

SETS OF UNIQUENESS FOR FUNCTIONS REGULAR IN THE UNIT CIRCLE.

By

LENNART CARLESON

of Uppsala.

1. For a large number of classes C of functions $f(z)$ regular in the unit circle, we have very complete knowledge concerning the existence of a boundary function

$$F(\theta) = \lim_{r \rightarrow 1} f(re^{i\theta}),$$

the classical result being that of Fatou. However, very little is known about the properties of this boundary function $F(\theta)$, and in particular about the sets E associated with the class C , having the property that $f(z)$ vanishes identically if $F(\theta) = 0$ on E . Let us call a closed set of this kind a *set of uniqueness* for the class C . If E is not a set of uniqueness, we speak of a set of multiplicity. Our whole knowledge in this direction seems to be contained in a classical result of F. and M. Riesz: E is a set of uniqueness for the class of bounded functions if and only if it has positive Lebesgue measure.

We shall here consider the problem of finding the sets of uniqueness for three different classes of functions, namely:

- 1°. Functions with high regularity in $|z| \leq 1$;
- 2°. Functions with a bounded Dirichlet integral;
- 3°. Absolutely convergent Taylor series.

The first class gives us information regarding the nature of the boundary function of analytic functions in general and shows clearly the decisive role of the integral

$$\int_0^{2\pi} \log |f(re^{i\theta})| d\theta.$$

The second class will be considered on account of its function theoretic interest, and we shall here be concerned mainly with the extremal problem associated with a set of multiplicity. The third class, finally, has been selected on account of its importance for general problems in function spaces. In all three cases, new properties of sets are of importance, and the non-metrical nature of the problem is evident. It should be noted that no complete solution in either case 2 or case 3 is given. It seems also highly improbable that there are any known set theoretical equivalences in these two cases.

I. Functions with high regularity.

2. Let E be an arbitrary closed set in $(0, 2\pi)$. For $t > 0$, let E_t be the closed set of points with distance $\leq t$ from E . We then consider the measure

$$\varphi(t) = \varphi_E(t) = m E_t.$$

The properties of this function as $t \rightarrow 0$, are decisive for the sets of uniqueness for functions with high regularity. The importance of this function for uniqueness problems of this type was recognized first by Beurling [1], who proved the direct part of the following theorem.

Theorem 1. *Let E be a closed set in $(0, 2\pi)$. If a function $f(z)$ belongs to one of the following classes*

(a) $f(z)$ satisfies a Lipschitz condition of order $\alpha > 0$;

(b) $f(z) = \sum_0^{\infty} a_n z^n$, $|a_n| = O(n^{-p})$, $p > 1$;

then E is a set of uniqueness if and only if the integral

$$(1) \quad \int_0^1 \frac{\varphi_E(t)}{t} dt$$

diverges.

We first prove a simple lemma which shows the connection between our condition and that in Beurling [1].

Lemma. *If $\{l_n\}$ denotes the lengths of the finite complementary intervals of E , the divergence of (1) is equivalent to the statement*

$$(2) \quad mE > 0 \quad \text{or} \quad \sum_{n=1}^{\infty} l_n \log \frac{1}{l_n} = \infty.$$

If $mE > 0$, then $\varphi(t) \geq \varphi(0) > 0$ and (1) diverges trivially. We therefore assume that $mE = 0$. If we introduce the function

$$\psi(t) = \sum_{l_v \leq t} l_v$$

we find by a partial integration that the divergence of the series in (2) is equivalent to the divergence of

$$(3) \quad \int_0^1 \frac{\psi(t)}{t} dt.$$

We now have

$$\psi(t) \leq \varphi(t) \leq \sum_{2^n t \leq 2\pi} 2^{-n+1} \psi(2^n t) + 2t.$$

Thus, if (3) diverges, the same is true for the integral (1). To prove the converse, we need only use the inequalities

$$\begin{aligned} \int_0^1 \frac{\varphi(t)}{t} dt &\leq 2 + \sum_0^\infty 2^{-n+1} \int_0^{2\pi 2^{-n}} \frac{\psi(2^n t)}{t} dt \\ &= 2 + 4 \int_0^{2\pi} \frac{\psi(t)}{t} dt. \end{aligned}$$

Let us first consider the non-constructive case when the integral (1) diverges. We assume that $f(z) \not\equiv 0$ belongs to the Lipschitz class of order α , and that $F(\theta) = 0$ on E . Since $f(z)$ is a bounded function, it follows from Jensen's formula that

$$(4) \quad \int_0^{2\pi} \log |F(\theta)| d\theta > -\infty.$$

Since f satisfies a Lipschitz condition, we deduce that $|F(\theta)| \leq M t^\alpha$ when $\theta \in E_t$, M being a fixed constant. Hence the integral (4) is less than

$$\int_0^\pi \log (M t^\alpha) d\varphi(t) = M \varphi(\pi) + \alpha \int_0^\pi \log t d\varphi(t).$$

The last integral diverges by hypothesis and consequently we have $f(z) \equiv 0$.

The non-trivial part of the proof consists in the construction of $f(z) \not\equiv 0$ with $F(\theta) = 0$ on E and with the regularity (b), when the integral (1) converges. Let $\omega_v = (\alpha_v, \beta_v)$ denote the complementary intervals of E , where $\alpha_1 = 0$ and $\beta_2 = 2\pi$, and define a realvalued function $h(t)$ by the conventions

$$h(t) = K \left\{ \log \frac{1}{\beta_v - t} + \log \frac{1}{t - \alpha_v} \right\}, \quad \alpha_v < t < \beta_v, \quad v = 1, 2, \dots$$

where $K > 1$ is a constant. By the lemma this function is defined almost everywhere in $(0, 2\pi)$ and

$$\int_0^{2\pi} |h(x)| dx < \infty.$$

We can thus form the function

$$f(z) = \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ix} + z}{e^{ix} - z} h(x) dx \right\},$$

so that $f(z)$ is analytic and bounded in $|z| < 1$. We shall prove that, for $K > K(p)$, $f(z)$ satisfies condition (b).

