\alpha \neq j} f^n_{i\alpha}(u) q_{\alpha j} \Big) e^{q_j u} \, du. \tag{4.3}$$

The limit functions f_{ij} define a process $\mathfrak{F} \equiv \{f_{ij}(t)\}$ such that $f'_{ij}(0) = q_{ij}$, and both the backward and forward equations hold for $\mathfrak{F}$. (These results are proved in [6] only when $\sum_{\alpha} q_{i\alpha} = 0$ for all i, but the proofs are still valid when $\sum_{\alpha} q_{i\alpha} \leq 0$: cf. also Hille [12]. In [6], the forward equations are only shown to hold in the wide sense; Theorem 4 shows that they must then hold in the strict sense.)

If $\{p_{ij}(t)\}$ is any process such that $p'_{ij}(0) = q_{ij}$, we shall call it a *Q-process* and

[1] Alternative constructions, leading to the same process as does Feller's construction, have been given by Reuter & Ledermann [23] and Kato [15].

the associated semigroup $\{P_t\}$ a *Q-semigroup*. It will then be appropriate to call $\mathfrak{F}$ the *minimal* Q-process, because $f_{ij}(t) \le p_{ij}(t)$ for any Q-process $\{p_{ij}(t)\}$. This can be proved (cf. Doob [4]) by writing the inequality (3.1) as

$$\frac{d}{dt}(p_{ij}(t)\, e^{q_i t}) \ge (\sum_{\alpha \neq i} q_{i\alpha}\, p_{\alpha j}(t))\, e^{q_i t},$$

and integrating which gives

$$p_{ij}(t) \ge \delta_{ij}\, e^{-q_i t} + e^{-q_i t} \int_0^t (\sum_{\alpha \neq i} q_{i\alpha}\, p_{\alpha j}(u))\, e^{q_j u}\, du.$$

On comparing this inequality with (4.2), an easy induction argument yields

$$p_{ij}(t) \ge f_{ij}^n(t) \qquad (n = 0, 1, 2, \ldots),$$

and hence
$$p_{ij}(t) \ge f_{ij}(t) = \lim_{n \to \infty} f_{ij}^n(t)$$

as asserted.

4.2. Minimal properties. There are two further "minimal" properties of $\mathfrak{F}$, or rather of the resolvent operator Φ_λ of the associated semigroup, which may conveniently be established now. Φ_λ is given (cf. (1.11) and (1.12)) by

$$(\Phi_\lambda x)_j \equiv \sum_\alpha x_\alpha\, \phi_{\alpha j}(\lambda) \qquad (x \in l), \tag{4.4}$$

where
$$\phi_{ij}(\lambda) \equiv \int_0^\infty f_{ij}(t)\, e^{-\lambda t}\, dt \qquad (\lambda > 0). \tag{4.5}$$

If we similarly write

$$\phi_{ij}^n(\lambda) \equiv \int_0^\infty f_{ij}^n(t)\, e^{-\lambda t}\, dt, \tag{4.6}$$

then $\phi_{ij}^n \uparrow \phi_{ij}$ as $n \to \infty$ because $f_{ij}^n \uparrow f_{ij}$. Also from (4.2) and (4.3) we find that $\phi_{ij}^0(\lambda) \equiv 0$ and that

$$(\lambda + q_i)\, \phi_{ij}^{n+1}(\lambda) = \delta_{ij} + \sum_{\alpha \neq i} q_{i\alpha}\, \phi_{\alpha j}^n(\lambda), \tag{4.7}$$

$$(\lambda + q_j)\, \phi_{ij}^{n+1}(\lambda) = \delta_{ij} + \sum_{\alpha \neq j} \phi_{i\alpha}^n(\lambda)\, q_{\alpha j}. \tag{4.8}$$

Hence if we write $\xi \equiv \Phi_\lambda x$ and define $\xi^n \in l$ by

$$(\xi^n)_j \equiv \sum_\alpha x_\alpha\, \phi_{\alpha j}^n(\lambda),$$

then $\xi_j = \lim_{n\to\infty} \xi_j^n$ (for each j). Now $\xi_j^0 \equiv 0$, and

$$\begin{aligned}\xi_j^{n+1} = \sum_\alpha x_\alpha \phi_{\alpha j}^{n+1} &= \sum_\alpha x_\alpha \left\{\frac{1}{\lambda+q_j}(\delta_{\alpha j} + \sum_{\beta\neq j} \phi_{\alpha\beta}^n q_{\beta j})\right\} \\ &= \frac{1}{\lambda+q_j}(x_j + \sum_{\beta\neq j}(\sum_\alpha x_\alpha \phi_{\alpha\beta}^n) q_{\beta j}) \\ &= \frac{1}{\lambda+q_j}(x_j + \sum_{\beta\neq j} \xi_\beta^n q_{\beta j});\end{aligned}$$

$$(\lambda+q_j)\,\xi_j^{n+1} = x_j + \sum_{\beta\neq j} \xi_\beta^n q_{\beta j}. \tag{4.9}$$

Thus

$$(\Phi_\lambda x)_j = \lim_{n\to\infty} \xi_j^n, \tag{4.10}$$

where $\xi_j^0 = 0$ and ξ_j^n is defined by the recurrence relations (4.9).

Similarly, if Φ_λ^* is the adjoint of Φ_λ, we can calculate $y\,\Phi_\lambda^*$ from

$$(y\,\Phi_\lambda^*)_i = \lim_{n\to\infty} \eta_i^n, \tag{4.11}$$

where $\eta_i^0 \equiv 0$ and η_i^n is defined by the recurrence relations

$$(\lambda+q_i)\,\eta_i^{n+1} = y_i + \sum_{\alpha\neq i} q_{i\alpha}\,\eta_\alpha^n. \tag{4.12}$$

[Here $y \in m$ and the adjoint operator Φ_λ^* is written on the right, so that

$$(y, \Phi_\lambda x) = (y\,\Phi_\lambda^*, x)$$

for all $y \in m$ and $x \in l$.]

Finally we observe that Feller's process $\mathfrak{F}$ satisfies the backward and forward equations, so that by Theorems 2 and 4 its infinitesimal generator Ω_F satisfies

$$Q_0 \subseteq \Omega_F \subseteq Q. \tag{4.13}$$

By taking adjoints in (4.13), we obtain

$$Q^* \subseteq \Omega_F^* \subseteq Q_0^*. \tag{4.14}$$

It seems difficult to give an explicit description of the operator Q^*, but fortunately we shall only have to deal with Q_0^*. Now $y\,(\in m)$ is in $\mathfrak{D}(Q_0^*)$ if and only if there exists $z \in m$ such that $(y, Q_0 x) = (z, x)$ for all $x \in \mathfrak{D}(Q_0)$, and then $y\,Q_0^* \equiv z$. From this definition it follows almost immediately that

$$(y\,Q_0^*)_i \equiv \sum_\alpha q_{i\alpha}\,y_\alpha, \tag{4.15}$$

the domain $\mathcal{D}(Q_0^*)$ consisting precisely of those elements $y \in m$ for which (4.15) defines a bounded sequence $(y\,Q_0^*)_i$.

Because Φ_λ is inverse to $\lambda I - \Omega_F$, Φ_λ^* will be inverse to $\lambda I - \Omega_F^*$. In particular, from (4.13) and (4.14), we shall have

$$(\lambda I - Q)\,\Phi_\lambda x = x \qquad (x \in l),$$

$$y\,\Phi_\lambda^*(\lambda I - Q_0^*) = y \qquad (y \in m).$$

This means that $\xi' = \xi \equiv \Phi_\lambda x$ and $\eta' = \eta \equiv y\,\Phi_\lambda^*$ are solutions of the respective equations

$$(\lambda I - Q)\,\xi' = x, \tag{4.16}$$

$$\eta'(\lambda I - Q_0^*) = y, \tag{4.17}$$

and of course $\xi \geq 0$ when $x \geq 0$ and $\eta \geq 0$ when $y \geq 0$. The following theorem states that they are *the least positive solutions* of these equations.

THEOREM 5.

(i) *If* $x \geq 0$ *is in* l, *and* $\xi' \geq 0$ *in* $\mathcal{D}(Q)$ *satisfies* (4.16), *then* $\xi' \geq \xi \equiv \Phi_\lambda x$.

(ii) *If* $y \geq 0$ *is in* m, *and* $\eta' \geq 0$ *in* $\mathcal{D}(Q_0^*)$ *satisfies* (4.17), *then* $\eta' \geq \eta \equiv y\,\Phi_\lambda^*$.

Proof. (i) We can write (4.16) as

$$(\lambda + q_j)\,\xi_j' = x_j + \sum_{\beta \neq j} \xi_\beta'\, q_{\beta j}.$$

On comparing this with (4.9), and using the fact that $\xi_j' \geq 0 = \xi_j^0$, we find (by induction) that $\xi_j' \geq \xi_j^n$ for each n and hence by (4.10) that

$$\xi_j' \geq \lim_{n\to\infty} \xi_j^n = \xi_j.$$

(ii) This is proved similarly by writing (4.17) as

$$(\lambda + q_i)\,\eta_i' = y_i + \sum_{\alpha \neq i} q_{i\alpha}\,\eta_\alpha'$$

and using (4.11) and (4.12).

§ 5. Uniqueness theorems: the backward equations

5.1. Uniqueness and the honesty of $\mathfrak{F}$. Now let a set $Q \equiv \{q_{ij}\}$ satisfying (4.1) be given, let the operator Q_0 be defined as in § 3.1, let the Feller process (semigroup) $\mathfrak{F}$ be constructed as in § 4.1, let "Q-process (semigroup)" denote any system for which

$p'_{ij}(0) = q_{ij}$, and let "the backward equations" mean the set of equations

$$(B_{ij}): \qquad p'_{ij}(t) = \sum_{\alpha} q_{i\alpha}\, p_{\alpha j}(t) \qquad (i, j \in E\,;\; t \geq 0)$$

(interpreted in the strict sence of § 3.1), where always the *given* q_{ij} are to be understood. We may then ask:

> *is* $\mathfrak{F}$ *the only process for which the backward equations hold, with the given coefficients* q_{ij}*?*

By Theorem 2, an equivalent form of this question is:

> *is* $\mathfrak{F}$ *the only contraction semigroup whose infinitesimal generator* Ω *is an extension of the given operator* Q_0*?*

The situation is particularly simple when the set Q is *conservative*, i.e. when

$$\sum_{\alpha} q_{i\alpha} = 0 \qquad (\text{all } i), \tag{5.1}$$

and we assume until further notice that (5.1) holds. By Theorem 3, *every* Q-process will then satisfy the backward equations, and our question now becomes:

> *is* $\mathfrak{F}$ *the only* Q*-process?*

THEOREM 6. *Let* $Q \equiv \{q_{ij}\}$ *satisfy* (4.1) *and* (5.1), *and let* $\mathfrak{F} \equiv \{f_{ij}(t)\}$ *be the minimal* Q*-process.*

(i) *If* $\mathfrak{F}$ *is honest, then it is the only* Q*-process.*

(ii) *If* $\mathfrak{F}$ *is dishonest. then there exist infinitely many* Q*-processes, including infinitely many honest* Q*-pvocesses.*

Proof. (i) If $\mathfrak{F}$ is honest, and $\mathfrak{P}$ is another Q-process, then $p_{ij}(t) \geq f_{ij}(t)$ by the minimality of $\mathfrak{F}$. But also $\sum_{\alpha} p_{i\alpha}(t) \leq 1 = \sum_{\alpha} f_{i\alpha}(t)$, so that $p_{ij}(t) = f_{ij}(t)$.

