Analytic capacity and differentiability properties of finely harmonic functions

by

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1. Introduction

Let f be a finely harmonic function defined on a finely open set V in the complex plane $\mathbf{C}$. In this paper we investigate the problem: To what extent is f differentiable in V?

There are of course several ways of interpreting the question. Debiard and Gaveau [4], [5] have proved the following: Let $K\subset\mathbf{C}$ be compact and let $H(K)$ denote the uniform closure on K of functions harmonic in a neighbourhood of K. Then $H(K)$ coincides with the set of functions continuous on K and finely harmonic on the fine interior K' of K. And if $g\in H(K)$ is the uniform limit of functions g_n harmonic in a neighbourhood of K, then ∇g_n converges in $L^2(m)$ to a limit ∇g, which does not depend on the sequence chosen. Here and later m denotes planar Lebesgue measure. In the other direction they give an example of a compact set K and a point $x_0\in K'$ such that $|\nabla g_n(x_0)|\to\infty$ as $n\to\infty$.

It was conjectured by T. J. Lyons (private communication) that $\{\nabla g_n(x)\}$ always converges outside a set of zero *logarithmic* capacity. In section 3 we prove that this conjecture fails: For any compact set E with zero *analytic* capacity, there exists a compact set K with $E\subseteq K'$ and functions g_n harmonic in a neighbourhood of K such that $g_n\to 0$ uniformly on K and $|\partial g_n/\partial\bar{z}|\to\infty$ uniformly on E (Theorem 1).

In section 4 we show that parts of the proof of Theorem 1 can be used to prove the following estimate for analytic capacity γ (Theorem 2): If E, F are compact sets and $0<\alpha<1$, then

$$\gamma(E)\leqslant A_\alpha[\gamma(E\setminus F)+C_\alpha(F)^{1/\alpha}],$$

where C_α is the capacity associated to the potential $|z|^{-\alpha}$ and A_α is a constant depending only on α. This result in turn implies that any compact set of Hausdorff dimension less than 1 is γ-negligible, i.e. negligible with respect to approximation by bounded analytic functions (Theorem 3).

In section 5 we prove a partial converse of Theorem 1. More precisely, we prove that if $\{f_n\}$ are functions harmonic in a neighbourhood of K converging uniformly to a function f such that $|f_n - f| < 2^{-n}$ on K, then for any rectifiable arc J the sequence $\{\nabla f_n\}$ converges a.e. on $J \cap K'$ w.r.t. arc length on J to a limit depending on f but not on the sequence $\{f_n\}$ (Theorem 4). We also prove that $\{\nabla f_n\}$ converges C_1-everywhere on K', where C_1 denotes Newtonian capacity (Theorem 5).

In section 6 we consider a different interpretation of the question above: Given a finely harmonic function f on a finely open set, what can be said about the set of points where f is finely differentiable? Fuglede [8] has proved, using a theorem of Mizuta [16], that f is always finely differentiable outside a set G with $C_h^*(G)=0$, where C_h^* is the outer capacity associated to the kernel $h(z)=|z|^{-1}\log(1/|z|)$. For completeness we include a proof of this result (Theorem 6). In the opposite direction, we prove that given any compact set E with $C_h(E)=0$, then we can find a finely open set $V \supseteq E$ and a finely harmonic function f on V such that f is not finely differentiable at any point of E (Theorem 7).

2. Some preliminaries

We refer the reader to Helms [11] for information about the fine topology, and to Fuglede [7] for the theory of finely harmonic functions; the latter is used only in section 6.

Let $k(z)$ denote one of the kernels

$$W(z) = \log\frac{1}{|z|},$$

$$R_\alpha(z) = |z|^{-\alpha}; \quad \text{where } 0 < \alpha \leq 1,$$

$$h(z) = |z|^{-1}\log\frac{1}{|z|}.$$

Then if $G \subseteq \{z : |z| \leq \frac{1}{2}\}$ is a Borel set, the capacity of G (associated to the kernel $k(z)$) is defined by

$$C_k(G) = \sup\{\mu(E); \mu \in \Gamma_k(G)\}$$

where $\Gamma_k(G)$ is the set of positive measures μ on G such that

$$\int k(x-y)\, d\mu(y) \leq 1 \quad \text{for all } x \in \mathbf{C}.$$

If $k=W$, the corresponding capacity is the *Wiener* capacity, which we denote by C_W. The *logarithmic* capacity, cap, is connected to the Wiener capacity as follows:

$$\operatorname{cap}(G)=\exp(-C_W(G)^{-1}).$$

In particular, cap and C_W have the same null sets. If $k(z)=R_\alpha(z)$ or $k(z)=h(z)$, the corresponding capacities are denoted by C_α and C_h, respectively. If $\alpha=1$ we get the Newtonian capacity C_1. We refer to Carleson [2] for more information about these capacities.

Let $g(t)$ be a continuous, increasing function on $[0,\infty)$ such that $g(0)=0$. Let E be a bounded, plane set. For $\delta>0$ we consider all coverings of E with a countable number of discs Δ_j with radii $\varrho_j\leq\delta$ and define

$$\Lambda_g^\delta(E)=\inf\left\{\sum_j g(\varrho_j)\right\},$$

the inf being taken over all such coverings. The limit

$$\Lambda_g(E)=\lim_{\delta\to 0}\Lambda_g^\delta(E)$$

is called the *Hausdorff measure* of E with respect to the measure function g. If $g(t)=t^\alpha$ for some $\alpha>0$, Λ_g is called α-dimensional Hausdorff measure and denoted by Λ^α. The *Hausdorff dimension* of the set E is the unique number r such that

$$\Lambda_\alpha(E)=\infty \quad \text{for all } \alpha<r$$

and

$$\Lambda_\alpha(E)=0 \quad \text{for all } \alpha>r.$$

A set function related to Λ_g is the *Hausdorff content*

$$M_g(E)=\inf\left\{\sum g(\varrho_j)\right\},$$

the inf being taken over all coverings of E with a countable number of discs Δ_j with radii ϱ_j. If $g(t)=t^\alpha$ we get the α-dimensional Hausdorff content M_α. Properties of Hausdorff measure and Hausdorff content can be found in Carleson [2] and Garnett [10].

Finally we recall the definition of analytic capacity: If $K\subset\mathbf{C}$ is compact, the *analytic capacity of* K, $\gamma(K)$, is defined by

$$\gamma(K) = \sup\{|f'(\infty)|;\, f\in B(K)\},$$

where $B(K)$ is the set of functions f analytic outside K such that $f(\infty)=0$, $|f(z)|\leq 1$ outside K.

For a general set E we define

$$\gamma(E) = \sup\{\gamma(K);\ K \text{ compact},\ K\subset E\}.$$

The analytic capacity plays a crucial role in problems involving approximation by analytic functions. See Gamelin [9], Garnett [10] and Vitushkin [17]. Some of the metric properties of analytic capacity are:

(1) If K is compact then $\gamma(K)\leq M_1(K)\leq \Lambda_1(K)$.

(2) There exists compact sets L such that

$$\gamma(L)=0 \quad\text{and}\quad \Lambda_1(L)>0.$$

(3) For linear, Borel sets E we have

$$\gamma(E) = \frac{1}{4}l(E),$$

where $l(E)$ is the length of E.

