Upper semi-continuous set-valued functions

by

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§ 1. Introduction

A map F from a metric space X to the power set of a metric space Y is said to be upper semi-continuous, if the set $\{x: F(x) \cap H \neq \varnothing\}$ is closed in X, whenever, H is a closed set in Y. Our first aim in this paper is to obtain information about the possible structure of such maps. One special case of an upper semi-continuous map is provided by the inverse image function $F = f^{-1}$, when f is a closed continuous map from Y to X, that is, when f is continuous and maps closed sets in Y to closed sets in X. In 1947 Vaĭnšteĭn [15] announced and in 1952 [16] gave the proof that, in this special case, each set $F(x) = f^{-1}(x)$, with x in X, has a compact boundary. In 1948, Choquet [17] considered upper semi-continuous set-valued functions, under the name strongly upper semi-continuous functions. Choquet expressed the opinion that the condition of strong upper upper semi-continuity is very restrictive; a view that we shall amply justify. He gave, without proof, the result that, if F is an upper semi-continuous map of a metric space X to a metric space Y, then, for each x_0 in X, it is possible to choose a compact set K contained in $F(x_0)$ with the property that for each neighbourhood G of K in Y, there is a neighbourhood U of x_0 in X with

$$F(U) \subset G \cup F(x_0).$$

Had Choquet given the proof of his result, it seems sure that the connection between this result and Vaĭnšteĭn's result would have been apparent. As it was, the connection remained undiscovered for many years.

Following up Vaĭnšteĭn's work, Taĭmanov [14] and Lašnev [7], show that, in the special case of the inverse image function F of a closed continuous function f, the set of x, for which $F(x)$ has a non-empty interior, is a sigma-discrete set in X. More recently, in 1977, in a manuscript [18], that has remained unpublished, S. Dolecki rediscovered Choquet's result, in a slightly different form. He gives some applications, writing with S. Rolewicz in [19] and extensions with A. Lechicki in [20].

In this paper we take the theory rather further. Recall that a family of sets in a

space is said to be discrete, if each point of the space has a neighbourhood that meets at most one set of the family, and that a family of sets is said to be σ-discrete if it is a countable union of discrete families. Further, a family $\{Q_\alpha\}_{\alpha \in A}$ is said to be discretely σ-decomposable, if, for each α in A, we have

$$Q_\alpha = \bigcup_{n=1}^{\infty} Q_\alpha^{(n)},$$

and each family $\{Q_\alpha^{(n)}\}_{\alpha \in A}$, $n=1, 2, \ldots$, is discrete.

THEOREM 1 (decomposition). *Let F be an upper semi-continuous map of a metric space X to the power set of a metric space Y. Let*

$$T = \bigcup_{x \in X} \{x\} \times F(x)$$

be the graph of F in $X \times Y$. For each x in X, write

$$E(x) = [(X \setminus \{x\}) \times (Y \setminus F(x))] \cap T,$$

and

$$K(x) = \mathrm{proj}_Y([\mathrm{cl}\, E(x)] \cap [\{X\} \times F(x)]),$$

where 'cl' denotes closure in $X \times Y$ and proj_Y the projection onto Y.

(a) *For each x in X, the set $K(x)$ is compact. The set*

$$K = \bigcup_{x \in X} \{x\} \times K(x)$$

is a $\mathscr{G}_\delta$-set in $X \times Y$. The set $\{x: K(x) \cap H \neq \varnothing\}$ is a $\mathscr{G}_\delta$-set in X, whenever H is closed in Y.

(b) *The sets of constancy of the restriction of F to*

$$\Xi = \{x: K(x) = \varnothing\},$$

i.e., the subsets of Ξ on which F takes a particular set as its value, form a disjoint family, that is discretely σ-decomposable in the completion X^* of X, the sets of the family being $\mathscr{F}_\sigma$-sets in X with union Ξ.

(c) *There is a σ-discrete family $\{P_\beta \times S_\beta\}_{\beta \in B}$ of rectangles that are relatively closed subsets of T, with each set P_β, $\beta \in B$, closed in X, and with*

$$\bigcup_{\beta \in B} P_\beta \times S_\beta = T \setminus K.$$

Here the set $K(x)$ is the set that must have been used by Choquet in his proof; it is the set reintroduced by Dolecki and called by him the active frontier of F.

A map f from a metric space X to a metric space Y is said to be a selector for a map F from X to the non-empty sets of Y, if $f(x)$ is in $F(x)$ for all x in X. Such a selector f is said to be of the first Borel class if $f^{-1}(H)$ is a $\mathscr{G}_\delta$-set in X for each closed set H in Y, and is said to be of the second Borel class if $f^{-1}(H)$ is an $\mathscr{F}_{\sigma\delta}$-set (that is, the countable intersection of countable unions of closed sets) in X for each closed set H in Y. Engelking [1], Theorem 1, has proved that if X and Y are metric spaces and F is an upper semi-continuous map on X, each of whose values is a non-empty complete and separable subset of Y, then F has a selector of the first Borel class. Our structure theorem enables us to prove a selection theorem for upper semi-continuous maps between metric spaces of extraordinary generality and precision.

THEOREM 2 (selection). *Let F be an upper semi-continuous map from a metric space X to the non-empty subsets of a metric space Y. Then F has a selector f of the second Borel class. Further it is possible to choose f, an $\mathscr{F}_\sigma$-set X_1 in X and its complementary $\mathscr{G}_\delta$-set $X_2 = X \setminus X_1$, so that the restrictions of f to X_1 and to X_2 are of the first Borel class. If F takes only compact values, then F has a selector f of the first Borel class.*

Provided we assume more about the space Y and the map F, we can obtain a nicely parameterized family of selectors filling out the whole space by means of the following representation theorem.

Recall that a function f is said to be closed if it maps closed sets to closed sets; and recall that if m is an infinite cardinal number, then $B(m)$ denotes the Baire space of weight m, that is, the product of a countable sequence of discrete spaces of cardinality m.

THEOREM 3 (representation). *Let m be an infinite cardinal and let Y be a complete metric space of weight m. Let F be an upper semi-continuous map of a metric space X to the non-empty closed subsets of Y. Then there is a map g from the cartesian product of X with the Baire space $B(m)$ to Y with the following properties.*

(a) *For all (x, σ) in $X \times B(m)$ we have $g(x, \sigma) \in F(x)$. For each x in X and each y in $F(x)$, there is a σ in $B(m)$ with $g(x, \sigma) = y$.*

(b) *The family $\{g(x, \cdot)\}_{x \in X}$ is an equicontinuous family of closed uniformly continuous functions.*

(c) *For each σ in $B(m)$, the function $g(\cdot, \sigma)$ is a selector for F and is of the first Borel class.*

Let T be a set in the cartesian product $X \times Y$ of two metric spaces X and Y. If we know that each section

$$(\{x\} \times Y) \cap T,$$

with x in X, is of some fixed Borel class, the information we have is too disorganized to enable us to say anything about the global nature of T. Recent results of Saint-Raymond [12] and of Louveau [8] show that the additional information that T is a Borel set has the remarkable effect of enabling a reorganization of the previously disorganized information, and leads to global information about T. Our next theorem shows that the additional information that T is the graph of an upper semi-continuous function has a similar effect.

The Borel sets of additive class α and of multiplicative class α in a metric space X are defined inductively for $0 \leq \alpha < \omega_1$. The sets of additive class zero are just the open sets of the space, and the sets of multiplicative class zero are just the closed sets of the space. When $1 \leq \alpha < \omega_1$, the sets of additive class α are just the countable unions of sets chosen from the sets of multiplicative class β with $0 \leq \beta < \alpha$; and the sets of multiplicative class α are just the countable intersections of sets chosen from the sets of additive class β with $0 \leq \beta < \alpha$. A set S in X is said to be a Souslin-$\mathscr{F}$ set, if it has a representation of the form

$$S = \bigcup_{\sigma} \bigcap_{n=1}^{\infty} F(\sigma|n),$$

where each set $F(\sigma|n)$ is closed, the union is taken over all σ in the space $\mathbf{N}^{\mathbf{N}}$, where $\mathbf{N} = \{1, 2, \ldots\}$, of all infinite sequences

$$\sigma = \sigma_1, \sigma_2, \sigma_3, \ldots$$

of positive integers, and

$$\sigma|n = \sigma_1, \sigma_2, \ldots, \sigma_n.$$

A set S in X is said to be a co-Souslin-$\mathscr{F}$ set, if its complement $X \setminus S$ is a Souslin-$\mathscr{F}$ set.

THEOREM 4 (graph structure). *Let F be an upper semi-continuous map of a metric space X to the power set of a metric space Y, and let*

$$T = \bigcup_{x \in X} \{x\} \times F(x)$$

be the graph of F.

(a) $F(x)$ *is a Borel set in* Y *of additive class* α, *with* $\alpha \geqslant 2$, *for all* x *in* X, *if, and only if,* T *is a Borel set of additive class* α *in* $X \times Y$.

(b) $F(x)$ *is a Borel set in* Y *of multiplicative class* α, *with* $\alpha \geqslant 2$, *for all* x *in* X, *if, and only if,* T *is a Borel set of multiplicative class* α *in* $X \times Y$.

(c) $F(x)$ *is a Souslin-$\mathscr{F}$ set in* Y *for all* x *in* X, *if, and only if,* T *is a Souslin-$\mathscr{F}$ set in* $X \times Y$.

(d) $F(x)$ *is a co-Souslin-$\mathscr{F}$ set in* Y *for all* $x \in X$, *if, and only if,* T *is a co-Souslin-$\mathscr{F}$ set in* $X \times Y$.

We draw attention to one consequence of the decomposition theorem that we have found useful as a tool in proving the selection theorem. A family $\{X_\alpha\}_{\alpha \in A}$ of sets in a metric space X is said to be an absolutely additive family of closed sets, if

$$\bigcup_{\alpha \in B} X_\alpha$$

is closed in X, for each subset B of A.

THEOREM 5. *Let* $\{X_\alpha\}_{\alpha \in A}$ *be an absolutely additive family of closed sets in a metric space* X. *Define a set-valued map* F *from* X *to the power set of* A *by*

$$F(x) = \{\alpha \in A : x \in X_\alpha\}$$

for each x *in* X. *Then the sets of constancy of* F, *i.e., the subsets of* X *on which* F *takes a particular set as its value, form a disjoint family that is discretely σ-decomposable in the completion* X^* *of* X, *each set of constancy being an* $\mathscr{F}_\sigma$-*set in* X, *and each set* X_α, $\alpha \in A$, *being the union of the sets of constancy that it constains.*

We have announced most of the results in this paper in [5].

§ 2. The structure of upper semi-continuous maps

In this section we use the assumptions and notation of Theorem 1 and we prove a sequence of lemmas establishing the results stated in Theorem 1. The first of these is the result of Choquet [17] rediscovered by Dolecki [18].

LEMMA 1. *The set* $K(x)$ *is compact, for each* x *in* X.