(ii) If $\mathfrak{F}$ is dishonest, let Ω_F denote its infinitesimal generator. We then define an operator Ω_c, with domain $\mathfrak{D}(\Omega_F)$, as follows.

Choose any $c \in l$ such that $c > 0$ and $0 < \|c\| \leq 1$, and put

$$\Omega_c\, x \equiv \Omega_F\, x - (e, \Omega_F\, x) c \qquad (x \in \mathfrak{D}(\Omega_F)). \tag{5.2}$$

Now
$$\mathfrak{D}(Q_0) \subseteq \mathfrak{D}(\Omega_F) = \mathfrak{D}(\Omega_c)$$
(cf. (4.13)), and if
$$x \in \mathfrak{D}(Q_0)$$
then
$$\Omega_F\, x = Q_0\, x,$$

and $$\Omega_c x = Q_0 x - (e, Q_0 x) c = Q_0 x$$

because (5.1) implies that $(e, Q_0 x) = 0$. Thus Ω_c is an extension of Q_0. It has been shown elsewhere (Reuter [22]) that Ω_c generates a contraction semigroup, $\mathcal{G}_c$. Thus for each choice of c we obtain a Q-process $\mathcal{P}_c$; moreover (cf. [22]) distinct choices of c lead to distinct processes, and the process will be honest when $\|c\| = 1$.

5.2. Doob's construction. The construction in (ii) of the proof of Theorem 6 is an analytical version of a well known probabilistic construction of Doob [4]. This may be seen as follows. In [22] it was shown that the transition semigroup $\mathcal{G}_c$ generated by Ω_c when $\|c\| = 1$ has a resolvent Ψ_λ, given in terms of the resolvent Φ_λ for $\mathcal{F}$ by

$$\Psi_\lambda x = \Phi_\lambda x + \frac{(z(\lambda), x)}{1 - (z(\lambda), c)} \Phi_\lambda c, \tag{5.3}$$

where $$z(\lambda) \equiv e - e \lambda \Phi_\lambda^*$$

so that $$1 - (z(\lambda), c) = \|\lambda \Phi_\lambda c\| > 0.$$

Now from (5.3), $$\Psi_\lambda c = [1 - (z(\lambda), c)]^{-1} \Phi_\lambda c$$

and so $$\Psi_\lambda x = \Phi_\lambda x + (z(\lambda), x) \Psi_\lambda c.$$

Putting $x = u^i$ in this equation and taking the jth component on each side, we obtain

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + [1 - \sum_\alpha \lambda \phi_{i\alpha}(\lambda)] \sum c_\beta \psi_{\beta j}(\lambda) \tag{5.4}$$

for all $\lambda > 0$. Now $1 - \sum \lambda \phi_{i\alpha}(\lambda)$ is the Laplace transform of $d_i'(t)$, where

$$d_i(t) \equiv 1 - \sum_\alpha f_{i\alpha}(t),$$

and so Lerch's theorem leads from (5.4) to the integral equation

$$p_{ij}(t) = f_{ij}(t) + \int_0^t d_i'(s) \left(\sum_\beta c_\beta p_{\beta j}(t-s)\right) ds \tag{5.5}$$

for the transition probabilities $p_{ij}(t)$ of $\mathcal{P}_c$. The argument is clearly reversible, so that $\mathcal{P}_c$ is the *only* process whose transition probabilities satisfy (5.5), and this provides a convenient way of identifying Doob's construction with ours.

Doob's construction starts from the fact that $f_{ij}(t)$ can be identified with the probability that a certain stochastic system (initially in state i) will be in state j at time t, having performed at most a finite number of jumps. Thus $d_i(t)$ is the probability that an infinite number of jumps will have occurred in $(0, t)$ and if $s \equiv \lim s_n$, where s_n is the epoch of the nth jump, then

$$d_i'(s)\,ds \qquad (0 < s < \infty)$$

will be the distribution of the random variable s. Now in Doob's construction[1] the system is assigned at the epoch s to one of the states $1, 2, 3, \ldots$ of E, with respective probabilities $c_1, c_2, c_3, \ldots$, and it then starts its career all over again. It will readily be seen that the transition probabilities $p_{ij}(t)$ for the resulting process will satisfy (5.5), so that the positive unit vector c which occurs in our construction corresponds to the probability distribution $[c_1, c_2, c_3, \ldots]$ which controls the assignment of the system to a fresh starting-point after it has "run out of instructions".

It is also interesting to observe (following Doob [4]) that the process generated by Ω_c does not satisfy the forward equations when $\mathfrak{F}$ is dishonest. A probabilistic deduction of this fact has been given by Doob: for an analytical deduction, we have only to observe that because $\mathfrak{F}$ is dishonest, there exists an $x \in \mathfrak{D}(\Omega_F)$ for which $(e, \Omega_F x) \neq 0$ (See [22]). But $\Omega_F x = Q x$, and so we shall have $\Omega_c x \neq Q x$ for this x, which shows that Ω_c is not a restriction of Q and (by Theorem 4) that the forward equations do not hold. We can go further than this and find a correct substitute for the forward equations. Because $u^i \in \mathfrak{D}(\Omega_c)$ we have

$$\begin{aligned} p_{ij}'(t) &= (v^j, \Omega_c P_t u^i) \\ &= (v^j, Q P_t u^i - (e, Q P_t u^i)\,c) \\ &= (Q P_t u^i)_j - (e, Q P_t u^i)\,c_j \\ &= \sum_\alpha p_{i\alpha}(t)\,q_{\alpha j} - c_j \sum_\beta \Big(\sum_\alpha p_{i\alpha}(t)\,q_{\alpha\beta}\Big). \end{aligned} \tag{5.6}$$

Observe that $-(e, Q P_t u^i) = -(e, \Omega_F P_t u^i) \geq 0$, because $0 \leq P_t u^i \in \mathfrak{D}(\Omega_c) = \mathfrak{D}(\Omega_F)$ (cf. (1.17)) so that the extra term

$$-c_j \sum_\beta \Big(\sum_\alpha p_{i\alpha}(t)\,q_{\alpha\beta}\Big)$$

in (5.6) is non-negative, and thus the old forward equation now holds as an inequality, as already noted at (3.2).

5.3. A uniqueness criterion. It is desirable to have a direct test, involving Q alone, for uniqueness of Q-semigroups. Theorem 6 as it stands does not fulfil this requirement: we need a criterion in terms of the q_{ij} for the honesty of $\mathfrak{F}$.

[1] The state space E will temporarily be labelled $1, 2, 3, \ldots$.

To obtain such a criterion, we introduce the vector $z(\lambda) \in m$ defined as in § 5.2 by

$$z(\lambda) \equiv e - e\lambda\Phi_\lambda^*. \tag{5.7}$$

We have

$$z_i(\lambda) = (z(\lambda), u^i) = (e, u^i - \lambda\Phi_\lambda u^i)$$

$$= \lambda \int_0^\infty d_i(t)\, e^{-\lambda t}\, dt, \tag{5.8}$$

where

$$d_i(t) \equiv 1 - \sum_\alpha f_{i\alpha}(t).$$

We can relate $z(\lambda)$ to the q_{ij} by observing that $e Q_0^* = 0$ (because Q is conservative) and that $e\Phi_\lambda^*(\lambda I - Q_0^*) = e$ (by (4.17)). It follows that

$$z(\lambda)(\lambda I - Q_0^*) = e(\lambda I - Q_0^*) - \lambda e \Phi_\lambda^*(\lambda I - Q_0^*)$$

$$= \lambda e - 0 - \lambda e = 0,$$

so that $z(\lambda)$ *belongs to the nullspace* $\mathfrak{N}(\lambda I - Q_0^*)$ *of* $\lambda I - Q_0^*$.

The final step towards an honesty criterion for $\mathfrak{F}$ is provided by

LEMMA 4. *If* $\lambda > 0$, $\xi \in \mathfrak{N}(\lambda I - Q_0^*)$, *and* $\|\xi\| \leq 1$, *then*

$$-z(\lambda) \leq \xi \leq z(\lambda). \tag{5.9}$$

Proof. From (5.8) we have

$$z_i(\lambda) = 1 - \lambda \sum_\alpha \phi_{i\alpha}(\lambda)$$

$$= \lim_{n\to\infty} \left(1 - \lambda \sum_\alpha \phi_{i\alpha}^n(\lambda)\right)$$

$$= \lim_{n\to\infty} z_i^n,$$

say. From the recurrence relations (4.7), we see that

$$z_i^0 = 1, \qquad (\lambda + q_i) z_i^{n+1} = \sum_{\alpha \neq i} q_{i\alpha} z_\alpha^n, \tag{5.10}$$

and that $z_i^n \downarrow z_i$ as $n \to \infty$. On the other hand, the fact that $\xi \in \mathfrak{N}(\lambda I - Q_0^*)$ can be stated as

$$(\lambda + q_i)\xi_i = \sum_{\alpha \neq i} q_{i\alpha} \xi_\alpha. \tag{5.11}$$

But $\xi_i \leq 1 = z_i^0$ (because $\|\xi\| \leq 1$), so that an easy induction based on (5.10) and (5.11) gives $\xi_i \leq z_i^n$ $(n = 0, 1, 2, \ldots)$, whence $\xi_i \leq z_i = \lim_{n\to\infty} z_i^n$. Similarly $-\xi_i \leq z$ and so $-z \leq \xi \leq z$ as asserted.

We can now derive the desired uniqueness criterion.

THEOREM 7. *Let Q be conservative, and consider the set of equations*

$$(U_\lambda): \qquad (\lambda + q_i)\,\xi_i = \sum_{\alpha \neq i} q_{i\alpha}\,\xi_\alpha. \tag{5.11}$$

Each of the following conditions is necessary and sufficient in order that there be only one Q-process.