(4) If E is a Borel set and $\Lambda_\alpha(E)>0$ for some $\alpha>1$, then $\gamma(E)>0$. Analytic capacity is related to logarithmic and Newtonian capacity by:

(5) If K is compact $C_1(K)\leq\gamma(K)\leq \operatorname{cap}(K)$, with equality on the right hand side if K is also connected.

3. Differentiation in terms of approximating sequences (I)

Throughout this article we will let $D=\{z;\ |z|<\frac{1}{4}\}$ and $\bar{D}$ the closure of D. If μ is a measure on E we put

$$P_\mu(z) = \int_E \log\frac{1}{|\zeta-z|}\,d\mu(\zeta)$$

and

$$\hat{\mu}(z) = \int_E \frac{1}{\zeta-z}\,d\mu(\zeta),$$

when the integrals converge. If K is compact, $C(K)$ and $C_R(K)$ denote the complex and real continuous functions on K respectively.

LEMMA 1. *Let E be a compact subset of D with $\gamma(E)=0$. Then for any $M>0$ we can find a real measure ϱ on a compact subset of $\bar{D}-E$ such that*

(*i*) $P_{|\varrho|}(z)\leq 1;\quad z\in E,$

and

(*ii*) $|\hat{\varrho}(z)|\geq M;\quad z\in E.$

Proof. Choose $\varepsilon>0$ with

$$\varepsilon^{-1}>2\cdot\log\frac{1}{\operatorname{dist}(E,\,\mathbf{C}\setminus D)}.$$

Since $\gamma(E)=0$, we can find an open set $V\supset E$, with $V\subset D$, such that $\gamma(\bar{V})<\varepsilon^2$ ([18], p. 16). Let $F=\bar{D}\setminus V$, $K=\bar{V}$. Let Ω denote the set of real measures ϱ on F such that $P_{|\varrho|}\leq 1$ on E. Assume the conclusion of the lemma is false. Then the sets of functions $A=\{\operatorname{Re}\hat{\varrho}(z);\ \varrho\in\Omega\}$ and $B=\{f\in C_R(E);\ f\geq M \text{ on } E\}$ are disjoint convex subsets of $C_R(E)$, and the separation theorem for convex sets yields a real measure μ on E with

$$\int g\,d\mu\leq 1\quad\text{for } g\in A,\qquad \int f\,d\mu\geq 1\quad\text{for } f\in B.$$

This implies that μ is positive, $\|\mu\|\geq\frac{1}{M}$ and

$$\left|\operatorname{Re}\int\int\frac{d\mu(\zeta)\,d\varrho(z)}{\zeta-z}\right|\leq 1\quad\text{for } \varrho\in\Omega. \tag{3.1}$$

Let G be the set of functions $g\in C_R(F)$ such that

$$g(z)\leq\sum_{i=1}^{n}\alpha_i\log\frac{1}{|z-z_i|}\quad\text{on } F,$$

for some $z_1,\ldots,z_n\in E$ and $\alpha_1,\ldots,\alpha_n>0$, $\Sigma_{i=1}^n\alpha_i\leq 1$. If ϱ is a real measure on F such that

$$\varrho(g)\leq 1\quad\text{for all } g\in G, \tag{3.2}$$

then $\varrho\geq 0$ and $\varrho\in\Omega$. Hence by (3.1) we have

$$\varrho(\operatorname{Re}\hat{\mu})\leq 1 \text{ and } \varrho(-\operatorname{Re}\hat{\mu})\leq 1. \tag{3.3}$$

So, by the separation theorem applied to the functions $\operatorname{Re}\hat{\mu}$ and $-\operatorname{Re}\hat{\mu}$ separately, we can conclude that there must exist $z_1,\ldots,z_n\in E$ and $\alpha_1,\ldots,\alpha_n>0$ with $\Sigma_{i=1}^n\alpha_i\leq 4$ and

$$|\text{Re}\,\hat{\mu}(z)| \leq \psi(z), \quad z \in F \tag{3.4}$$

where

$$\psi(z) = \sum_{i=1}^{n} \alpha_i \log \frac{1}{|z-z_i|}.$$

Define

$$f(z) = \exp\left(\varepsilon \int \frac{d\mu(\zeta)}{\zeta - z}\right) - 1; \quad z \in \mathbf{C} \setminus V$$

and extend f continuously to $\mathbf{C}$ so that

$$|f(z)| \leq 1 + \exp[\varepsilon\psi(z)] \text{ for } z \in D.$$

Then f is analytic outside K, $f(\infty)=0$ and

$$|f'(\infty)| = \lim_{z\to\infty} |zf(z)| = \varepsilon\|\mu\| \geq \frac{\varepsilon}{M}. \tag{3.5}$$

Let θ be a C^1 function on $\mathbf{R}$ with $0 \leq \theta(x) \leq 1$ for $x \in \mathbf{R}$, $\theta(x)=1$ if $x \leq \varepsilon^{-1}$, $\theta(x)=0$ for $x \geq 2\varepsilon^{-1}$ and $\theta'(x) \leq 4\varepsilon$ for $x \in \mathbf{R}$. Put

$$\varphi(z) = \theta(\psi(z)).$$

Then φ is C^1 on $\mathbf{C}$.
Define

$$g(\zeta) = f(\zeta)\,\varphi(\zeta) + \frac{1}{\pi} \int \frac{f(z)}{z-\zeta} \frac{\partial\varphi}{\partial\bar{z}}\, dm(z)$$

and put

$$h = f - g. \tag{3.6}$$

By the choice of ε we have $\psi < \varepsilon^{-1}$ outside D, so

$$\text{supp}\, \frac{\partial\varphi}{\partial\bar{z}} \subseteq \bar{D}.$$

Since $\bar{\partial}g = \bar{\partial}(f\varphi) - f\bar{\partial}\varphi = \varphi\bar{\partial}f$, we see that g is analytic outside $K \cap \{\psi \leq 2\varepsilon^{-1}\}$ and h is analytic outside $K \cap \{\psi \geq \varepsilon^{-1}\}$. Moreover, $g(\infty)=0$ and both g and h are continuous on $\mathbf{C}$. (See Gamelin [9], Lemma II.1.7.)

Note that

$$\left|\frac{\partial\varphi}{\partial\bar z}\right| \leqslant 4\varepsilon\left|\frac{\partial\psi}{\partial\bar z}\right| \leqslant 4\varepsilon\sum_{i=1}^{n}\frac{\alpha_i}{|z-z_i|} \quad \text{for all } z,$$

and

$$|f(z)| \leqslant e^2+1, \text{ on supp } \frac{\partial\varphi}{\partial\bar z}.$$

Now let

$$\zeta\in\bar D\cap\{\psi\leqslant 2\varepsilon^{-1}\}.$$

Then

$$|f(\zeta)|\leqslant e^2+1$$

and so

$$\begin{aligned}|g(\zeta)| &\leqslant e^2+1+\frac{4\varepsilon}{\pi}(e^2+1)\sum_{i=1}^{n}\alpha_i\int_D\frac{dm(z)}{|z-\zeta|\,|z-z_i|}\\ &\leqslant N_1+N_2\,\varepsilon\sum_{i=1}^{n}\alpha_i\log\frac{1}{|\zeta-z_i|}\\ &= N_1+N_2\,\varepsilon\psi(\zeta)\leqslant N_1+2N_2=N_3,\end{aligned}$$

where N_1, N_2, ... denote constants.