We have first to prove that $F(\theta)$ vanishes on E . It is evident that $F(\theta) = 0$ if θ belongs to the boundary of any complementary interval of E . For an arbitrary $\theta \in E$, let I_η denote the interval $\theta - \eta < x < \theta + \eta$. We find that

$$\lim_{\eta \rightarrow 0} \frac{1}{2\eta} \int_{I_\eta} h(x) dx \geq \lim_{\eta \rightarrow 0} \frac{K}{2\eta} \log \frac{1}{2\eta} \cdot 2\eta = \infty,$$

from which it follows that $F(\theta) = 0$. If we introduce the notation

$$\varrho_v(\theta) = |\theta - \alpha_v| \cdot |\beta_v - \theta|,$$

we have

$$(5) \quad |F(\theta)| = \varrho_v(\theta)^K \text{ for } \theta \in \omega_v.$$

Let O be the complement of E with respect to $(0, 2\pi)$. We shall show that a constant M exists such that

$$(6) \quad \overline{\lim}_{r \rightarrow 1} |f'(re^{i\theta})| < M, \quad \theta \in O,$$

provided that $K > 2$. By differentiation, we find

$$f'(z) = -\frac{1}{\pi} \int_0^{2\pi} \frac{e^{ix} h(x) dx}{(e^{ix} - z)^2} \cdot f(z).$$

Suppose that $z = re^{i\theta}$ and that θ belongs to ω_v . If ω' denotes the set $|\theta - x| > \frac{1}{2} \varrho_v(\theta)$, we have

$$\overline{\lim}_{r \rightarrow 1} \left| \int_{\omega'} \frac{e^{ix} h(x) dx}{(e^{ix} - z)^2} \right| \leq \frac{\text{Const.}}{\varrho_v(\theta)^2} \int_0^{2\pi} |h(x)| dx.$$

On account of (5), it is therefore sufficient to consider the integral taken over the complement ω of ω' , where $h(x)$ has a very simple form. In the interval ω we expand $h(x)$ in its Taylor series. We have

$$\log \frac{1}{t - \alpha_\nu} = \log \frac{1}{\theta - \alpha_\nu} + \frac{\theta - t}{\theta - \alpha_\nu} + (\theta - t)^2 O(\varrho_\nu(\theta)^{-2})$$

and a similar expression corresponding to β_ν . If we use these expansions, we can again integrate over the whole interval $(0, 2\pi)$, thus making an error of the same order of magnitude as before. We finally get

$$\overline{\lim}_{r \rightarrow 1} |f'(r e^{i\theta})| < \text{Const. } \varrho_\nu(\theta)^{K-2}, \theta \in \omega_\nu,$$

and assertion (6) follows.

In general, it is not possible to infer that $f'(z)$ is bounded in $|z| < 1$ from an inequality of type (6). In our case, however, it is possible. Namely, if θ and θ' both belong to ω_ν , it follows from (6) that

$$(7) \quad |F(\theta) - F(\theta')| \leq M |\theta - \theta'|,$$

and if $\theta \in \omega_\nu$ and $\theta' \in \omega_\mu$, $\nu \neq \mu$, we find in the same way

$$|F(\theta) - F(\theta')| \leq M |\theta - \alpha_\nu| + M |\theta' - \beta_\mu| \leq M |\theta - \theta'|,$$

say. If at least one of θ or θ' belongs to E , (7) is verified trivially. Now, since $f(z)$ is a bounded function, it must coincide with the Cauchy integral of $F(\theta)$, and since this function satisfies a Lipschitz condition of order 1, the same must be true for $f(z)$ considered as a function in $|z| < 1$. Hence, for some constant M' we have

$$|f(z) - f(z')| \leq M' |z - z'|, |z| < 1, |z'| < 1,$$

which evidently implies that $|f'(z)| \leq M'$.

We can now use exactly the same argument to show that if $K > 3$, then

$$\overline{\lim}_{r \rightarrow 1} |f''(r e^{i\theta})| < \text{Const. } \varrho_\nu(\theta)^{K-3}, \theta \in \omega_\nu.$$

The only difference is that we have to use one more term in the Taylor development of $h(x)$. As before, it then follows that $|f''(z)| \leq M''$ in $|z| < 1$. Finally we find that if $K > p + 1$, $f^{(p)}(z)$ is bounded in $|z| < 1$; this implies that $f(z)$ satisfies condition (b). The theorem is thus completely proved.

3. The situation here is remarkable in that if the integral (1) diverges, only functions with very weak regularity can vanish on E without vanishing identically, while if the integral converges, we can construct non-trivial functions with very high regularity. In order that a less restrictive condition than the divergence of (1) should imply $f(z) \equiv 0$, we have to impose strong conditions on the function.

Theorem 2. A function $f(z) = \sum_0^{\infty} a_n z^n$ whose coefficients a_n satisfy

$$\left\{ \log \frac{1}{|a_n|} \right\}^{-1} = O(n^{-p}), \quad 0 < p < \frac{1}{2},$$

vanishes identically if $F(\theta)$ vanishes on a perfect set E and

$$(8) \quad \int_0^1 \frac{\varphi_E(t)}{t^{1-p}} dt = \infty.$$

Under our assumption, the limit function $F(\theta)$ has derivatives of all orders and $F^{(k)}(\theta)$ satisfies an inequality

$$|F^{(k)}(\theta)| \leq \sum_1^{\infty} n^k |a_n| \leq M^k \Gamma\left(\frac{k}{p}\right)$$

for some constant M . Since E is a perfect set, it follows that $F^{(k)}(\theta) = 0$, if $\theta \in E$, for $k = 0, 1, \dots$. Let θ belong to $\omega_r = (\alpha_r, \beta_r)$, a complementary interval of length l_r . Then

$$\begin{aligned} |F(\theta)| &\leq \inf_k \left| \int_{\alpha_r}^{\beta_r} \frac{(t-\theta)^{k-1}}{(k-1)!} F^{(k)}(t) dt \right| \\ &\leq \inf_k M^k \Gamma\left(\frac{k}{p}\right) \frac{l_r^k}{k!} \leq \exp \left\{ -M' l_r^{-\frac{p}{p-1}} \right\}. \end{aligned}$$

We therefore have

$$\int_0^{2\pi} \log |F(\theta)| d\theta \leq -M' \sum_1^{\infty} l_r^{\frac{1-2p}{1-p}}.$$

In the same way as before, we can prove that the last series diverges in view of our assumption (8), unless $mE > 0$; in this case, however, the theorem is trivially true.