(1_λ) *For some one $\lambda > 0$, (U_λ) has no bounded solution other than $\xi_i = 0$ (all i).*

(2_λ) *For some one $\lambda > 0$, (U_λ) has no bounded non-negative solution other than $\xi_i = 0$ (all i).*

Proof. (i) If there is only one Q-process, then (by Theorem 6) $\mathfrak{F}$ it honest and $d_i(t) \equiv 0$. Hence, from (5.10), $z(\lambda) = 0$ for every $\lambda > 0$. Lemma 4 now shows that, for every $\lambda > 0$, (U_λ) can have no bounded solution other than $\xi_i = 0$.

(ii) It is now enough to show that (2_λ), which is weaker than (1_λ), implies that there is only one Q-process. Now $\xi_i \equiv z_i(\lambda)$ defines a bounded non-negative solution of (u_λ), so that if (2_λ) is satisfied (for some one $\lambda > 0$), then $z_i(\lambda) = 0$, or

$$\int_0^\infty d_i(t)\, e^{-\lambda t}\, dt = 0$$

for this λ. Since $d_i(t)$ is continuous and non-negative, this implies that $d_i(t) \equiv 0$. Thus $\mathfrak{F}$ is honest, and by Theorem 6 there is only one Q-process.

The equations (U_λ) and the nullspace $\mathfrak{N}(\lambda I - Q_0^*)$ have previously occurred in the work of Kato [15], who proved that conditions (1_λ) or (2_λ) of Theorem 7 were equivalent to the honesty of a certain process $\mathfrak{K} \equiv \{k_{ij}(t)\}$ associated with a given conservative set Q. He also showed that $\mathfrak{K}$, like $\mathfrak{F}$, has a certain minimal property: it is the minimal process whose generator is an extension of Q_0. Our result goes further than Kato's in two respects: he does not prove that the generator of every Q-process is an extension of Q_0, nor that honesty of $\mathfrak{K}$ is equivalent to uniqueness of Q-semigroups, when Q is conservative. Also the exact relation between $\mathfrak{K}$ and $\mathfrak{F}$ is left in doubt in [15]. They are in fact identical, because on the one hand $\mathfrak{K}$ is a Q-process so that $k_{ij}(t) \geq f_{ij}(t)$ (by the minimality of $\mathfrak{F}$), and on the other hand $\Omega_F \supseteq Q_0$ so that $f_{ij}(t) \geq k_{ij}(t)$ (by Kato's minimality theorem for $\mathfrak{K}$).

5.4. The non-conservative case. The uniqueness criterion in Theorem 7 can be extended to the general case when Q is not necessarily conservative, but we must then restrict ourselves to processes for which the backward equations hold (equivalently, for which $\Omega \supseteq Q_0$). Let us define D_i by

$$\sum_{\alpha} q_{i\alpha} + D_i = 0, \tag{5.12}$$

so that $0 \le D_i < \infty$, and $D_i > 0$ for at least one i when Q is not conservative. Now for any process satisfying the backward equations, Theorem 3 shows that the "deficiencies" $d_i \equiv d_i'(0)$ must satisfy $d_i = D_i$, so that such a process cannot be honest if Q is non-conservative. In particular, this applies to $\mathfrak{F}$, and so the honesty of $\mathfrak{F}$ can no longer be relevant to the uniqueness problem when Q is non-conservative. We will therefore use a device for reducing the uniqueness problem for Q to the uniqueness problem for a certain related conservative set $\tilde{Q}$, defined below.

We enlarge the set of states E to $\tilde{E}$ by adjoining an extra state, labelled θ, and then define an enlarged set $\tilde{Q} \equiv \{\tilde{q}_{ij} : i, j \in \tilde{E}\}$ by

$$\left.\begin{aligned} &\tilde{q}_{ij} \equiv q_{ij} \quad (i, j \in E), \\ &\tilde{q}_{i\theta} \equiv D_i \quad (i \in E), \\ &\tilde{q}_{\theta} \equiv -\tilde{q}_{\theta\theta} = 0, \qquad \tilde{q}_{\theta j} = 0 \quad (j \in E). \end{aligned}\right\} \tag{5.13}$$

By (5.12), the set $\tilde{Q}$ will be conservative.

Now suppose that $\mathfrak{P} \equiv \{p_{ij}(t) : i, j \in E\}$ is a process, generated by the operator Ω (on l), such that $\Omega \supseteq Q_0$; as at (1.8) and (1.9), define

$$d_i(t) \equiv 1 - \sum_{\alpha} p_{i\alpha}(t), \qquad d_i \equiv d_i'(0). \tag{5.14}$$

Then (by Theorem 3) we must have $d_i = D_i$. If we enlarge $\mathfrak{P}$ to $\tilde{\mathfrak{P}} \equiv \{\tilde{p}_{ij}(t) : i, j \in \tilde{E}\}$ by defining[1]

$$\left.\begin{aligned} &\tilde{p}_{ij}(t) \equiv p_{ij}(t) \quad (i, j \in E), \\ &\tilde{p}_{i\theta}(t) \equiv d_i(t) \quad (i \in E), \\ &\tilde{p}_{\theta\theta}(t) \equiv 1, \qquad \tilde{p}_{\theta j}(t) \equiv 0 \quad (j \in E), \end{aligned}\right\} \tag{5.15}$$

it can be verified without difficulty that $\tilde{\mathfrak{P}}$ is an honest process, and (because $d_i = D_i$) that $\tilde{p}_{ij}'(0) = \tilde{q}_{ij}$. In other words $\tilde{\mathfrak{P}}$ is an honest $\tilde{Q}$-process.

Conversely, suppose that $\tilde{\mathfrak{P}} \equiv \{\tilde{p}_{ij}(t) : i, j \in \tilde{E}\}$ is *any* honest $\tilde{Q}$-process. Because $\tilde{Q}$ is conservative, $\tilde{\mathfrak{P}}$ will satisfy the backward equations and in particular the equations

$$(\tilde{B}_{\theta j}): \qquad \tilde{p}_{\theta j}'(t) = \sum_{\alpha \in \tilde{E}} \tilde{q}_{\theta\alpha} \tilde{p}_{\alpha j}(t) \qquad (j \in \tilde{E}).$$

These at once give $\tilde{p}_{\theta\theta}(t) = 1$ and $\tilde{p}_{\theta j}(t) = 0$ $(j \in E)$, and it now follows easily that

$$\mathfrak{P} \equiv \{p_{ij}(t) : i, j \in E\},$$

[1] As in KENDALL [16].

where $p_{ij}(t) \equiv \tilde{p}_{ij}(t)$, is a Q-process. Moreover, because $\tilde{\mathcal{P}}$ is honest

$$d_i(t) \equiv 1 - \sum_{\alpha \in E} p_{i\alpha}(t) = \tilde{p}_{i\theta}(t),$$

so that $d_i = \tilde{q}_{i\theta} = D_i$ and therefore the generator Ω of $\mathcal{P}$ is an extension of Q_0. Thus we have set up a $1-1$ *correspondence between Q-processes with $\Omega \supseteq Q_0$* on the one hand, and *honest $\tilde{Q}$-processes* on the other. By Theorem 6, either there is only one honest $\tilde{Q}$-process or there are infinitely many, and the two cases can be distinguished as in Theorem 7 by considering the set of equations

$$(\tilde{U}_\lambda): \qquad (\lambda + \tilde{q}_i)\, \tilde{\xi}_i = \sum_{\alpha \neq i} \tilde{q}_{i\alpha}\, \tilde{\xi}_\alpha \qquad (i \in \tilde{E}). \tag{5.16}$$

But if $\tilde{\xi}$ is a solution of $(\tilde{U}_\lambda)$, then $\tilde{\xi}_\theta = 0$ (because $\tilde{q}_{\theta\alpha} = 0$) and so if we define $\xi_i \equiv \tilde{\xi}_i$ $(i \in E)$ we obtain a solution ξ of (U_λ) (cf. (5.13)). Conversely, from a solution ξ of (U_λ) we can obtain a solution $\tilde{\xi}$ of $(\tilde{U}_\lambda)$ by defining $\tilde{\xi}_i \equiv \xi_i$ $(i \in E)$, $\tilde{\xi}_\theta = 0$. Thus there is a $1-1$ *correspondence between solutions of $(\tilde{U}_\lambda)$ and of (U_λ)*, in particular $(\tilde{U}_\lambda)$ has non-zero bounded solutions if and only if (U_λ) does. We have therefore established the desired uniqueness criterion:

THEOREM 8. *Let $Q \equiv \{q_{ij}\}$ satisfy* (4.1) (*but not necessarily* (5.1)) *and let $\lambda > 0$. Then the equivalent conditions* (1_λ) *and* (2_λ) *of Theorem 7 are necessary and sufficient in order that there be exactly one process satisfying the backward equations.*

If the conditions do not hold, then there exist infinitely many such processes.

It should be stressed once again that the processes covered by Theorem 8 are always Q-processes, i.e. have $p'_{ij}(0) = q_{ij}$, but that when Q is non-conservative there may exist Q-processes which do not satisfy the backward equations (Kolmogorov [19]) and so are not covered by Theorem 8.

§ 6. Uniqueness theorems: the forward equations

6.1. We now take an arbitrary, *not necessarily conservative*, set $Q \equiv \{q_{ij}\}$ satisfying (4.1), and consider processes which satisfy the forward equations

$$(F_{ij}): \qquad p'_{ij}(t) = \sum_\alpha p_{i\alpha}(t)\, q_{\alpha j} \qquad (i, j \in E;\ t \geq 0)$$

in the strict sense; equivalently (by Theorem 4), processes whose generators Ω are restrictions of the operator Q defined in § **3.1**. These are always Q-processes[1] (satisfy

[1] They will therefore automatically satisfy the backward equations when Q is conservative, but they may fail to do so otherwise. See § **8.1**.

$p'_{ij}(0) = q_{ij}$), and there is always *one* such process: the minimal Q-process $\mathfrak{F}$. Necessary conditions for the existence of more than one such process are easily found.

LEMMA 5. *If there exist two distinct processes such that $\Omega \subseteq Q$, then*

(i) *the minimal Q-process $\mathfrak{F}$ is dishonest, and*

(ii) *the nullspace $\mathcal{N}(\lambda I - Q)$, for each $\lambda > 0$, contains a positive element of l.*

Proof. Let $\mathfrak{D}$ be a process, distinct from $\mathfrak{F}$, such that $\Omega \subseteq Q$, and let Ψ_λ denote its resolvent operator.

If $\mathfrak{F}$ were honest then (as remarked in § **5.4**) Q must be conservative, and the Q-process $\mathfrak{D}$ must (by Theorem 6) coincide with $\mathfrak{F}$, contrary to assumption. This proves (i).