(The inequality

$$\int_D\frac{dm(z)}{|z-\zeta|\,|z-z_i|}\leqslant \text{const.}\cdot\log\frac{1}{|\zeta-z_i|}$$

can for exampel be seen by splitting D into 4 parts:

$$D_1=D\cap\{z;\ |z-\zeta|\leqslant\tfrac12|\zeta-z_i|\},\quad D_2=D\cap\{z;\ |z-z_i|\leqslant\tfrac12|\zeta-z_i|\},$$

$$D_3=(D\setminus D_1)\cap\{z;\ |z-\zeta|\leqslant|z-z_i|\},\quad D_4=(D\setminus D_2)\cap\{z;|z-\zeta|\geqslant|z-z_i|\}.)$$

Therefore by the maximum principle

$$|g(\zeta)|\leqslant N_3 \quad \text{on } \mathbf{C},$$

and therefore

$$|g'(\infty)| \leq N_3 \gamma(K) \leq N_3 \varepsilon^2.$$

Also

$$|h'(\infty)| = \frac{1}{\pi} \left| \int f(\zeta) \frac{\partial \varphi}{\partial \bar{z}} dm(z) \right| \leq \frac{4\varepsilon(e^2+1)}{\pi} \sum \alpha_i \int_{\psi \geq \varepsilon^{-1}} \frac{dm(z)}{|z-z_i|}.$$

Now if Δ is a disc of radius r, then

$$\int_\Delta \frac{1}{|z-z_i|} dm(z) \leq 2\pi r = 2\pi \cdot M_1(\Delta)$$

so

$$|h'(\infty)| \leq 32\varepsilon(e^2+1) M_1(\{\psi \geq \varepsilon^{-1}\}) \leq N_4 \varepsilon^2, \tag{3.8}$$

by Corollary 1, p. 202 in Landkof [12].

Combining (3.5)–(3.8) we get

$$\frac{\varepsilon}{M} \leq |f'(\infty)| \leq |g'(\infty)| + |h'(\infty)| \leq (N_3+N_4)\varepsilon^2,$$

which is a contradiction if ε is small enough.

THEOREM 1. *Let E be a compact set with $\gamma(E)=0$. Then we can find a compact set K with $E \subseteq K'$ and a sequence f_n of real-valued functions, each harmonic in a neighbourhood of K, such that $f_n \to 0$ uniformly on K and $|\partial f_n/\partial \bar{z}| \to \infty$ uniformly on E.*

Proof. We may assume $E \subseteq D$. By Lemma 1 we can find a sequence of real measures ϱ_n, each supported on a compact subset of $\bar{D} \setminus E$, with

$$P_{|\varrho_n|} \leq 2^{-n} \quad \text{and} \quad |\hat{\varrho}_n| \geq 3^n \quad \text{on } E.$$

We may also assume that ϱ_n is a finite linear combination of point masses. Let

$$\varrho = \sum_n |\varrho_n|$$

and let

$$K = \{z; P_\varrho(z) \leq 2\} \cap \bar{D}.$$

Then, since P_ϱ is lower semicontinuous, K is compact. Moreover, $K\cap \operatorname{supp}(\varrho_n)=0$ for all n. Since P_ϱ is finely continuous and $P_\varrho\leq 1$ on E, we must have $E\subseteq K'$.

Now let

$$f_n = 2^{-n} P_{\varrho_n}.$$

Then f_n is harmonic in a neighbourhood of K, $|f_n|\leq 2^{1-n}$ on K, and if $z\in E$

$$\left|\frac{\partial f_n}{\partial \bar{z}}(z)\right| = 2^{-n}|\hat{\varrho}_n(z)| \geq \left(\frac{3}{2}\right)^n.$$

That completes the proof of Theorem 1.

Remark. For finely homomorphic functions the situation is different. T. J. Lyons ([13], [14]) has proved that if $f\in C(K)$ is finely holomorphic in K' (i.e. $f\in A_f(K)$) then $f'(z)$ exists for all $z\in K'$, in the sense that if we choose any sequence of functions $f_n\in A_f(K)$ extending holomorphically to a neighbourhood of z and converging uniformly to f on K, then $\lim_{n\to\infty} f_n'(z)$ exists and depends only on f, not on the sequence chosen.

4. An estimate for analytic capacity

Part of the proof of Lemma 1 can be adapted to yield the following estimate for analytic capacity. As mentioned in section 2 we let C_α denote the capacity associated to the potential $|z|^{-\alpha}$.

THEOREM 2. *Let E and F be compact,* $0<\alpha<1$. *Then*

$$\gamma(E)\leq N(\alpha)\,[\gamma(E\setminus F)+C_\alpha(F)^{1/\alpha}],$$

where $N(\alpha)$ is a constant depending only on α.

Proof. We may assume that F has a smooth boundary, since in general we can find a set $F_0\supseteq F$ with smooth boundary and $C_\alpha(F_0)-C_\alpha(F)$ as small as we wish. Then by Theorem 3, p. 17, in Carleson [2] there is a positive measure μ on F with $\|\mu\|=C_\alpha(F)$ and, writing

$$\psi(z) = \int \frac{d\mu(\zeta)}{|\zeta - z|^\alpha},$$

$\psi(z)=1$ for $z\in F$ and $\psi(z)\leq 1$ everywhere.

Let θ be a C^1 function on $\mathbf{R}$ with $0\leqslant\theta\leqslant 1$, $\theta(x)=1$ for $x\leqslant\frac{1}{2}$, $\theta(x)=0$ for $x\geqslant\frac{3}{4}$ and $|\theta'(x)|\leqslant 8$ everywhere. Put

$$\varphi(z)=\theta(\psi(z)).$$

Then φ is C^1 on $\mathbf{C}$.

Let f be analytic outside E and satisfy

$$|f|\leqslant 1,\ f(\infty)=0 \text{ and } f'(\infty)=\gamma(E).$$

Define f to be 0 on E and put

$$g(\zeta)=\varphi(\zeta)f(\zeta)+\frac{1}{\pi}\int\frac{f(z)}{z-\zeta}\frac{\partial\varphi}{\partial\bar{z}}\,dm(z).$$

Then g is analytic outside $E\cap\{\psi\leqslant\frac{3}{4}\}$, and therefore outside a compact subset of $E\setminus F$. Put

$$h=f-g.$$

Then h is analytic outside $E\cap\{\psi\geqslant\frac{1}{2}\}$. (See Gamelin [9], Lemma I.1.7.)

