The existence of surfaces of constant mean curvature with free boundaries

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1. Introduction

Suppose S is a surface in $\mathbf{R}^3$ diffeomorphic to the standard sphere S^2 by a smooth diffeomorphism $\Psi: \mathbf{R}^3 \to \mathbf{R}^3$ of class C^4, and let $H \in \mathbf{R}$. In this paper we give a sufficient condition for the existence of an unstable disc-type surface of constant mean curvature H with boundary on S and intersecting S orthogonally along its boundary. In isothermal coordinates such a surface may be parametrized by a map $X \in C^2(B; \mathbf{R}^3) \cap C^1(\bar{B}; \mathbf{R}^3)$ of the unit disc

$$B = B_1(0) = \{(u, v) = w \in \mathbf{R}^2 \mid u^2 + v^2 < 1\}$$

into $\mathbf{R}^3$ satisfying the following conditions:

$$\Delta X = 2H X_u \wedge X_v \tag{1.1}$$

$$|X_u|^2 - |X_v|^2 = 0 = X_u \cdot X_v \tag{1.2}$$

$$X(\partial B) \subset S, \tag{1.3}$$

$$\partial_n X(w) \perp T_{X(w)} S, \quad \forall\, w \in \partial B. \tag{1.4}$$

Here $X_u = (\partial/\partial u) X$, etc., "$\wedge$" denotes the exterior product in $\mathbf{R}^3$, "$\cdot$" denotes the scalar product, n is the outward unit normal on ∂B, "$\perp$" means orthogonal, and $T_Q S$ denotes the tangent space to S at $Q \in S$.

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For brevity, solutions to (1.1)–(1.4) will be called H-surfaces (supported by S). Moreover, if such a solution is not degenerate to a constant map, by slight abuse of terminology it will be called unstable. Indeed, if S is strictly convex, any non-constant minimal surface ($H=0$) supported by S is strictly unstable as a critical point of Dirichlet's integral, cp. Section 2.

Physically, solutions to (1.1)–(1.4) may arise as minimal partitioning hypersurfaces inside S dividing off a prescribed portion of the volume enclosed by S. Such partition problems have been proposed as models for capillarity phenomena and liquid crystals. In this context the curvature constant H appears as a Lagrange multiplier. By using geometric measure theory the partition problem can be solved in vast generality, cp. [12], [25]. However, little is known about partition surfaces of a prescribed topological type, cp. [13].

Let L be minimal with the property that

$$S \subset \overline{B_L(Q)} \tag{1.5}$$

for some $Q \in \mathbf{R}^3$, where $\overline{B_L(Q)}$ as usual denotes the closed ball of radius L around Q (in $\mathbf{R}^3$). By translation we may assume $Q=0$.

THEOREM 1.1. *Suppose S satisfies the above assumption, and let L be given by* (1.5). *Then there exists a set $\mathcal{H}$ of curvature such that*

$$0 \in \mathcal{H}, \quad \text{and} \quad \mathcal{H} \text{ is dense in } \left[-\frac{1}{L}, \frac{1}{L}\right]$$

with the property that for any $H \in \mathcal{H}$ there is a regular, non-constant solution X to (1.1)–(1.4), *satisfying the maximum modulus estimate*

$$\|X\|_{L^\infty} \leq L. \tag{1.6}$$

Remarks. (i) Considering the limitation of the range of admissible curvatures, our theorem may appear as a natural extension of Hildebrandt's existence result [8] for the Plateau problem. However, note that our solutions will (in general) be unstable, and (in general) the only stable solutions to (1.1)–(1.4) will be the trivial constant solutions $X \equiv X_0 \in S$. For the Plateau problem the existence of unstable H-surfaces (for suitably small $|H|$) was established only in 1982 independently by Brezis-Coron [1] and the author [20] – with an addendum by Steffen [19]. In [23] finally an existence result was derived showing that for any value $H \neq 0$ unstable H-surfaces will exist whenever there is a stable surface of constant mean curvature H spanning the given contour.

Similarly, it is expected that there will be unstable H-surfaces supported by S for *any* value of H different from 0. But technical complications due to the existence of spheres of constant mean curvature H inside S for large $|H|$ prevent us from proving a more general result. Conceivably, combining our methods with a variant of the "sphere-attaching lemma" of Wente [26, p. 285 ff.] will lead to complete existence results. However, we will not pursue this further.

(ii) Theorem 1.1 generalizes the existence result [21] for minimal surfaces ($H=0$). It is tempting to conjecture that for surfaces S which have mean curvature $\geqslant H$ with respect to the interior normal one can even find embedded discs of constant mean curvature H inside S, as Grüter and Jost have shown in the case of minimal surfaces; cp. [7], [10]. Note that by the maximum principle for any $H\in\mathbf{R}$ any H-surface X supported by S will lie inside S whenever S satisfies the condition:

$$\text{Any } Q\in S \text{ lies on the boundary of some ball of radius } 1/|H| \text{ containing } S, \tag{1.7}$$

and provided $\|X\|_{L^\infty}\leqslant L\leqslant 1/|H|$.

Problem (1.1)–(1.4) poses numerous technical difficulties. In particular, by invariance of (1.1)–(1.4) under the non-compact group of conformal transformations of the disc it is impossible that the Palais-Smale condition([1]) be satisfied in any variational problem associated with (1.1)–(1.4) where this group of symmetries acts.

The following chapter is devoted to setting up the variational problem corresponding to (1.1)–(1.4). In Chapter 3 we study the evolution problem associated with (1.1)–(1.4) and prove local existence and uniqueness of solutions to the "parabolic form" of (1.1)–(1.4). Our approach will be based on the methods introduced in [24]. Ideas from [24] will also be used to investigate the asymptotics and possible singularities of the flow. Finally, in Chapter 4 the proof of Theorem 1.1 will be given.

Although for minimal surfaces our approach may seem somewhat more involved than e.g. the approximation method used in [21] also in this case we believe that a direct method may have its advantages.

Moreover, the construction and analysis of the flow associated with (1.1)–(1.4) which constitute the major part of this work may be of some interest in itself.

Ultimately, we hope that results like Theorem 3.1 or Theorem 3.2 may be instrumental in proving higher multiplicity results for unstable H-surfaces. By analogy with the problem of closed geodesics on S it is expected that (for sufficiently small $|H|$, at

([1]) Recall that a C^1-functional E on a manifold $\mathcal{M}$ satisfies the Palais-Smale condition if any sequence $\{X_m\}$ in $\mathcal{M}$ such that $|E(X_m)|\leqslant c$ uniformly while $|dE(X_m)|\to 0$ is relatively compact.

least) there exist even three geometrically distinct unstable solutions to (1.1)–(1.4). However, to this moment only partial results are available, cp. [10], [18].

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2. The variational problem

Solutions to the above problem (1.1)–(1.4) will be characterized as saddle points of a suitable functional E_H.

Let L^p, $H^{m,p}$, $C^{m,\alpha}$, denote the usual Lebesgue-, Sobolev-, and Hölder-spaces. Domain and range will be specified like $L^p(\Omega, \mathbf{R}^n)$ if necessary. For $X \in H^{1,2}(B; \mathbf{R}^3), \Omega \subset B$, let

$$D(X; \Omega) = \frac{1}{2} \int_\Omega |\nabla X|^2 \, dw$$

be the *Dirichlet integral* of X over Ω. For brevity $D(X; B) = D(X)$.

For $X \in H^{1,2} \cap L^\infty(B; \mathbf{R}^3)$, moreover, let

$$V(X) = \frac{1}{3} \int_B X_u \wedge X_v \cdot X \, dw$$

be the *volume* of X. $V(X)$ measures the (algebraic) volume of the cone with vertex at 0 swept out by the surfaces X, while $D(X)$ may be thought of as a measure for the area of X. V and D are related by a Sobolev-type inequality, the *isoperimetric inequality* for closed surfaces in $\mathbf{R}^3$, cp. Radó [14], Wente [26], of which we state the following version:

THEOREM 2.1. *Let $X, Y \in H^{1,2} \cap L^\infty(B; \mathbf{R}^3)$ satisfy $X - Y \in H_0^{1,2}(B; \mathbf{R}^3)$, i.e. $X = Y$ on ∂B in the $H^{1,2}$-sense. Then*

$$36\pi[V(X) - V(Y)]^2 \leq [D(X) + D(Y)]^3.$$

The constant 36π is best possible.

Remark that equation (1.1) formally equals the Euler-Lagrange equations of the functional

$$\hat{E}_H(X) = D(X) + 2H\,V(X)$$

with respect to compactly supported variation vectors $\varphi \in C_0^\infty(B;\mathbf{R}^3)$. I.e. $\hat{E}_H$ is the functional corresponding to (1.1) for fixed Dirichlet (or Plateau) boundary data, cp. [8].