Proof. Let x^* be a fixed point of X and let $\{k_i\}$ be any sequence of distinct points of $K(x^*)$. Then

$$(x^*, k_i) \in \operatorname{cl} E(x^*)$$

for $i \geqslant 1$, and, for each $i \geqslant 1$, we can choose (x_i, y_i) in $E(x^*)$ with

$$\varrho((x_i, y_i), (x^*, k_i)) < 1/i.$$

As

$$E(x^*) = [(X \setminus \{x^*\}) \times (Y \setminus F(x^*))] \cap T,$$

we have $x_i \neq x^*$ and $y_i \notin F(x^*)$ but $(x_i, y_i) \in T$, for $i \geqslant 1$.

First suppose that the sequence $\{k_i\}$ has no convergent subsequence. Then the sequence $\{y_i\}$ has no convergent subsequence and the set

$$H = \{y_1, y_2, \ldots\}$$

is closed in Y. Hence the set

$$\{x : F(x) \cap H \neq \varnothing\}$$

is a closed set in X containing the sequence $\{x_i\}$ converging to x^*. Thus

$$F(x^*) \cap H \neq \varnothing,$$

contrary to the condition $y_i \notin F(x^*)$ for $i \geqslant 1$.

Now $\{k_i\}$ must have a convergent subsequence. We suppose that $\{k_i\}$ itself converges to a point y^* in Y. Now the set

$$H^* = \{y^*, y_1, y_2, \ldots\}$$

is closed in Y. It again follows that

$$F(x^*) \cap H^* \neq \varnothing.$$

Now, as $y_i \notin F(x^*)$ for $i \geqslant 1$, we conclude that $y^* \in F(x^*)$. As (x^*, y^*) is the limit of the sequence of points (x_i, y_i) in $E(x^*)$, we see that $y^* \in K(x^*)$. Thus each sequence of points of $K(x^*)$ has a subsequence that converges to a point of $K(x^*)$, and $K(x^*)$ is compact.

LEMMA 2. *The set K has the representation*

$$K = \bigcap_{r=1}^{\infty} K^{(r)},$$

with

$$K^{(r)} = \{(x,y) \in X \times Y : (\exists \xi)(\exists \eta)(0 < \varrho_1(x,\xi) < 2^{-r} \ \& \ \varrho_2(y,\eta) < 2^{-r} \ \& \ (\xi,\eta) \in T \ \& \ (x,\eta) \notin T\}$$

an open set in $X \times Y$ for each $r \geqslant 1$, where ϱ_1 is the metric on X and ϱ_2 is the metric on Y.

Proof. To see that the set $K^{(r)}$ is open, consider any point (x', y') in $K^{(r)}$ and choose a point (ξ', η') satisfying the conditions

$$0 < \varrho_1(x', \xi') < 2^{-r}, \quad \varrho_2(y', \eta') < 2^{-r}, \quad (\xi', \eta') \in T \quad \text{and} \quad (x', \eta') \notin T.$$

By the upper semi-continuity of F, the set of x with

$$(x, \eta') \notin T$$

is open in X. Hence the same point (ξ', η') satisfies the defining conditions for $K^{(r)}$ for all points (x, y) sufficiently close to (x', y'). Thus $K^{(r)}$ is open for each $r \geq 1$.

Consider any point (x^*, y^*) in K. The $(x^*, y^*) \in T$. Also, (x^*, y^*) is the limit of a sequence of points, say $(\xi^{(s)}, \eta^{(s)})$, $s \geq 1$, of $E(x^*)$. Then

$$\xi^{(s)} \neq x^* \quad \text{and} \quad (\xi^{(s)}, \eta^{(s)}) \in T \quad \text{but} \quad (x^*, \eta^{(s)}) \notin T,$$

for $s \geq 1$. If r is fixed, and s is suffciently large,

$$0 < \varrho_1(x^*, \xi^{(s)}) < 2^{-r}, \quad \varrho_2(y^*, \eta^{(s)}) < 2^{-r}, \quad (\xi^{(s)}, \eta^{(s)}) \in T \quad \text{and} \quad (x^*, \eta^{(s)}) \in T.$$

Hence, for each $r \geq 1$,
$$(x^*, y^*) \in K^{(r)},$$
and
$$(x^*, y^*) \in \bigcap_{r=1}^{\infty} K^{(r)}.$$

Thus
$$K \subset \bigcap_{r=1}^{\infty} K^{(r)}.$$

On the other hand, suppose that $(x^*, y^*) \in \bigcap_{r=1}^{\infty} K^{(r)}$. Then there will be a sequence $(\xi^{(r)}, \eta^{(r)})$ of points with

$$0 < \varrho_1(x^*, \xi^{(r)}) < 2^{-r}, \quad \varrho_2(y^*, \eta^{(r)}) < 2^{-r}, \quad (\xi^{(r)}, \eta^{(r)}) \in T \quad \text{and} \quad (x^*, \eta^{(r)}) \notin T.$$

for each $r \geq 1$. The set
$$H^* = \{y^*, \eta^{(1)}, \eta^{(2)}, \ldots\}$$

is closed in Y. Just as in the proof of Lemma 1, it follows that $y^* \in F(x^*)$. So (x^*, y^*), being a point of T that is the limit of the sequence $(\xi^{(r)}, \eta^{(r)})$ of points in $E(x^*)$ is in $\{x^*\} \times K(x^*)$ and so is in K. Thus

$$K = \bigcap_{r=1}^{\infty} K^{(r)},$$

as required.

LEMMA 3. *If H is any closed subset of Y, write*

$$\Xi(H) = \{x : K(x) \cap H = \varnothing\}.$$

Then the sets of constancy of the restriction of $F(x) \cap H$ to $\Xi(H)$ form a disjoint family that is discretely σ-decomposable in the completion X^* of X, each set of constancy being an $\mathscr{F}_\sigma$-set in X.

Proof. Write

$$\Xi(H) = \{x : K(x) \cap H = \varnothing\}.$$

Consider a point ξ of $\Xi(H)$. Then

$$[\operatorname{cl} E(\xi)] \cap [\{\xi\} \times (F(\xi) \cap H)] = \varnothing.$$

Let $C(\xi; 1/i)$ denote the cylinder of all points (x, y) with

$$\varrho_1(x, \xi) < 1/i, \quad y \in Y,$$

where ϱ_1 is the metric on X. Suppose that, for each $i \geq 1$, the open cylinder $C(\xi; 1/i)$ meets

$$\operatorname{cl} [E(\xi) \cap \{X \times H\}].$$

Then for each $i \geq 1$, we can choose a point (x_i, y_i) in

$$C(\xi; 1/i) \cap [E(\xi) \cap \{X \times H\}].$$

As

$$E(\xi) = [(X \setminus \{\xi\}) \times (Y \setminus F(\xi))] \cap T,$$

we have

$$\varrho_1(x_i, \xi) < 1/i, \quad x_i \neq \xi, \quad y_i \in H, \quad y_i \notin F(\xi),$$

and

$$y_i \in F(x_i).$$

If the sequence $\{y_i\}$ had a subsequence converging to a point η^* of Y, then the corresponding points (x_i, y_i) of $E(\xi)$ would converge to the point (ξ, η^*) in $X \times H$. As $K(\xi) \cap H = \varnothing$, we would have to have $\eta^* \notin F(\xi)$. Now the points of the subsequence of $\{y_i\}$ together with η^* would form a closed set L in Y and

$$\{x : F(x) \cap L \neq \varnothing\}$$

would be a closed set in X containing a subsequence of the points $\{x_i\}$ converging to ξ but not containing ξ. Thus $\{y_i\}$ has no convergent subsequence. We again get a contradiction, by the same argument, on taking L to be the closed set

$$L = \{y_1, y_2, \ldots\}.$$

We conclude that, for each ξ in $\Xi(H)$, there is an $i \geq 1$ such that $C(\xi; 1/i)$ does not meet

$$\mathrm{cl}\,[E(\xi) \cap \{X \times H\}].$$

In particular, for all x with $\varrho_1(x, \xi) < 1/i$, we have

$$[\{x\} \times (Y \setminus F(\xi))] \cap T \cap \{X \times H\} = \varnothing,$$

so that

$$(Y \setminus F(\xi)) \cap F(x) \cap H = \varnothing,$$

and

$$F(x) \cap H \subset F(\xi).$$

For each ξ in $\Xi(H)$, let $i(\xi)$ be the least integer i such that

$$F(x) \cap H \subset F(\xi)$$

for all x with $\varrho_1(x, \xi) < 1/i$. Then

$$C(\xi; 1/i(\xi)) \cap T \cap (X \times H) \subset B(\xi; 1/i(\xi)) \times F(\xi),$$

with $B(\xi; 1/i(\xi))$ the set of all x with $\varrho_1(x, \xi) < 1/i(\xi)$.

For each ξ in $\Xi(H)$, let $Q_H(\xi)$ denote the set of all x in X satisfying the conditions:

(a) $F(\xi) \cap H \subset F(x)$;

and

(b) $C(x; 1/i(\xi)) \cap T \cap (X \times H) \subset B(x; 1/i(\xi)) \times F(\xi)$.

Note that $\xi \in Q_H(\xi)$. Further the set

$$\bigcap_{y \in F(\xi) \cap H} \{x : F(x) \cap \{y\} \neq \varnothing\}.$$

of those x in X satisfying the condition (a) is closed in X, by the upper semi-continuity of F. We prove that the set of those x in X satisfying the condition (b) is also closed. Suppose that $\{x_j\}$ is a sequence of points of X all satisfying the condition (b), and

converging to some point, x^* say, of X. If x^* did not satisfy the condition (b), there would be a point (x, y) of T with

$$\varrho_1(x^*, x) < 1/i(\xi), \quad y \in H, \quad \text{and} \quad y \notin F(\xi).$$

Then, provided j is sufficiently large, we would have

$$\varrho_1(x_j, x) < 1/i(\xi)$$

as well as

$$(x, y) \in T, \quad y \in H, \quad y \notin F(\xi),$$

and the point x_j would not satisfy condition (b), contrary to its choice. Hence the set of x in X satisfying the condition (b) is closed in X. It now follows that $Q_H(\xi)$ is closed.

Note that condition (b) implies, in particular, that

$$F(x) \cap H \subset F(\xi).$$

It follows that $Q_H(\xi)$ is just the set of all x such that

$$F(x) \cap H = F(\xi) \cap H$$

and such that

$$F(\sigma) \cap H \subset F(\xi)$$

for all σ in $B(x; 1/i(\xi))$. Note further that $Q_H(\xi)$ is determined once $F(\xi) \cap H$ and $i(\xi)$ are known; it only depends on ξ through the dependence of $F(\xi) \cap H$ and of $i(\xi)$ on ξ.