Note that

$$\left|\frac{\partial\varphi}{\partial\bar{z}}\right|\leqslant 8\left|\frac{\partial\psi}{\partial\bar{z}}\right|\leqslant 16\,\alpha\int\frac{d\mu(w)}{|w-z|^{\alpha+1}}.$$

Thus for $\zeta\in\mathbf{C}$ we have

$$\begin{aligned}|g(\zeta)|&\leqslant 1+\frac{16\,\alpha}{\pi}\int\int\frac{dm(z)d\mu(w)}{|z-\zeta|\,|w-z|^{\alpha+1}}\\&\leqslant 1+N_1(\alpha)\int\frac{d\mu(w)}{|w-\zeta|^{\alpha}}=1+N_1(\alpha)\,\psi(\zeta)\leqslant 1+N_1(\alpha).\end{aligned}$$

($N_1(\alpha)$, $N_2(\alpha)$, ..., denote constants depending only on α). Therefore

$$|g'(\infty)|\leqslant[1+N_1(\alpha)]\,\gamma(E\setminus F). \tag{4.2}$$

Also, for $\zeta\in\mathbf{C}$ we have

$$|h(\zeta)|\leqslant 2+N_1(\alpha),$$

so

$$|h'(\infty)| \leq [2+N_1(\alpha)]\,\gamma(\{\psi \geq \tfrac{1}{2}\}). \tag{4.3}$$

Now

$$\gamma(\{\psi \geq \tfrac{1}{2}\} \leq M_1\{\psi \geq \tfrac{1}{2}\}) \leq N_2(\alpha)\, C_\alpha(\{\psi \geq \tfrac{1}{2}\})^{1/\alpha}. \tag{4.4}$$

(The last inequality follows from the existence of a positive measure μ on E such that $\mu(\Delta(z, r)) \leq r$ for all $z \in \mathbf{C}$, $r>0$ and $\mu(E) \geq N \cdot \mathbf{M}_1(E)$, where N is a constant.)

Moreover, by Lemma 2.4, p. 149 in Landkof [12]

$$C_\alpha(\{\psi \geq \tfrac{1}{2}\}) \leq 4C_\alpha(F). \tag{4.5}$$

So, by combining (4.1)–(4.5) we conclude that

$$\gamma(E) = f'(\infty) \leq |h'(\infty)| + |g'(\infty)| \leq N_3(\alpha)\left[\gamma(E \setminus F) + C_\alpha(F)^{1/\alpha}\right],$$

and the proof is complete. Theorem 2 is an improvement on Theorem 6.1 of Davie [3]. One application of Theorem 2 is the following:

A subset G of the extended complex plane S^2 is called *γ-negligible* if for some constant $M>0$ whenever f is a bounded Borel function on S^2, analytic on some open set V, we can find bounded Borel functions on S^2 analytic on an open set containing $V \cup G$ such that $|f_n| \leq M|f|$ on S^2 and $f_n \to f$ pointwise on V. In other words, the γ-negligible sets are the negligible sets in connection with bounded pointwise approximation by analytic functions. A bounded, plane set G is γ-negligible if and only if

$$\gamma(T \cup G) \leq M\gamma(T), \tag{4.6}$$

for some constant $M>0$ and all plane sets T (see Davie [3]). Using Theorem 2 we get:

THEOREM 3. *Let K be a compact set of Hausdorff dimension less than* 1. *Then K is γ-negligible.*

Proof. Let L be a compact subset of T. Then by Theorem 2

$$\gamma(L \cup K) \leq A(\alpha)\left[\gamma(L \setminus K) + C_\alpha(K)^{1/\alpha}\right].$$

By assumption there exists $\alpha<1$ such that $\Lambda_\alpha(K)=0$, and this implies that $C_\alpha(K)=0$ (see Theorem 1, p. 28 in Carleson [2]), so the result follows from (4.6).

5. Differentiation in terms of approximating sequences (II)

We now set out to prove a converse of Theorem 1 in section 3. To get a more general result we will use the following theorem of Calderon [1]:

There exists $\alpha>0$ such that if φ is a real function on $\mathbf{R}$ with $|\varphi'|\geqslant\alpha$ and $z(t)=t+i\varphi(t)$, then, writing

$$Qf(t)=\sup_{\varepsilon>0}\left|\int_{|s-t|\geqslant\varepsilon}\frac{f(s)}{z(s)-z(t)}\,ds\right|,$$

we have

$$|\{t;\ Qf(t)>\lambda\}|\leqslant\frac{N_1\|f\|_1}{\lambda},\quad\text{for all } f\in L^1(\mathbf{R}),\ \lambda\in\mathbf{R}.$$

Here $|\ \ |$ denotes 1-dimensional Lebesgue measure and, as before, N_1, N_2, ... denote constants.

We require the following corollary:

COROLLARY 1. *Let φ, z be as above. Let $a>0$ and write*

$$Tf(t)=\int\frac{f(s)\,ds}{ia+z(s)-z(t)}.$$

Then

$$|\{t;|Tf(t)|>\lambda\}|\leqslant\frac{N_2\|f\|_1}{\lambda}\quad\text{for all } f\in L^1(\mathbf{R}),\ \lambda>0.$$

Proof.

$$|Tf(t)|\leqslant\left|\int_{|s-t|\leqslant a}\frac{f(s)\,ds}{ia+z(s)-z(t)}\right|$$
$$+a\left|\int_{|s-t|\geqslant a}\frac{f(s)\,ds}{\{ia+z(s)-z(t)\}\{z(s)-z(t)\}}\right|+\left|\int_{|s-t|\geqslant a}\frac{f(s)\,ds}{z(s)-z(t)}\right|$$
$$\leqslant\frac{N_3}{a}\int_{|s-t|\leqslant a}|f(s)|\,ds+N_4\,a\int_{|s-t|\geqslant a}\frac{|f(s)|}{(s-t)^2}\,ds+\left|\int_{|s-t|\geqslant a}\frac{f(s)\,ds}{z(s)-z(t)}\right|.$$

The first two terms have L^1-norms bounded by $N_5\|f\|_1$, so the result follows from Calderon's theorem.

COROLLARY 2. *Let φ, z be as above, let $a>0$ and let ϱ be a (complex) measure on* $\mathbf{R}$. *Then*

$$\left|\left\{t;\ \left|\int \frac{d\varrho(s)}{ia+z(s)-z(t)}\right| > \lambda\right\}\right| \leqslant \frac{N_2\|\varrho\|}{\lambda};\quad \lambda>0.$$

Proof. This follows from Corollary 1 by approximating ϱ by absolutely continuous measures.