Note that $V(X)$ for fixed boundary data equals the (algebraic) volume between X and a fixed reference surface X_0 satisfying the required boundary condition.

In our case the boundary data of admissible surfaces X are allowed to vary freely on the supporting surface S. Therefore it will be necessary to correct the volume term by subtracting the volume of a suitable reference surface $\tilde{X}$ *on* S, varying with X. In this way we will arrive at a functional whose Euler-Lagrange equations give all of (1.1)–(1.4).

This program will now be carried out in detail.

Admissible functions. As will become apparent later, a natural class on which to study problem (1.1)–(1.4) is the class

$$\mathscr{C}(S)=\{X\in H^{1,2}(B;\mathbf{R}^3)\,|\,X(\partial B)\subset S\ \text{ a.e.}\}$$

of $H^{1,2}$-surfaces with boundary on S.

Sometimes, it will also be convenient to consider the subclass

$$\mathscr{C}_2(S)=\mathscr{C}(S)\cap H^{2,2}(B;\mathbf{R}^3).$$

By using arguments of Schoen-Uhlenbeck [16] it is easily verified that $\mathscr{C}_2(S)$ is dense in $\mathscr{C}(S)$.

Moreover, $\mathscr{C}_2(S)$ is a manifold (while $\mathscr{C}(S)$ is not) with tangent space at $X\in\mathscr{C}_2(S)$ given by

$$T_X\mathscr{C}_2(S)=\{\varphi\in H^{2,2}(B;\mathbf{R}^3)\,|\,\varphi(w)\in T_{X(w)}S,\ \forall\, w\in\partial B\}.$$

Extension operators. Corresponding to $\mathscr{C}(S)$, resp. $\mathscr{C}_2(S)$ consider the (sub-)classes

$$\tilde{\mathscr{C}}(S)=\{\tilde{X}\in C(S)\,|\,\tilde{X}(B)\subset S\},\quad \tilde{\mathscr{C}}_2(S)=\tilde{\mathscr{C}}(S)\cap H^{2,2}(B;\mathbf{R}^3)$$

of $\mathscr{C}(S)$-, resp. $\mathscr{C}_2(S)$-surfaces contained in S, with tangent space at $\tilde{X}\in\mathscr{C}_2(S)$

$$T_{\tilde{X}}\tilde{\mathscr{C}}_2(S)=\{\varphi\in T_{\tilde{X}}\mathscr{C}_2(S)\,|\,\varphi(w)\in T_{X(w)}S,\ \forall\, w\in\bar{B}\}.$$

Call $\tilde{X}\in\tilde{\mathscr{C}}(S)$ an *extension* of $X\in\mathscr{C}(S)$ iff $X=\tilde{X}$ on ∂B. An *extension operator* is a smooth map $\eta:\mathscr{D}(\eta)\subset\mathscr{C}(S)\to\tilde{\mathscr{C}}(S)$ with open domain $\mathscr{D}(\eta)\subset\mathscr{C}(S)$ and smooth restriction $\eta:\mathscr{D}(\eta)\cap\mathscr{C}_2(S)\to\tilde{\mathscr{C}}_2(S)$ such that $\eta(X)$ is an extension of X for every $X\in\mathscr{D}(\eta)$.

Given $X \in \mathscr{C}(S)$ and an extenion $\tilde{X}_0 \in \mathscr{C}(S)$ the set of extensions of X may be divided into countably many components by indexing different extensions $\tilde{X}$ with the topological degree of the mapping

$$\hat{X}(w)=\begin{cases} \tilde{X}(w), & |w|\leq 1 \\ \tilde{X}_0\left(\dfrac{w}{|w|^2}\right), & |w|>1 \end{cases}$$

from $\overline{\mathbf{R}^2} \cong S^2$ into $S \cong S^2$. As in [2, Lemma 1] this degree equals the quantity

$$\frac{V(\tilde{X})-V(\tilde{X}_0)}{\operatorname{vol}(S)} \in \mathbf{Z} \tag{2.1}$$

(up to possible change of sign), where vol(S) denotes the 3-dimensional measure of the region enclosed by S.

By attaching a branched k-fold covering of a sphere suitably, it is clear that from one map $\varphi_0: S^2 \to S^2$ we can obtain a map φ_k of degree

$$\deg(\varphi_k)=\deg(\varphi_0)+k$$

for any k, which also coincides with φ_0 outside an arbitrarily small open set.

In this way it is clear that given $X \in \mathscr{C}(S)$ and *one* extension operator η_0 with $X \in \mathscr{D}(\eta_0)$ there will exist countably many such extension operators η_k, $k \in \mathbf{Z}$, with

$$\frac{V(\eta_k(X))-V(\eta_0(X))}{\operatorname{vol}(S)}=k. \tag{2.2}$$

The following lemma hence guarantees a rich choice of possible extensions of any given $X \in \mathscr{C}(S)$:

LEMMA 2.1. *For any $X \in \mathscr{C}(S)$ there is an extension operator η_0 (and hence a countable family η_k, $k \in \mathbf{Z}$, of extension operators satisfying* (2.2)) *defined in a neighborhood of X.*

Proof. First assume that S is convex.

Given $X \in \mathscr{C}(S)$, replace X by the harmonic surface Y:

$$\Delta Y=0 \text{ in } B; \quad Y=X \text{ on } \partial B.$$

By standard regularity results Y is as smooth as the data, so $Y \in \mathscr{C}(S)$. Moreover, Y depends smoothly on X.

Now let P be a point lying interior to S not on $\overline{Y(B)}$ (cf. Lemma A.1 in the Appendix) and project Y from P onto S to obtain $\tilde{X}$, i.e. for any $w \in B$ let $\tilde{X}(w)$ be the unique point of intersection of the half line from P through $Y(w)$ with S.

Since $P \notin \overline{\mathrm{im}(Y)}$ this projection preserves regularity, and $\tilde{X} \in \mathscr{C}(S)$, if $X \in \mathscr{C}(S)$, resp. $\tilde{X} \in \mathscr{C}_2(S)$, if $X \in \mathscr{C}_2(S)$.

In order to extend the composition mapping

$$X \to Y \to \tilde{X}$$

smoothly to a neighborhood of X it is sufficient to extend the projection $Y \to \tilde{X}$ to harmonic surfaces close to Y in $\mathscr{C}(S)$. This is clearly possible if for all harmonic $Z \in \mathscr{C}(S)$ sufficiently close to Y we have $P \notin \overline{\mathrm{im}(Z)}$. So suppose by contradition that for a sequence $\{Z_m\}_m$ of harmonic surfaces in $\mathscr{C}(S)$, $Z_m \to Y$ in H^1, we have $Z_m(w_m) \to P$ for some $w_m \in \bar{B}$. Since $w_m \in B$, and we may reparametrize Z_m by conformal maps $\tau_m \in \mathscr{C}^1(\bar{B};\bar{B})$ to obtain a family $\tilde{Z}_m = Z_m \circ \tau_m$ with $\tilde{Z}_m(0) \to P$.

But by conformal invariance of Dirichlet's integral, and since $D(Z_m)$ for an H^1-convergent sequence is surely bounded, even after this change of parameter

$$D(\tilde{Z}_m) \leq c$$

uniformly, and we may assume that a subsequence $\tilde{Z}_m \rightharpoonup \tilde{Z}$ weakly in $H^{1,2}(B;\mathbf{R}^3)$. Also the equation $\Delta Z_m = 0$ is conformally invariant, whence $\Delta \tilde{Z}_m = 0$ and $\tilde{Z}_m \to \tilde{Z}$ uniformly locally in B. In particular $\tilde{Z}_m(0) \to \tilde{Z}(0) = P$.

By invariance of Dirichlet's integral and of harmonicity again, and since $Z_m \to Y$ in $H^{1,2}(B;\mathbf{R}^3)$, it also follows that

$$\tilde{Z}_m - Y \circ \tau_m \to 0 \quad \text{in} \quad H^{1,2}(B;\mathbf{R}^3)$$

and uniformly locally on B.

Considering the sequence of harmonic surface $Y_m = Y \circ \tau_m \in \mathscr{C}(S)$ we thus find that

$$Y_m(0) \to \tilde{Z}(0) = P.$$

On the other hand $\overline{Y_m(B)} = \overline{Y(B)} \not\ni P$ for all m, and therefore we obtain the desired contradiction,

This concludes the construction in the case of a convex surface S. If S is only diffeomorphic to S^2 by a diffeomorphism $\Psi : \mathbf{R}^3 \to \mathbf{R}^3$, we carry out the above construction for the images $\Psi(X) \in \mathscr{C}(S^2)$ and apply Ψ^{-1} to obtain the desired extensions in $\mathscr{C}(S)$. □

Variational integrals. Now we are able to define a family of variational integrals giving rise to the Euler equations (1.1)–(1.4).