For each $j \geq 1$, let $\Xi_j(H)$ denote the set of all ξ in $\Xi(H)$ with $i(\xi) = j$. We show that, for each $j \geq 1$, the family

$$\{Q_H(\xi): \xi \in \Xi_j(H)\},$$

is a family that is discrete in the closure X^* of X, each set being closed in X. Here we use the convention that two identical sets are not to be distinguished because they have different indices. Suppose that, for some point x of X^*, each neighbourhood of x in X^* meets at least two of the sets $Q_H(\xi)$ with $\xi \in \Xi_j(H)$. Then we can find points ξ, ξ' of $\Xi_j(H)$ such that

$$\varrho_1(x, \xi) < 1/(4j),$$

$$\varrho_1(x, \xi') < 1/(4j).$$

Now

$$\varrho_1(\xi, \xi') < 1/(2j).$$

So $\xi' \in B(\xi; 1/i(\xi))$ as $i(\xi)=j$. Hence

$$F(\xi') \cap H \subset F(\xi).$$

Similarly

$$F(\xi) \cap H \subset F(\xi').$$

Hence

$$F(\xi) \cap H = F(\xi') \cap H$$

and

$$i(\xi) = i(\xi').$$

Now $Q_H(\xi')$ coincides with $Q_H(\xi)$ and these sets are not distinct. Thus the family

$$\{Q_H(\xi): \xi \in \Xi_j(H)\}$$

is a family that is discrete in X^*, each set being closed in X. Hence the family

$$\{Q_H(\xi): \xi \in \Xi(H)\}$$

is σ-discrete in X^*, each set being closed in X. Further $F(x) \cap H$ is constant on each set of this family.

Now the sets of constancy of $F(x) \cap H$ on $\Xi(H)$ are obtained by choosing some ξ^* in $\Xi(H)$ and, for each possible value of j, choosing ξ_j^* in $\Xi_j(H)$ with

$$F(\xi_j^*) = F(\xi^*),$$

and then taking the union of the corresponding sets $Q_H(\xi_j^*)$. Thus the sets of constancy of $F(x) \cap H$ on $\Xi(H)$ form a disjoint family that is discretely σ-decomposable in X^*, these sets of constancy being $\mathscr{F}_\sigma$-sets in X with union $\Xi(H)$.

Each set of constancy of $F(x) \cap H$ on $\Xi(H)$ is a countable union of closed sets $Q_H(\xi)$, with at most one ξ in each set Ξ_j. By the standard reduction theorem [6], p. 350, this sequence of closed sets has a disjoint refinement by $\mathscr{F}_\sigma$-sets. The family of all the $\mathscr{F}_\sigma$-sets in all such refinements is a disjoint family of $\mathscr{F}_\sigma$-sets with union $\Xi(H)$ and with $F(x) \cap H$ constant on each set of the family, and this family is σ-discrete in X^*.

LEMMA 4. *The set*

$$\{x: K(x) \cap H \neq \varnothing\}$$

is a $\mathscr{G}_\delta$-set in X, whenever H is a closed set in Y.

Proof. By the last paragraph of the proof of Lemma 3, the set $\Xi(H)$, being the union of a σ-discrete family of $\mathscr{F}_\sigma$-sets in X, is itself an $\mathscr{F}_\sigma$-set. Thus

$$\{x: K(x) \cap H \neq \varnothing\} = X \setminus \Xi(H)$$

is a $\mathscr{G}_\delta$-set in X.

LEMMA 5. *The sets of constancy of the restriction of F to*

$$\Xi = \{x: K(x) = \varnothing\}$$

form a disjoint family of $\mathscr{F}_\sigma$-sets with union Ξ, and this family is discretely σ-decomposable in X^*.

Proof. Take $H = Y$ in Lemma 3.

LEMMA 6. *There is a σ-discrete family $\{P_\beta \times S_\beta\}_{\beta \in B}$ of rectangles that are relatively closed in T, with each set $P_\beta, \beta \in B$, closed in X, and with*

$$\bigcup_{\beta \in B} P_\beta \times S_\beta = T \setminus K.$$

Proof. As Y is a metric space, we can choose a σ-discrete family $\{H_\gamma\}_{\gamma \in \Gamma}$ of closed sets in Y forming a base for the open sets of Y. By the antepenultimate paragraph of the proof of the Lemma 3, for each γ in Γ we can choose a σ-discrete family $\{Q_{\gamma\alpha}\}_{\alpha \in A(\gamma)}$ of closed sets with union $\Xi(H_\gamma)$ with $F(x) \cap H_\gamma$ constant on each set of the family. Take

$$B = \bigcup_{\gamma \in \Gamma} \{\gamma\} \times A(\gamma), \quad P_{(\gamma, \alpha)} = Q_{\gamma\alpha},$$

for $(\gamma, \alpha) \in B$, and

$$S_{(\gamma, \alpha)} = F(\xi) \cap H_\gamma, \quad \text{with } \xi \in Q_{\gamma\alpha}, \quad \text{for } (\gamma, \alpha) \in B.$$

Then the rectangle

$$P_{(\gamma, \alpha)} \times S_{(\gamma, \alpha)}$$

is the intersection of the closed rectangle

$$P_{(\gamma, \alpha)} \times H_\gamma$$

with T, for each (γ, α) in B. It is easy to verify that the family $\{P_\beta \times S_\beta\}_{\beta \in B}$ is σ-discrete. It is clear that

$$P_\beta \times S_\beta \subset T \setminus K$$

for each β in B. It remains only to prove that each point of $T \setminus K$ belongs to some $P_\beta \times S_\beta$ with $\beta \in B$. Consider any point (x^*, y^*) in $T \setminus K$. Then

$$y^* \in F(x^*) \quad \text{but } y^* \notin K(x^*).$$

As $K(x^*)$ is compact, we can choose γ^* in Γ so that

$$y^* \in H_{\gamma^*}, \quad K(x^*) \cap H_{\gamma^*} = \varnothing.$$

Then

$$x^* \in \Xi(H_{\gamma^*})$$

and we can choose α^* in $A(\gamma^*)$ with

$$x^* \in P_{(\gamma^*, \alpha^*)} = Q_{\gamma^* \alpha^*}.$$

Now

$$(x^*, y^*) \in P_{\beta^*} \times S_{\beta^*},$$

with $\beta^* = (\gamma^*, \alpha^*)$. Thus

$$\bigcup_{\beta \in B} P_\beta \times S_\beta = T \setminus K,$$

as required.

Proof of Theorem 1. The result follows from Lemmas 1, 2, 4, 5 and 6.

§ 3. Absolutely additive families of closed sets

In this section we give the very short proof of Theorem 5 stated in the introduction.

Proof of Theorem 5. Take Y to be the set A with the discrete topology. The map F has graph

$$\bigcup_{\alpha \in A} X_\alpha \times \{\alpha\}.$$

For any subset B of $A = Y$, the set

$$\{x : F(x) \cap B \neq \varnothing\} = \bigcup_{\alpha \in B} X_\alpha$$

is closed in X. Thus F is upper semi-continuous. As Y is discrete

$$\operatorname{cl} E(x) = \operatorname{cl}\left([(X\backslash\{x\})\times(Y\backslash F(x))]\cap T\right) \subset X\times(Y\backslash F(x)),$$

so that

$$K(x) = \varnothing,$$

for all x in X. The required result now follows immediately from part (b) of Theorem 1.

§ 4. Selectors for upper semi-continuous maps

Before we prove Theorem 2, stated in the introduction, it will be convenient to prove two lemmas.

LEMMA 7. *Let F be an upper semi-continuous map of a metric space X to the power set of a metric space Y. Let L be a closed subset of Y and let Ξ be a subset of X with*

$$K(x)\cap L = \varnothing \quad \text{but} \quad F(x)\cap L \neq \varnothing,$$

for all x in Ξ. Then the restriction of $F(x)\cap L$ to Ξ has a selector f, whose sets of constancy form a disjoint family that is discretely σ-decomposable in X^*, each set of constancy being a relative $\mathcal{F}_\sigma$-set in Ξ.

Proof. In the notation of Lemma 3, the sets of constancy of $F(x)\cap L$ restricted to

$$\Xi(L) = \{x\colon K(x)\cap L = \varnothing\},$$

form a disjoint family, say $\{Q_\alpha\}_{\alpha\in A}$, that is discretely σ-decomposable in X^*, each set of constancy $Q_\alpha, \alpha\in A$, being an $\mathcal{F}_\sigma$-set in X. The family $\{\Xi\cap Q_\alpha\}_{\alpha\in A}$ remains discretely σ-decomposable in X^*, each set $\Xi\cap Q_\alpha$ being a relative $\mathcal{F}_\sigma$-set in Ξ. We suppress any empty sets in this family $\{\Xi\cap Q_\alpha\}_{\alpha\in A}$ and choose a representative point $\xi_\alpha, \sigma\in A$, from each non-empty member. Then $F(\xi_\alpha)\cap L \neq \varnothing$ and we define f on $\Xi\cap Q_\alpha, \alpha\in A$, by taking f to be any point of $F(\xi_\alpha)\cap L$. The sets of constancy for this selector f defined on Ξ will be certain unions

$$\bigcup_{\alpha\in B} \Xi\cap Q_\alpha,$$

with B a subset of A. As $\{Q_\alpha\}_{\alpha\in A}$ is a discretely σ-decomposable family of $\mathcal{F}_\sigma$-sets, each such set of constancy for f is a relative $\mathcal{F}_\sigma$-set in Ξ. As any disjoint family of unions of subfamilies of a discrete family is discrete, it follows that the sets of constancy of f form a discretely σ-decomposable family in X^*. This proves the lemma.

LEMMA 8. *Let F be an upper semi-continuous map of a metric space X to the power set of a metric space Y. Let $\{L_\gamma\}_{\gamma\in\Gamma}$ be a σ-discrete family of closed sets in Y with union L. Let Φ be a subset of X with*

$$F(x)\cap L \neq \varnothing,$$

for each x in Φ. Then there is a disjoint family $\{X_\gamma\}_{\gamma\in\Gamma}$ that is discretely σ-decomposable in X^*, with each set $X_\gamma, \gamma\in\Gamma$, a relative $\mathscr{F}_\sigma$-set in Φ, with

$$\bigcup_{\gamma\in\Gamma} X_\gamma = \Phi,$$

and with

$$X_\gamma \subset \{x: F(x)\cap L_\gamma \neq \varnothing\},$$

for each γ in Γ.

Proof. As $\{L_\gamma\}_{\gamma\in\Gamma}$ is a σ-discrete family of closed sets, we can write Γ as a disjoint union

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma(n)$$

with each family $\{L_\gamma\}_{\gamma\in\Gamma(n)}, n=1, 2, \ldots$, a discrete family of closed sets. Such a family is an absolutely additive family of closed sets.