LEMMA 2. *Let φ, z be as above. Then there is a constant N_6 such that for any compact subset E of* $\mathbf{R}$ *with $|E|>0$, there is a positive measure μ on $z(E)$ with*

$$\|\mu\| \geqslant N_6|E| \quad\text{and}\quad |\hat{\mu}(z)| \leqslant 1,\quad z\in \mathbf{C}\setminus z(E).$$

Proof. It suffices to show that, given $a>0$, there is a positive measure μ on $z(E)$ with

$$\|\mu\| \geqslant N_6|E| \quad\text{and}\quad |\hat{\mu}(z)| \leqslant 1 \quad \text{for } z=t+iy,\ |y-\varphi(t)|>a. \tag{5.1}$$

Then we can take a sequence $a_n \to 0$ and a weak-star cluster point of the corresponding μ's. So suppose there is an $a>0$ such that (5.1) does not hold for any positive measure μ on $z(E)$. Then there is no positive measure σ on E with

$$\|\sigma\| \geqslant N_6|E| \quad\text{and}\quad \left|\int \frac{d\sigma(s)}{\varepsilon ia+z(s)-z(t)}\right| \leqslant 1 \quad \text{for } t\in\mathbf{R},\ \varepsilon=\pm 1.$$

Let C_0 ($\mathbf{R}$) be the continuous functions on $\mathbf{R}$ vanishing at ∞. Consider the space $S=C_0(\mathbf{R})+C_0(\mathbf{R})$, with norm $\|(x, y)\|=\max(\|x\|, \|y\|)$. Apply the separation theorem for convex sets to the unit ball B and the set $K=\{(f_\sigma^{+1}, f_\sigma^{-1});\ \sigma$ positive measure on E, $\|\sigma\|\geqslant N_6|E|\}$, where

$$f_\sigma^\varepsilon(t) = \int \frac{d\sigma(s)}{\varepsilon ia+z(s)-z(t)};\quad \varepsilon=\pm 1.$$

Then we get complex measures ϱ, τ on $\mathbf{R}$ such that $\|\varrho\|+\|\tau\|\leqslant 1$ and

$$\operatorname{Re}\left[\int \frac{d\varrho(s)}{ia+z(s)-z(t)} + \int \frac{d\tau(s)}{-ia+z(s)-z(t)}\right] > (N_6|E|)^{-1} \quad \text{for } t\in E.$$

But this contradicts Corollary 2 if $N_6=1/3A_2$.

We will also need the following result, which is an adaptation of Theorem 9.9 in Fuglede [7]:

LEMMA 3. *Let K be compact and let $z_0 \in K'$. Then we can find a fine neighbourhood $L \subseteq K$ of z_0 and $M>0$ such that whenever f is a real harmonic function in a neighbourhood of K with $|f| \leq 1$ there, we can find a measure μ supported on a compact subset of* $\mathbf{C} \setminus L$, *with $\|\mu\| \leq M$ and*

$$P_\mu = f \quad \text{on } L.$$

Proof. Let U, V be open sets with $K \subseteq V$, $\bar{V} \subseteq U$. We may assume $U \subseteq D = \{|z| < \frac{1}{4}\}$. Let φ be a C^2 function with compact support in U, with $\varphi = 1$ on V. Choose a bounded C^2 superharmonic function p on U with $p(z) > 1$ for $z \in U$ and

$$\int p \, d\nu < p(z_0),$$

where ν is the Keldysh measure for z_0 w.r.t. K. (This is possible since $z_0 \in K'$.)

Let U_n be a decreasing sequence of open sets with smooth boundary, such that

$$\bar{U}_n \subseteq V \quad \text{and} \quad \bigcap_{n=1}^{\infty} U_n = K.$$

Define p_n on U by

$$p_n = \begin{cases} p & \text{on } U \setminus U_n \\ \widetilde{p|\partial U_n} & \text{on } U_n \end{cases}$$

where $\widetilde{p|\partial U_n}$ is the harmonic extension of $p|\partial U_n$ to U_n. Then p_n is continuous, superharmonic and $p_n \uparrow q$, where q is superharmonic on U. Moreover,

$$q(z_0) = \int p \, d\nu < p(z_0) \quad \text{and} \quad q(z) \leq p(z) \quad \text{for } z \in U.$$

Choose $\lambda > (p(z_0) - q(z_0))^{-1}$ and let

$$L = \{z \in U; \lambda(p(z) - q(z)) > 1\}.$$

Then $L \subseteq K$ and L is a fine neighbourhood of z_0.

Also let

$$L_n = \{z \in U; \lambda(p(z) - p_n(z)) > 1\}.$$

Then L_n is open and $L \subseteq L_n \subseteq U_n$.

Now let f be a real harmonic function on an open set $W \supseteq K$, with $|f| \leq 1$. Choose n so that $\bar{U}_n \subseteq W$. Define u on U by

$$u = \begin{cases} \min\{\lambda p, f+(\lambda+1)p_n\} & \text{on } W \\ \lambda p \quad \text{on } U \setminus \bar{U}_n \end{cases}$$

The two definitions agree on $W \setminus \bar{U}_n$, since $p=p_n$ there. The function u is superharmonic in U and

$$f = u-(\lambda+1)p_n \quad \text{in } L_n.$$

Since u and p_n are harmonic in L_n, so is u.

Since u and p_n are superharmonic, $\nabla^2 u$ and $\nabla^2 p_n$ are positive measures on U. Let $\sigma=\nabla^2(\varphi u)$, $\tau=\nabla^2(\varphi p_n)$ and $\mu=\sigma-(\lambda+1)p_n$. Then σ and τ have no mass on L_n, $P_\sigma=\varphi u$ and $P_\tau=\varphi p_n$. So $P_\mu=f$.

Now $\sigma|V$ is positive, so $P_{\sigma|V}(z) \geq \log 2 \cdot \|\sigma|V\|$, $z \in U$ (using $U \subseteq D$). Outside a compact subset of V we have $\nabla^2(\varphi u)=\lambda\nabla^2(\varphi p)$. Hence $\|\sigma|U\setminus V\| \leq N_1$ and $|P_{\sigma|U\setminus V}(z)| \leq N_2$, $z \in U$ (N_1, N_2, ... denote constants independent of f). So $\|\sigma|V\| \leq (\log 2)^{-1}$ [sup $|\varphi u|+N_2]=N_3$ and $\|\sigma\| \leq N_1+N_3$. Similarly $\|\tau\| \leq N_4$. That completes the proof.

We are now ready for a partial converse of Theorem 1. A weaker form of this result (with straight line segments instead of rectifiable arcs) could be proved without the use of Calderon's theorem, but using the weak-type (1,1) estimate for the Hilbert transform instead.

THEOREM 4. *Let $K \subset \mathbf{C}$ be compact and $J \subset \mathbf{C}$ a rectifiable arc. Let f_n be real functions harmonic in a neighbourhood of K such that $f_n \to f$ uniformly on K, with $|f_n(z)-f(z)|<2^{-n}$ for $z \in K$. Then ∇f_n converges* a.e. *on $J \cap K'$ with respect to arc length on J to a limit depending on f but not on the sequence $\{f_n\}$.*

Proof. Suppose ∇f_n does not converge a.e. on $J \cap K'$. Let F be a compact subset of $J \cap K'$ of positive length such that $\nabla f_n(z)$ does not converge for any $z \in F$. Parametrize J by $z=\theta(s)$, $s=$arc length, $0 \leq s \leq S$. Then θ' exists a.e. and $|\theta'|=1$ a.e. By Egoroff's theorem there is a compact set $E_0 \subseteq [0, S]$ such that $|E_0|>0$, $\varphi(E_0) \subseteq F$ and

$$\frac{\theta(s+t)-\theta(s)}{t} \to \theta'(s) \quad \text{as } t \to 0, \text{ uniformly for } s \in E_0.$$

Then one can readily find, after rotating the coordinates if necessary, a function $z(t)$ as in Calderon's theorem and a compact set $E \subseteq \mathbf{R}$ such that $|E|>0$ and $z(E) \subseteq F_0$.