For $X \in \mathscr{C}(S)$ and an extension operator η defined in a neighborhood of X, let

$$E_H(X) = D(X) + 2H\,[V(X) - V(\eta(X))]. \tag{2.3}$$

Since $V(\eta(X))$ by (2.1)–(2.2) is only determined up to integral multiples of vol(S) this definition actually gives a countable family of functionals. However, the differential of E_H (at a point $X \in \mathscr{C}_2(S)$) will be independent of the particular extension operator. In particular, critical points of E_H will not depend on η:

LEMMA 2.2. *Let $X \in \mathscr{C}_2(S)$, η any extension operator defined in a neighborhood of X. Then X is a critical point of E_H on $\mathscr{C}_2(S)$, i.e.*

$$\langle dE_H(X), \varphi \rangle = 0, \quad \forall \varphi \in T_X \mathscr{C}_2(S),$$

iff X solves (1.1)–(1.4).

Proof. Note that $H_0^{1,2} \cap H^{2,2}(B;\mathbf{R}^3) \subset T_X\mathscr{C}_2(S)$ and that by (2.1)

$$V(\eta(X+\varphi)) = V(\eta(X)), \quad \forall \varphi \in H_0^{1,2} \cap H^{2,2}(B;\mathbf{R}^3), \tag{2.4}$$

provided X and $X+\varphi$ are connected in $\mathscr{D}(\eta)$. The equation

$$\langle dE_H(X), \varphi \rangle = 0, \quad \forall \varphi \in H_0^{1,2} \cap H^{2,2}(B;\mathbf{R}^3),$$

thus is equivalent to (1.1).

Next observe that $T_{\tilde{X}}\mathscr{C}_2(S) \subset T_X\mathscr{C}_2(S)$, where $\tilde{X} = \eta(X)$, and that by antisymmetry of the volume form

$$\tilde{X}_u \wedge \tilde{X}_v \cdot \varphi \equiv 0, \quad \forall \varphi \in T_{\tilde{X}} \tilde{\mathscr{C}}_2(S),$$

since $\tilde{X}_u(w), \tilde{X}_v(w), \varphi(w) \in T_{\tilde{X}(w)}S$, a.e. on B.

Hence by partial integration

$$\begin{aligned} \langle dV(\tilde{X}), \varphi \rangle &= \int_B \tilde{X}_u \wedge \tilde{X}_v \cdot \varphi \, dw - \int_{\partial B} (v \cdot \tilde{X}_u - u \cdot \tilde{X}_v) \wedge \tilde{X} \cdot \varphi \, do \\ &= \int_{\partial B} \partial_\tau \tilde{X} \wedge \tilde{X} \cdot \varphi \, do, \quad \forall \varphi \in T_{\tilde{X}} \mathscr{C}_2(S), \end{aligned} \tag{2.5}$$

where $\partial_\tau \tilde{X}$ is the tangent derivative of $\tilde{X}$ along ∂B (in the counter clock-wise direction). Since $\tilde{X}=X$ on ∂B: $\partial_\tau \tilde{X}=\partial_\tau X$ on ∂B, and also $\langle d\eta(X), \varphi\rangle=\varphi$ on ∂B; and since finally

$$T_{\tilde{X}}\mathscr{C}_2(S)+H_0^{1,2}\cap H^{2,2}(B;\mathbf{R}^3)=T_X\mathscr{C}_2(S),$$

we may combine (2.4)–(2.5) in the single statement

$$\langle dV(\eta(X)), \varphi\rangle = \int_{\partial B} \partial_\tau X\wedge X\cdot\varphi\, do, \quad \forall\varphi\in T_X\mathscr{C}_2(S). \tag{2.6}$$

Performing an integration by parts similar to (2.5) for $\langle dV(X), \varphi\rangle$ we hence infer that

$$\langle d[V(X)-V(\eta(X))], \varphi\rangle = \int_B X_u\wedge X_v\cdot\varphi\, dw \tag{2.7}$$

while the boundary integrals cancel.

Integrating by parts and using (1.1) we deduce the natural boundary condition

$$\begin{aligned}\langle dE_H(X), \varphi\rangle &= \int_B [-\Delta X+2HX_u\wedge X_v]\cdot\varphi\, dw+\int_{\partial B}\partial_n X\cdot\varphi\, do\\ &= \int_{\partial B}\partial_n X\cdot\varphi\, do=0, \quad \forall\varphi\in T_X\mathscr{C}_2(S),\end{aligned} \tag{2.8}$$

where $\partial_n X$ is the derivative of X with respect to the outward normal on ∂B. I.e. $\partial_n X$ is (weakly) orthogonal to S along ∂B.

Introducing polar coordinates (r, ϕ) on B from (2.8) we see that

$$\frac{\partial}{\partial r}X\cdot\frac{\partial}{\partial\phi}X=0 \quad \text{a.e.} \quad \text{on } \partial B.$$

But then the holomorphic function

$$\Phi(r,\phi)=\left(r^2\left|\frac{\partial}{\partial r}X\right|^2-\left|\frac{\partial}{\partial\phi}X\right|^2\right)-2ir\frac{\partial}{\partial r}X\cdot\frac{\partial}{\partial\phi}X$$

is real on ∂B (hence on $\bar{B}$) and therefore constant (by the Cauchy-Riemann equations). Inspection at $r=0$ shows that $\Phi\equiv 0$, *i.e.* X is conformal and satisfies (1.2).

Boundary regularity and strong orthogonality now follow from [5], [6]. □

Mini-max characterization of unstable H-surfaces. Define a family P of paths in $\mathscr{C}(S)$ connecting constant (point-) mappings on S (which we may regard as local minimizers of our functional E_H; cp. (2.11) below) as follows:

$$P=\{p\in C^0([0.1];\mathscr{C}(S))|p(0)\equiv p_0\in S, p(1)\equiv p_1\in S, \|p(s)\|_{L^\infty}\leq L, \text{ and}$$
$$|V(\eta(p(0)))-V(\eta(p(1)))| = \mathrm{vol}(S) \text{ for any choice of extension}$$
$$\text{operators } \eta \text{ such that } E_H \text{ is continuous along } p\}.$$

The volume condition defining the class P may be visualized as follows: If we extend $p(0)$ by $\eta(p(0))=p(0)=p_0\in\tilde{\mathscr{C}}(S)$ any continuous choice of extensions of $p(t)$ at $t=1$ must give a map covering S (of degree ± 1).

Also let

$$\beta_H=\inf_{p\in P}\sup_{0\leq t\leq 1}\left[E_H(p(t))-\frac{E_H(p(0))+E_H(p(1))}{2}\right] \tag{2.9}$$

where E_H may be defined using any choice of extension operators such that E_H is continuous along p.

LEMMA 2.3. *$P\neq\emptyset$, and for any $H\in\mathbf{R}$ we have the estimate*

$$\beta_H\leq c(1+|H|)$$

where c denotes a constant depending only on S.

Proof. To construct a comparison path $p\in P$ let $\Psi\colon \mathbf{R}^3\to\mathbf{R}^3$ be the diffeomorphism in the hypotheses of the theorem mapping S to S^2. Let $q(t)$ be the family of plane parallel surfaces bounded by circles of constant latitude on S^2:

$$q(t;u,v)=r(2t-1)(u,v,0)+(0,0,2t-1)$$

where $r(s)=\sqrt{1-s^2}$, and let $p(t)=\Psi^{-1}(q(t))$. Clearly $(p(0),p(1))$ are constants $\in S$. Moreover, $p(s)$ lies "inside" S and hence satisfies $\|p(s)\|_{L^\infty}\leq L$. Finally, if we let $\tilde{p}(s)=\Psi^{-1}(\tilde{q}(s))$, where $\tilde{q}(s)$ is a parametrization of the spherical cap "below" $q(t)$, clearly $V(\tilde{p}(0))=0$, $|V(\tilde{p}(1))|=\mathrm{vol}(S)$. For any other continuous choice of extension both these quantities could at most change by the same integral multiple of $\mathrm{vol}(S)$. Hence $p\in P, \beta_H<\infty$. The bound follows immediately from the form of E_H since p does not depend on H. □

LEMMA 2.4. *$\beta_H>|H|\,\mathrm{vol}(S)$.*

Proof. Note that by definition of P

$$|H|\,\mathrm{vol}(S)=\sup\{E_H(p(0)),E_H(p(1))\}-\frac{E_H(p(0))+E_H(p(1))}{2}.$$

The claim hence amounts to show the following:

There exists $\delta>0$ such that

$$\sup_{0\leq t\leq 1} E_H(p(t)) \geq \sup\{E_H(p(0)), E_H(p(1))\}+\delta \tag{2.10}$$

for any $p\in P$.