Write

$$Z_\gamma = \{x: F(x)\cap L_\gamma \neq \varnothing\},$$

for each γ in Γ. As F is upper semi-continuous, each set Z_γ is closed in X, and, further, each family $\{Z_\gamma\}_{\gamma\in\Gamma(n)}$, $n=1, 2, \ldots$, is an absolutely additive family of closed sets in X. By Theorem 5, stated in the introduction and proved in § 3, for each $n\geqslant 1$, the sets of constancy of the map γ_n from X to the subsets $\Gamma(n)$, defined by

$$\gamma_n(x) = \{\gamma: \gamma\in\Gamma(n) \quad \text{and} \quad x\in Z_\gamma\},$$

for x in X, form a disjoint family $\{Q_\alpha\}_{\alpha\in A(n)}$, that is discretely σ-decomposable in the completion X^* of X, each set of constancy being an $\mathscr{F}_\sigma$-set in X. Let $A^*(n)$ be the subset of $A(n)$ for which $\gamma_n(x)$ is non-empty for x in Q_α. Then

$$Q^{(n)} = \bigcup_{\alpha\in A^*(n)} Q_\alpha$$

is an $\mathscr{F}_\sigma$-set for each $n \geqslant 1$ and

$$\bigcup_{n=1}^{\infty} Q^{(n)} \supset \Phi.$$

Let $\{P^{(n)}\}$ be a disjoint sequence of $\mathscr{F}_\sigma$-sets with

$$P^{(n)} \subset Q^{(n)}, \; n \geqslant 1, \quad \text{and} \quad \bigcup_{n=1}^{\infty} P^{(n)} = \bigcup_{n=1}^{\infty} Q^{(n)}.$$

Then

$$\{P^{(n)} \cap Q_\alpha : \alpha \in A(n) \quad \text{and} \quad n \geqslant 1\}$$

is a disjoint family that is discretely σ-decomposable in X^*, each set being an $\mathscr{F}_\sigma$-set in X, and with union containing Φ. For each pair α, n, with $\alpha \in A^*(n)$, for which the set of constancy Q_α corresponds to a non-empty set $\gamma_n(x)$ in $\Gamma(n)$, we choose such an index $\delta(\alpha, n)$ in $\Gamma(n)$ with $x \in Z_{\delta(\alpha,n)}$.

For each γ in Γ, $\gamma \in \Gamma(n)$ for some n, and we take

$$X_\gamma = \bigcup \{\Phi \cap P^{(n)} \cap Q_\alpha : \delta(\alpha, n) = \gamma\}.$$

It is easy to verify that this family $\{X_\gamma\}_{\gamma \in \Gamma}$ satisfies our requirements.

Proof of Theorem 2. The metric space Y has a closed σ-discrete base for its open sets. Using this base we can choose a system of index sets

$$\Gamma(\varphi),$$

$$\Gamma(\gamma_1), \quad \gamma_1 \in \Gamma(\varphi),$$

$$\Gamma(\gamma_1, \gamma_2), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1),$$

$$\ldots,$$

$$\Gamma(\gamma_1, \gamma_2, \ldots, \gamma_n), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), \quad \ldots, \quad \gamma_n \in \Gamma(\gamma_1, \ldots, \gamma_{n-1}),$$

$$\ldots,$$

and a corresponding system of closed sets

$$L(\varphi) = Y,$$

$$L(\gamma_1), \quad \gamma_1 \in \Gamma(\varphi),$$

$$L(\gamma_1, \gamma_2), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1),$$

$$\ldots,$$

$$L(\gamma_1, \gamma_2, \ldots, \gamma_n), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), \quad \ldots, \quad \gamma_n \in \Gamma(\gamma_1, \ldots, \gamma_{n-1}),$$

$$\ldots,$$

with the following properties. For each $n \geq 1$, and each choice of γ_1 in $\Gamma(\varphi)$, γ_2 in $\Gamma(\gamma_1), \ldots, \gamma_{n-1}$ in $\Gamma(\gamma_1, \gamma_2, \ldots, \gamma_{n-2})$ the family

$$\{L(\gamma_1, \gamma_2, \ldots, \gamma_n) : \gamma_n \in \Gamma(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})\}$$

is a σ-discrete family of closed sets, each of diameter at most 2^{-n}, and with union the set $L(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$. Here we are using the letter φ to represent a sequence of indices of zero length, so that $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ must be identified with φ when $n=1$.

To simplify the notation it will be convenient to use γ to denote a sequence

$$\gamma = \gamma_1, \gamma_2, \ldots$$

in the set Π of such sequences satisfying

$$\gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), \quad \gamma_3 \in \Gamma(\gamma_1, \gamma_2), \ldots,$$

and to use

$$\gamma|n = \gamma_1, \gamma_2, \ldots, \gamma_n$$

to denote the first n terms of the sequence γ. We will use this notation $\gamma|n$ even in the cases when the subsequent elements $\gamma_{n+1}, \gamma_{n+2}, \ldots$ are unknown or irrelevent.

We start an inductive construction by writing

$$L(\varphi) = Y, \quad X(\varphi) = X,$$
$$\Xi(\varphi) = \{x : K(x) \cap L(\varphi) = \varnothing\}.$$

As F maps to the non-empty sets of Y, by Lemma 7, the restriction of F to $\Xi(\varphi)$ has a selector $f(\varphi; x)$, whose sets of constancy form a disjoint family that is discretely σ-decomposable in X^*, each set of constancy being a relative $\mathscr{F}_\sigma$-set in $\Xi(\varphi)$. By Lemma 3, $\Xi(\varphi)$ is itself an $\mathscr{F}_\sigma$-set in X, and so each set of constancy of $f(\varphi; x)$.

We use induction on n to choose, for each γ in Π, and for each $n \geq 1$, sets

$$X(\gamma|n), \ \Xi(\gamma|n),$$

and a function $f(\gamma|n; x)$ from $\Xi(\gamma|n)$ to Y with the following properties.

(a) The family

$$\{X(\gamma|n) : \gamma \in \Gamma\},$$

regarded as indexed by the finite sequences $\gamma_1, \gamma_2, \ldots, \gamma_n$, rather than by the infinite

sequences $\gamma_1, \gamma_2, \ldots$, is a disjoint family that is discretely σ-decomposable in X^*, each set of the family being an $\mathscr{F}_\sigma$-set, and the union of the family being $X(\gamma|n-1)$.

(b) For each x in $X(\gamma|n)$, we have

$$F(x) \cap L(\gamma|n) \neq \varnothing.$$

(c) $\qquad\qquad \Xi(\gamma|n) = \{x \in X(\gamma|n): K(x) \cap L(\gamma|n) = \varnothing\}.$

(d) The function $f(\gamma|n; x)$ is a selector for the restriction of $F(x) \cap L(\gamma|n)$ to $\Xi(\gamma|n)$, and the sets of constancy of this selector form a disjoint family that is discretely σ-decomposable in X^*, each set of constancy being an $\mathscr{F}_\sigma$-set.

The conditions (a) to (d) are all satisfied for $n=0$ and for each γ in Π.

Now suppose that $n \geqslant 1$, and that conditions (a) to (d) have been satisfied for all smaller values of n and for all γ in Π. We can confine our attention to those γ in Γ with $\gamma|n$ fixed, say equal to $\delta|n-1$. Then

$$\{L(\gamma|n): \gamma \in \Gamma \quad \text{and} \quad \gamma|n-1 = \delta|n-1\}$$

is a σ-discrete family of closed sets in Y with union $L(\delta|n-1)$. By condition (b), for each x in $X(\delta|n-1)$, we have

$$F(x) \cap L(\delta|n-1) \neq \varnothing.$$

By Lemma 8, with $\Phi = X(\delta|n-1)$, there is a disjoint family that is discretely σ-decomposable in X^*, with each set $X(\gamma|n)$ a relative $\mathscr{F}_\sigma$-set in $X(\delta|n-1)$, with $X(\delta|n-1)$ as union for the family, and with

$$F(x) \cap L(\gamma|n) \neq \varnothing,$$

for each x in $X(\gamma|n)$. As $X(\delta|n-1)$ is an $\mathscr{F}_\sigma$-set, so is each set $X(\gamma|n)$. Having obtained the sets $X(\gamma|n)$, satisfying (a) and (b) we define $\Xi(\gamma|n)$ by condition (c). The functions $f(\gamma|n; x)$ satisfying the condition (d) are now obtained, by use of Lemma 7, as in the case of $f(\varphi; x)$.

For each $n \geqslant 1$, and each δ in Π, the family

$$\{X(\gamma|n): \gamma \in \Pi \quad \text{and} \quad \gamma|n-1 = \delta|n-1\}$$

is discretely σ-decomposable in X^*, and is contained in $X(\delta|n-1)$. It follows, by induction on n, that the family

$$\{X(\gamma|n): \gamma \in \Pi\}$$

is discretely σ-decomposable, for each $n \geq 1$. As $\Xi(\gamma|n) \subset X(\gamma|n)$, for all n and all γ in Π, the family

$$\{\Xi(\gamma|n): \gamma \in \Pi\}$$

is discretely σ-decomposable for each $n \geq 1$. As each set of the family is an $\mathscr{F}_\sigma$-set so is the union.

$$\Xi^{(n)} = \{\Xi(\gamma|n): \gamma \in \Pi\},$$

for $n \geq 1$. The set

$$\Xi^{(0)} = \Xi(\varphi)$$

is also an $\mathscr{F}_\sigma$-set. By the reduction theorem, we can choose a disjoint sequence $Z^{(0)}, Z^{(1)}, Z^{(2)}, \ldots$ of $\mathscr{F}_\sigma$-sets with

$$Z^{(n)} \subset \Xi^{(n)}, \quad n \geq 0,$$

$$\bigcup_{n=0}^{\infty} Z^{(n)} = \bigcup_{n=0}^{\infty} \Xi^{(n)}.$$

We now define sets $Z(\gamma|n)$ and $\Psi(\gamma|n)$ by the formulae

$$Z(\gamma|n) = \Xi(\gamma|n) \cap Z^{(n)},$$

$$\Psi(\gamma|n) = X(\gamma|n) \setminus \left(\bigcup_{r=0}^{n} Z(\gamma|r) \right),$$

for $n \geq 0$ and $\gamma \in \Pi$. The family

$$\{Z(\gamma|n): \gamma \in \Pi \quad \text{and} \quad n \geq 0\}$$

is a disjoint discretely σ-decomposable family in X^*, each set being an $\mathscr{F}_\sigma$-set in X.

We take

$$S_0 = T = \bigcup_{x \in X} (\{x\} \times F(x)),$$

$$S_n = [\bigcup \{\{x\} \times \{f(\gamma|r;x)\}: x \in Z(\gamma|r), \gamma \in \Pi \quad \text{and} \quad 0 \leq r \leq n\}] \cup [\bigcup \{\Psi(\gamma|n) \times L(\gamma|n): \gamma \in \Pi\}]$$

for $n \geq 1$, and

$$S = \bigcap_{n=0}^{\infty} S_n.$$

We prove that S is the graph of a selector f for F, satisfying our requirements.