Finally, we shall treat the most regular case, which leads to the following

problem: what condition must a function satisfy in order that $f(z_v) = 0$ for any infinite sequence $\{z_v\}$ in $|z| \leq 1$ should imply $f(z) \equiv 0$?

Theorem 3. *If $|a_n| \leq \varrho_n$, where $\left\{\log \frac{1}{\varrho_n}\right\}$ is a concave sequence and*

$$\sum_1^{\infty} \frac{\log \varrho_n}{n^{3/2}} = -\infty,$$

then $f(z) \equiv 0$, if $f(z)$ has Taylor coefficients a_n and vanishes on an infinite set in $|z| \leq 1$.

This theorem is essentially a consequence of Carleman's fundamental theorem on quasi analytic functions. We can obtain a simple proof in the following way. The zeros of $f(z)$ have a limit point, which we assume to be $z = 1$. Then $F^{(k)}(0) = 0$, $k = 0, 1, \dots$. The function

$$\psi(\zeta) = \sum_1^{\infty} a_n n^\zeta$$

is analytic for $\operatorname{Re}\{\zeta\} > 0$ and $\psi(k) = 0$, $k = 1, 2, \dots$. Furthermore,

$$|\psi(\xi + i\eta)| \leq \sum_1^{\infty} \varrho_n n^\xi = e^{m(\xi)}.$$

A theorem of Ostrowski and our assumption on ϱ_n implies that

$$\int_0^{\infty} \exp\left(-\frac{m(\xi)}{2\xi}\right) d\xi = \infty.$$

Then $\psi(\zeta) \equiv 0^1$ and $a_n = 0$, $n = 0, 1, \dots$

II. Functions with a bounded Dirichlet integral.

1. Sets of uniqueness.

4. In this chapter, we shall consider two different problems for the class D of functions with a bounded Dirichlet integral: we shall study the sets of uniqueness and, when E is not a set of uniqueness, investigate the associated extremal function.

We first observe that Theorem 1 has a non-trivial application to this class D : a set E is not a set of uniqueness if the integral (1) converges. This result shows an interesting difference between conformal mappings which are schlicht and non-

¹ Compare, Carleson [2].

schlicht. Under a univalent mapping of the unit circle onto a domain of finite area, at most a set of logarithmic capacity zero on the circumference of the circle can be mapped onto a single point¹, while there exist Riemann surfaces of finite area having boundary points which project onto a single point and correspond to a set on the circle with dimension arbitrarily close to 1.

For the class D , $F(\theta)$ is determined except on a set of capacity zero. It is therefore to be expected that sets of capacity zero can be neglected in considering uniqueness problems. We have, in fact, the following result.

Theorem 4. *If E is a closed set of capacity zero, there exists a function $f \in D$ with $F(\theta) = 0$ on E .*

This theorem is known, but since there seems to be no proof in print, we shall sketch a proof here.

Let E_n , $n = 1, 2, \dots$, denote a finite sum of closed intervals, which contains E , and let $u_n(z)$ be the equilibrium potential corresponding to E_n ,

$$u_n(e^{i\theta}) = V_n, \quad \theta \in E_n.$$

By assumption, we can choose E_n so that $\sum_1^\infty V_n^{-1} < \infty$. Then

$$U(z) = \sum_{n=1}^{\infty} \frac{u_n(z)}{V_n}$$

is a harmonic function with a finite Dirichlet integral which tends to infinity on E . Defining $f(z)$ by the relation $\log|f| = -U$, we see that f is a function of the desired kind.

5. We now turn to the uniqueness problem. The essential properties of the sets of multiplicity are contained in the previous results, but we have to impose further regularity conditions on the set in question. Let $C_\alpha(E)$ denote capacity of order α .² We shall assume that for some $\alpha > 0$, $C_\alpha(E)$ is positive and that the set is homogeneous in the following sense. Given a point $x \in E$, let I_η denote the interval $(x - \eta, x + \eta)$ and assume that for each $x \in E$

$$(9) \quad C_\alpha(E \cap I_\eta) > m \eta$$

for a fixed constant $m > 0$.

¹ For this and other results concerning the class D , which are cited without reference, see Beurling [1].

² For definitions, see Carleson [1].

Theorem 5. *If (9) holds, then E is a set of uniqueness if and only if the integral (1) diverges.*

It is essential that some condition of type (9) be introduced in order to ensure that the divergence of (1) is not due to the presence of long complementary intervals in the neighbourhood of an isolated closed subset of E , which is metrically very thin (e. g. has capacity zero). It is to be noted that assumption (9) is actually very weak.

Let us first set down the following known facts concerning potentials. Let μ be a distribution of unit mass on the set E and suppose that the corresponding potential of order α is less than V . Then, if α_n and β_n are the Fourier-Stieltjes coefficients of μ , we have

$$(10) \quad \sum_1^{\infty} \frac{1}{n^{1-\alpha}} (\alpha_n^2 + \beta_n^2) \leq \text{Const. } V$$

where the constant depends only on α .