Now let $x \geq 0$. Then $\Psi_\lambda x \geq \Phi_\lambda x$, and also $(\lambda I - Q)(\Psi_\lambda x - \Phi_\lambda x) = 0$ because both Ω and Ω_F are restrictions of Q. But $\mathfrak{D} \neq \mathfrak{F}$, so that $\Psi_\lambda \neq \Phi_\lambda$ and therefore $\Psi_\lambda x \neq \Phi_\lambda x$ for some $x > 0$. For this x, $\Psi_\lambda x - \Phi_\lambda x$ is a positive element in $\mathcal{N}(\lambda I - Q)$, and this proves (ii).

We shall see later that these conditions are also sufficient. The proof of this will require some properties of $\mathcal{N}(\lambda I - Q)$ and of $\mathcal{N}^+(\lambda I - Q)$, the set of non-negative vectors in $\mathcal{N}(\lambda I - Q)$.

LEMMA 6. *Let $\lambda > 0$, $\mu > 0$, and*

$$A(\mu, \lambda) = I + (\mu - \lambda)\Phi_\lambda. \tag{6.1}$$

Then $A(\mu, \lambda)$ has the (bounded) inverse $A(\lambda, \mu)$, maps $\mathcal{N}(\mu I - Q)$ on to $\mathcal{N}(\lambda I - Q)$, and maps $\mathcal{N}^+(\mu I - Q)$ on to $\mathcal{N}^+(\lambda I - Q)$.

Proof. Because $(\lambda I - \Omega_F)\Phi_\lambda = I$, we have

$$\begin{aligned} A(\mu, \lambda) &= I - \lambda\Phi_\lambda + \mu\Phi_\lambda = -\Omega_F\Phi_\lambda + \mu\Phi_\lambda \\ &= (\mu I - \Omega_F)\Phi_\lambda. \end{aligned} \tag{6.2}$$

Hence

$$A(\lambda, \mu) A(\mu, \lambda) = (\lambda I - \Omega_F)\Phi_\mu(\mu I - \Omega_F)\Phi_\lambda = I;$$

interchanging λ and μ, $A(\mu, \lambda) A(\lambda, \mu) = I$. Thus $A(\mu, \lambda)$ has inverse $A(\lambda, \mu)$.

Next, if $x \in \mathcal{N}(\mu I - Q)$, then

$$A(\mu, \lambda)x = x + (\mu - \lambda)\Phi_\lambda x \in \mathcal{D}(Q)$$

because $\Phi_\lambda x \in \mathcal{D}(\Omega_F) \subseteq \mathcal{D}(Q)$, and

$$\begin{aligned} Q A(\mu, \lambda) x &= Q x + (\mu - \lambda) Q \Phi_\lambda x \\ &= \mu x + (\lambda - \mu)(-\Omega_F \Phi_\lambda x) \\ &= \mu x + (\lambda - \mu)(x - \lambda \Phi_\lambda x) = \lambda A(\mu, \lambda) x; \end{aligned}$$

thus $A(\mu, \lambda) x \in \mathcal{N}(\lambda I - Q)$, and $A(\mu, \lambda)$ maps $\mathcal{N}(\mu I - Q)$ *into* $\mathcal{N}(\lambda I - Q)$. But the inverse $A(\lambda, \mu)$ maps $\mathcal{N}(\lambda I - Q)$ into $\mathcal{N}(\mu I - Q)$, so that $A(\mu, \lambda)$ in fact maps $\mathcal{N}(\mu I - Q)$ *on to* $\mathcal{N}(\lambda I - Q)$.

It will now be enough to prove that $A(\mu, \lambda)$ maps $\mathcal{N}^+(\mu I - Q)$ into $\mathcal{N}^+(\lambda I - Q)$. Let $x \geq 0$ and $x \in \mathcal{N}(\mu I - Q)$; we need only check that $y \equiv A(\mu, \lambda) x \geq 0$. This is trivial if $\lambda \leq \mu$, because then $\Phi_\lambda x \geq 0$ and

$$y = x + (\mu - \lambda) \Phi_\lambda x \geq 0;$$

on the other hand, if $\lambda > \mu$, we note that

$$(\lambda I - Q) x = (\lambda - \mu) x \geq 0$$

and so by the minimal property (Theorem 5) of Φ_λ,

$$x \geq \Phi_\lambda((\lambda - \mu) x),$$

i.e. $y \equiv x + (\mu - \lambda) \Phi_\lambda x \geq 0$, as required.

The isomorphism between $\mathcal{N}(\lambda I - Q)$ and $\mathcal{N}(\mu I - Q)$, described above, will be exhibited again (in a slightly different way) in §**7.1**. The lemma shows that if $\mathcal{N}^+(\lambda I - Q)$ is non-trivial for *one* $\lambda > 0$, then it is non-trivial for *all* $\lambda > 0$.

6.2. We now give a construction which will show that the necessary conditions stated in Lemma 5 are also sufficient.

THEOREM 9. *Suppose that $\mathcal{F}$ is dishonest and that $\mathcal{N}^+(\lambda I - Q) \neq \{0\}$ for one $\lambda > 0$ (equivalently, by Lemma 6, for all $\lambda > 0$). Then there exist infinitely many processes, including at least one honest process, such that $\Omega \subseteq Q$.*

Proof. Define an operator Ω as follows. Fix $\mu > 0$, choose a non-zero element y in $\mathcal{N}^+(\mu I - Q)$ such that $(e, \mu y) \leq 1$, and let $\mathcal{D}(\Omega)$ be the set of $\xi \in l$ which can be written in the form

$$\xi = \eta - (e, \Omega_F \eta) y \tag{6.3}$$

for some η in $\mathcal{D}(\Omega_F)$. [For a given ξ in $\mathcal{D}(Q)$ there is at most one such η, namely $\eta = \Phi_\mu(\mu I - Q)\xi$.] Clearly $\mathcal{D}(\Omega) \subseteq \mathcal{D}(Q)$, so that we may define Ω as the restriction of Q to $\mathcal{D}(\Omega)$:

$$\Omega \xi \equiv Q \xi, \qquad \xi \in \mathcal{D}(\Omega). \tag{6.4}$$

We now show by using the Hille–Yosida theorem[1] that Ω generates a contraction semigroup. We must verify that:

(i) $\mathcal{D}(\Omega)$ is dense in l;

(ii) for each $\lambda > 0$ and $x \in l$, the equation

$$\lambda \xi - \Omega \xi = x \tag{6.5}$$

has a unique solution $\xi \equiv \Psi_\lambda x$ in $\mathcal{D}(\Omega)$, and $\Psi_\lambda x \geq 0$ when $x \geq 0$;

(iii) $$(e, \Omega \xi) \leq 0 \text{ when } \xi \geq 0, \ \xi \in \mathcal{D}(\Omega). \tag{6.6}$$

If $\mathcal{D}(\Omega)$ were not dense we could find $z \neq 0$ in m such that $(z, \xi) = 0$ for all $\xi \in \mathcal{D}(\Omega)$. The general element ξ of $\mathcal{D}(\Omega)$ is given by (6.3), where η varies over $\mathcal{D}(\Omega_F) = \mathcal{R}(\Phi_\mu)$ and so has the form $\eta = \Phi_\mu x$ for some $x \in l$. Thus we should have

$$(z, \Phi_\mu x) - (e, \Omega_F \Phi_\mu x)(z, y) = 0 \tag{6.7}$$

for all $x \in l$. Using $-\Omega_F \Phi_\mu = I - \mu \Phi_\mu$ we write (6.7) as

$$(z \Phi_\mu^* + (z, y)(e - e \mu \Phi_\mu^*), x) = 0, \qquad \text{all } x \in l,$$

and deduce that

$$z \Phi_\mu^* + (z, y)(e - e \mu \Phi_\mu^*) = 0. \tag{6.8}$$

If $(z, y) = 0$, then (6.8) gives $z \Phi_\mu^* = 0$, whence $z = 0$ contrary to assumption; if $(z, y) \neq 0$, (6.8) shows that $e \in \mathcal{R}(\Phi_\mu^*) = \mathcal{D}(\Omega_F^*)$ and by operating on (6.8) with $\mu I - \Omega_F^*$ we get

$$z - (z, y) e \Omega_F^* = 0,$$

$$(z, y)\{1 - (e \Omega_F^*, y)\} = 0. \tag{6.9}$$

But $e \Omega_F^* = e Q_0^* \leq 0$ because $(e Q_0^*)_i = \sum_\alpha q_{i\alpha}$, and so $(e \Omega_F^*) y) \leq 0$ and $1 - (e, \Omega_F^*, y) \neq 0$, whence (6.9) gives $(z, y) = 0$ contrary to assumption. We have now shown that no $z \neq 0$ can annihilate $\mathcal{D}(\Omega)$, which proves (i).

Now consider the equation (6.5). From the definitions (6.3) and (6.4) of Ω, and the fact that the representation (6.3) of $\xi \in \mathcal{D}(\Omega)$ is unique, we see that (6.6) is equivalent to

$$\lambda \{\eta - (e, \Omega_F \eta) y\} - Q \{\eta - (e, \Omega_F \eta) y\} = x, \qquad \eta \in \mathcal{D}(\Omega_F),$$

or to

$$(\lambda I - \Omega_F) \eta = x + (\lambda - \mu)(e, \Omega_F \eta) y. \tag{6.10}$$

From (6.10), η necessarily has the form

$$\eta = \Phi_\lambda (x + \varrho y) \tag{6.11}$$

[1] In a form which differs slightly from the usual one: see [17], [22].

for some ϱ, and this η will satisfy (6.10) if and only if

$$x+\varrho y = x+(\lambda-\mu)(e, \Omega_F \Phi_\lambda (x+\varrho y))y$$

or (because $\Omega_F \Phi_\lambda = \lambda \Phi_\lambda - I$)

$$\varrho\{1+(\lambda-\mu)(e, y-\lambda\Phi_\lambda y)\} = (\mu-\lambda)(e, x-\lambda\Phi_\lambda x). \tag{6.12}$$

In (6.12), the coefficient of ϱ is positive because

$$\begin{aligned}1+(\lambda-\mu)(e, y-\lambda\Phi_\lambda y) &\geq 1-\mu(e, y-\lambda\Phi_\lambda y) \\ &> 1-\mu(e, y) \geq 0.\end{aligned}$$

Thus there is exactly one solution ϱ to (6.12), and hence exactly one solution ξ to (6.5), given by (6.3), (6.11) and (6.12). A simple calculation gives

$$\xi \equiv \Psi_\lambda x = \Phi_\lambda x + \sigma(e, x-\lambda\Phi_\lambda x) A(\mu, \lambda) y, \tag{6.13}$$

where
$$\sigma^{-1} = 1+(\lambda-\mu)(e, y-\lambda\Phi_\lambda y) > 0 \tag{6.14}$$

and $A(\mu, \lambda)$ is defined by (6.1). When $x \geq 0$, then $(e, x-\lambda\Phi_\lambda x) \geq 0$ and also $A(\mu, \lambda) y \geq 0$ (by Lemma 6), so that

$$\Psi_\lambda x \geq \Phi_\lambda x \geq 0.$$

This concludes the proof of (ii).