Consider any point x^* in X. By condition (a), there is a unique γ^* in Π such that

$$x^* \in X(\gamma^*|n)$$

for $n=0,1,2,\dots$. First suppose that x^* belongs to the set

$$X_1 = \bigcup_{n=0}^{\infty} \Xi^{(n)} = \bigcup_{n=0}^{\infty} Z^{(n)}.$$

Then for a unique $n^* \geq 0$ we have

$$x^* \in Z^{(n^*)} = \cup\{Z(\gamma|n^*): \gamma \in \Pi\}.$$

As the family

$$\{X(\gamma|n^*): \gamma \in \Pi\}$$

is disjoint, and $x^* \in X(\gamma^*|n^*)$, we must have

$$x^* \in Z(\gamma^*|n) \subset \Xi(\gamma^*|n^*).$$

Now

$$f(\gamma^*|n^*; x^*) \in F(x^*) \cap L(\gamma^*|n^*).$$

As $x^* \in Z^{(n)}$, just for $n=n^*$, we have

$$x^* \in X(\gamma^*|r) \setminus \bigcup_{s=0}^{r} Z(\gamma^*|s),$$

for $0 \leq r < n^*$. Thus

$$(x^*, f(\gamma^*|n^*; x^*)) \in \Psi(\gamma^*|r) \times L(\gamma^*|r),$$

for $0 \leq r < n^*$, and

$$(x^*, f(\gamma^*|n^*; x^*)) \in S_r$$

for $0 \leq r < n^*$. Hence

$$(x^*, f(\gamma^*|n^*; x^*)) \in S.$$

Further, for $n \geq n^*$, the only point of the form (x^*, y) in S_n is this point with $y = F(\gamma^*|n^*; x^*)$. In this case $(x^*, f(\gamma^*|n; x^*))$ is the unique point of the set

$$(\{x^*\} \times Y) \cap S.$$

We write

$$f(x^*) = f(\gamma^*|n^*; x^*).$$

Now consider the case when x^* belongs to the set

$$X_2 = X \setminus X_1.$$

Then

$$x^* \notin \bigcap_{n=0}^{\infty} \Xi^{(n)},$$

and, for each $n \geq 0$,

$$x^* \in X(\gamma^*|n) \setminus \Xi(\gamma^*|n),$$

so that

$$K(x^*) \cap L(\gamma^*|n) \neq \varnothing.$$

As $K(x^*)$ is compact, while $L(\gamma^*|n)$ is a closed set of diameter at most 2^{-n}, for $n \geq 1$, there is a unique point y^* in the set

$$\bigcap_{n=0}^{\infty} K(x^*) \cap L(\gamma^*|n).$$

As

$$x^* \in X(\gamma^*|n) \setminus \Xi(\gamma^*|n)$$

for $n \geq 0$, we have

$$x^* \in X(\gamma^*|n) \setminus Z(\gamma^*|n)$$

for all $n \geq 0$, and

$$x^* \in \Psi(\gamma^*|n)$$

for all $n \geq 0$. Hence

$$(x^*, y^*) \in \Psi(\gamma^*|n) \times L(\gamma^*|n) \subset S_n,$$

for all $n \geq 0$, and $(x^*, y^*) \in S$. So y^* is the unique point y in Y for which $(x^*, y) \in S$. We write

$$f(x^*) = y^*.$$

Now we have shown that f is a well-defined function from X to Y with graph S. As

X_1 is an $\mathscr{F}_\sigma$-set and X_2 is a $\mathscr{G}_\delta$-set it will suffice that the restrictions of f to X_1 and to X_2 are of the first Borel class.

Consider any closed set H in Y. Write

$$P = \{x : F(x) \cap H \neq \varnothing\},$$

$$Q = \{x : f(x) \in H\},$$

$$P_1 = X_1 \cap P, \quad P_2 = X_2 \cap P,$$

$$Q_1 = X_1 \cap Q, \quad Q_2 = X_2 \cap Q.$$

By the upper semi-continuity of F, the set P is closed in X. We need to prove that Q_1 is a relative $\mathscr{G}_\delta$-set in X_1 and that Q_2 is a relative $\mathscr{G}_\delta$-set in X_2. It will clearly suffice to prove that $P_1 \setminus Q_1$ is an $\mathscr{F}_\sigma$-set in X_1 and that $P_2 \setminus Q_2$ is a relative $\mathscr{F}_\sigma$-set in X_2.

We recall that the function $f(\gamma|n; x)$ is chosen, by Lemma 7, as a selector for $F(x) \cap L(\gamma|n)$ on $\Xi(\varphi|n)$, its sets of constancy forming a disjoint family that is discretely σ-decomposable in X^*, each set of constancy being an $\mathscr{F}_\sigma$-set in X. Later in the construction, we only make use of the restriction of $f(\gamma|n; x)$ to the $\mathscr{F}_\sigma$-set $Z(\gamma|n)$. For each $n \geq 0$ and each γ in Π, let

$$\{\Theta(\gamma|n; \alpha)\}_{\alpha \in A(\gamma|n)}$$

be the family of sets of constancy of the restriction of $f(\gamma|n; x)$ to $Z(\gamma|n)$, and let $\theta(\gamma|n; \alpha)$ denote the value that $f(\gamma|n; x)$ takes on $\Theta(\gamma|n; \alpha)$. We identify $P_1 \setminus Q_1$ with the set

$$D_1 = P_1 \cap \bigcup \{\Theta(\gamma|n; \alpha) : \theta(\gamma|n; \alpha) \notin H, \ \gamma \in \Pi, \ n \geq 0 \text{ and } \alpha \in A(\gamma|n)\}.$$

If $x \in D_1$, then $x \in P_1$ and, for some $\gamma \in \Pi$, $n \geq 0$ and $\alpha \in A(\gamma|n)$, we have $x \in \Theta(\gamma|n; \alpha)$ and $\theta(\gamma|n; \alpha) \notin H$. Thus

$$f(x) = f(\gamma|n; x) = \theta(\gamma|n; \alpha) \notin H,$$

and $x \notin Q_1$, so that $x \in P_1 \setminus Q_1$. On the other hand, if $x \in P_1 \setminus Q_1$, then $x \in P_1$ and so, for some $n \geq 0$, $\gamma \in \Pi$ and $\alpha \in A(\gamma|n)$, we have

$$x \in \Theta(\gamma|n; \alpha).$$

As $x \notin Q_1$, we have $f(x) \notin H$. So

$$\theta(\gamma|n; \alpha) = f(\gamma|n; x) = f(x) \notin H.$$

Hence $x \in D_1$. Now

$$P_1 \setminus Q_1 = D_1.$$

As the family

$$\bigcup\{\Theta(\gamma|n;\alpha)\colon \theta(\gamma|n;\alpha)\notin H,\ \gamma\in\Pi,\ n\geqslant 0 \text{ and } \alpha\in A(\gamma|n)\}$$

is a discretely σ-decomposable family of $\mathscr{F}_\sigma$-sets, the set $P_1\setminus Q_1$ is itself an $\mathscr{F}_\sigma$-set.

We now identify $P_2\setminus Q_2$ with the set

$$D_2 = P_2 \cap \bigcup\{X(\gamma|n)\colon L(\gamma|n)\cap h=\varnothing,\ \gamma\in\Pi \text{ and } n\geqslant 0\}.$$

If $x\in D_2$, then, for some $\gamma\in\Pi$ and $n\geqslant 0$, we have

$$x\in X(\gamma|n) \quad\text{and}\quad L(\gamma|n)\cap H=\varnothing.$$

As $x\in X_2$, we have $f(x)\in L(\gamma|n)$. Hence $f(x)\notin H$ and $x\notin Q_2$, so that $x\in P_2\setminus Q_2$. On the other hand, if $x\in P_2\setminus Q_2$, then there is a unique γ^* in Π with

$$x\in X(\gamma^*|n)$$

for $n\geqslant 0$. As $x\in X_2$, we have $f(x)\in L(\gamma^*|n)$ for all $n\geqslant 0$. As $x\notin Q_2$, we have $f(x)\notin H$. As H is closed, and $L(\gamma^*|n)$ has diameter at most 2^{-n}, we have

$$L(\gamma^*|n^*)\cap H=\varnothing$$

for n^* sufficiently large. Hence $x\in D_2$. Thus

$$P_2\setminus Q_2 = D_2.$$

As D_2 is the intersection of P_2 with the union of a discretely σ-decomposable family of $\mathscr{F}_\sigma$-sets, D_2 is an $\mathscr{F}_\sigma$-set relative to X_2. This completes the proof for a general upper semi-continuous F.

We now consider the case when F takes only compact values. The discussion of this special case is much simpler than the general case. There is no need to define the compact sets $K(x)$ by the formula

$$K(x) = \mathrm{proj}_Y([\mathrm{cl}\,E(x)]\cap[\{x\}\times F(x)]),$$

it suffices simply to take

$$K(x) = F(x)$$

for all x in X. The families

$$\{L(\gamma_1,\gamma_2,\dots,\gamma_n)\colon \gamma_n\in\Gamma(\gamma_1,\gamma_2,\dots,\gamma_{n-1})\}$$

with their index sets can be chosen just as in the opening paragraphs of this proof. An inductive construction is stated by taking

$$L(\varphi) = Y, \quad X(\varphi) = X.$$

We need no set $\Xi(\varphi)$, but it may be convenient to think of $\Xi(\varphi)$ and the similar sets $\Xi(\gamma|n)$ as all replaced by the empty set. Sets $X(\gamma|n)$ for γ in Π and $n \geqslant 1$, are defined, satisfying the conditions (a) and (b), inductively, as before, by use of Lemma 8. We need none of the sets $\Xi(\gamma|n)$, $\gamma \in \Pi$, $n \geqslant 1$, nor $\Xi^{(n)}$, $n \geqslant 1$, nor $Z^{(n)}$ with $n \geqslant 1$, and can simply take

$$\Psi(\gamma|n) = X(\gamma|n)$$

for $n \geqslant 0$, $\gamma \in \Pi$, and

$$S_n = \{\Psi(\gamma|n) \times L(\gamma|n) : \gamma \in \Pi\},$$
$$S = \bigcap_{n=0}^{\infty} S_n.$$

Now, for each x^* in X, there is a unique γ^* in Π with

$$x^* \in \bigcap_{n=0}^{\infty} X(\gamma^*|n)$$

and then a unique y^* in

$$\bigcap_{n=0}^{\infty} K(x^*) \cap L(\gamma^*|n).$$

We take $f(x^*)$ to be this unique point Y^*. It follows, without difficulty, by simplification of the main proof, that f is a selector for F of the first Borel class.

COROLLARY. *The sets of constancy of the restriction of f to X_1 are $\mathcal{F}_\sigma$-sets and the family of these sets is discretely σ-decomposable in X^*. The restriction of f to X_2 is a selector for K on X_2.*

§ 5. Selector representations for upper semi-continuous set-valued maps

In this section we prove the representation theorem, Theorem 3, stated in the introduction. Recall that a continuous function is said to be proper if it maps closed sets to closed sets and the inverse of each point in the range is compact. Morita [11], p. 36, has

shown that if Y is a metric space, then there exists a cardinal number m, a subset Δ of $B(m)$, and a proper map f, of Δ onto Y. We will need the following refinement of Morita's result.