Now suppose that a function f in D exists with $F(\theta) = 0$ on E and that $f \not\equiv 0$. We shall prove that if (1) diverges, this assumption leads to a contradiction. As before, we may suppose that $mE = 0$. The function $h(x) = |F(x)|$ has a Fourier series

$$(11) \quad h(x) \sim \frac{1}{2} a_0 + \sum_0^{\infty} (a_n \cos nx + b_n \sin nx)$$

such that

$$\sum_1^{\infty} n (a_n^2 + b_n^2) < \infty.$$

Let us consider the function $h_t(x) = \frac{1}{2t} \int_{x-t}^{x+t} h(y) dy$, $t > 0$. The corresponding Fourier

coefficients are those of $h(x)$ multiplied by $\frac{\sin nt}{nt}$. For a distribution μ satisfying (10), it follows by Schwarz's inequality that

$$(12) \quad \int_E h_t(x) d\mu(x) = \int_E (h_t(x) - h(x)) d\mu(x) < \text{Const. } V^{\frac{1}{2}} t^{\frac{\alpha}{3}}.$$

Let k_n denote the number of complementary intervals of E with lengths lying between 2^{-n+1} and 2^{-n} . It follows from our assumption that (1) diverges, that

$$\sum_{n=1}^{\infty} \frac{n k_n}{2^n} = \infty.$$

Let $\omega_1, \omega_2, \dots, \omega_{k_n}$, be the intervals in question. Let δ_i , $i = 1, 2, \dots, 2k_n$, be the intervals of length 2^{-n} with midpoints x_i at the endpoints of the intervals ω_v . Each δ_i intersects at most one δ_j , $i \neq j$. Let $\gamma > 0$ be a constant to be determined later; consider those intervals δ_i for which

$$(13) \quad h_\tau(x_i) > 2^{-\gamma n}, \quad \tau = 2^{-n},$$

holds, and let S be the set formed by these intervals. The inequalities (13) imply that $h_{2^\tau}(x) > 2^{-\gamma n-1}$ holds for x belonging to S . From the general relation (12) it follows that

$$C_\alpha(ES) < \text{Const. } 2^{(2\gamma - \frac{1}{3})n}.$$

Let N be the number of intervals δ_i constituting S . In order to estimate N , we must use assumption (9). If μ_i is the equilibrium distribution for potentials of order α of $E\delta_i$, we construct the set function

$$\mu = N^{-1} \sum_{\delta_i \in S} \mu_i.$$

For the corresponding α -potential u , we have by (9) the estimate

$$u \leq \text{Const.} \left\{ 2^n N^{-1} + N^{-1} 2^{n\alpha} \sum_1^N \nu^{-\alpha} \right\} \leq \text{Const. } 2^n N^{-1}.$$

Hence

$$N 2^{-n} < \text{Const. } 2^{(2\gamma - \frac{1}{3})n}$$

and

$$N < \text{Const. } 2^{pn}, \quad \text{where } p < 1 \text{ if } \gamma < \frac{\alpha}{3}.$$

If the inequality (13) does not hold for the endpoints of an interval ω_v , we have

$$\frac{1}{m \omega_v} \int_{\omega_v} \log h(x) dx \leq \log \left\{ \frac{1}{m \omega_v} \int_{\omega_v} h(x) dx \right\} \leq -\gamma n + \text{Const.}$$

The number of indices ν , for which the inequality above is true, is greater than $k_n - 2N \geq k_n - \text{Const. } 2^{pn}$. Hence

$$\sum_{\nu=1}^{k_n} \int_{\omega_\nu} \log h(x) dx \leq -\gamma n 2^{-n} (k_n - \text{Const. } 2^{pn}) + \text{Const.} \sum_1^{k_n} m \omega_\nu.$$

We finally get since $p < 1$

$$\int_0^{2\pi} \log h(x) dx \leq -\gamma \sum_1^\infty n k_n 2^{-n} + \text{Const.}$$

Since this series diverges and $\gamma > 0$ we have obtained the desired contradiction.

The general situation concerning sets of uniqueness for the class D can be summarized in the following way: when we weaken the metrical assumptions on the sets, we must strengthen the assumptions on the complementary intervals. Finally, let us note the following very special consequence of theorem 5: there are sets of uniqueness of measure zero and even of arbitrarily small dimension. A construction can also very easily be carried out in the same way as in the proof of theorem 8 below.

2. The extremal problem.

6. In this section, we shall study the extremal problem associated with a set of multiplicity. We shall consider only the most regular case, where the integral (1) converges. Theorem 4 here shows that E has positive logarithmic capacity.

In D , we introduce the scalar product of two functions $f(z) = \sum_0^\infty a_n z^n$ and $g(z) = \sum_0^\infty b_n z^n$ by the relation

$$(f, g) = \sum_1^\infty n a_n \bar{b}_n.$$

Then $\|f\|^2 = (f, f)$ is the Dirichlet integral of $f(z)$ if we disregard a factor π^{-1} . We denote by D_E the linear, convex subspace of D for which $f(0) = 1$ and

$$(14) \quad \lim_{r \rightarrow 1} f(r e^{i\theta}) = 0, \quad \theta \in E,$$

except perhaps on a set of (inner) capacity zero (= "a.e."). We shall first prove the following result.

Theorem 6. D_E is a closed subspace of D .

Let $\{f_n(z)\}$ be a sequence of functions in D_E and suppose that $f(z) \in D$ exists such that $\|f - f_n\| \rightarrow 0, n \rightarrow \infty$. We must prove that (14) holds "a.e.". If μ is an arbitrary distribution of unit mass on E with finite energy integral, it is sufficient to prove that the boundary function of f cannot be different from zero on a set where μ does not vanish. Exactly as in the proof of theorem 5, it follows by developing $|f(e^{ix}) - f_n(e^{ix})|$ in a Fourier series that

$$\lim_{n \rightarrow \infty} \lim_{r \rightarrow 1} \int_0^{2\pi} |f(r e^{ix}) - f_n(r e^{ix})| d\mu(x) = 0,$$

if we observe that $\|f - f_n\| \rightarrow 0$. Since $\lim_{r \rightarrow 1} f_n(r e^{ix}) = 0$ "a.e." on E , it follows that f has the same property.