Let t_0 be a point of density of E w.r.t. Lebesgue measure on $\mathbf{R}$. Since $z(t_0) \in K'$, there exists, by Lemma 3, a fine neighbourhood $L \subseteq K$ of $z(t_0)$ and a constant M such that for every function g harmonic in a neighbourhood of K we can find a measure μ with support in $\mathbf{C} \setminus L$ such that

$$g(z) = P_\mu(z) \quad \text{in } L, \quad \|\mu\| \leq M \cdot \sup_K |g|$$

Since L is a fine neighbourhood of $z(t_0)$ and t_0 is a point of density for E, there exists a compact set $Q \subseteq E$ with $|Q|>0$ and $z(Q) \subseteq L$. For each n let σ_n be a measure with support in $\mathbf{C} \setminus L$ such that

$$f_n - f_{n+1} = P_{\sigma_n}; \quad \|\sigma_n\| \leq M \cdot 2^{1-n}.$$

Write

$$h_n(z) = \frac{\partial}{\partial \bar{z}} (f_n(z) - f_{n+1}(z)) = \bar{\hat{\sigma}}_n(z), \tag{5.2}$$

and let

$$Q_n = \{t \in Q; |h_n(z(t))| > 2^{-n/2}\} \tag{5.3}$$

We can find a compact set $R_n \subseteq Q_n$ and a $w_n \in \mathbf{C}$ such that

$$|w_n| = 1, \; |R_n| \geq \frac{1}{3} |Q_n| \quad \text{and} \quad \operatorname{Re}[w h_n(z(t))] > 2^{-(n/2)-1} \quad \text{on } R_n. \tag{5.4}$$

By Lemma 2 there is a positive measure μ_n on $z(R_n)$ with

$$\|\mu\| \geq \frac{N_6}{3} |Q_n| \quad \text{and} \quad |\hat{\mu}_n(z)| \leq 1, \quad z \in \mathbf{C} \setminus z(R_n). \tag{5.5}$$

Combining (5.2) and (5.5) we get

$$\frac{N_6}{3} \cdot 2^{-n/2-1} |Q_n| \leq \left| \int h_n \, d\mu_n \right| = \left| \int \int \frac{d\mu_n(z) \, d\sigma_n(\zeta)}{\zeta - z} \right| \leq \|\sigma_n\| \leq M \cdot 2^{1-n}.$$

So

$$|Q_n| \leq N_7 \cdot 2^{-n/2}.$$

Therefore

$$\left|\bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty} Q_n\right)\right| = 0,$$

so for a.a. $t\in Q$ there exists m with $t\notin \bigcup_{n=m}^{\infty} Q_n$.

For such a t we have, for $r\geq m$ and $k\geq 0$,

$$\left|\frac{\partial f_{r+k}}{\partial \bar{z}}(z(t)) - \frac{\partial f_r}{\partial \bar{z}}(z(t))\right| \leq \sum_{j=r}^{r+k-1} |h_j(z(t))| \leq \sum_{j=r}^{r+k-1} 2^{-j/2} \leq 2^{2-r/2},$$

so that $\{\partial f_n/\partial \bar{z}\}$ converges a.e. on Q.

This contradiction shows that $\{\nabla f_n\}$ converges a.e. on $J\cap K'$.

If $f=0$ then the same argument applied to f_n instead of $f_n - f_{n+1}$ shows that $\nabla f_n \to 0$ a.e. on $J\cap K'$. So the limit is independent of the choice of f_n (up to sets of zero length). That completes the proof.

Basically the same proof also gives the following:

THEOREM 5. *Let $K\subset \mathbf{C}$ be compact and let f_n be real functions, harmonic in a neighbourhood of K, such that $f_n\to f$ uniformly on K, with $\sup_K |f_n - f| < 2^{-n}$. Then ∇f_n converges C_1–everywhere on K' to a limit depending on f but not on the sequence $\{f_n\}$.*

(C_1 is the Newtonian capacity, defined in the introduction.)

Proof. Suppose there exists a Borel set G with $C_1(G)>0$ and such that ∇f_n does not converge for any $z\in G$. By Doob's quasi Lindelöf principle for the fine topology (see Doob [6]) and the subadditivity of C_1 (see Carleson [2], p. 24) we conclude that there exists a point $z_0\in G$ such that

$$C_1(G\cap U) > 0$$

for all finely open sets U with $z_0\in U$.

As in the proof of Theorem 4 we find a fine neighbourhood L of z_0, $L\subset K$, a constant M and measures σ_n with support in $\mathbf{C}\setminus L$ with

$$f_n - f_{n+1} = P_{\sigma_n} \quad \text{on } L, \quad \|\sigma_n\| \leq M\cdot 2^{1-n}.$$

So we put

$$h_n(z) = \frac{\partial}{\partial \bar{z}}(f_n(z) - f_{n+1}(z)) = \bar{\hat{\sigma}}_n(z),$$

and

$$Q_n = \{z \in G \cap L;\ |h_n(z)| > 2^{-n/2}\}.$$

Then we can find a compact set $R_n \subseteq Q_n$ and $w_n \in \mathbf{C}$ with $|w_n| = 1$, such that

$$C_1(R_n) \geq \frac{1}{4} C_1(Q_n) \quad \text{and} \quad \operatorname{Re}[wh_n(z)] > 2^{-n/2-1} \quad \text{on } R_n.$$

Choose a positive measure μ_n on R_n such that

$$\mu_n(R_n) \geq \frac{1}{2} C_1(R_n) \quad \text{and} \quad \int \frac{d\mu_n(\zeta)}{|\zeta - z|} \leq 1 \quad \text{for } z \notin R_n.$$

Then

$$2^{-n/2-4} \cdot C_1(Q_n) \leq \left| \int h_n \, d\mu_n \right| = \left| \int \int \frac{d\mu_n(z)}{\zeta - z} \, d\sigma_n(\zeta) \right| \leq \|\sigma_n\|.$$

Therefore

$$C_1(Q_n) \leq 32M \cdot 2^{n/2},$$

and we conclude that $\{\partial f_n / \partial \bar{z}\}$ converge C_1–everywhere on $G \cap L$.

This contradiction proves the theorem.

If we combine Theorem 1 and Theorem 4 we get the following:

COROLLARY 3. *Let J be a rectifiable arc and E a Borel subset of J of positive length. Then $\gamma(E) > 0$.*

Corollary 3 was conjectured by Denjoy and recently proved by Marshall [15], also using Calderon's theorem.

In the light of Theorem 1 and Theorem 4 it is natural to conjecture that, in the circumstances of Theorem 4, $\{\nabla f_n\}$ converges on K', except on a set of zero analytic capacity. One could prove this by the method of proof of Theorem 4 provided one could show that

(a) analytic capacity is subadditive, i.e.

$$\gamma(S \cup T) \leq \gamma(S) + \gamma(T),$$

for all Borel sets S, T. (See Davie [3].)

(b) Any compact set E with $\gamma(E) > 0$ admits a non-zero positive measure μ with

$$|\hat{\mu}(z)| \leq 1 \quad \text{for all } z \in \mathbf{C} \setminus E.$$

(See Zalcman [18], p. 20.)

(The proof actually requires $\|\mu\| \geq A\gamma(E)$ where $A>0$ is independent of E, but one can in fact show that (b) implies this.) The validity of (a) and (b) is open.