LEMMA 9. *Let m be an infinite cardinal and let Y be a complete metric space of weight m. Then there is a closed subset Δ of the Baire space $B(m)$ and a uniformly continuous proper map of Δ onto Y.*

Proof. For each $n \geqslant 1$, the space Y is covered by its open balls of radius 2^{-n}. Since Y is paracompact (cf. [2], p. 349) this open cover of Y has a locally finite refinement by non-empty open sets, say by the family $\{G^{(n)}(\xi): \xi \in \Xi(n)\}$. As Y has weight m, the index set $\Xi(n)$ has cardinal not exceeding m. For each ξ in $\Xi(n)$, write $F^{(n)}(\xi) = \mathrm{cl}\, G^{(n)}(\xi)$. Then the family $\{F^{(n)}(\xi): \xi \in \Xi(n)\}$ is a locally finite family of at most m non-empty closed sets, each set having diameter at most 2^{-n+1}, the union of the family being Y. Here 'cl' denotes closure in Y.

Let $\varkappa$ be the least ordinal with cardinal m. We will use $\gamma_1, \gamma_2, \dots$ to denote ordinals with $0 \leqslant \gamma_i < \varkappa$, for $i \geqslant 1$. Rather as in the proof of Theorem 2, we choose a system of index sets

$$\Gamma(\varphi),$$

$$\Gamma(\gamma_1), \quad \gamma_1 \in \Gamma(\varphi),$$

$$\Gamma(\gamma_1, \gamma_2), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1),$$

$$\dots,$$

$$\Gamma(\gamma_1, \gamma_2, \dots, \gamma_n), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), \quad \dots, \quad \gamma_n \in \Gamma(\gamma_1, \dots, \gamma_{n-1}),$$

$$\dots,$$

and a corresponding system of closed sets

$$L(\varphi) = Y$$

$$L(\gamma_1), \quad \gamma_1 \in \Gamma(\varphi),$$

$$L(\gamma_1, \gamma_2), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1),$$

$$\dots,$$

$$L(\gamma_1, \gamma_2, \dots, \gamma_n), \quad \gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), \quad \dots, \quad \gamma_n \in \Gamma(\gamma_1, \dots, \gamma_{n-1}),$$

$$\dots,$$

with the following properties.

(a) Each index set $\Gamma(\gamma_1, \gamma_2, ..., \gamma_n)$ is a non-empty initial segment of an ordinal not exceeding $\varkappa$.

(b) For each sequence $\gamma_1, \gamma_2, ..., \gamma_{n-1}$, with

$$\gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), ...,, \quad \gamma_{n-1} \in \Gamma(\gamma_1, ..., \gamma_{n-2}),$$

the family

$$\{L(\gamma_1, \gamma_2, ..., \gamma_{n-1}, \delta): \delta \in \Gamma(\gamma_1, ..., \gamma_{n-1})\}$$

is a locally finite family of non-empty closed sets of diameter at most 2^{-n+1}, with union $L(\gamma_1, \gamma_2, ..., \gamma_{n-1})$.

To simplify the notation, it will be convenient to use γ to denote a sequence

$$\gamma = \gamma_1, \gamma_2, ...$$

in the set Δ of all such sequences satisfying

$$\gamma_1 \in \Gamma(\varphi), \quad \gamma_2 \in \Gamma(\gamma_1), \quad \gamma_3 \in \Gamma(\gamma_1, \gamma_2), ...,$$

and to use

$$\gamma|n = \gamma_1, \gamma_2, ..., \gamma_n$$

to denote the first n terms of the sequence γ. We will use this notation $\gamma|n$ even in cases when the subsequent elements $\gamma_{n+1}, \gamma_{n+2}, ...$ are unknown or irrelevent. For each $\gamma \in \Delta$, and each $n \geqslant 1$, we use $\Delta(\gamma|n)$ to denote the set of finite sequences $\delta_1, \delta_2, ..., \delta_n, \delta_{n+1}$, with

$$\delta_1 = \gamma_1, \quad \delta_2 = \gamma_2, \quad ..., \quad \delta_n = \gamma_n, \quad \delta_{n+1} \in \Gamma(\gamma|n).$$

To start the inductive construction we take $\Gamma(\varphi)$ to be the set of ordinals less than the least ordinal $\psi(\varphi)$ with cardinal equal to the cardinal of the family $\{F^{(1)}(\xi): \xi \in \Xi(1)\}$ of closed sets. In this case $\Delta(\varphi) = \Gamma(\varphi)$, and we take

$$\{L(\delta_1): \delta_1 \in \Gamma(\varphi)\} = \{L(\delta|1): \delta|1 \in \Delta(\varphi)\}$$

to be a wellordering of these non-empty closed sets. When $\gamma \in \Delta$, and $n \geqslant 0$, and when the set $L(\gamma|n)$ has been chosen, we take $\Gamma(\gamma|n)$ to be the set of ordinals less than the least ordinal $\psi(\gamma|n)$ with cardinal equal to the cardinal of the family of non-empty closed sets amongst the family

$$\{L(\gamma|n) \cap F^{(n+1)}(\xi): \xi \in \Xi(n+1)\}.$$

We take the family

$$\{L(\gamma_1, \ldots, \gamma_n, \delta): 0 \le \delta < \psi(\gamma|n)\} = \{L(\delta|n+1): \delta|n+1 \in \Delta(\gamma|n)\}$$

to be a wellordering of these non-empty closed sets.

It is easy to verify that this inductive procedure leads to the construction of the index sets and the families of sets satisfying the conditions (a) and (b). Further, expressing these conditions in terms of the simplified notation we have the following results.

(c) For each γ in Δ, and for each $n \ge 0$, the index set $\Delta(\gamma|n)$ is non-empty and consistes of finite sequences $\delta|n+1$, with $\delta \in \Delta$ and $\delta|n = \gamma|n$.

(d) For each γ in Δ, and for each $n \ge 0$, the family

$$\{L(\delta|n+1): \delta|n+1 \in \Delta(\gamma|n)\}$$

is a locally finite family of non-empty closed sets of diameter at most 2^{-n+1}, with union $L(\gamma|n)$.

Now Δ is the set of sequences δ in $B(m)$ with

$$\delta|n+1 \in \Delta(\delta|n), \quad \text{for } n \ge 0.$$

Hence $B(m) \setminus \Delta$ is the union of the sets

$$\{\delta: \delta|1 \notin \Delta(\varphi)\}, \quad \{\delta: \delta|n \in \Delta(\delta|n-1) \text{ and } \delta|n+1 \notin \Delta(\delta|n)\}, \quad n \ge 1.$$

Thus $B(m) \setminus \Delta$ is open and Δ is closed in $B(m)$.

We note that, if $\delta \in \Delta$, then the sequence

$$L(\delta|n), \quad n = 1, 2, \ldots,$$

is a decreasing sequence of non-empty closed sets with diameters decreasing to zero. As Y is complete we can define $h(\delta)$, for δ in Δ, to be the unique point in

$$\bigcap_{n=1}^{\infty} L(\delta|n).$$

The diameter condition ensures that h is a uniformly continuous map. The covering condition ensures that h maps Δ onto Y.

Consider any point y in Y. As the families

$$\{L(\delta|n+1): \delta|n+1 \in \Delta(\gamma|n)\},$$

$0 \leqslant n \leqslant k$, are locally finite, they are point finite and the closed set $h^{-1}(y)$ in $B(m)$ meets only a finite number of the Baire intervals of $B(m)$ of order $k+1$. As this holds for each $k \geqslant 0$, the set $h^{-1}(y)$ is compact in $B(m)$.

Consider any sequence $\delta^{(1)}, \delta^{(2)}, \ldots$ of points of Δ, and suppose that the sequence

$$y^{(1)} = h(\delta^{(1)}), \quad y^{(2)} = h(\delta^{(2)}), \ldots$$

converges to a point y^* of Y. We show that the sequence $\delta^{(1)}, \delta^{(2)}, \ldots$ has a subsequence converging to a point δ^* in Δ. We choose a nested sequence

$$N(1) \supset N(2) \supset \ldots$$

of infinite sequences of positive integers with the property that $\delta^{(i)}$ belongs to a single Baire interval of order k for all i in $N(k)$, $k \geqslant 1$. As the family $\{L(\delta|1): \delta|1 \in \Delta(\varphi)\}$ is locally finite y^* has a neighbourhood $U(1)$ that meets only finitely many of the sets $L(\delta|1)$, $\delta|1 \in \Delta(\varphi)$. Provided $n(1)$ is sufficiently large, we have

$$h(\delta^{(i)}) \in U(1)$$

for $i \geqslant n(1)$. Hence $\delta^{(i)}|1$ takes only finitely many values for $i \geqslant n(1)$. Now we can choose an infinite sequence $N(1)$ of positive integers so that $\delta^{(i)}|1$ takes the same value for all i in $N(1)$. Proceeding in this way we can choose $N(1), N(2), \ldots$ inductively to satisfy our requirements. We then take N to be a diagonal sequence. This ensures that $\delta^{(i)}$ converges to some δ^* in Δ as i tends to infinity through N. By the continuity of h, this ensures that $y^* = h(\delta^*)$ with δ^* a limit of the sequence $\delta^{(1)}, \delta^{(2)}, \ldots$.

Thus h maps any closed set in Δ into a closed set in Y.

As h is a continuous closed map with compact inverse images for points of Y, it is a proper map. This proves the lemma.

Proof of Theorem 3. By Lemma 9, there is a closed uniformly continuous function η mapping a closed subset Δ of $B(m)$ onto Y. Then $\eta^{-1} \circ F$ is an upper semi-continuous map of the metric space X to the non-empty closed subsets of $B(m)$. Provided we have the special case of the theorem, when Y coincides with $B(m)$, we can choose a map g from $X \times B(m)$ to $B(m)$ satisfying (a), (b) and (c) for $\eta^{-1} \circ F$ in place of F. It is easy to verify that $\eta \circ g$ then satisfies (a), (b) and (c) for F. Thus it suffices to prove the special case of the theorem when $Y = B(m)$; the rest of this proof will consider only this special case.

Let $\varkappa$ be the least ordinal with cardinal m. We take $B(m)$ to be the space of sequences

$$\sigma = \sigma_1, \sigma_2, \ldots,$$

with $0 \leqslant \sigma_i < \varkappa$. For each $n \geqslant 1$, we write

$$\sigma|n = \sigma_1, \sigma_2, \ldots, \sigma_n,$$

and we use φ to denote the empty sequence of zero length. We take the distance between two distinct elements σ, τ of $B(m)$ to be 2^{-r}, where r is the first integer with $\sigma_r \neq \tau_r$. It will be convenient to write

$$\mathbf{I} = \mathbf{I}(\varphi) = B(m),$$

and

$$\mathbf{I}(\sigma|n) = \{\tau \in \mathbf{I}: \tau_i = \sigma_i \quad \text{for } 1 \leqslant i \leqslant n\},$$

for $\sigma \in \mathbf{I}$ and $n \geqslant 1$. For each $n \geqslant 1$, the family

$$\{\mathbf{I}(\sigma|n): \sigma \in \mathbf{I}\}$$

is a discrete partition of $\mathbf{I}$ into clopen sets. Similarly, if $n \geqslant 1$ and $\tau \in \mathbf{I}$, the family

$$\{\mathbf{I}(\sigma|n): \sigma|n-1 = \tau|n-1 \quad \text{and} \quad \sigma \in \mathbf{I}\}$$

is a discrete partition of $\mathbf{I}(\tau|n-1)$ into clopen sets.