We may assume $E_H(p(0))\geq E_H(p(1))$. Choose an extension such that $\eta(p(0))=p(0)=p_0$, $E_H(p(0))=0$.

To complete the proof of Lemma 2.4 we need the following

LEMMA 2.5. *There exist $\alpha>0, c$ depending only on S such that η may be extended smoothly to surfaces $X\in\mathscr{C}(S)$ with $D(X)<\alpha$ in such a way that there results:*

$$D(\eta(X))\leq c\cdot D(X).$$

Proof. For convex S replace X by its associate harmonic surface Y as in Lemma 2.1 and project from a fixed point Q lying in the interior of S to obtain $\tilde{X}$. This is possible since by [21, proof of Lemma 3.7, p. 553]:

$$[\operatorname{dist}(Y,S)]^2\leq c\cdot D(Y)\leq c\cdot D(X)<c\alpha,$$

cp. also (A.2)–(A.3) in the Appendix. Hence for suitable choice of α

$$\operatorname{dist}(Y,S)<\frac{1}{2}\operatorname{dist}(Q,S),$$

for all Y with $D(Y)<\alpha$, and the map projecting Y onto $\tilde{X}$ will be bounded independent of Y.

In the general case we use the diffeomorphism Ψ to transfer the above extension operator from S^2 to S. □

LEMMA 2.6. $\inf_{p\in P}\sup_{0\leq t\leq 1} D(X)\geq\alpha>0.$

Proof. Otherwise, for a certain $p\in P$ we would have $D(X)<\alpha$, uniformly for $X=p(t)$, $0\leq t\leq 1$. The map η constructed in Lemma 2.5 is an admissible extension with $V(\eta(p(0)))=V(\eta(p(1)))=0$, a contradiction. □

Proof of Lemma 2.4 (completed). Now use the isoperimetric inequality Theorem 2.1 to estimate for any $X=p(t), \tilde{X}=\eta(X)$:

$$
\begin{aligned}
E_H(X) &= D(X)+2H[V(X)-V(\tilde{X})] \\
&\geqslant D(X)-\frac{|H|}{3\sqrt{\pi}}[D(X)+D(\tilde{X})]^{3/2} \\
&\geqslant D(X)\big(1-c\sqrt{H^2D(X)}\big)\geqslant \frac{1}{2}D(X),
\end{aligned} \tag{2.11}
$$

provided $D(X)<\alpha_H$ for some suitable chosen $\alpha_H\in\,]0,\alpha[$, where η, α, c are as in Lemma 2.5 (2.10) and the lemma thus is seen to be a consequence of Lemma 2.6. □

The deformation lemma. Recall that for a C^1-functional E on a Hilbert manifold $\mathcal{M}$ satisfying the Palais-Smale condition one can easily construct a continuous deformation of $\mathcal{M}$ from a (pseudo-) gradient flow for E with the property that E decreases uniformly along the trajectories of this deformation away from critical points of E.

With the minimax-characterization (2.9) and the bound Lemma 2.4 of a possible critical value of E_H at our disposal, a deformation of $\mathscr{C}(S)$ having the above property would immediately lead to a proof of our Theorem 1.1—and even show the existence of a non-constant solution to (1.1)–(1.4) for *all* H.

However, as we remarked in the introduction, by conformal invariance it is impossible that the Palais-Smale condition holds for our functional E_H, and the construction of a suitable deformation becomes a delicate matter—to be dealt with in the next section.

3. The evolution problem

Notations. For *a* domain $\Omega\subset\mathbf{R}^2$, $-\infty\leqslant s<t\leqslant\infty$ let

$$\Omega_s^t=\Omega\times\,]s,t[,\quad \Omega_s^0=\Omega_s,\quad \Omega_0^t=\Omega^t.$$

For such Ω, s, t introduce the space

$$V(\Omega_s^t)=\{X\in C^0([s,t];H^{1,2}(\Omega,\mathbf{R}^3))\,|\,|\nabla^2X|, |\partial_tX|\in L^2(\Omega_s^t)\},$$

where the derivatives are taken in the distribution sense.

Also let

$$\mathbf{R}_+^2=\{(u,v)\in\mathbf{R}^2|v>0\}$$

denote the upper half-plane.

c denotes a generic constant depending only on S and a bound for H, occasionally numbered for clarity.

In this section we study existence and uniqueness of solutions to the time dependent problem

$$\partial_t X - \Delta X + 2H X_u \wedge X_v = 0 \tag{3.1}$$

in a cylinder B^T with initial condition

$$X(0) = X_0 \in \mathscr{C}(S), \tag{3.2}$$

subject to the free boundary condition

$$X(w,t) \in S \quad \text{a.e. on } (\partial B)^T \tag{3.3}$$

and to the orthogonality condition

$$\partial_n X(w,t) \perp T_{X(w,t)} S \quad \text{a.e. on } (\partial B)^T. \tag{3.4}$$

Moreover, we analyze the regularity of solutions to (3.1)–(3.4), and study their behavior for $t \to \infty$ and in the neighborhood of possible singularities in the same way as we did for the evolution problem associated with harmonic mappings of Riemanian surfaces, cp. [24]. The results we obtain are completely analogous to those of [24]. In fact our derivation of these results reveals the deep connections between the two problems. In particular we establish:

THEOREM 3.1. *For any $H \in \mathbf{R}$, any $X_0 \in \mathscr{C}(S)$ there exists a unique solution $X \in \bigcap_{T<\bar{T}} V(B^T)$ of problem* (3.1)–(3.4), *defined and regular on $\bar{B} \times]0, \bar{T}[$ where $\bar{T} > 0$ is characterized by the condition that*

$$\lim_{T \to \bar{T}} \sup_{(w,t) \in B^T} D(X(t); B_R(w) \cap B) \geq \bar{\varepsilon} \tag{3.5}$$

for all $R>0$, with a constant $\bar{\varepsilon}>0$ depending only on S and H.

If in addition $|H| < 1/L$, $\|X_0\|_{L^\infty} \leq L$, the solution X will satisfy the maximum modulus estimate

$$\|X\|_{L^\infty(B^{\bar{T}})} \leq L. \tag{3.6}$$

Moreover, if (3.5) *holds for some $\bar{T} \leq \infty$ and*

$$\int_0^{\bar{T}} \int_B |\partial_t X|^2 \, dw \, dt + \sup_{0 \leq t \leq \bar{T}} D(X(t)) < \infty, \tag{3.7}$$

there exists a sequence $t_m \nearrow \bar{T}$ *and sequences* $R_m \searrow 0, w_m \in B$ *such that the rescaled functions* (*after a possible rotation of coordinates*)

$$X_m(w) \equiv X(w_m + R_m w, t_m) \to \bar{X} \quad \text{in} \quad H^{2,2}_{\mathrm{loc}}(\mathbf{R}^2_+; \mathbf{R}^3), \tag{3.8}$$

where $\bar{X}: \mathbf{R}^2_+ \to \mathbf{R}^3$ *is conformal to a non-constant, regular solution to* (1.1)–(1.4).

Finally, if for some $R>0$ *and all* $\bar{T} \leq \infty$ (3.7) *holds while* (3.5) *is not achieved, X is globally regular and there exists a sequence* $t_m \to \infty$ *such that* $X(t_m) \in \mathscr{C}_2(S)$ *and*

$$X(t_m) \to \bar{X} \quad \text{in} \quad H^{2,2}(B; \mathbf{R}^3)$$

strongly, where $\bar{X}$ *is a solution to* (1.1)–(1.4).

For minimal surfaces ($H=0$) condition (3.7) is automatically satisfied (cp. Lemma 3.6) and we obtain global existence of (distribution) solutions to (3.1)–(3.4):

THEOREM 3.2. *Suppose* $H=0$. *Then for any* $X_0 \in \mathscr{C}(S)$ *there exists a* (*distribution*) *solution X to* (3.1)–(3.4) *which is defined and regular on* $\bar{B} \times]0, \infty]$ *with exception of at most finitely many points* $(w^{(k)}, T^{(k)}), T^{(k)} \leq \infty, 1 \leq k \leq K$, *and unique in this class.*

The singularities are characterized by the condition that

$$\limsup_{t \nearrow T^{(k)}} D(X(t); B_R(w^{(k)} \cap B) \geq \bar{\varepsilon}$$

for all $R>0$, *and for each k there exist sequences* $t_m^{(k)} \nearrow T^{(k)}$, $w_m^{(k)}, R_m^{(k)} \searrow 0$ *and surfaces* $\bar{X}^{(k)}$ *such that the rescaled functions* (*after a possible rotation of coordinates*)

$$X_m^{(k)}(w) \equiv X(w_m^{(k)} + R_m^{(k)} w, t_m^{(k)}) \to \bar{X}^{(k)} \quad \text{in} \quad H^{2,2}_{\mathrm{loc}}(\mathbf{R}^2_+, \mathbf{R}^3),$$

where $\bar{X}^{(k)}$ *is conformal to a non-constant, regular minimal surface solving* (1.1)–(1.4) *for* $H=0$.