7. We now come to the extremal problem of finding an f in D_E realizing the lower bound

$$\inf_{f \in D_E} \|f\| = M,$$

and we shall show that the lower bound is attained for a function with very regular behaviour.

Theorem 7. *There exists a unique function $f(z)$ in D_E for which $\|f\| = M$ holds. Its derivative $f'(z)$ is analytic and single valued in the whole plane except on the set E regarded as a subset of the circle $|z| = 1$.*

We first have to prove that $M > 0$. Let $g \in D_E$ have Taylor coefficients b_n and suppose again that μ has a finite energy integral and that $\mu(E) = 1$. Then

$$0 = \lim_{r \rightarrow 1} \int_E g(re^{ix}) d\mu(x) = 1 + \lim_{r \rightarrow 1} \int_E \sum_1^{\infty} b_n r^n e^{inx} d\mu(x).$$

The last limit is less than a multiple of $\|g\|$, as an application of Schwarz's inequality shows. Hence $\|g\|$ is bounded from below, and it follows that $M > 0$.

If now $\|f_n\| \rightarrow M$, $f_n \in D_E$, then by a standard argument in the theory of Hilbert space, D_E being convex, we have

$$\|f_n - f_m\| \rightarrow 0, \quad n, m \rightarrow \infty.$$

Since D_E is closed, an extremal function $f \in D_E$ exists and is unique.

The rest of the proof will be devoted to establishing the last statement of the theorem. We start with the following lemma.

Lemma. *Let $\varphi(x)$ be integrable on $(0, 1)$ and suppose that for all $\psi(x)$ with continuous second derivative in $(0, 1)$ and such that*

$$\varrho(\psi) = \int_0^1 |\psi(x)| dx \leq 1, \quad \psi(0) = \psi(1) = 0,$$

$\varphi(x)$ satisfies the relation

$$L(\psi) = \int_0^1 \varphi(x) \psi'(x) dx \leq 1.$$

Then we can redefine $\varphi(x)$ on a set of measure zero so that $\varphi'(x)$ exists a.e. and satisfies $|\varphi'(x)| \leq 1$.

It follows from our assumptions that $L(\psi) \leq \varrho(\psi)$. If ψ is an arbitrary function which is integrable on $(0, 1)$, then functions $\{\psi_n\}$ in the domain of definition of L exist such that $\varrho(\psi - \psi_n) \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see that

$$L(\psi) = \lim_{n \rightarrow \infty} L(\psi_n)$$

exists and is independent of the particular sequence $\{\psi_n\}$ chosen. Clearly, it extends $L(\psi)$ to a linear functional on $L^1(0, 1)$. Hence a bounded function $\lambda(x)$ exists such that

$$L(\psi) = \int_0^1 \psi(x) \lambda(x) dx, \quad |\lambda(x)| \leq 1.$$

Let x_0 and x be two fixed points, $x_0 < x$, and define for $\varepsilon > 0$ a continuous function $\psi_\varepsilon(t)$ so that $\psi_\varepsilon(t) = 0$ for $t < x_0$ and $t > x + \varepsilon$, $\psi_\varepsilon(t) = 1$ for $x_0 + \varepsilon < t < x$ and let $\psi_\varepsilon(t)$ be linear for other values of t . Then for almost all x and x_0 , we have

$$\varphi(x_0) - \varphi(x) = \lim_{\varepsilon \rightarrow 0} L(\psi_\varepsilon) = \int_{x_0}^x \lambda(t) dt.$$

We define $\varphi(x)$ so that this holds for all x and find $\varphi'(x) = -\lambda(x)$ a.e.; this proves the lemma.

8. We return to the proof of theorem 7. For all complex numbers t and every $g \in D_E$, $f + tzg$ belongs to D_E . Since f solves our extremal problem, it follows in the usual way that

$$(15) \quad (f, zg) = 0, \quad g \in D_E.$$

We consider a point ξ in the complement of E . By the method used in the proof of theorem 1, we construct a function $g \in D_E$ such that its boundary function has absolute value 1 in an interval I around ξ and has continuous second derivative in $|z| \leq 1$. By a conformal mapping of the image domain, we construct a similar function $h = h_1 + ih_2$, belonging to D_E , such that $h(0) = 0$ and such that its real part h_1 vanishes on an interval $\omega \subset I$, $\xi \in \omega$. We may also assume that $h_2 \neq 0$ on ω . We observe that h is analytic on ω by Schwarz's reflection principle.

Let now $p = p_1 + ip_2$ be an arbitrary analytic function in $|z| < 1$, continuous in $|z| \leq 1$, and consider the function

$$hp = \sigma = \sigma_1 + i\sigma_2.$$

On ω , we have $\sigma_1 = -h_2 p_2$. We consider a subinterval ω' of ω , which still contains ξ . If we determine p_2 so that σ_1 assumes prescribed values $\psi(x)$ on ω' , where ψ has continuous second derivative on ω' , vanishes at the endpoints of ω' , and satisfies

$$(16) \quad \int_{\omega'} |\psi(x)| dx \leq 1,$$

and if we let p_2 vanish outside of ω' , then σ has a uniformly continuous derivative on $|z| = 1$ outside of ω . Since $h(0) = 0$, we have by (15) $(f, \sigma) = 0$, and if $f = u + iv$, this relation can be written in the form

$$\iint_{|z| < 1} \left(\frac{\partial v}{\partial x} \frac{\partial \sigma_2}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial \sigma_2}{\partial y} \right) dx dy = 0, \quad z = x + iy.$$

By assumption, σ_2 has a continuous derivative in $|z| \leq 1$. We integrate by parts and find

$$\lim_{r \rightarrow 1} \int_0^{2\pi} v \frac{\partial \sigma_2}{\partial r} d\theta = 0, \quad z = r e^{i\theta}.$$

If we observe that $\frac{\partial \sigma_2}{\partial r} = -\frac{1}{r} \frac{\partial \sigma_1}{\partial \theta}$ and that $\frac{\partial \sigma_1}{\partial \theta}$ vanishes on $\omega - \omega'$ and is uniformly bounded outside of ω , we obtain

$$\left| \int_{\omega'} v \frac{\partial \sigma_1}{\partial \theta} d\theta \right| < \text{Const.}$$

for all σ_1 satisfying (16). It follows now from the lemma that $\frac{\partial v}{\partial \theta}$ exists a.e. on ω' and is bounded. A similar argument shows that $\frac{\partial u}{\partial \theta}$ is bounded in a neighbourhood of ξ . If we now let ξ vary, we find that $f'(z)$ is bounded on $|z| = 1$ except on an arbitrary open set containing E .