For each $n \geqslant 1$ and each σ in $\mathbf{I}$ we write

$$X(\sigma|n) = \{x: F(x) \cap \mathbf{I}(\sigma|n) \neq \varnothing\}.$$

As F is an upper semi-continuous map from X to $\mathbf{I}$, for $n \geqslant 1$, the family

$$\{X(\sigma|n): \sigma \in \mathbf{I}\}$$

is an absolutely additive family of closed sets in X. Define a map $F(n; x)$, from X to the power set of the sequences $\sigma|n$ of length n, by the formula

$$F(n; x) = \{\sigma|n: \sigma \in \mathbf{I} \quad \text{and} \quad F(x) \cap \mathbf{I}(\sigma|n) \neq \varnothing\}.$$

By Theorem 5, the sets of constancy of the map $F(n; x)$ are $\mathscr{F}_\sigma$-sets in X and the family of these sets is discretely σ-decomposable in X^*. When $0 < n < m$, the sets of constancy of $F(m; x)$ form a refinement of those of $F(n; x)$. Hence we may choose a system of index sets

$$A(\varphi),$$

$$A(\alpha_1), \quad \alpha_1 \in A(\varphi),$$

$$A(\alpha_1, \alpha_2), \quad \alpha_1 \in A(\varphi), \quad \alpha_2 \in A(\alpha_1),$$

$$\ldots,$$

$$A(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \alpha_1 \in A(\varphi), \quad \alpha_2 \in A(\alpha_1), \quad \ldots, \quad \alpha_n \in A(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}),$$

$$\ldots,$$

and we use

$$Q(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \alpha_1 \in A(\varphi), \quad \alpha_2 \in A(\alpha_1), \quad \ldots, \quad \alpha_n \in A(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}),$$

to denote the sets of constancy of $F(n; x)$, with

$$Q(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \bigcup (Q(\alpha_1, \ldots, \alpha_n): \alpha_n \in A(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})).$$

Let $\Sigma(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the set of sequences $\sigma|n$ of length n that is the constant value of $F(n; x)$ taken for x in $Q(\alpha_1, \alpha_2, \ldots, \alpha_n)$. As $F(x)$ is non-empty for each x in X, so $F(n; x)$ is non-empty for each x in X and each $n \geq 1$.

To simplify the notation, let A denote the set of all sequences

$$\alpha_1, \alpha_2, \alpha_3, \ldots,$$

subject to

$$\alpha_1 \in A(\varphi), \quad \alpha_2 \in A(\alpha_1), \quad \alpha_3 \in A(\alpha_1, \alpha_2), \ldots.$$

Write

$$\alpha|n = \alpha_1, \alpha_2, \ldots, \alpha_n,$$

when $\alpha \in A$ and $n \geq 1$. In this notation, the sets of constancy of $F(n; x)$ are the sets $Q(\alpha|n)$ with $\alpha \in A$, and the value of $F(n; x)$ on $Q(\alpha|n)$ is $\Sigma(\alpha|n)$.

For each x in X, let $\alpha(x)$ denote the unique α in A with

$$x \in Q(\alpha(x)|n), \quad \text{for } n \geq 1.$$

In making selections in **I** it is usual to give preference to points that are lower in the lexicographical order on **I**. In this construction we need to introduce a family of lexicographical orders on **I** parametrized by a variable χ chosen from **I**. For each χ in **I**, we introduce a modified ordering $<(\chi)$ of **I** by taking:

$$\sigma <(\chi) \tau, \quad \text{if} \quad \sigma < \tau \quad \text{and} \quad \tau_1 \neq \chi_1;$$

$$\tau <(\chi) \sigma, \quad \text{if} \quad \tau_1 = \chi_1 \quad \text{but} \quad \sigma_1 \neq \chi_1;$$

$$\sigma <(\chi)\tau \quad \text{if} \quad \sigma < \tau, \sigma_1 = \tau_1 = \chi_1 \quad \text{and} \quad \tau_2 \neq \chi_2;$$

$$\tau <(\chi)\sigma, \quad \text{if} \quad \sigma_1 = \tau_1 = \chi_1 \quad \text{and} \quad \tau_2 = \chi_2 \quad \text{but} \quad \sigma_2 \neq \chi_1;$$

$$\ldots;$$

$$\sigma <(\chi)\tau, \quad \text{if} \quad \sigma < \tau, \sigma_1 = \tau_1 = \chi_1, \ldots, \quad \sigma_r = \tau_r = \chi_r \quad \text{and} \quad \tau_{r+1} \neq \chi_{r+1};$$

$$\tau <(\chi)\sigma, \quad \text{if} \quad \sigma_1 = \tau_1 = \chi_1, \ldots, \quad \sigma_r = \tau_r = \chi_r \quad \text{and} \quad \tau_{r+1} = \chi_{r+1} \quad \text{but} \quad \sigma_{r+1} \neq \chi_{r+1};$$

$$\ldots.$$

Note that the question of whether or not $\sigma <(\chi)\tau$ is determined by a knowledge of $\sigma|r$ and of $\tau|r$, where r is the first integer with $\sigma_r \neq \tau_r$. This ordering is designed to provide preference for the point χ of $\mathbf{I}$ with a minimal rearrangement of the order relationships for σ and τ that do not start with an initial partial sequence $\chi_1, \chi_2, \ldots$. The order $<(\chi)$ will also be used to relate finite sequences. If σ, τ are given with $\sigma|n \neq \tau|n$, then either

$$\xi <(\chi)\eta \quad \text{for all } \xi, \eta \text{ in } \mathbf{I} \text{ with } \xi|n = \sigma|n \quad \text{and} \quad \eta|n = \tau|n,$$

or

$$\eta <(\chi)\xi \quad \text{for all } \xi, \eta \text{ in } \mathbf{I} \text{ with } \xi|n = \sigma|n \quad \text{and} \quad \eta|n = \tau|n.$$

In the first case we write $\sigma|n <(\chi)\tau|n$, and in the second case we write $\tau|n <(\chi)\sigma|n$. For a fixed n, the relation $<(\chi)$ on the sequences of length n is a well-ordering. For each α in A and each $n \geq 1$, we use $\gamma(\alpha|n, \chi)$ to denote the first element of $\Sigma(\alpha|n)$ under the order $<(\chi)$.

We can now define the required map g from $X \times \mathbf{I}$ to $\mathbf{I}$. For each x in X and each χ in $\mathbf{I}$, there is a unique $g(x, \chi)$ in $\mathbf{I}$ satisfying

$$g(x, \chi)|n = \gamma(\alpha(x)|n, \chi)$$

for each $n \geq 1$.

Using the nesting properties of the sets of constancy and the properties of the ordering $<(\chi)$ it is easy to verify that $g(x, \chi)$ is uniquely and consistently defined in this way.

For each x in X and each χ in $\mathbf{I}$, we have

$$g(x, \chi)|n \in \Sigma(\alpha(x)|n),$$

so that

$$g(x, \chi)|n \in F(n; x),$$

and

$$F(x) \in \mathbf{I}(g(x, \chi)|n) \neq \varnothing.$$

Now the sets

$$F(x) \cap \mathbf{I}(g(x, \chi)|n),$$

$n = 1, 2, \ldots$, form a decreasing sequence of non-empty closed sets in $\mathbf{I}$, and the diameter of the nth set is at most 2^{-n}. As $F(x)$ is closed in the complete space $\mathbf{I}$, it follows that

$$g(x, \chi) \in F(x).$$

Further, if $\chi \in F(x)$, then, for each $n \geq 1$, the first element in $\Sigma(\alpha(x)|n)$ under the ordering $<(\chi)$ will be $\chi|n$. Thus, in this case, when $\chi \in F(x)$, we have

$$g(x, \chi) = \chi.$$

Thus g satisfies the condition (a) in the statement of the theorem.

For each χ in $\mathbf{I}$ the function $g(x, \chi)$ is a selector for $F(x)$ on X. Let $\mathbf{H}$ be any closed subset of $\mathbf{I}$ and take

$$P = \{x : F(x) \cap \mathbf{H} \neq \varnothing\},$$

$$Q = \{x : g(x, \chi) \in \mathbf{H}\}.$$

By the upper semi-cintinuity of F, the set P is closed in X. To prove that Q is a $\mathcal{G}_\delta$-set in X it suffices to prove that $P \backslash Q$ is an $\mathcal{F}_\sigma$-set in X. By the method used to discuss the restriction of F to X_2, in the proof of Theorem 2, we verify that

$$P \backslash Q = P \cap \bigcup\{Q(\alpha|n) : \gamma(\alpha|n, \chi) \cap \{\eta|n : \eta \in \mathbf{H}\} = \varnothing, \ \alpha \in A \text{ and } n \geq 1\}.$$

Thus $P \backslash Q$ is an $\mathcal{F}_\sigma$-set. Hence $g(x, \chi)$ is of the first Borel class, for each fixed χ. This shows that g satisfies the required condition (c).

For each x in X, we already know that $g(x, \chi)$ is a surjective map of $\mathbf{I}$ to $F(x)$. If χ' differs from χ first in the nth place, the orderings $<(\chi')$ and $<(\chi)$ coincide, when they are applied to finite sequences $\sigma|r$ with $1 \leq r < n$. Thus

$$g(x, \chi')|r = g(x, \chi)|r \quad \text{when } 1 \leq r < n.$$

Hence

$$\varrho(g(x, \chi'), g(x, \chi)) \leq \varrho(\chi', \chi)$$

for all x in X and χ, χ' in $\mathbf{I}$. We need to prove that the map $g(x, \chi)$ is a closed map from $\mathbf{I}$ to $\mathbf{I}$ when x is fixed in X. Let $\mathbf{H}$ be any closed set in $\mathbf{I}$. Suppose that $\eta^{(1)}, \eta^{(2)}, \ldots$ is any sequence of points of $\mathbf{H}$ and that the sequence

$$g(x, \eta^{(1)}), g(x, \eta^{(2)}), \ldots$$

converges to a point, γ say, in $\mathbf{I}$. We need to find an η in $\mathbf{H}$ with $g(x, \eta)=\gamma$, in order to prove that g maps the closed set $\mathbf{H}$ in $\mathbf{I}$ into a closed set in $\mathbf{I}$. As

$$g(x, \eta^{(i)}) \in F,$$

for all $i \geq 1$, and as $F(x)$ is closed, we have $\gamma \in F(x)$.