Similar results will hold for arbitrary $H \in R$. However, in general we cannot guarantee the uniform boundedness (3.7) of $D(X(t))$ and solutions might cease to exist (even in the distribution sense) after a finite time and might pass through denumerably many singularities on their intervals of existence.

Although results as powerful as Theorem 3.2 are not needed to complete the proof of Theorem 1.1, they may be of some interest in themselves. A short of proof is supplied at the end of this chapter.

In the following various constants $\varepsilon_j>0$ depending only on S and H will be introduced. We agree to let $\bar{\varepsilon}$ be the least of these numbers.

A basic inequality. Note the following Sobolev-type inequality

LEMMA 3.3. *For any smooth and bounded domain $\Omega\subset\mathbf{R}^2$, any function $\varphi\in H^{1,2}(\Omega)$*

$$\int_\Omega |\varphi|^4\,dw \leq c\cdot\int_\Omega |\varphi|^2\,dw\left\{\int_\Omega |\nabla\varphi|^2\,dw+\frac{1}{\mu(\Omega)}\int_\Omega |\varphi|^2\,dw\right\}$$

with a constant c depending only the shape of Ω. (I.e. c is invariant under scaling $w\mapsto Rw$.)

By a covering argument Lemma 3.3 implies the following estimate for functions in $V(\Omega_s^\prime)$:

LEMMA 3.4. *There exists a constant c such that for any $T\leq\infty$, any $X\in V(B^T)$, any $R\in\,]0,1]$ there holds the estimate*

$$\int_{B^T} |\nabla X|^4\,dw\,dt \leq c\operatorname*{ess\,sup}_{(w,t)\in B^T} D(X(t);B_R(w)\cap B)\left\{\int_{B^T} |\nabla^2 X|^2\,dw\,dt+R^{-2}\int_{B^T}|\nabla X|^2\,dw\,dt\right\}. \tag{3.9'}$$

Moreover, for any $w_0\in B$, any $R\in\,]0,1]$, any $X\in V(B^T)$, and any function $\zeta\in C_0^\infty(B_R(w_0)$ depending only on the distance $|w-w_0|$ and non-increasing as a function of this distance there holds the estimate

$$\int_{B^T} |\nabla X|^4\zeta^2\,dw\,dt \leq c\operatorname*{ess\,sup}_{0\leq t\leq T} D(X(t);B_R(w_0)\cap B)\left\{\int_{B^T} |\nabla^2 X|^2\zeta^2\,dw\,dt+R^{-2}\int_{B^T}|\nabla X|^2\zeta^2\,dw\,dt\right\}. \tag{3.9''}$$

For details cf. [11, II. Theorem 2.2 and Remark 2.1, p. 63 f.] and [24, Lemma 3.1, 3.2].

In view of (3.9′), (3.9″) the quantity

$$\varepsilon(R)=\sup_{(w,t)\in B^T} D(X(t);B_R(w)\cap B) \tag{3.10}$$

associated with a function $X\in V(B^T)$ will play an important role in our estimates.

L^∞-a-priori bounds.

LEMMA 3.5. *Suppose* $|H|<1/L$, $\|X_0\|_{L^\infty}\leq L$. *Then for any continuous solution* $X\in V(B^{\bar{T}})$ *of* (3.1)–(3.4) *there holds the estimate*

$$\|X(t)\|_{L^\infty}\leq L$$

for all $t\in[0,\bar{T}]$.

Proof. Multiply (3.1) by X to obtain the differential inequality

$$\begin{aligned}\frac{1}{2}(\partial_t-\Delta)|X|^2 &= -2HX\cdot X_u\wedge X_v-|\nabla X|^2\\ &\leq |\nabla X|^2(|H|\,\|X\|_{L^\infty(B^T)}-1)\leq 0,\end{aligned}$$

for all $T>0$ such that $X(t)$ is defined on $[0,T]$ and $\|X\|_{L^\infty(B^T)}\leq 1/|H|$.

The parabolic maximum principle then implies that for such T in fact $\|X\|_{L^\infty(B^T)}\leq L$ and by continuity of the flow $X(t)$ the set of such T is both open and closed in the interval $[0,\bar{T}]$. □

L^2-a-priori bounds. No quantitative bounds on H are required to obtain L^2-a-priori bounds for solutions of (3.1)–(3.4). First consider the time derivative of X:

LEMMA 3.6. *Suppose* $H\in\mathbf{R}$, $X_0\in\mathscr{C}(S)$, *and let* $X\in V(B^T)$ *be a solution to* (3.1)–(3.4). *Then* $D(X(t)), E_H(X(t))$ *are absolutely continuous in* $t\in[0,T]$, *and there holds the estimate*

$$\int_{B^T}|\partial_t X|^2\,dw\,dt+E_H(X(T))\leq E_H(X_0).$$

Proof. Differentiate and integrate by parts—observing that $\partial_n X\cdot\partial_t X=0$ in the distribution sense on $(\partial B)^T$ by (3.3)–(3.4)—to obtain for a.e. $t\in[0,T]$:

$$\frac{d}{dt}D(X(t))=\int_B\nabla X\nabla\partial_t X\,dw=-\int_B\Delta X\partial_t X\,dw\in L^1([0,T]),$$

resp. (cp. (2.7), (2.8))

$$\begin{aligned}\frac{d}{dt}E_H(X(t)) &= \int_B[-\Delta X+X_u\wedge X_v]\cdot\partial_t X\,dw\\ &= -\int_B|\partial_t X|^2\,dw\in L^1([0,T]).\end{aligned}$$ □

Let us now bound the second spatial derivatives of X:

LEMMA 3.7. *For any $H\in\mathbf{R}$ there exist constants $c_1, \varepsilon_1>0$ depending only on S and H such that for any solution $X\in V(B^T)$ of* (3.1)–(3.4) *and any $R\in]0,1]$ there holds the estimate*

$$\int_{B^T} |\nabla^2 X|^2\, dw\, dt \leq c_1 D(X_0) + c_1 T R^{-2} \sup_{0\leq t\leq T} D(X(t)),$$

provided $\varepsilon(R)\leq \varepsilon_1$, cp. (3.10).

Proof. Multiplying (3.1) by $-\Delta X$ and integrating by parts on account of (3.3)–(3.4) we obtain

$$\int_{B^T} \partial_t\left(\frac{|\nabla X|^2}{2}\right) dw\, dt + \int_{B^T} |\Delta X|^2\, dw\, dt \leq |H| \int_{B^T} |\nabla X|^2 |\Delta X|\, dw\, dt.$$

$$\leq \frac{1}{2}\int_{B^T} |\Delta X|^2\, dw\, dt + c\int_{B^T} |\nabla X|^4\, dw\, dt.$$

By (3.9′), (3.10) this implies

$$D(X(T)) - D(X_0) + \frac{1}{2}\int_{B^T} |\Delta X|^2\, dw\, dt \leq c\varepsilon_1 \int_{B^T} |\nabla^2 X|^2\, dw\, dt + cTR^{-2} \sup_{0\leq t\leq T} D(X(t)). \tag{3.11}$$

Let (r, ϕ) denote polar coordinates on B. Multiplying (3.1) by $-(\partial^2/\partial\phi^2)X$ and integrating by parts we find

$$\begin{aligned}\int_{B^T} \partial_t\left(\frac{1}{2}\left|\frac{\partial}{\partial\phi}X\right|^2\right) dw\, dt + \int_{B^T} \left|\nabla\frac{\partial}{\partial\phi}X\right|^2 dw\, dt \leq & \left|\int_{(\partial B^T)} \partial_n X \cdot \frac{\partial^2}{\partial\phi^2} X\, d\phi\, dt\right| \\ & + |H| \int_{B^T} |\nabla X|^2 |\nabla^2 X|\, dw\, dt.\end{aligned} \tag{3.12}$$

For $X\in S$ now let $G(X)$ be the outer normal to S. If $S\in C^m$, the Gauss map $G\in C^{m-1}$. By (3.4)

$$\partial_n X = (\partial_n X\cdot G(X))\cdot G(X) \quad \text{a.e.} \quad \text{on } (\partial B)^T.$$