9. We are now compelled to use another variational argument. Suppose that $\tau(z)$ is harmonic in $|z| < 1$ and continuously differentiable in $|z| \leq 1$ and that $\tau(0) = 0$. If $\tau'(z)$ is the conjugate function of τ , normalized by $\tau'(0) = 0$, then for every real t , the function $g = f e^{t(\tau + i\tau')}$ belongs to D_E . If we set $U(z) = |f(z)|^2$ and $V(z) = |g(z)|^2$, then

$$4\pi \|g\|^2 = \iint_{|z| < 1} \Delta V dx dy \geq \iint_{|z| < 1} \Delta U dx dy = 4\pi \|f\|^2.$$

An ordinary variational argument shows that, $U_x = \frac{\partial U}{\partial x}$ etc.,

$$\iint_{|z| < 1} \{ \tau \Delta U + 2(U_x \tau_x + U_y \tau_y) \} dx dy = 0.$$

With the aid of Green's formula, we rewrite this in the form

$$\lim_{r \rightarrow 1} \int_0^{2\pi} \left(\frac{\partial U}{\partial r} \tau + \frac{\partial \tau}{\partial r} U \right) d\theta = 0, \quad z = r e^{i\theta}.$$

Let $u(z)$ be the harmonic function in $|z| < 1$ which is equal to U on $|z| = 1$. We may then, on account of the regularity of τ , replace U by u in the last term above, and we obtain after one more application of Green's formula

$$(17) \quad \lim_{r \rightarrow 1} \int_0^{2\pi} \left(\frac{\partial U}{\partial r} + \frac{\partial u}{\partial r} \right) \tau d\theta = 0.$$

Let us first consider the particular case when τ vanishes on $|z| = 1$ in a neighbourhood of E . Since $f'(z)$ is bounded when $\tau \neq 0$, it follows from (17) that

$$(18) \quad \frac{\partial U}{\partial r} + \frac{\partial u}{\partial r} = \lambda \text{ a.e. outside of } E,$$

where λ is a certain constant. To determine λ , let us formally set $\tau \equiv 1$ in (17). The integral then tends to $4\pi \|f\|^2$. Let S_ε be a finite sum of intervals containing E such that $mS_\varepsilon = \varepsilon$. In (17) we choose $\tau = 1$ on S_ε and $\tau = -1$ in a suitable interval of length ε in the complement of S_ε , and set $\tau = 0$ on the rest of $|z| = 1$. The formula may be applied although τ is not uniformly continuous, and we see that

$$\lim_{\varepsilon \rightarrow 0} \lim_{r \rightarrow 1} \int_{S_\varepsilon} \left(\frac{\partial U}{\partial r} + \frac{\partial u}{\partial r} \right) d\theta = 0.$$

Outside of S_ε , the integrand tends boundedly to λ , and hence we have proved that $\lambda = 2 \|f\|^2$.

10. Let us now assume that $f(z) = \sum_0^\infty a_n z^n$. The general relation (15) gives us $(f, z^p f) = 0$ for $p \geq 1$. If we set $a_n = 0$ for $n < 0$ and define c_p by the equality

$$c_p = \sum_1^\infty n \overline{a_n} a_{n+p},$$

it is apparent that $c_p = 0$ for $p < 0$. We form the series

$$F(z) = \sum_{p=0}^\infty c_p z^p.$$

Let us again consider a subinterval ω of the complement of E . If we use the fact that $c_p = 0$ for $p < 0$, we see that F has a representation

$$F(z) = \lim_{\rho \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\overline{\partial f(\zeta)}}{\partial \rho} f(\zeta) \operatorname{Re} \left\{ \frac{\zeta + z}{\zeta - z} \right\} d\varphi, \quad \zeta = \rho e^{i\varphi}.$$

Since $f'(z)$ is bounded on ω , it follows by an application of Green's formula in the sector $\arg z \notin \omega$, that $F(z)$ is bounded on ω and that

$$\lim_{r \rightarrow 1} F(re^{i\theta}) = e^{-i\theta} f(e^{i\theta}) \lim_{r \rightarrow 1} \overline{f'(re^{i\theta})}$$

almost everywhere on ω . Let G be the closed subset of ω where $f(e^{i\theta})$ vanishes. Since all functions involved are bounded on ω , it follows by a known theorem on analytic continuation that $f'(z)$ can be analytically continued over $\omega - G$ by the function

$$f'(z) = \frac{1}{z} \frac{\overline{F\left(\frac{1}{\bar{z}}\right)}}{f\left(\frac{1}{\bar{z}}\right)}, \quad |z| > 1.$$

It now remains to prove that G is empty and that $f(z)$ does not vanish in $|z| < 1$. To prove that G is empty, let us assume that $f(e^{i\alpha}) = 0$ for $\alpha \in \omega$. $\frac{\partial U}{\partial r}$ is continuous on $\omega - G$ and tends to zero on G , since f' is bounded. Hence the relation (18) must hold everywhere on ω , since $\frac{\partial u}{\partial r}$ is bounded there. At the point α , $\frac{\partial U}{\partial r} = 0$ while $\frac{\partial u}{\partial r} < 0$. This would imply $\lambda < 0$, which contradicts our determination of λ . Therefore G is empty, and $f'(z)$ is meromorphic outside of E .