6. Fine differentiability

Let f be a real function defined on a finely open, plane set V. We say that f is *finely differentiable* at a point $z_0 \in V$ if there exists a vector $\nabla f(z_0) \in \mathbf{R}^2$ (called the fine gradient of f at z_0) such that

$$\frac{|f(z)-f(z_0)-\langle z-z_0, \nabla f(z_0)\rangle|}{|z-z_0|} \to 0$$

when z converge finely to z_0. (If $z=a+ib$, $w=u+iv$ then $\langle z, w\rangle = au+bv$.)

If f is finely harmonic in V, to what extent is f finely differentiable in V? As mentioned in the introduction, Fuglede [8] has proved, using a result of Mizuta [16], that f is finely differentiable outside a C_h^*-null set where C_h^* is the outer capacity associated to the kernel $h(z)=|z|^{-1}\log 1/|z|$. For completeness we first give a proof of Fuglede's result (Theorem 6), and then we proceed to prove that this result is the best possible (Theorem 7).

THEOREM 6. (Fuglede). *Let f be finely harmonic on a finely open set V. Then f is finely differentiable at each point, except on a C_h^*-null set.*

Proof. Suppose not. We may assume $V \subseteq D$. Then there is a set $E \subseteq V$ with $C_h^*(E)>0$ such that f is not finely differentiable at any point of E. By Doob's quasi-Lindelöf principle ([6]) we can find $z_0 \in E$ such that $C_h^*(E \cap W)>0$ for every finely open set W containing z_0. By Theorem 9.9 in Fuglede [7], there is a finely open set U containing z_0, with $U \subseteq V$, and a real measure μ such that

$$f = P_\mu \quad \text{on } U.$$

Since $C_h^*(U \cap E)>0$ we can find $z \in U \cap E$ such that $\int h(\zeta-z)\,d|\mu|(\zeta)<\infty$. We shall obtain a contradiction by showing that P_μ is differentiable at z:

Fix $\varepsilon>0$ and write $\mu=\sigma+\tau$, where $\int h(\zeta-z)\,d|\sigma|(\zeta)<\varepsilon$ and $z \notin \operatorname{supp}\tau$.

Let $A_n=\{\zeta; 2^{-n-1}<|\zeta-z|\leq 2^{-n}\}$. Then

$$\sum_{n=1}^{\infty} n2^n |\sigma|(A_n) < N_1 \varepsilon, \quad \text{where } N_1 \text{ is a constant.}$$

Let $\sigma_n = \sigma|A_n$ and put

$$\Omega_n = \left\{\zeta; 2^{-n-2} < |\zeta - z| < 2^{-n+1} \quad \text{and} \quad P_{|\sigma_n|}(\zeta) \geq \varepsilon 2^{-n}\right\}.$$

Then

$$C_W(\Omega_n) \leq \frac{2^n |\sigma|(A_n)}{\varepsilon},$$

where C_W is the Wiener capacity defined in the introduction.

Therefore

$$\sum n C_W(\Omega_n) < \infty,$$

so if we put $L = V \setminus \bigcup_n \Omega_n$, L is a fine neighbourhood of z_0 by Wiener's criterion.

Now let $\zeta \in L$. Then $\zeta \in A_n$ for some n.

Writing $B_n = A_{n-1} \cup A_n \cup A_{n+1}$ we have

$$\left|\frac{P_\sigma(\zeta) - P_\sigma(z)}{\zeta - z}\right| \leq \frac{1}{|\zeta - z|} \int_{\mathbf{C} \setminus B_n} \left|\log\frac{1}{|w-\zeta|} - \log\frac{1}{|w-z|}\right| d|\sigma|(w)$$
$$+ \frac{1}{|\zeta - z|} \int_{B_n} \left(\log\frac{1}{|w-\zeta|} + \log\frac{1}{|w-z|}\right) d|\sigma|(w)$$

The first term is bounded by

$$N_2 \int \frac{d|\sigma|(w)}{|w-z|} \leq N_3 \varepsilon$$

and the second by

$$\left[P_{|\sigma_{n-1}| + |\sigma_n| + |\sigma_{n+1}|}(\zeta) + (n+1)\,|\sigma|(B_n)\right] 2^{n+1} \leq N_4 \varepsilon,$$

since $\zeta \notin \Omega_{n-1} \cup \Omega_n \cup \Omega_{n+1}$.

So

$$\left|\frac{P_\sigma(\zeta) - P_\sigma(z)}{\zeta - z}\right| \leq N_5 \varepsilon,$$

and since P_τ is differentiable at z we conclude that P_μ is finely differentiable at z.

Next we turn to the converse of Theorem 6. We need the following lemma:

LEMMA 4. *Let $E \subseteq D$ be compact such that $C_h(E)=0$. Then there is a positive measure μ on D consisting of a countable sum of point masses such that*

$$\int h(\zeta - z)\, d\mu(\zeta) = \infty \quad \text{and} \quad \int \frac{d\mu(\zeta)}{|\zeta - z|} \leq 1, \quad \text{for all } z \in E.$$

Proof. It suffices to construct, for a given $M>0$, a positive measure σ on a compact subset of $D \setminus E$ such that

$$\int h(\zeta - z)\, d\sigma(\zeta) \geq M \quad \text{and} \quad \int \frac{d\mu(\zeta)}{|\zeta - z|} \leq 1, \quad \text{for } z \in E.$$

For then we can modify σ so that it is a finite sum of point masses.

Then we can take a sequence $M_n = 3^n$ with corresponding σ_n and define $\mu = \Sigma 2^{-n} \sigma_n$.

So let M be given. Choose $\varepsilon>0$. Let F be a compact neighbourhood of E such that $F \subseteq D$, $C_h(F)<\varepsilon$ and F has smooth boundary. By Theorem 3 of Carleson [2], p. 17, there is a positive measure ν on F with

$$\|\nu\| = C_h(F) \quad \text{and} \quad \int h(\zeta - z)\, d\nu(\zeta) = 1, \quad z \in F. \tag{6.1}$$

Choose $\delta < \operatorname{dist}(E, \mathbf{C} \setminus F)$ and let

$$\psi(z) = \frac{1}{\pi \delta^2} \cdot \int_{|\zeta - z| \leq \delta} d\nu(\zeta). \tag{6.2}$$

Then

$$\psi(z) \leq \frac{\varepsilon}{\pi \delta^2} \quad \text{for } z \in \mathbf{C}, \tag{6.3}$$

$$\int \psi(z)\, dm(z) = \frac{1}{\pi \delta^2} \cdot \int_F \left(\int_{|\zeta - z| \leq \delta} dm(z) \right) d\nu(\zeta) = C_h(F) \tag{6.4}$$

and for $z \in E$ we have, by (6.1),

$$\int h(w - z)\, \psi(w)\, dm(w) = 1.$$

(To see this, note that from (6.1) we have, for $z \in E$

$$\pi\delta^2 = \int_{\Delta(z,\delta)} \left(\int_{\mathbf{C}} h(\zeta - w)\, d\nu(\zeta) \right) dm(w) = \int_{\mathbf{C}} \left(\int_{|w-z| \leq \delta} h(\zeta - w)\, dm(w) \right) d\nu(\zeta).$$