Noting that by (3.3)

$$G(X)\cdot\frac{\partial}{\partial\phi}X = 0. \quad \text{a.e.}$$

an integration by parts with respect to ϕ yields

$$\int_{\partial B} \partial_n X \cdot \frac{\partial^2}{\partial\phi^2} X \, d\phi = \int_{\partial B} (\partial_n X \cdot G(X)) \left(\frac{\partial}{\partial\phi} X \cdot \nabla G(X) \cdot \frac{\partial}{\partial\phi} X\right) d\phi$$

for a.e. t. Extend G of class C^1 to $\mathbf{R}^3$. By the divergence theorem then

$$\begin{aligned}\int_{\partial B} \partial_n X \cdot \frac{\partial^2}{\partial\phi^2} X \, d\phi &= \int_B \Delta X \left\{ G(X) \left(\frac{\partial}{\partial\phi} X \cdot \nabla G(X) \cdot \frac{\partial}{\partial\phi} X\right)\right\} dw \\ &\quad + \int_B \nabla X \cdot \nabla \left\{ G(X) \left(\frac{\partial}{\partial\phi} X \cdot \nabla G(X) \cdot \frac{\partial}{\partial\phi} X\right)\right\} dw \\ &\leq c \int_B |\nabla^2 X| \, |\nabla X|^2 + |\nabla X|^4 \, dw.\end{aligned}$$

Hence (3.12) may be estimated for arbitrary $\delta \in]0,1[$:

$$\begin{aligned}&\frac{1}{2}\int_B \left|\frac{\partial}{\partial\phi} X(T)\right|^2 dw - \frac{1}{2}\int_B \left|\frac{\partial}{\partial\phi} X_0\right|^2 dw + \int_{B^T} \left|\nabla \frac{\partial}{\partial\phi} X\right|^2 dw\, dt \\ &\quad \leq (c(\delta)\,\varepsilon_1 + \delta) \int_{B^T} |\nabla^2 X|^2 \, dw\, dt + c(\delta)\, TR^{-2} \sup_{0\leq t\leq T} D(X(t)).\end{aligned} \tag{3.13}$$

Since

$$|\nabla^2 X|^2 \leq |\Delta X|^2 + 4\left|\nabla \frac{\partial}{\partial\phi} X\right|^2 \tag{3.14}$$

from (3.11), (3.13) we obtain

$$\int_{B^T} |\nabla^2 X|^2 \, dw\, dt \leq (c(\delta)\,\varepsilon_1 + 4\delta) \int_{B^T} |\nabla^2 X|^2 \, dw\, dt + cD(X_0) + c(\delta)\, TR^{-2} \sup_{0\leq t\leq T} D(X(t))$$

and the claim follows for $\varepsilon_1, \delta > 0$ sufficiently small. □

A variant of the above proof also gives the following

LEMMA 3.8. *Suppose $H \in \mathbf{R}$. Then there exist constants $c_2, \varepsilon_2 > 0$ depending only on S and H such that for any solution $X \in V(B^T)$ of* (3.1)–(3.4), *any $R \in]0,1[$, any $w_0 \in B$ there holds:*

$$D(X(T); B_R(w_0) \cap B) \leq 2D(X_0, B_{2R}(w_0) \cap B) + c_2\, TR^{-2} \varepsilon(R),$$

provided $\varepsilon(R) \leq \varepsilon_2$.

Proof. Let $\zeta \in C_0^\infty(B_{2R}(w_0))$ be a non-increasing function of the distance $|w-w_0|$ such that $\zeta\equiv 1$ on $B_R(w_0)$, $|\nabla\zeta|\leq cR^{-1}$. Testing (3.1) with $-\Delta X\zeta^2$, $-(\partial^2/\partial\phi^2)X\cdot\zeta^2$ resp. and going through the proof of Lemma 3.7 using (3.9″) instead of (3.9′), analogous to (3.11), (3.13) we obtain the estimate

$$\begin{aligned}\int_{B^T} \partial_t\left(\frac{1}{2}|\nabla X|^2\zeta^2+\frac{1}{2}\left|\frac{\partial}{\partial\phi}X\right|^2\zeta^2\right)+|\nabla^2 X|^2\zeta^2\,dw\,dt &\leq (c\varepsilon_2+\delta)\int_{B^T}|\nabla^2X|^2\zeta^2\,dw\,dt\\ &+c(\delta)\,TR^{-2}\sup_{0\leq t\leq T} D(X(t);B_{2R}(w_0)\cap B)+\delta\int_{B^T}|\partial_t X|^2\zeta^2\,dw\,dt\end{aligned} \tag{3.15}$$

for any $\delta>0$.

The last term results from estimating terms like

$$\int_{B^T}\partial_t X\,\nabla X\zeta\nabla\zeta\,dw\,dt\leq\delta\int_{B^T}|\partial_t X|^2\zeta^2\,dw\,dt+c(\delta)\int_{B^T}|\nabla X|^2|\nabla\zeta|^2\,dw\,dt,$$

while

$$\int_{B^T}|\nabla X|^2|\nabla\zeta|^2\,dw\,dt\leq cTR^{-2}\sup_{0\leq t\leq T}D(X(t);B_{2R}(w_0)\cap B).$$

On the other hand, by (3.1)

$$|\partial_t X|\leq c(|\nabla^2 X|+|\nabla X|^2);$$

hence

$$\int_{B^T}|\partial_t X|^2\zeta^2\,dw\,dt\leq c\int_{B^T}|\nabla^2X|^2\zeta^2\,dw\,dt+c\int_{B^T}|\nabla X|^4\zeta^2\,dw\,dt. \tag{3.16}$$

Now again apply estimate (3.9″):

$$\int_{B^T}|\nabla X|^4\zeta^2\,dw\,dt\leq\varepsilon_2\int_{B^T}|\nabla^2X|^2\zeta^2\,dw\,dt+cTR^{-2}\sup_{0\leq t\leq T}D(X(t),B_{2R}(w_0)\cap B).$$

Also note the trivial estimate

$$\sup_{0\leq t\leq T}D(X(t);B_{2R}(w_0)\cap B)\leq c\varepsilon(R).$$

Choosing δ sufficiently small and adding (3.15)–(3.16) we thus arrive at the estimate:

$$\int_{B^T} \partial_t\left(2|\nabla X|^2\zeta^2+\left|\frac{\partial}{\partial\phi}X\right|^2\zeta^2\right)+|\nabla^2X|^2\zeta^2\,dw\,dt\leq c\varepsilon_2\int|\nabla^2X|^2\zeta^2\,dw\,dt+c\varepsilon(R)\,TR^{-2}.$$

On account of the simple inequality for $t\in[0,T]$:

$$D(X(t);B_R(w_0)\cap B)\leq\frac{1}{4}\int_B\left(2|\nabla X(t)|^2\zeta^2+\left|\frac{\partial}{\partial\phi}X(t)\right|^2\zeta^2\right)dw\leq 2D(X(t),B_{2R}(w_0)\cap B)$$

the claim now follows if $\varepsilon_2>0$ is chosen small enough. □

For later reference we note the following useful inequality which results as a by-product of the preceding proof:

Remark 3.9. Suppose $H\in\mathbf{R}$. Then for any solution $X\in V(B^T)$ of (3.1)–(3.4), any $R\in\,]0,1[$, any $w_0\in B$ there holds the estimate

$$\int_0^T\int_{B_R(w_0)\cap B}|\nabla^2X|^2\,dw\,dt\leq 4c_2(1+TR^{-2})\,\varepsilon(R),$$

provided $\varepsilon(R)\leq\varepsilon_2$.

Higher regularity. So far we had found analogies with the evolution problem for harmonic mappings of surfaces on a technical level. In order to obtain higher regularity (and later local existence) for the flow (3.1)–(3.4) we now make use of a more profound relationship which has already played an important role in regularity analysis of free boundaries of surfaces; cp. [9, p. 241]. To see this relation consider the reflection of a given solution $X\in V(B^T)$ to (3.1)–(3.4) in S:

By compactness and regularity of S there exists a δ-neighborhood $U_\delta(S)$ of S such that any point $P\in U_\delta(S)$ has a unique projection $\pi_S(P)\in S$, defined by

$$|P-\pi_S(P)|=\min_{Q\in S}|P-Q|.$$

Now let

$$\mathcal{R}(P)=2\pi_S(P)-P$$

denote the reflection of a point $P\in U_\delta(S)$ in S. Note that $\mathcal{R}$ is involutory, $\mathcal{R}^2=\mathrm{id}$. Moreover, $\mathcal{R}\in C^{m-1}$ if $S\in C^m$, $m\geq 1$.