Let us finally assume that $f'(z)$ has a pole, i.e. that $f(z_0) = 0$ for some z_0 in the unit circle. It is then very easy to prove¹ that if

$$g(z) = z_0 \frac{1 - z\bar{z}_0}{z_0 - z} f(z),$$

then $g(z)$ belongs to D_E and $\|g\| < \|f\|$. This contradicts the minimality of f .

Exactly the same argument can be used to prove that the extremal problem associated with a sequence of points $\{z_n\}$ in $|z| < 1$, $f(z_n) = 0$, has a solution with similar properties. The essential difference is that in this case, $f'(z)$ has simple poles at the points \bar{z}_n^{-1} .

¹ See Lokki [1], p. 27.

III. Absolutely convergent Taylor series.

11. The problem of determining the sets of uniqueness for the class A of absolutely convergent Taylor series is of great interest for functional analysis. We shall here mention a closure problem which is equivalent to the uniqueness problem.

Let H^2 be the Hilbert space of square integrable analytic functions $F(z) = \sum_0^{\infty} a_n z^n$ in $|z| < 1$ with the norm

$$\|F\|^2 = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 d\theta = \sum_0^{\infty} |a_n|^2.$$

Let $F(z)$ be a given function and consider its rotations $F_t(z) = F(e^{it}z)$. For H^2 , the Wiener closure problem is trivial: $\{F_t(z)\}$, $0 \leq t < 2\pi$, is fundamental on H^2 if and only if $a_n \neq 0$, $n = 0, 1, 2, \dots$. The relevant problem on H^2 is the following: determine the closure sets E for which it is true that $\{F_t(z)\}$, $t \in E$, is fundamental on H^2 , whenever $a_n \neq 0$, $n = 0, 1, \dots$. It is obvious that E is a closure set if and only if E is a set of uniqueness for A . Let us in this connection observe that for certain subspaces of H^2 the complete solution of the closure problem is given in theorem 1. If, for example, S_α is characterized by the convergence of $\sum_1^{\infty} |a_n|^2 n^\alpha$, $0 < \alpha < \infty$, then the closure sets corresponding to S_α are independent of α and are exactly those sets for which the integral (1) diverges.

12. We shall first show that there exist closed sets of uniqueness with measure zero. Since no property of continuity is involved in the definition of A , this is far from obvious and depends on the arithmetical nature of the class.

Theorem 8. *There exist closed sets E of uniqueness for the class A with vanishing Lebesgue measure.*

Lemma. *Let p be a given number, $0 < p < \frac{1}{2}$. Then there exists a closed set F of measure zero such that its Lebesgue function $\mu(x)$ has the property*

$$\mu_n = \int_F e^{inx} d\mu(x) = O(n^{-p}), \quad n \rightarrow \infty.$$

For such constructions, see e.g. Salem [1].

Let H denote the sequence $0, (\log 2)^{-1}, (\log 3)^{-1}, \dots$. Then if F is the set described in the lemma, we define

$$E = F + H = \{x + y; x \in F, y \in H\}.$$

E is obviously closed and has measure zero, since it is contained in the union of a countable number of translations of F . We shall prove that E is a set of uniqueness.

Suppose that $f(z) = \sum_0^\infty a_n z^n$ belongs to A and vanishes on E . We then construct a function $\varphi(\zeta)$, $|\zeta| < 1$, by the relation

$$\varphi(\zeta) = \int_F f(e^{it}\zeta) d\mu(t) = \sum_0^\infty a_n \mu_n \zeta^n = \sum_0^\infty A_n \zeta^n.$$

Since $f(e^{is}) = 0$ for s belonging to E , $\varphi(e^{iy})$ vanishes on H . Furthermore

$$\sum_1^\infty |A_n| n^p = \sum_1^\infty |a_n| |\mu_n| n^p \leq \text{Const.} \sum_1^\infty |a_n| < \infty.$$

This implies that $\varphi(\zeta)$ satisfies a Lipschitz condition of order p and hence must vanish identically by choice of H . In particular, $A_0 = a_0 = 0$. If we repeat the argument on $z^{-1}f(z)$, $z^{-2}f(z)$, \dots , we see that $f(z) \equiv 0$.

It was shown in Salem [1] that we may choose for F a Cantor set. It is readily verified that for these sets the integral (1) converges, which implies that F is not a set of uniqueness for the class A . A union of a countable number of sets of multiplicity may thus be a set of uniqueness. A similar remark is true also for the class D .

13. We now proceed to the proof of a theorem which shows clearly the arithmetical nature of the problem. We shall consider the following natural extension of the uniqueness problem: to determine the closed sets E such that every continuous function on E is the boundary function of an element in A . In order to formulate our result, we need a new concept.

Let E be a closed set in $(0, 2\pi)$, and consider the class $\Gamma = \Gamma_E$ of functions of bounded variation on $(0, 2\pi)$ which are constant outside of E . Let Γ° be the subclass of Γ of functions normalized by

$$\int_E |d\mu(x)| = 1.$$

We then define the *index of linear dependence* within E as

$$p(E) = \inf_{\mu \in \Gamma^\circ} \sup_{n \geq 0} \left| \int_E e^{inx} d\mu(x) \right|.$$

Obviously $0 \leq p(E) \leq 1$. If $p(E) = 1$, we call E a *Kronecker set*. If $mE > 0$, then the class Γ contains absolutely continuous functions and by the Riemann-Lebesgue theorem we have in this case $p(E) = 0$.

To justify our terminology, we choose for E a sequence $A = \{\lambda_\nu\}$. Then $p(A) = 1$ if and only if the numbers $\{\lambda_\nu\}$ are linearly independent modulo 2π . This is an easy consequence of Kronecker's approximation theorem. For a finite sequence of linearly dependent numbers, $p(E)$ is always positive and is found to depend on how large the integers n_ν in the relations $\sum n_\nu \lambda_\nu \equiv 0$ have to be.

Without going into the proof we point out that *there are perfect sets E such that $p(E) = 1$.*

The theorem which we shall prove can now be formulated as follows.