Note that

$$g(x, \eta^{(i)})|1 = \gamma_1$$

for all sufficiently large i. Suppose that for some infinite sequence of such sufficiently large i, we have

$$\eta_1^{(i)} \neq \gamma_1.$$

Then, for these values of i, we have

$$F(x) \cap \mathbf{I}(\eta_1^{(i)}) = \varnothing,$$

as otherwise we would necessarily have

$$g(x, \eta^{(i)})|1 = \eta_1^{(i)} \neq \gamma_1.$$

So in finding the first point of $\Sigma(\alpha(x)|n)$ under the ordering $<(\eta^{(i)})$ we are concerned only with points in $\mathbf{I} \setminus \mathbf{I}(\eta_1^{(i)})$ and so the ordering $<(\eta^{(i)})$ effectively coincides with the ordering $<(o)$ with $o=0, 0, 0, \ldots$. Thus, for all i in this infinite sequence

$$g(x, \eta^{(i)})$$

is independent of i and must coincide with γ. Thus $g(x, \eta^{(i)})=\gamma$, for some $\eta^{(i)}$ in H. Now suppose, on the other hand, that for all sufficiently large i, we have $\eta_1^{(i)}=\gamma_1$. Note that

$$g(x, \eta^{(i)})|2 = \gamma|2,$$

for all sufficiently large i. Supposet that, for some infinite sequence of such sufficiently large i, we have

$$\eta_2^{(i)} \neq \gamma_2.$$

For these values of i, we have

$$F(x) \cap \mathbf{I}(\eta^{(i)}|2) = \varnothing.$$

In finding the first point of $\Sigma(\alpha(x)|n)$, for $n \geqslant 2$, the ordering $<(\eta^{(i)})$ effectively coincides with the ordering $<(\gamma_1, o)$ with $\gamma_1, o = \gamma_1, 0, 0, 0, \ldots$. Again, for all i in the sequence,

$$g(x, \eta^{(i)})$$

is independent of i and so must coincide with γ, yielding $g(x, \eta^{(i)}) = \gamma$ for some $\eta^{(i)}$ in $\mathbf{H}$. So we may suppose that

$$\eta^{(i)}|2 = \gamma|2$$

for all sufficiently large i.

Proceeding in this way, we either find an η in $\mathbf{H}$ with $g(x, \eta) = \gamma$, or we find that, for each $n \geqslant 1$,

$$\eta^{(i)}|n = \gamma|n$$

for all sufficiently large i. In this second case, $\eta^{(i)}$ converges to γ. As $\mathbf{H}$ is closed we have $\gamma \in F(x) \cap \mathbf{H}$ and $g(x, \gamma) = \gamma$ with $\gamma \in \mathbf{H}$. Thus the image of $\mathbf{H}$ is closed. Hence g satisfies the required condition (b) and the proof is complete.

§ 6. Upper semi-continuous maps whose values are Borel sets, Souslin-$\mathscr{F}$ sets or co-Souslin-$\mathscr{F}$ sets

We need the following lemmas for the proof of Theorem 4. The first (Lemma 10) is a special case of results of Montgomery (cf. [10], Theorems 1 and 2 and the following remark), who proved that a subset E of a metric space X, which is locally of one of the four types below (that is, each point x of E has an open neighbourhood U such that $E \cap U$ is of the given type in X), is itself of that type in X. Kuratowski [6], pp. 358–362, gives Montgomery's proof, Michael [9], Proposition 4.2 and the preceeding remark, and Stone [13], Lemma 4, give other proofs for the Borel classes via transfinite induction and locally finite open refinements of open covers, and Hansell [3], Lemma 2, gives an easy transfinite induction proof of the special case, that is, of parts (a) and (b) of Lemma 10 below: parts (c) and (d) are straight forward.

LEMMA 10. *Let $\{E_\gamma\}_{\gamma \in \Gamma}$ be a discrete family of sets in a metric space X, and write*

$$E = \bigcup_{\gamma \in \Gamma} E_\gamma.$$

(a) *If $0 \leqslant \alpha < \omega_1$ and each set E_γ, $\gamma \in \Gamma$, is of additive Borel class α in X, then E is of additive Borel class α in X.*

(b) *If $0 \leqslant \alpha < \omega_1$ and each set E_γ, $\gamma \in \Gamma$, is of multiplicative Borel class α in X, then E is of multiplicative Borel class α in X.*

(c) *If each set E_γ, $\gamma \in \Gamma$, is a Souslin-$\mathscr{F}$ set in X, then E is a Souslin-$\mathscr{F}$ set in X.*

(d) *If each set E_γ, $\gamma \in \Gamma$, is a co-Souslin-$\mathscr{F}$ set in X, then E is a co-Souslin-$\mathscr{F}$ set in X.*

Since the families of additive Borel class α sets in X, Souslin-$\mathscr{F}$ sets in X, and co-Souslin-$\mathscr{F}$ sets in X are countably additive, it follows immediately that (a), (c) and (d) of Lemma 10 hold for σ-discrete families $\{E_\gamma\}_{\gamma \in \Gamma}$. In general (b) of Lemma 10, of course, does not hold for σ-discrete families, but the following lemma ensures that it does hold for the σ-discrete families that we consider. Lemma 11 is a slight variant of our Lemma 4 in [4]. Recall that an ambiguous Borel class α set in X is one that is both additive Borel class α in X and multiplicative Borel class α in X.

LEMMA 11. *Let Y be a subset of a metric space X and suppose that*

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

with each set Y_n of multiplicative Borel class α, with $2 \leqslant \alpha < \omega_1$, in X. If each set Y_n, considered as a subset of Y, is of ambiguous Borel class β less than α in Y, then Y is of multiplicative Borel class α in X.

Proof. If α is a limit ordinal, we can choose a sequence

$$\beta_1 \leqslant \beta_2 \leqslant \dots$$

of ordinals less than α with the property that α is the least ordinal exceeding β_i, for all $i \geqslant 1$. If α is of the form $\gamma + 1$ with $1 \leqslant \gamma$, we can take

$$\beta_i = \gamma, \quad i \geqslant 1,$$

and again α is the least ordinal exceeding β_i, for all $i \geqslant 1$. Now, in each case, for each $n \geqslant 1$, we can choose a sequence

$$Y_{nk}, \quad k = 1, 2, \dots,$$

of sets with Y_{nk} of additive class β_k in X and with

$$Y_n = \bigcap_{k=1}^{\infty} Y_{nk}, \quad n \geqslant 1.$$

Since each Y_n, considered as a subset of Y, is of ambiguous class β less than α in Y, as straight forward transfinite induction argument gives a set Y_n^* of ambiguous class β in X with

$$Y_n = Y \cap Y_n^*.$$

Write $Y_0 = \varnothing$ and

$$W_n = Y_n^* \setminus \bigcup_{i=0}^{n-1} Y_i^*,$$

for $n \geq 1$. Then we have

$$Y \cap W_n = Y_n \setminus \bigcup_{i=0}^{n-1} Y_i, \quad n \geq 1,$$

and

$$Y_n \cap W_n = Y_n \setminus \bigcup_{i=0}^{n-1} Y_i, \quad n \geq 1.$$

Consider the set

$$Z = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} (W_n \cap Y_{nk})$$

in X. Now for each n, k, the set $W_n \cap Y_{nk}$ is of additive class $\max(\beta, \beta_k)$. As $\alpha \geq 2$, this set Z is of multiplicative Borel class α in X.

It remains to identify Z with Y. If $y \in Y$, then for just one $n \geq 1$, we have

$$y \in Y_n \setminus \bigcup_{i=0}^{n-1} Y_i = W_n \cap Y_n.$$

Then, for each n and each $k \geq 1$,

$$y \in W_n \cap Y_{nk},$$

and so $y \in Z$. On the other hand, if $z \in Z$, then for each $k \geq 1$,

$$z \in \bigcup_{n=1}^{\infty} (W_n \cap Y_{nk}).$$

As the sets $W_1, W_2, \ldots$ are disjoint, there is an n, independent of k, such that

$$z \in W_n \cap Y_{nk}$$

for $k \geq 1$. Hence

$$z \in \bigcap_{k=1}^{\infty} Y_{nk} = Y_n \subset Y.$$

Thus $Z = Y$ as required.

Summarizing we have

LEMMA 12. *Let $\{E_\gamma\}_{\gamma \in \Gamma}$ be a σ-discrete family of sets in a metric space X, and write*

$$E = \bigcup_{\gamma \in \Gamma} E_\gamma.$$

(a) *If $0 \leqslant \alpha < \omega_1$ and each set E_γ, $\gamma \in \Gamma$, is of additive Borel class α in X, then E is of additive Borel class α in X.*

(b) *If $2 \leqslant \alpha < \omega_1$ and each set E_γ, $\gamma \in \Gamma$, is of multiplicative Borel class α in X, and if each E_γ, $\gamma \in \Gamma$, considered as a subset of E, is of ambiguous Borel class β less than α in E, then E is of multiplicative Borel class α in X.*

(c) *If each set E_γ, $\gamma \in \Gamma$, is a Souslin-$\mathscr{F}$ set in X, then E is a Souslin-$\mathscr{F}$ set in X.*

(d) *If each set E_γ, $\gamma \in \Gamma$, is a co-Souslin-$\mathscr{F}$ set in X, then E is a co-Souslin-$\mathscr{F}$ set in X.*

Proof of Theorem 4. We confine our attention to the case (b); the other cases are similar but slightly simpler.

We use the notation and results of Theorem 1 and its proof. By the result (a) of Theorem 1, the set K is a $\mathscr{G}_\delta$-set and so is a set of multiplicative class α as $\alpha > 1$. By the result (c) of Theorem 1, we have

$$T \setminus K = \bigcup_{\beta \in B} P_\beta \times S_\beta,$$

where the family $\{P_\beta \times S_\beta\}_{\beta \in B}$ is σ-discrete, and each rectangle $P_\beta \times S_\beta$ is relatively closed in T, and so also in $T \setminus K$, and each set P_β is closed in X. We may clearly suppose that

$$P_\beta \neq \varnothing,$$

for each β in B, and we can then choose p_β in P_β for each β in B. As

$$\{p_\beta\} \times S_\beta = (\{p_\beta\} \times Y) \cap (P_\beta \times S_\beta).$$

is a relatively closed subset of

$$(\{p_\beta\} \times Y) \cap T = (\{p_\beta\} \times F(p_\beta)),$$

the set S_β is of multiplicative Borel class α, for each β in B. Hence $P_\beta \times S_\beta$ is also of multiplicative Borel class α, for each β in B, as well as being relatively closed in

$$T \setminus K = \bigcup_{\beta \in B} P_\beta \times S_\beta.$$

By Lemma 12, the set

$$T \setminus K$$

and so also

$$T = (T \setminus K) \cup K,$$

is of multiplicative Borel class α in $X \times Y$.

COROLLARY. *If we know that each set $F(x)$, with $x \in X$, is an $\mathscr{F}_\sigma$-set in Y, we can conclude that T is the union of a $\mathscr{G}_\delta$-set with an $\mathscr{F}_\sigma$-set.*

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