For a solution $X \in V(B^T)$ to (3.1)–(3.4) now consider the set

$$\tilde{D} := \left\{ (w,t) \in (\mathbf{R}^2 \setminus \bar{B})^T \,|\, X\left(\frac{w}{|w|^2}, t\right) \in U_\delta(S) \right\}$$

and define an extension to $\hat{D} = \bar{B}^T \cup \tilde{D}$ by letting

$$\hat{X}(w,t) = \begin{cases} X(w,t), & \text{if } w \in \bar{B} \\ \mathscr{R}\left(X\left(\frac{w}{|w|^2}, t\right)\right), & \text{if } (w,t) \in \tilde{D}. \end{cases}$$

By the interior regularity results for (3.1) $\tilde{D}$ is open in $\mathbf{R}^2 \times]0, T[$ and it is meaningful to consider (distributional) derivatives of $\hat{X}$ on $\hat{D}$. Using the relation

$$\mathscr{R}\left(\hat{X}\left(\frac{w}{|w|^2}, t\right)\right) = X(w,t)$$

for $(w,t) \in B^T$ with $(w/|w|^2, t) \in \tilde{D}$ it is elementary to verify the following facts:

LEMMA 3.10. *Suppose $X \in V(B^T)$ solves (3.1)–(3.4) for some $H \in \mathbf{R}$. Then $\hat{X}$ satisfies*

$$\partial_t \hat{X}, \nabla^2 \hat{X} \in L^2_{\mathrm{loc}}(\hat{D}) \tag{3.17}$$

and $\hat{X}$ is a weak solution to a system

$$\partial_t \hat{X} - a(w)\, \Delta \hat{X} + \hat{\Gamma}(w, \hat{X})(\nabla \hat{X}, \nabla \hat{X}) = 0 \tag{3.18}$$

in $\hat{D}$ with uniformly Lipschitz continuous coefficients $a \geq 1$ and a bounded bilinear form $\hat{\Gamma}$ whose coefficients are measurable in w and of class C^1 in $\hat{X}$.

Proof. To obtain (3.17) note the pointwise estimates

$$|\partial_t \hat{X}(w,t)| \leq \left| \nabla \mathscr{R}(X) \cdot \partial_t X\left(\frac{w}{|w|^2}, t\right) \right| \leq c \left| \partial_t X\left(\frac{w}{|w|^2}, t\right) \right|, \quad \text{etc.} \tag{3.19}$$

for all $(w,t) \in \tilde{D}$.

Moreover, it is easy to see that $\hat{X}(t)$ is of class $H^{2,2}$ for a.e. $t \in [0, t]$ on its domain $\hat{D}(t)$. Indeed, it is clear that $\hat{X}(t) \in H^{2,2}$ separately on B and $\hat{D}(t) \setminus \bar{B} = \tilde{D}(t)$ for a.e. t. But (3.3)–(3.4) imply that for any $\varphi \in C_0^\infty(\hat{D}(t))$:

$$\begin{aligned} \int_{\hat{D}(t)} \hat{X} \nabla^2 \varphi \, dw &= \int_{\tilde{D}(t)} \hat{X} \nabla^2 \varphi \, dw + \int_B \hat{X} \nabla^2 \varphi \, dw \\ &= \int_{\tilde{D}(t)} \nabla^2 \hat{X} \varphi \, dw + \int_B \nabla^2 \hat{X} \varphi \, dw \end{aligned}$$

while the boundary terms cancel. Hence the L^2-function $\nabla^2\hat{X}(t)$ (defined on $\tilde{D}(t)\cup B$) is the 2nd distributional derivative of $\hat{X}(t)$, and $\hat{X}(t)\in H^{2,2}(\hat{D}(t);\mathbf{R}^3)$ for a.e. t.

(3.18) now follows from (3.1) upon differentiating

$$X(w,t)=\mathcal{R}\left(\hat{X}\left(\frac{w}{|w|^2},t\right)\right)$$

by the chain rule. In particular, note that the coefficients of $\hat{\Gamma}$ involve the Christoffel symbols of the metric

$$\tilde{g}_{ij}(\hat{X})=(\nabla\mathcal{R}^i(\hat{X})\cdot\nabla\mathcal{R}^j(\hat{X}))_{1\leq i,j\leq 3},$$

and (3.18) is the evolution equation for H-surfaces from $\hat{D}$ into $\mathbf{R}^2$ with metric

$$\hat{g}_{ij}(w,X)=\begin{cases}\delta_{ij}, & \text{if } |w|\leq 1\\ \tilde{g}_{ij}(X), & \text{if } |w|>1.\end{cases}$$

Since $\mathcal{R}\in C^{m-1}$ for $S\in C^m$, $\tilde{g}_{ij}\in C^{m-2}$, and the Christoffel symbols $\tilde{\Gamma}$ associated with $\tilde{g}_{ij}$ will be of class C^{m-3}. Hence, if $S\in C^4$, the coefficients $\hat{\Gamma}$ in (3.18) will be locally uniformly bounded and measurable in w and of class C^1 in X.

Finally, a is given by

$$a(w)=\begin{cases}1, & |w|\leq 1\\ |w|^4, & |w|>1.\end{cases}$$

Remark 3.11. For later reference we note another useful implication of (3.3)–(3.4). Let $\delta>0$ be as in the definition of $\mathcal{R}$, and let $\varphi\in C_0^\infty$ satisfy $0\leq\varphi\leq 1$, $\varphi(s)\equiv 1$ if $|s|<\delta/2$, while $\varphi(s)\equiv 0$ if $|s|\geq\delta$. Then for any solution $X\in V(B^T)$ with extension $\hat{X}$ the function

$$\hat{\varphi}(w,t)=\begin{cases}1, & \text{if } |w|\leq 1\\ \varphi(\hat{X}(w,t)-\pi_\delta(\hat{X}(w,t))), & \text{if } (w,t)\in\tilde{D}\\ 0, & \text{if } (w,t)\notin\hat{D}\end{cases}\tag{3.20}$$

belongs to $H^{2,2}_{\text{loc}}(\mathbf{R}^2)$ for a.e. t and satisfies a.e.

$$|\nabla\hat{\varphi}|\leq C|\nabla\hat{X}|,\quad |\nabla^2\hat{\varphi}|\leq C|\nabla^2\hat{X}|\tag{3.21'}$$

moreover, the distributional derivative $\partial_t\hat{\varphi}\in L^2_{\text{loc}}$ and

$$|\partial_t\hat{\varphi}|\leq C|\partial_t\hat{X}|\quad\text{a.e.}\tag{3.21''}$$

LEMMA 3.12. *Let $H\in\mathbf{R}$. There exists a constant $\varepsilon_3>0$ depending only on S and H such that the following is true:*

Any weak solution $X \in V(B^T)$ to (3.1)–(3.4) with initial data $X_0 \in \mathscr{C}(S)$ is Hölder continuous on $\bar{B}\times]0,T]$, and on any subinterval $[\tau, T]$, $\tau>0$, the Hölder norm of X is uniformly bounded in terms of T, τ and the number

$$R=\sup\{R\in]0,1]|\varepsilon(R)\leqslant\varepsilon_3\}.$$

If $X_0 \in \mathscr{C}_2(S)$, the solution X is Hölder continuous on $\bar{B}\times[0,T]$ and its Hölder norm is bounded in terms of T, R, and the $H^{2,2}$-norm of X_0.

Proof. We proceed as in [24, Lemma 3.10] using the extended system (3.18).

First we derive uniform bounds for smooth solutions for the L^2-norm of $\partial_t X(t)$ for a.e. $t>0$.

Let $\hat{\varphi}$ be the function constructed in Remark 3.11, $w_0 \in B$, and let $\zeta \in C_0^\infty(B_{2R}(w_0))$ be a radially symmetric function as in the proof of Lemma 3.8. Differentiate (3.18) with respect to time to obtain the differential inequality

$$|\partial_t^2 \hat{X}-a(w)\,\Delta\partial_t\hat{X}|\leqslant c|\nabla\partial_t\hat{X}|\,|\nabla\hat{X}|+c|\partial_t\hat{X}|\,|\nabla\hat{X}|^2$$

on $\hat{D}$. Testing this inequality with the function $\partial_t\hat{X}\hat{\varphi}^2\zeta^2$ and integrating over a time interval $[t_0, t_1]\subset[0,T]$ we infer that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^2}|\partial_t\hat{X}(t)|^2\hat{\varphi}^2\zeta^2\,dw\,\Big|_{t=t_0}^{t_1}+\int_{(\mathbf{R}^2)_{t_0}^{t_1}}a(w)|\nabla\partial_t\hat{X}|^2\hat{\varphi}^2\zeta^2\,dw\,dt\\
&=\int_{(\mathbf{R}^2)_{t_0}^{t_1}}\partial_t\left(\frac{|\partial_t\hat{X}|^2\hat{\varphi}^2\zeta^2}{2}\right)+a(w)|\nabla\partial_t\hat{X}|^2\hat{\varphi}^2\zeta^2\,dw\,dt\\
&\leqslant c\int_{(\mathbf{R}^2)_{t_0}^{t_1}}\{|\partial_t\hat{X}|^3\hat{\varphi}\zeta^2+|\nabla\partial_t\hat{X}|\,|\partial_t\hat{X}|\hat{\varphi}^2\zeta^2+|\nabla\partial_t\hat{X}|\,|\partial_t\hat{X}||\nabla\hat{X}|\hat{\varphi}\zeta^2\\
&\qquad+|\nabla\partial_t\hat{X}|\,|\partial_t\hat{X}|\hat{\varphi}^2\zeta|\nabla\zeta|+|\partial_t X|^2|\nabla\hat{X}|^2\hat{\varphi}\zeta^2\}\,dw\,dt.
\end{aligned}$$

Note that we have used (3.21′), (3.21″) to estimate derivatives of $\hat{\varphi}$.