To prove (iii), suppose that

$$\xi = \eta - (e, \Omega_F \eta) y \geq 0,$$

where $\eta \in \mathcal{D}(\Omega_F)$. If $(e, \Omega_F \eta) > 0$, then we should have $\eta \geq 0$ and so $(e, \Omega_F \eta) \leq 0$ contrary to assumption. Therefore $(e, \Omega_F \eta) \leq 0$. Now

$$\begin{aligned}\Omega \xi &= Q\xi = \Omega_F \eta - (e, \Omega_F \eta)\mu y, \\ (e, \Omega\xi) &= (e, \Omega_F \eta)\{1-(e, \mu y)\},\end{aligned} \tag{6.15}$$

and so $(e, \Omega\xi) \leq 0$, which proves (iii).

We have now shown that Ω generates a contraction semigroup whose resolvent Ψ_λ is given by (6.13); in particular

$$\Psi_\mu x = \Phi_\mu x + (e, x-\mu\Phi_\mu x) y.$$

Because $\mathcal{F}$ is dishonest, $(e, x-\mu\Phi_\mu x) \neq 0$ for at least one x, and therefore distinct choices of $y \in \mathcal{N}^+(\mu I - Q)$ lead to distinct resolvents Ψ_μ and so to distinct semigroups. Finally, $\mathcal{F}$ is dishonest and therefore $(e, \Omega_F \eta) \neq 0$ for some η in $\mathcal{D}(\Omega_F)$. On the other hand Ω generates a transition semigroup if and only if $(e, \Omega\xi) = 0$ for all ξ in

$\mathcal{D}(\Omega)$, and from (6.15) this is so if and only if $(e, \mu y) = 1$. We can always choose such y in $\mathcal{N}^+(\mu I - Q) \neq \{0\}$, and therefore our construction can be made to yield at least one transition semigroup. This concludes the proof of Theorem 9.

It would be interesting to have a probabilistic interpretation for the above construction (when $(e, \mu y) = 1$); I have not succeeded in finding one.

6.3. It is now possible to describe the amount of non-uniqueness which prevails amongst processes such that $\Omega \subseteq Q$. Denote by n^+ the *maximum number of linearly independent elements of* $\mathcal{N}^+(\lambda I - Q)$. As the notation implies, n^+ does not depend on λ (by Lemma 6); by the statement "$n^+ = \infty$" we understand simply that $\mathcal{N}^+(\lambda I - Q)$ contains finite linearly independent sets with arbitrarily many members.

THEOREM 10.

(i) *If $\mathfrak{F}$ is honest, or if $\mathfrak{F}$ is dishonest but $n^+ = 0$, there is exactly one process such that $\Omega \subseteq Q$, namely $\mathfrak{F}$.*

(ii) *If $\mathfrak{F}$ is dishonest and $n^+ = 1$, there are infinitely many processes with $\Omega \subseteq Q$, and exactly one of them is honest.*

(iii) *If $\mathfrak{F}$ is dishonest and $n^+ > 1$, there are infinitely many processes, including infinitely many honest ones, such that $\Omega \subseteq Q$.*

Proof. Statement (i) follows from Lemma 5; statements (ii) and (iii) follow from the construction used in the proof of Theorem 9, *except* for the fact that there is exactly one honest process with $\Omega \subseteq Q$ in case (ii). We already know from Theorem 9 that there is at least one: to see that it is unique, denote its resolvent by Ψ_λ and let $y(\lambda)$ be the unique element of $\mathcal{N}^+(\lambda I - Q)$ with $(e, \lambda y(\lambda)) = 1$. When $x \geq 0$, then (cf. Lemma 5) $\Psi_\lambda x - \Phi_\lambda x$ is in $\mathcal{N}^+(\lambda I - Q)$ and is therefore a non-negative multiple of $y(\lambda)$, say

$$\Psi_\lambda x - \Phi_\lambda x = \varrho\, y(\lambda) \qquad (\varrho = \varrho(x, \lambda) \geq 0).$$

But $\lambda \Psi_\lambda$ is a transition operator, so that

$$\begin{aligned}(e, x) &= (e, \lambda \Psi_\lambda x) = (e, \lambda \Phi_\lambda x) + \varrho\,(e, \lambda y(\lambda)) \\ &= (e, \lambda \Phi_\lambda x) + \varrho.\end{aligned}$$

Thus

$$\Psi_\lambda x = \Phi_\lambda x + (e, x - \lambda \Phi_\lambda x)\, y(\lambda) \tag{6.16}$$

when $x \geq 0$, and hence for all x; Ψ_λ is unique, and hence so is the corresponding process.

It should be noted for applications of Theorem 10 that $\mathcal{N}^+(\lambda I - Q)$ consists of the non-negative solutions $y = \{y_\alpha\}$ of

$$(V_\lambda): \qquad (\lambda + q_j) y_j = \sum_{\alpha \neq j} y_\alpha q_{\alpha j} \tag{6.17}$$

with $\sum y_\alpha < \infty$, and that n^+ can be found by looking at (V_λ) for any *one* $\lambda > 0$. Compare this with the way in which the equations (U_λ), (5.13), occur in Theorems 7 and 8.

§ 7. Some further results

7.1. Restrictions of Q. In § **6.2** we constructed some operators $\Omega \subseteq Q$ which generate contraction semigroups. No description of *all* such operators seems to be known, but every such Ω must satisfy certain conditions, closely connected with the fact (see Lemma 6) that $\mathcal{N}(\lambda I - Q)$ and $\mathcal{N}(\mu I - Q)$ are isomorphic whenever $\lambda, \mu > 0$. If Ψ_λ is the resolvent operator for such an Ω, then the operator

$$C(\lambda) \equiv \Psi_\lambda (\lambda I - Q), \tag{7.1}$$

defined on $\mathcal{D}(Q)$, maps $\mathcal{D}(Q)$ into $\mathcal{D}(\Omega) \subseteq \mathcal{D}(Q)$, leaves $\mathcal{D}(\Omega)$ elementwise fixed, and has nullspace $\mathcal{N}(\lambda I - Q)$. Thus $C(\lambda)$ projects $\mathcal{D}(Q)$ on to $\mathcal{D}(\Omega)$, and $\mathcal{D}(Q)$ can be written as a direct sum

$$\mathcal{D}(Q) = \mathcal{D}(\Omega) \oplus \mathcal{N}(\lambda I - Q), \tag{7.2}$$

the canonical decomposition of $y \in \mathcal{D}(Q)$ being $y = C(\lambda) y + (I - C(\lambda)) y$. From (7.2) we see that, for each $\lambda > 0$, $\mathcal{N}(\lambda I - Q)$ is isomorphic with $\mathcal{D}(Q)/\mathcal{D}(\Omega)$. It follows that $\mathcal{N}(\mu I - Q)$ and $\mathcal{N}(\lambda I - Q)$ are also isomorphic, and the isomorphism can be obtained by mapping $y \in \mathcal{N}(\mu I - Q)$ on the coset $y + \mathcal{D}(\Omega)$ of $\mathcal{D}(Q)/\mathcal{D}(\Omega)$ and then applying $I - C(\lambda)$ to this coset. The resulting mapping from $\mathcal{N}(\mu I - Q)$ to $\mathcal{N}(\lambda I - Q)$ is

$$\begin{aligned} y \to y - C(\lambda) y &= y - \Psi_\lambda (\lambda I - Q) y \\ &= y + (\mu - \lambda) \Psi_\lambda y \end{aligned}$$

because $Q y = \mu y$. When $\Omega \equiv \Omega_F$, so that $\Psi_\lambda \equiv \Phi_\lambda$, this is precisely the mapping $A(\mu, \lambda)$ in (6.1).

The fact that $\mathcal{N}(\lambda I - Q) \cong \mathcal{D}(Q)/\mathcal{D}(\Omega)$ has two consequences. First, fixing a particular Ω (say Ω_F), we see that the structure of $\mathcal{N}(\lambda I - Q)$, and in particular its *dimension* n, is independent of λ. Secondly, $\mathcal{D}(Q)/\mathcal{D}(\Omega)$ does not depend on the particular Ω considered, in particular it must always have the same dimension n. Thus the extent to which Q must be restricted to obtain an Ω is in a sense fixed, once for all: what we lack at present is a general description of the form which the restriction must take. For the special Ω constructed in § **6.2**, whose domain is described at (6.3), it is not hard to show that $\xi \in \mathcal{D}(\Omega)$ if and only if $\xi \in \mathcal{D}(Q)$ and

$$\xi - \Phi_\mu (\mu I - Q)\xi = (e, (I - \mu \Phi_\mu)(\mu I - Q)\xi) y; \tag{7.3}$$

thus Ω is obtained from Q by imposing the "lateral condition" (7.3) (in Feller's terminology). It would be interesting to have a probabilistic interpretation of this condition.

7.2. Extensions of Q_0. Now suppose that $\Omega \supseteq Q_0$ generates a contraction semigroup, with resolvent Ψ_λ. Then $\Omega^* \subseteq Q_0^*$, the mapping

$$E(\lambda) \equiv (\lambda I - Q_0^*)\Psi_\lambda^* \tag{7.4}$$

(analogous to $C(\lambda)$ in (7.1)) projects $\mathcal{D}(Q_0^*)$ into $\mathcal{D}(\Omega^*)$, and $\mathcal{D}(Q_0^*)$. can be written as a direct sum

$$\mathcal{D}(Q_0^*) = \mathcal{D}(\Omega^*) \oplus \mathcal{N}(\lambda I - Q_0^*). \tag{7.5}$$

Hence $\mathcal{N}(\lambda I - Q_0^*) \cong \mathcal{D}(Q_0^*)/\mathcal{D}(\Omega^*)$, and also there is a 1–1 mapping

$$z \to z + z(\mu - \lambda)\Psi_\lambda^*$$

from $\mathcal{N}(\mu I - Q_0^*)$ on to $\mathcal{N}(\lambda I - Q_0^*)$. (For the particular choice $\Omega \equiv \Omega_F$, $\Psi_\lambda \equiv \Phi_\lambda$, this mapping is simply the adjoint $A^*(\mu, \lambda)$ of $A(\mu, \lambda)$ (see (6.3)); one can show just as in Lemma 6 that $A^*(\mu, \lambda)$ also maps $\mathcal{N}^+(\mu I - Q_0^*)$ on to $\mathcal{N}^+(\lambda I - Q_0^*)$.) It follows (as in § **7.1**) that the *dimension* n^* of $\mathcal{N}(\lambda I - Q_0^*)$ is independent of λ, and that $\mathcal{D}(Q_0^*)/\mathcal{D}(\Omega^*)$ must have dimension n^* for every Ω considered here.