Theorem 9. *If E is a closed set such that $p(E) > 0$, then every continuous function $\varphi(x)$ defined on E is the boundary function of an element in A , i.e. $\varphi(x)$ has a representation*

$$(19) \quad \varphi(x) = \sum_0^\infty a_n e^{in x}, \quad x \in E, \quad \sum_0^\infty |a_n| < \infty.$$

Corollary. *E is not a set of uniqueness for the class A if $p(E) > 0$.*

To prove the corollary, we need only choose $\varphi(x) = e^{-ix}$ in the theorem above. We obtain a function in A which vanishes at the origin and is equal to 1 on the set E .

The proof of the theorem depends on the following observation.

Lemma. *If $p(E) > 0$ and $\varphi(x)$ is continuous on E , then there exists a sequence $\{A_n\}$, $\lim_{n \rightarrow \infty} A_n = \infty$, such that for all $\mu \in \Gamma_E$*

$$\left| \int_E \varphi(x) d\mu(x) \right| \leq \sup_{n \geq 0} \frac{1}{A_n} \left| \int_E e^{in x} d\mu(x) \right|.$$

For a given $\mu \in \Gamma_E$, let $B = B_\mu$ be the lefthand side above and $\psi(n) = \psi_\mu(n)$ the n :th Fourier-Stieltjes coefficient of μ . Let us assume, as we may, that $|\varphi(x)| \leq 1$. If now $M = 2p(E)^{-1}$, we choose the integer n_1 so that if

$$C_1: \quad B_\mu = M \text{ and } |\psi_\mu(\nu)| \leq 1, \quad \nu = 0, 1, \dots, n_1,$$

then

$$\inf_{\mu \in C_1} \sup_{\nu \geq 0} |\psi_\mu(\nu)| = L_1 > 4.$$

To prove that n_1 exists, let us assume that for every $n > 1$, a function $\mu_n \in \Gamma_E$ exists satisfying $B_{\mu_n} = B_n = M$, $|\psi_n(\nu)| \leq 1$, $\nu = 0, 1, \dots, n$, and $|\psi_n(\nu)| \leq 8$ for all $\nu \geq 0$. From the definition of $p(E)$, it follows that

$$\int_E |d\mu_n(x)| \leq 8p(E)^{-1}.$$

There is a sequence $\{n_i\}$ such that $\mu_{n_i} \rightarrow \mu$, where $\mu \in \Gamma_E$. Since $B_{n_i} \rightarrow B_\mu$ and $\psi_{n_i} \rightarrow \psi_\mu$, we have $B_\mu = M$ and $|\psi_\mu(\nu)| \leq 1$, $\nu = 0, 1, \dots$. We thus find

$$M \leq \int_E |\varphi(x)| |d\mu(x)| \leq \int_E |d\mu(x)| \leq p(E)^{-1},$$

which contradicts the definition of M .

Let us assume that the classes C_1, C_2, \dots, C_{k-1} with corresponding integers n_1, n_2, \dots, n_{k-1} have been determined. We then choose n_k so that if

$$C_k: \quad \mu \in C_{k-1}, |\psi_\mu(\nu)| \leq 2^{k-1}, \quad \nu = n_{k-1} + 1, \dots, n_k,$$

then

$$\inf_{\mu \in C_k} \sup_{\nu \geq 0} |\psi_\mu(\nu)| = L_k > 2^{k+1}.$$

The existence of n_k is proved exactly as in the case $k=1$; if n_k did not exist, there would exist a function $\mu \in C_{k-1}$ such that $|\psi_\mu(\nu)| \leq 2^k$ for all $\nu \geq 0$, and this contradicts the definition of C_{k-1} .

We can now choose the sequence $\{A_n\}$. We define ($n_0 = 0$)

$$A_\nu = M^{-1} 2^k, \quad \nu = n_k + 1, \dots, n_{k+1}; \quad k = 0, 1, \dots$$

If now $|\psi_\mu(\nu)| \leq A_\nu$ for all ν and $\int_E \varphi(x) d\mu(x) = 1$, then $M\mu$ belongs to all of the classes C_k , and, for all k , we must have

$$\int_E |d\mu(x)| \geq \sup_{\nu \geq 0} |\psi_\mu(\nu)| \geq L_k > 2^k.$$

This is clearly impossible, and the lemma is proved.

Suppose now that $\{\lambda_\nu\}$ is a sequence which is dense on E and consider the infinite system of equations

$$\sum_{n=0}^{\infty} x_n \cdot \frac{e^{in\lambda_\nu}}{A_n} = \varphi(\lambda_\nu)$$

$$\sum_0^{\infty} |x_n| \leq K,$$

where the sequence $\{A_n\}$ is as determined in the lemma, for a given continuous function $\varphi(x)$. By a theorem of F. Riesz this system has a solution $\{x_n\}$ if for every sequence $\{h_\nu\}_1^r$,

$$\left| \sum_1^r h_\nu \varphi(\lambda_\nu) \right| \leq \sup_{n \geq 0} \frac{K}{A_n} \left| \sum_{\nu=1}^r h_\nu e^{in\lambda_\nu} \right|.$$

It follows from the lemma that a solution $\{x_n\}$ exists for K sufficiently large. By continuity, we must then have

$$\sum_{n=0}^{\infty} \frac{x_n}{A_n} e^{inx} = \varphi(x), \quad x \in E,$$

and the theorem is proved.

References.

- BEURLING, A. [1]: Ensembles exceptionnels. Acta Math. 72 (1940).
CARLESON, L. [1]: On a class of meromorphic functions and its associated exceptional sets. Uppsala 1950.
———. [2]: On Bernstein's approximation problem. Proc. Amer. Math. Soc. 2 (1951).
LOKKI, O. [1]: Über analytische Funktionen, deren Dirichlet Integral endlich ist und die in gegebenen Punkten vorgeschriebene Werte annehmen. Ann. Acad. Fenn. 39 (1947).
SALEM, R. [1]: On singular monotonic functions whose spectrum has a given Hausdorff dimension. Arkiv f. mat. 1 (1951).