Substituting $w' = \zeta + z - w$ in the inner integral we get

$$\pi\delta^2 = \int_{\mathbf{C}} \left(\int_{|\zeta - w'| \leq \delta} h(w' - z)\, dm(w') \right) d\nu(\zeta) = \int_{\mathbf{C}} \left(\int_{|\zeta - w'| \leq \delta} d\nu(\zeta) \right) h(w' - z)\, dm(w').$$

Hence

$$\int \psi(w')\, h(w' - z)\, dm(w') = 1 \quad \text{for } z \in E.)$$

Since $C_h(F) < \varepsilon$, it follows that if we choose ε small enough ($\varepsilon < 1/30M^2e^M$ will do), we have

$$\int \frac{\psi(\zeta)\, dm(\zeta)}{|\zeta - z|} \leq \frac{1}{2M} \quad \text{for all } z \in E. \tag{6.5}$$

Let U be open, $E \subseteq U \subseteq D$, with $M(U)$ so small that

$$\int_U h(\zeta - z)\, dm(\zeta) \leq \frac{\pi\delta^2}{2\varepsilon}, \quad \text{for } z \in \mathbf{C}.$$

Then

$$\int_U h(\zeta - z)\, \psi(\zeta)\, dm(\zeta) \leq \frac{1}{2}, \quad \text{using (6.3).}$$

Now put $\sigma = 2M\psi m|_{\mathbf{C} \setminus U}$. Then σ has the desired properties.

For the proof of our last result, we will need the following lemma:

LEMMA 5. *Let μ be a positive measure on D and let $a > 0$. Put*

$$A = \{z \in D; P_\mu(z) > a\}.$$

Then

(i) $C_W(A) \leq a^{-1} \|\mu\|$.

If in addition μ has no mass outside A, then

(ii) $C_W(A) \leq a^{-1}\|\mu\|$.

Proof. Let σ be any positive measure on A, $P_\sigma \leq 1$. Then

$$\|\sigma\| \leq a^{-1} \int P_\mu \, d\sigma = a^{-1} \int P_\sigma \, d\mu \leq a^{-1}\|\mu\|,$$

which proves (i).

If μ has no mass outside A, we obtain, since $P_{a^{-1}\mu} \leq 1$ outside A,

$$\begin{aligned} C_W(A) &= \sup\{\sigma(A); \sigma \text{ positive measure on } A,\ P_\sigma \leq 1 \text{ outside } A\} \\ &\geq a^{-1}\mu(A) = a^{-1}\|\mu\|, \end{aligned}$$

which proves (ii).

THEOREM 7. *Let E be a compact set with $C_h(E)=0$. Then we can find a finely open set $V \supseteq E$ and a finely harmonic function f on V such that f is not finely differentiable at any point of E.*

Proof. Assume $E \subseteq D$. Let μ be as given by Lemma 4. Then $P_\mu < 1$ on E, because $\log t < t$ for all $t > 0$.

Let

$$V = \{z \in D; P_\mu(z) < 1\}$$

and put

$$f = P_\mu.$$

Then V is finely open, $E \subseteq V$ and f is finely harmonic on V.

Fix $z \in E$. We show that f is not finely differentiable at z; in fact we show that if L is a fine neighbourhood of z with $L \subseteq V$ then $(f(\zeta)-f(z))/(\zeta-z)$ cannot be bounded for $\zeta \in L$.

Suppose on the contrary that

$$\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right| \leq M \quad \text{for } \zeta \in L. \tag{6.6}$$

Let A_n, B_n be as in the proof of Theorem 6. If $\zeta \in A_n$ we have

$$\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right| \geqslant \frac{1}{|\zeta-z|}\int_{B_n}\log\frac{1}{|w-\zeta|}\,d\mu(w)-\frac{1}{|\zeta-z|}\int_{B_n}\log\frac{1}{|w-z|}\,d\mu(w)$$
$$-\frac{1}{|\zeta-z|}\int_{\mathbf{C}\setminus B_n}\left|\log\frac{1}{|w-\zeta|}-\log\frac{1}{|w-z|}\right|d\mu(w) \tag{6.7}$$
$$\geqslant 2^n\int_{A_n}\log\frac{1}{|w-\zeta|}\,d\mu(w)-N_1\,n2^n\mu(B_n)-N_2,$$

using the fact that $\int d\mu(w)/|w-z|\leqslant 1$.

Let

$$\Omega_n=\left\{\zeta;P_{\mu|A_n}(\zeta)>2^{-n}(M+N_1\,n2^n\mu(B_n)+N_2)\right\}.$$

Then $\Omega_n\subseteq B_n$ if we choose $N_1\geqslant 3\log 2$. Since μ is a sum of point masses, $\mu|A_n$ has no mass outside Ω_n and therefore

$$C_W(\Omega_n)=\frac{2^n\mu(A_n)}{M+N_1\,n2^n\mu(B_n)+N_2},$$

using Lemma 5.

Since $\Sigma n2^n\mu(A_n)=\infty$ and $\Sigma 2^n\mu(A_n)<\infty$ it follows that $\Sigma nC_W(\Omega_n)=\infty$. This can be seen as follows: Let $a_n=n2^n\mu(A_n)$, $C_n=nC_W(\Omega_n)$, $n\geqslant 3$, $b=N_1^{-1}(M+N_2)$.

Then $\Sigma a_n=\infty$, $\Sigma n^{-1}a_n<\infty$. Suppose $\Sigma C_n<\infty$. Then there exists n_0 such that

$$n>n_0 \;\Rightarrow\; C_n<(b+9)^{-1}N_1^{-1}.$$

So for $n>n_0$,

$$(b+9)\,a_n\leqslant 3a_{n-1}+a_n+a_{n+1}+b.$$

Hence if $n>n_0$ and $a_n\geqslant 1$,

$$3a_{n-1}+a_{n+1}\geqslant 8a_n.$$

So either

$$a_{n+1}\geqslant 2a_n \quad\text{or}\quad a_{n-1}\geqslant 2a_n.$$

In the former case repeated application gives

$$a_{n+k} \geq 2^k a_n \quad \text{for all } k \geq 1,$$

which contradicts $\Sigma n^{-1} a_n < \infty$.

In the latter case we get $a_{n-k} \geq 2^k a_n$ as long as $n-k \geq n_0$. So

$$a_n \leq 2^{n_0-n} a_{n_0} \quad \text{if } n > n_0 \text{ and } a_n \geq 1.$$

This implies that $a_n < 1$ for n large enough, and therefore

$$C_n \geq \frac{a_n}{(5+b)N_1}$$

for n large enough, which contradicts $\Sigma C_n < \infty$. But by (6.6) and (6.7) we see that $A_n \cap L \cap \Omega_n = \emptyset$, hence $A_n \cap \Omega_n \subseteq A_n \setminus L$. Since $\Sigma n C_W(B_n \cap \Omega_n) = \infty$, we have $\Sigma n C_W(A_n \cap \Omega_n) = \infty$ and therefore $\Sigma n C_W(A_n \setminus L) = \infty$, contradicting the assumption that L is a fine neighbourhood of z.

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