There are also relations between Ω and Q_0; these are most conveniently expressed in terms of $\bar{Q}_0$, the *least closed extension* of Q_0. The existence of $\bar{Q}_0$ is guaranteed because $Q_0 \subseteq \Omega_F$ and Ω_F is closed; its definition, we recall, is that $x \in \mathcal{D}(\bar{Q}_0)$ and $\bar{Q}_0 x = y$ whenever there exist $x_n \in \mathcal{D}(Q_0)$ such that $x_n \to x$ and $Q_0 x \to y$ (strongly). Here y is uniquely defined, indeed $y = \Omega_F x = Qx$ because Ω_F is closed. It is easily shown that

$$\mathcal{R}(\lambda I - \bar{Q}_0) = \overline{\mathcal{R}(\lambda I - Q_0)},$$

that $\bar{Q}_0^* = Q_0^*$, and that $\lambda I - \bar{Q}_0$ is $1-1$ from $\mathcal{D}(\bar{Q}_0)$ on to $\mathcal{R}(\lambda I - \bar{Q}_0)$, with inverse Φ_λ; proofs of these facts are left to the reader. Also if $\Omega \supseteq Q_0$ generates a contraction semigroup then $\Omega \supseteq \bar{Q}_0$, because Ω is closed. Now $\lambda I - \Omega$ is $1-1$ and maps $\mathcal{D}(\bar{Q}_0)$ and $\mathcal{D}(\Omega)$ on to $\mathcal{R}(\lambda I - \bar{Q}_0)$ and l, so that $\mathcal{D}(\Omega)/\mathcal{D}(\bar{Q}_0)$ and $l/\mathcal{R}(\lambda I - \bar{Q}_0)$ are (algebraically) isomorphic. On the other hand, the adjoint space of $l/\mathcal{R}(\lambda I - \bar{Q}_0)$ is precisely the annihilator in $l^*(=m)$ of $\mathcal{R}(\lambda I - \bar{Q}_0) = \overline{\mathcal{R}(\lambda I - Q_0)}$, which is $\mathcal{N}(\lambda I - Q_0^*)$; hence $l/\mathcal{R}(\lambda I - \bar{Q}_0)$ and $\mathcal{N}(\lambda I - Q_0^*)$ have the same dimension n^*, provided that we equate all infinite dimensions. Thus the spaces

$$\mathcal{D}(Q_0^*)/\mathcal{D}(\Omega^*), \qquad \mathcal{D}(\Omega)/\mathcal{D}(\bar{Q}_0), \qquad \mathcal{N}(\lambda I - Q_0^*)$$

all have the same dimension n^*. In particular, the amount by which $\bar{Q}_0$ must be extended to obtain an Ω is in a sense fixed, once for all, but once again a description of the most general method of obtaining such extensions is not known.

7.3. It will be clear that many questions have been left unanswered in this paper. For a *conservative* set $Q \equiv \{q_{ij}\}$ satisfying $\sum_\alpha q_{i\alpha} = 0$ for all i, we at least have necessary and sufficient conditions for uniqueness of Q-processes (processes with $p'_{ij}(0) = q_{ij}$); for a non-conservative Q we have uniqueness criteria for two special kinds of processes, those whose generators satisfy either $\Omega \supseteq Q_0$ or $\Omega \subseteq Q$, but there may exist Q-processes belonging to neither class (see § **8.3**). A general uniqueness criterion for Q-processes when Q is non-conservative has still to be found.

Also when Q is conservative there always exists an honest Q-process. When Q is non-conservative, no process with $\Omega \supseteq Q_0$ can be honest; also we know that either there is only one (dishonest) process with $\Omega \subseteq Q$, or there are infinitely many such processes including at least one honest one. It would be desirable to find under what conditions on Q there exists at least one honest Q-process.

Of course, a solution to the main outstanding problem of finding *all* Q-processes would answer the two more special questions posed above.

§ 8. Examples

8.1. Non-uniqueness for the Kolmogorov equations. Two problems were treated in §§ **5** and **6**: given $Q \equiv \{q_{ij}\}$,

(B) find a process satisfying the backward equations

(equivalently, such that $\Omega \supseteq Q_0$);

(F) find a process satisfying the forward equations

(equivalently, such that $\Omega \subseteq Q$).

We remind the reader that a solution to both problems is always provided by the minimal Q-process $\mathcal{F}$, that necessary and sufficient conditions for $\mathcal{F}$ to be the only solution to (B) or to (F) are given in Theorems 8 and 10, and that when Q is conservative every Q-process (in particular every solution to (F)) is a solution to (B).

In many cases the solution to both problems is unique. For example, if the q_{ij} are bounded so that

$$\sum_{i \neq \alpha} q_{i\alpha} \leq q_i \leq A \qquad (\text{all } i),$$

let $\lambda > 0$ and let $\zeta \in m$ and $\eta \in l$ be non-negative solutions to

$$(U_\lambda): \qquad (\lambda + q_i)\,\zeta_i = \sum_{\alpha \neq i} q_{i\alpha}\,\zeta_\alpha,$$

$$(V_\lambda): \qquad (\lambda + q_j)\,\eta_j = \sum_{\alpha \neq j} \eta_\alpha\, q_{\alpha j}.$$

Then

$$0 \leq \zeta_i \leq \frac{1}{\lambda + q_i}\,(\sum_{\alpha \neq i} q_{i\alpha})\,\|\zeta\| \leq \frac{q_i}{\lambda + q_i}\,\|\zeta\| \leq \frac{A}{\lambda + A}\,\|\zeta\|,$$

$$\|\zeta\| \leq \frac{A}{\lambda + A}\,\|\zeta\|,$$

so that $\zeta = 0$. Also, summing over j in (V_λ),

$$\lambda \sum \eta_j + \sum q_j \eta_j = \sum_j (\sum_{\alpha \neq j} \eta_\alpha q_{\alpha j}) = \sum_\alpha (\sum_{j \neq \alpha} q_{\alpha j})\,\eta_\alpha \leq \sum_\alpha q_\alpha \eta_\alpha,$$

whence $\sum \eta_j \leq 0$ and $\eta = 0$. Thus (U_λ) and (V_λ) have only trivial solutions: both (B) and (F) have $\mathfrak{F}$ as their only solution.

A familiar example in which (B) does not have a unique solution whilst (F) does is the birth process (Feller [7], p. 392–3). Label E as 1, 2, 3, ..., let $b_n > 0$ $(n \geq 1)$ and $\sum_1^\infty b_n^{-1} < \infty$, and define the conservative set Q by

$$q_{ii} = -b_i, \qquad q_{i,i+1} = +b_i, \qquad q_{ij} = 0 \text{ otherwise.} \tag{8.1}$$

Then (U_λ) becomes

$$(\lambda + b_i)\,\zeta_i = b_i\,\zeta_{i+1} \qquad (i = 1, 2, \ldots),$$

which has the solution

$$\zeta_0 = 1, \qquad \zeta_i = \prod_{n=1}^{i-1}\left(1 + \frac{\lambda}{b_n}\right) \qquad (i > 1);$$

ζ_i is positive, and is bounded because $\prod_1^\infty \left(1 + \frac{\lambda}{b_n}\right) < \infty$. Hence the solution to (B) is not unique. On the other hand (V_λ) becomes

$$(\lambda + b_1)\,\eta_1 = 0, \qquad (\lambda + b_j)\,\eta_j = b_{j-1}\,\eta_{j-1} \qquad (j > 1),$$

so that $\eta = 0$ and the solution to (F) is unique.

Next we illustrate the possibility that (B) may have a unique solution when (F) does not. This can happen only when Q is non-conservative, because if Q is conservative and (B) has a unique solution then $\mathfrak{F}$ is the only Q-process and in particular the only solution to (F). We give a slightly simplified form of an example due to

Kolmogorov [19] (and treated by semigroup methods in [18]). Label E as $1, 2, \ldots,$ let $a_n > 0$ $(n \geq 1)$ and $\sum_1^\infty a_n^{-1} < \infty$, and define Q by

$$q_{ii} = -a_i \ (i \geq 1), \qquad q_{i,i-1} = +a_i \ (i > 1), \qquad q_{ij} = 0 \text{ otherwise}; \tag{8.2}$$

note that Q is non-conservative because $\sum q_{1\alpha} = -a_1 < 0$. Now (U_λ) becomes

$$(\lambda + a_1)\zeta_1 = 0, \qquad (\lambda + a_i)\zeta_i = a_{i-1}\zeta_{i-1} \qquad (i > 1),$$

so that $\zeta = 0$ and the solution to (B) is unique; but (V_λ) becomes

$$(\lambda + a_j)\eta_j = a_{j+1}\eta_{j+1} \qquad (j \geq 1)$$

which has the solution (unique to within a constant factor)

$$\eta_j = a_j^{-1} \prod_{n=1}^{j-1} \left(1 + \frac{\lambda}{a_n}\right).$$

Now $\sum a_j^{-1}$ and $\prod \left(1 + \frac{\lambda}{a_n}\right)$ both converge, so that $\sum \eta_j$ converges, η is in l, and $\mathfrak{N}^+(\lambda I - Q)$ consists of positive multiples of η; also $\mathfrak{F}$ is dishonest because Q is non-conservative. By Theorem 10, there are infinitely many solutions to (F), and exactly one of them is honest: its generator Ω can be shown to be the restriction of Q to the subset of $\mathfrak{D}(Q)$ on which $(e, Qx) = 0$. A probabilistic description of a very similar process (in which the state 2 is made absorbing by putting $a_2 = 0$) can be found in [18].

8.2. There remains the possibility, even when Q is conservative, that neither (B) nor (F) has a unique solution. To illustrate this, we take an example due to Lévy [21] and discussed in more detail by Kendall ([17], § 3). We now label E as $\ldots, -1, 0, 1, \ldots,$ let $b_n > 0$ and $\sum_{-\infty}^{\infty} b_n^{-1} < \infty$, and define the conservative set Q exactly as at (8.1) (but i and j now range over all integers, not merely integers ≥ 1). Then (U_λ) and (V_λ) have positive solutions $\zeta \in m$ and $\eta \in l$, given by

$$\zeta_i = \prod_{-\infty}^{i-1} \left(1 + \frac{\lambda}{b_n}\right),$$

$$\eta_j = b_j^{-1} \prod_{j+1}^{\infty} \left(1 + \frac{\lambda}{b_n}\right).$$

Thus $\mathfrak{F}$ is dishonest and $\mathfrak{N}^+(\lambda I - Q) \neq \{0\}$, so that there are infinitely many solutions to (F) (and these are at the same time solutions to (B), since Q is conservative). As in the preceding example, there is exactly one *honest* process which solves (F), and its generator Ω is Q restricted by the side-condition $(e, Qx) = 0$. It is this particular process which is treated in [17] and there called "the flash": informally, this may be described as a birth process in which the system runs through the states in the order $\ldots, -1, 0, 1, 2, \ldots,$ the mean time spent in state j being b_j^{-1}. The system will run out of instructions after a finite time (having "reached $+\infty$") and then "returns to $-\infty$" and resumes its journey.

It is now easy to make up a conservative set Q for which there are infinitely many honest processes satisfying the forward equations[1] (and so *also* the backward equations). By Theorem 10, $\mathcal{N}(\lambda I - Q)$ must contain at least two independent positive vectors, and also $\mathcal{J}$ must be dishonest. It is almost obvious that we can achieve this simply by combining two sets Q of the type just considered. To be specific, let us now assign labels (s, n) to the states, where $s=1, 2$ and $n=\ldots, -1, 0, 1, \ldots$; denote the (s, n)th coordinate of $x \in l$ by x_n^s, let $b_n^s > 0$ and $\sum_n 1/b_n^s < \infty$, and define Q by

$$q_{(r,m)(s,n)} = \begin{cases} +b_m^r & \text{if } r=s \text{ and } n=m+1, \\ -b_m^r & \text{if } r=s \text{ and } n=m, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

Then the operator Q is given by

$$(Qx)_n^s = b_{n-1}^s x_{n-1}^s - b_n^s x_n^s$$

and its domain $\mathcal{D}(Q)$ consists of all x with

$$\sum_{s,n} |b_{n-1}^s x_{n-1}^s - b_n^s x_n^s| < \infty$$

(such x being automatically in l). We note that when $x \in \mathcal{D}(Q)$ the limits

$$L^s x = \lim_{n \to -\infty} b_n^s x_n^s, \qquad U^s x = \lim_{n \to +\infty} b_n^s x_n^s \qquad (s=1, 2)$$

exist, and that

$$(e, Qx) = \sum_{s=1}^{2} (L^s x - U^s x).$$

The reader will easily verify that both (U_λ) and (V_λ) have two independent positive solutions:[2] for instance one solution ζ to (U_λ) is given by

$$\zeta_n^1 = \prod_{\alpha=n}^{\infty} \left(1 + \frac{\lambda}{b_\alpha^1}\right)^{-1}, \qquad \zeta_n^2 = 0.$$

[1] The existence of such sets Q was discovered by KENDALL [17]. The present example, also due to KENDALL (unpublished), is simpler but less drastic.

[2] In KENDALL's example [17], both equations have countably many independent positive solutions.

Thus there must (by Theorem 10) be infinitely many honest processes satisfying the forward equations, and so *also* the backward equations; the construction of § **6.2** gives one such process for each element of norm λ^{-1} in $\mathcal{N}^+(\lambda I - Q)$, and hence gives a one-parameter family of such honest processes. However, we will now show that there exists a family of such honest processes specified by *two* parameters ϱ_1 and ϱ_2, each lying between 0 and 1 (inclusive). The generator Ω of the typical process of this family is the restriction of Q to the subset of $D(Q)$ on which

$$L^1 x = \varrho_1 U^1 x + (1 - \varrho_2) U^2 x,$$

$$L^2 x = (1 - \varrho_1) U^1 x + \varrho_2 U^2 x.$$

It can be verified (by using the Hille-Yosida theorem, following the pattern laid down in [18] and [17]) that Ω, for each choice of ϱ_1 and ϱ_2, generates an honest process such that $Q_0 \subseteq \Omega \subseteq Q$. It can also be shown that the construction of § **6.2** leads only to those processes of the above type for which $\varrho_1 + \varrho_2 = 1$.

An informal description of these processes can be obtained by calculating the resolvent operator and thus the Laplace transforms ψ_{ij} of the transition probability functions p_{ij} (cf. [17]). The states of the system are of two types, $(1, n)$ and $(2, n)$, belonging to "flashes" of the kind described at the beginning of this paragraph; the mean times spent in $(1, n)$ and $(2, n)$ are $1/b_n^1$ and $1/b_n^2$. The system, initially in some state $(1, n)$ or $(2, n)$, runs through flash 1 or flash 2. The constants ϱ_1 and ϱ_2 specify the behaviour of the system when it has come to the end of a flash ("arrived at $+\infty$"): from "$+\infty$" in flash 1 it goes immediately

$$\left\{\begin{array}{l}\text{to "}-\infty\text{" in flash 1 with probability } \varrho_1 \\ \text{to "}-\infty\text{" in flash 2 with probability } 1-\varrho_1\end{array}\right\},$$

and similarly it goes from the end of flash 2 to the beginning of flash 2 or flash 1 with probabilities ϱ_2 and $1 - \varrho_2$. In the special case when $\varrho_1 + \varrho_2 = 1$, the behaviour of the system after it has run out of instructions does not depend on whether it was previously in flash 1 or flash 2: it then begins flash 1 or flash 2 with probabilities ϱ_1 and ϱ_2. This indicates why the construction of § **6.2** does not in general yield *all* processes satisfying the forward equations: it fails to distinguish between different ways of "escaping to infinity" or "running out of instructions".

8.3. Failure of the Kolmogorov equations. When Q is conservative, the backward equations hold for every Q-process; the forward equations will fail whenever $\mathfrak{F}$ is dishonest and we apply the construction of § **5.1** to $\mathfrak{F}$.

When Q is non-conservative, the backward equations cannot hold for an honest Q-process. The last example in § **8.1** illustrates this possibility, and indeed there the forward equations do hold.

Processes for which the backward and forward equations *both* fail can of course be constructed in a trivial manner by combining two independent processes of the two kinds just mentioned. A simpler and perhaps more interesting example is the following. The states are labelled ω and $\ldots, -1, 0, 1, \ldots$; we then take $b_n > 0$ with $\sum 1/b_n < \infty$ and define Q by

$$\left.\begin{array}{lll} q_{ii} = -b_i, & q_{i,i+1} = +b_i & (i = \cdots, -1, 0, 1, \ldots), \\ q_{\omega\omega} = -1, & q_{ij} = 0 \text{ otherwise.} & \end{array}\right\} \tag{8.4}$$

Then Q is non-conservative, because $\sum_\alpha q_{\omega\alpha} = -1$, and (U_λ), (V_λ) have non-trivial positive solutions given by

$$\zeta_\omega = 0, \qquad \zeta_i = \prod_{-\infty}^{i-1} \left(1 + \frac{\lambda}{b_n}\right),$$

$$\eta_\omega = 0, \qquad \eta_j = b_j^{-1} \prod_{j+1}^{\infty} \left(1 + \frac{\lambda}{b_n}\right).$$

Thus there will be (as before) exactly one honest Q-process satisfying the forward equations, but we will now show that there is *also* an honest Q-process which does not satisfy the forward equations (nor, of course, the backward equations, Q being non-conservative). Its generator Ω has domain $\mathcal{D}(\Omega)$ consisting of all

$$x \equiv \{x_\omega;\ \ldots, x_{-1}, x_0, x_1, \ldots\}$$

such that

$$\sum_{-\infty}^{\infty} |b_{j-1} x_{j-1} - b_j x_j| < \infty \tag{8.5}$$

and

$$L x \equiv \lim_{n \to -\infty} b_n x_n = x_\omega \tag{8.6}$$

((8.5) implies that this limit exists, and therefore that $x \in l$).

Ω is defined by

$$\left.\begin{array}{l} (\Omega x)_\omega \equiv U x - x_\omega, \\ (\Omega x)_j \equiv b_{j-1} x_{j-1} - b_j x_j, \end{array}\right\} \tag{8.7}$$

where $U x \equiv \lim_{n \to +\infty} b_n x_n$. It can be shown that Ω generates a transition semigroup (an honest process); the necessary calculations are again similar to those in [17], and will be omitted. Now Ω is *not* a restriction of Q, because the vector x given by

$$x_\omega = 1, \; x_j = b_j^{-1},$$

is in $\mathcal{D}(\Omega)$, but $(Qx)_\omega = -1$ whilst $(\Omega x)_\omega = 0$. Thus the process generated by Ω does not satisfy the forward equations; on the other hand it *is* a Q-process, as may be verified by calculating the resolvent operator Ψ_λ and then using (3.15) and (3.16).

Once again it is possible to describe the above process $\mathcal{P}$, and also the (unique) honest Q-process $\mathcal{P}'$ which satisfies the forward equations, in an informal manner. For both processes, we have a flash (with states $\ldots, -1, 0, 1, \ldots$ and mean time b_j^{-1} in state j) "coupled" to an extra state ω. For the process $\mathcal{P}$, the system runs through the flash and then jumps at once to state ω; it remains at ω for a mean time 1, and then jumps back to the beginning of the flash and repeats its previous performance. For $\mathcal{P}'$, the system remains in the "flash" states once it reaches them, i.e. as in § **8.2** it jumps from the end of the flash to the beginning; if it is in state ω, it remains there for a mean time 1, then jumps to the flash and never returns to ω. The reader who is familiar with Doob's analysis [4] of the backward and forward equations in terms of sample function discontinuities will note that the sample functions for $\mathcal{P}$ have a right-hand discontinuity which is not a jump when the system leaves ω, and a similar left-hand discontinuity when it reaches ω; this explains the simultaneous failure of the forward and backward equations in probabilistic terms.

8.4. The birth-and-death process. This, our final example, can be made to exhibit each of the three combinations of uniqueness and non-uniqueness for problems (B) and (F) which were illustrated by the examples in § **8.1**. A further reason for discussing it here is that much attention has already been devoted to uniqueness problems for the birth-and-death process the most complete results being those recently announced by Karlin & McGregor [14]. We shall show that the uniqueness criteria of Theorems 7 and 10 can be applied to give a very simple alternative derivation of these results.

Let E be labelled as $0, 1, 2, \ldots$, and let

$$b_0 \geq 0, \qquad b_n > 0, \qquad a_n > 0 \qquad (n \geq 1);$$

we do *not* require that $b_0 > 0$. (The b_n and a_n will be "birth rates" and "death rates", the state label n is the size of the "population"; when $b_0 = 0$, the state 0 will be absorbing so that population cannot recover once it has become extinct; when $b_0 > 0$, the population may "revive" through "immigration" at a rate b_0. See Feller ([7], pp. 371–3) for an explicit description of the system.) The conservative set Q is now defined by

$$\left.\begin{aligned} q_{00} &= -b_0, \qquad q_{ii} = -(a_i+b_i) \qquad (i\geq 1),\\ q_{i,i+1} &= +b_i\,(i\geq 0), \qquad q_{i,i-1} = +a_i \qquad (i\geq 1),\\ q_{ij} &= 0 \qquad \text{otherwise.}\end{aligned}\right\} \tag{8.8}$$

The equations (U_λ) and (V_λ) are

$$\left.\begin{aligned} (\lambda+b_0)\,\zeta_0 &= b_0\,\zeta_1,\\ (\lambda+a_n+b_n)\,\zeta_n &= a_n\,\zeta_{n-1}+b_n\,\zeta_{n+1} \qquad (n\geq 1);\end{aligned}\right\} \tag{8.9}$$

$$\left.\begin{aligned} (\lambda+b_0)\,\eta_0 &= a_1\,\eta_1,\\ (\lambda+a_n+b_n)\,\eta_n &= b_{n-1}\,\eta_{n-1}+a_{n+1}\,\eta_{n+1} \qquad (n\geq 1).\end{aligned}\right\} \tag{8.10}$$

We rewrite (8.9) (for $n\geq 1$) as

$$b_n\,(\zeta_{n+1}-\zeta_n)=\lambda\,\zeta_n+a_n\,(\zeta_n-\zeta_{n-1}). \tag{8.11}$$

When $b_0=0$, $\zeta_0=0$ and ζ_1 is arbitrary, and $\zeta_2, \zeta_3, \ldots$ are then uniquely determined; when $b_0>0$, ζ_0 is arbitrary and $\zeta_1, \zeta_2, \ldots$ are then determined. We take $\zeta_1=1$ when $b_0=0$ and $\zeta_0=1$ $\left(\text{hence } \zeta_1=1+\dfrac{\lambda}{b_0}\right)$ when $b_0>0$; in either case, $\zeta_1>\zeta_0\geq 0$ and an easy induction from (8.11) shows that $\zeta_{n+1}>\zeta_n$ $(n\geq 1)$. We have then to decide whether ζ_n is bounded; if it is, then (U_λ) has a non-trivial solution.

For the second set of equations, (8.10), we sum the first $(n+1)$ equations and write $\sigma_n\equiv\eta_0+\cdots+\eta_n$. This leads to

$$\left.\begin{aligned} (\lambda+a_1+b_0)\,\sigma_0 &= a_1\,\sigma_1,\\ a_{n+1}\,(\sigma_{n+1}-\sigma_n) &= \lambda\,\sigma_n+b_n\,(\sigma_n-\sigma_{n-1}) \qquad (n\geq 1).\end{aligned}\right\} \tag{8.12}$$

Taking $\sigma_0=1$, we see by induction that $\sigma_{n+1}>\sigma_n$ [$(n\geq 0)$. We have to decide whether (8.10) has a non-negative solution] with $\sum\eta_j<\infty$ i.e. whether $\sigma_n=\eta_0+\cdots+\eta_n$ is bounded; if it is, then (V_λ) has a non-trivial solution.

It is now clear that (8.11) and (8.12) can be treated together by means of

LEMMA 7. *Suppose that $f_n>0$, $g_n>0$ for $n\geq 1$, that $0\leq z_0<z_1<z_2<\cdots$, and that*

$$z_{n+1}-z_n=f_n\,z_n+g_n\,(z_n-z_{n-1}) \tag{8.13}$$

for $n\geq 1$. Then z_n is bounded if and only if

$$\sum_{n=1}^{\infty}(f_n+g_n\,f_{n-1}+\cdots+g_n\,g_{n-1}\ldots g_2\,f_1+g_n\ldots g_2\,g_1)<\infty.$$

Proof. Repeated applications of (8.13) give

$$z_{n+1}-z_n=f_n z_n+g_n f_{n-1} z_{n-1}+\cdots+ \\ +g_n g_{n-1}\ldots g_2 f_1 z_1+g_n\ldots g_2 g_1(z_1-z_0).$$

Hence

$$z_{n+1}-z_n\le(f_n+g_n f_{n-1}+\cdots+g_n\ldots g_2 f_1+g_n\ldots g_2 g_1)z_n \\ =F_n z_n \quad \text{say};$$

on the other hand,

$$z_{n+1}-z_n\ge F_n(z_1-z_0).$$

These two inequalities give

$$z_1+(z_1-z_0)\sum_{k=1}^{n-1}F_k\le z_n\le z_1\prod_{k=1}^{n-1}(1+F_k)\qquad(n>1),$$

and this implies that z_n is bounded if and only if $\sum_1^\infty F_k<\infty$, as asserted.

To apply the lemma to (8.11) take $z_n=\zeta_n$, $f_n=\lambda/b_n$, $g_n=a_n/b_n$. Then

$$F_n=\lambda\left(\frac{1}{b_n}+\frac{a_n}{b_n b_{n-1}}+\cdots+\frac{a_n\ldots a_2}{b_n\ldots b_2 b_1}\right)+\frac{a_n\ldots a_1}{b_n\ldots b_1}. \tag{8.14}$$

The convergence of $\sum F_n$ does not, of course, depend on the choice of $\lambda>0$, nor is it affected by omitting the last term $(a_n\ldots a_1)/(b_n\ldots b_n)$ in (8.14). Thus ζ_n in (8.11) is bounded if and only if $R<\infty$, where

$$R\equiv\sum_{n=1}^\infty\left(\frac{1}{b_n}+\frac{a_n}{b_n b_{n-1}}+\cdots+\frac{a_n\ldots a_2}{b_n\ldots b_2 b_1}\right). \tag{8.15}$$

Similarly we apply the lemma to (8.12) by taking $z_n=\sigma_n$, $f_n=\lambda/a_{n+1}$ and $g_n=b_n/a_{n+1}$. Then

$$F_n=\lambda\left(\frac{1}{a_{n+1}}+\frac{b_n}{a_{n+1}a_n}+\cdots+\frac{b_n\ldots b_2}{a_{n+1}\ldots a_2}\right)+\frac{b_n\ldots b_1}{a_{n+1}\ldots a_2}; \tag{8.16}$$

again the convergence of $\sum F_n$ does not depend on the choice of λ, and also we may change the last term in (8.16) to $(b_n\ldots b_1)/(a_{n+1}\ldots a_2 a_1)$. Thus σ_n in (8.12) is bounded (equivalently, η_n in (8.10) satisfies $\sum\eta_n<\infty$) if and only if $S<\infty$, where

$$S\equiv\sum_{n=1}^\infty\left(\frac{1}{a_{n+1}}+\frac{b_n}{a_{n+1}a_n}+\cdots+\frac{b_n\ldots b_1}{a_{n+1}\ldots a_2 a_1}\right). \tag{8.17}$$

Finally, we show that R and S are *both* finite if and only if T is finite, where

$$T\equiv\sum_{n=1}^\infty\left(\frac{a_n\ldots a_2}{b_n\ldots b_2 b_1}+\frac{b_n\ldots b_1}{a_{n+1}\ldots a_2 a_1}\right). \tag{8.18}$$

Certainly T is finite if R and S are, because $T \leq R+S$. On the other hand, we may write

$$R = \sum_1^\infty \frac{a_n \dots a_2}{b_n \dots b_2 b_1}\left(1+\frac{b_1}{a_2}+\dots+\frac{b_1 \dots b_{n-1}}{a_2 \dots a_n}\right),$$

$$S = \sum_1^\infty \frac{b_n \dots b_1}{a_{n+1} \dots a_2 a_1}\left(1+\frac{a_1}{b_1}+\dots+\frac{a_1 \dots a_n}{b_1 \dots b_n}\right).$$

If T is finite, then $\sum (a_n \dots a_2)/(b_n \dots b_1)$ and $\sum (b_n \dots b_1)/(a_{n+1} \dots a_1)$ both converge, and the factors

$$1+\dots+\frac{b_1 \dots b_{n-1}}{a_2 \dots a_n}, \qquad 1+\dots+\frac{a_1 \dots a_n}{b_1 \dots b_n}$$

are both bounded; hence R and S are finite.

A complete classification of birth-and-death processes can now be given.

THEOREM 11. *Let the conservative set* $\boldsymbol{Q}$ *be defined by* (8.10), *and let* R, S *and* T *be defined by* (8.16)–(8.18).

(i) *If* $R=\infty$, *there is exactly one* $\boldsymbol{Q}$*-process; it is honest and satisfies the forward equations.*

(ii) *If* $R<\infty$ *and* $S=\infty$, *there are infinitely many* $\boldsymbol{Q}$*-processes. Only one of these satisfies the forward equations, but it is dishonest.*

(iii) *If* $R<\infty$ *and* $S<\infty$, *equivalently if* $T<\infty$, *there are infinitely many* $\boldsymbol{Q}$*-processes satisfying the forward equations. Exactly one of these is honest.*

Remarks (1). The backward equations hold in all cases because $\boldsymbol{Q}$ is conservative.

(2). The criteria have been stated so as not to involve b_0. They can (as should be clear from their derivation) be modified so as not to involve any pre-assigned finite subset of the coefficients a_n and b_n.

(3). The criterion "$R=\infty$" for uniqueness of $\boldsymbol{Q}$-processes is due to Dobrušin [3], who obtained it from Feller's conditions [6] for the honesty of the minimal $\boldsymbol{Q}$-process $\mathfrak{F}$; the method by which we have derived it is due to Kendall (unpublished).

(4). The criterion "$T<\infty$" for the existence of infinitely many $\boldsymbol{Q}$-processes satisfying the forward equations was found (for $b_0>0$) by Karlin & McGregor [14].

(5). The first example (with $b_0=0$) of case (iii) was found by Ledermann & Reuter [20]. They showed, by explicit construction of the transition probabilities $p_{ij}(t)$, that for a suitable choice of the a_n and b_n there can exist an honest $\boldsymbol{Q}$-process satisfying the forward equations even when $\mathfrak{F}$ is dishonest. It is clear from the definition (8.18) of T and from (iii) of Theorem 11 that this is most likely to occur when both a_n

and b_n grow rapidly with n, with b_n slightly larger than a_n. Taking

$$b_{n+1}/b_n = 1 + \varrho\, n^{-1} + O(n^{-2}) \qquad (\varrho > 0),$$

$$b_n/a_n = 1 + \sigma n^{-1} + O(n^{-2}),$$

as in [20], simple computations show that case (iii) occurs if and only if $\varrho - 1 > \sigma > 1$ (i.e. for a slightly larger range than that used in [20], Th. 10); a suitable choice of a_n, b_n would be $a_n = n^4$, $b_n = n^2(n+1)^2$ (so that $\varrho = 4$, $\sigma = 2$).

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