Next recall that integrals of $\hat{X}$ and its derivatives over $\hat{D}$ may be estimated by corresponding integrals of X over B; cp. (3.19). Hence we may replace the domain $\mathbf{R}^2$ by B and omit $\hat{\varphi}$ in all integrals at the expense of enlarging our constants c. If we then also apply Hölder's inequality we obtain

$$\int_B |\partial_t X(t_1)|^2\zeta^2\,dw + \int_{B_{t_0}^{t_1}} |\nabla\partial_t X|^2\zeta^2\,dw\,dt \le c\int_B |\partial_t X(t_0)|^2\zeta^2\,dw + \frac{1}{2}\int_{B_{t_0}^{t_1}} |\nabla\partial_t X|^2\zeta^2\,dw\,dt$$
$$+c\left(\int_{B_{t_0}^{t_1}} |\partial_t X|^2\zeta^2 + |\nabla X|^4\zeta^2\,dw\,dt\right)^{1/2}\left(\int_{B_{t_0}^{t_1}} |\partial_t X|^4\zeta^2\,dw\,dt\right)^{1/2} \tag{3.22}$$
$$+c\int_{B_{t_0}^{t_1}} |\partial_t X|^2(\zeta^2+|\nabla\zeta|^2)\,dw\,dt.$$

Since by (3.1) there holds

$$|\partial_t X| \le |\nabla^2 X| + c|\nabla X|^2 \tag{3.23}$$

we may estimate

$$\int_{B_{t_0}^{t_1}} |\partial_t X|^2\zeta^2 + |\nabla X|^4\zeta^2\,dw\,dt \le c\int_{B_{t_0}^{t_1}} |\nabla^2 X|^2\zeta^2 + |\nabla X|^4\zeta^2\,dw\,dt$$
$$\le c\int_{B_{t_0}^{t_1}} |\nabla^2 X|^2\zeta^2\,dw\,dt + c(t_1-t_0)R^{-2}\sup_{t_0\le t\le t_1} D(X(t); B_{2R}(w_0\cap B).$$
$$\le c(t_1-t_0)R^{-2}\varepsilon_3 + c\varepsilon_3.$$

For the last inequality we have also used (3.9″), Remark 3.9, and our assumption that $\varepsilon(R)\le\varepsilon_3$.

Moreover, note that we may also apply (3.9″) to estimate the term

$$\int_{B_{t_0}^{t_1}} |\partial_t X|^4\zeta^2\,dw\,dt \le c\operatorname*{ess\,sup}_{t_0\le t\le t_1}\int_{B_{2R}(w_0)\cap B} |\partial_t X(t)|^2\,dw$$
$$\times\left(\int_{B_{t_0}^{t_1}} |\nabla\partial_t X|^2\zeta^2\,dw\,dt + R^{-2}\int_{B_{t_0}^{t_1}} |\partial_t X|^2\zeta^2\,dw\,dt\right)$$

appearing on the right of (3.22).

Going back to (3.22) we may now write

$$\int_B |\partial_t X(t_1)|^2\zeta^2\,dw + \int_{B_{t_0}^{t_1}} |\nabla\partial_t X|^2\zeta^2\,dw\,dt \le c\int_B |\partial_t X(t_0)|^2\zeta^2\,dw + c[(1+(t_1-t_0)R^{-2})\,\varepsilon_3]^{1/2}$$
$$\times\left[\sup_{t_0\le t\le t_1}\int_{B_{2R}(w_0)\cap B} |\partial_t X(t)|^2\,dw\left(\int_{B_{t_0}^{t_1}} |\nabla\partial_t X|^2\zeta^2\,dw\,dt + R^{-2}\int_{B_{t_0}^{t_1}} |\partial_t X|^2\zeta^2\,dw\,dt\right)\right]^{1/2}$$

$$+c(t_1-t_0)(1+R^{-2})\sup_{t_0\leq t\leq t_1}\int_{B_{2R}(w_0)\cap B}|\partial_t X(t)|^2\,dw$$

$$\leq c\int_B|\partial_t X(t_0)|^2\zeta^2\,dw+c[(1+(t_1-t_0)R^{-2})\varepsilon_3]^{1/2}\int_{B_{t_0}^{t_1}}|\nabla\partial_t X|^2\zeta^2\,dw\,dt$$

$$+c[\varepsilon_3+(t_1-t_0)R^{-2}]\sup_{t_0\leq t\leq t_1}\int_{B_{2R}(w_0)\cap B}|\partial_t X(t)|^2\,dw.$$

I.e. for sufficiently small $\varepsilon_3>0, t_1-t_0\leq\varepsilon_3 R^2$ there holds

$$\int_{B_R(w_0)\cap B}|\partial_t X(t_1)|^2\,dw\leq c\int_{B_{2R}(w_0)\cap B}|\partial_t X(t_0)|^2\,dw+c\varepsilon_3\sup_{t_0\leq t\leq t_1}\int_{B_{2R}(w_0)\cap B}|\partial_t X(t)|^2\,dw.$$

This inequality will hold for any $w_0\in B$ and any $t_0, t_1\in[0,T]$ such that $t_1-t_0\leq\varepsilon_3 R^2$. Fix $0\leq\bar{t}_0\leq\bar{t}_2\leq\bar{t}_0+\varepsilon_3 R^2$, $\bar{t}_1=\frac{1}{2}(\bar{t}_0+\bar{t}_2)$, and for $t_0\in[\bar{t}_0,\bar{t}_1]$ let $w_0\in B$, $t_1\in[t_0,\bar{t}_2]$ be defined such that

$$2\int_{B_R(w_0)\cap B}|\partial_t X(t_1)|^2\,dw\geq\operatorname*{ess\,sup}_{(\bar{w},\bar{t})\in B_{t_0}^{\bar{t}_2}}\int_{B_R(\bar{w})\cap B}|\partial_t X(\bar{t})|^2\,dw.$$

Covering B with balls of radius R, for sufficiently small $\varepsilon_3>0$ and suitably chosen $t_0\in[\bar{t}_0,\bar{t}_1]$ we then obtain that

$$\begin{aligned}c^{-1}R^2\int_B|\partial_t X(\bar{t}_2)|^2\,dw&\leq\int_{B_R(w_0)\cap B}|\partial_t X(t_1)|^2\,dw\leq c\int_{B_{2R}(w_0)\cap B}|\partial_t X(t_0)|^2\,dw\\&\leq c\inf_{\bar{t}_0\leq t\leq\bar{t}_1}\int_B|\partial_t X(t)|^2\,dw\leq\frac{c}{\bar{t}_1-\bar{t}_0}\int_{B_{\bar{t}_0}^{\bar{t}_2}}|\partial_t X|^2\,dw\,dt.\end{aligned}\tag{3.24}$$

By (3.23) and Lemma 3.7 finally

$$\int_B|\partial_t X(t_1)|^2\,dw\leq\frac{c}{t_1-t_0}$$

for all $t_0, t_1\in[0,T]$ such that $0<t_1-t_0\leq\varepsilon_3 R^2$, with a constant $c=c(T,R)$. I.e. for all $t\in\,]0,T]$ there holds

$$\int_B|\partial_t X(t)|^2\,dw\leq c(1+t^{-1}).\tag{3.25'}$$

If $X_0 \in \mathscr{C}_2(S)$ from (3.23)–(3.24) we obtain

$$\int_B |\partial_t X(t)|^2 \, dw \leq c \tag{3.25''}$$

uniformly, with c depending in addition on $\|X_0\|_{H^{2,2}(B,\mathbf{R}^3)}$.

Now we derive pointwise estimates for $\int_B |\nabla^2 X(t)|^2 \, dw$, using (3.25′), (3.25″). Note that (3.18) implies

$$|\Delta \hat{X}(t)| \leq c(|\partial_t \hat{X}(t)| + |\nabla \hat{X}(t)|^2).$$

Testing with $\Delta \hat{X}(t)\, \hat{\varphi}^2$ and integrating by parts we find that for a.e. $t \in]0, T]$:

$$\int_{\mathbf{R}^2} |\nabla^2 \hat{X}|^2 \hat{\varphi}^2 \, dw \leq c \int_{\mathbf{R}^2} (|\partial_t \hat{X}|^2 + |\nabla \hat{X}|^4)\, \hat{\varphi}^2 \, dw + c \int_{\mathbf{R}^2} |\nabla^2 \hat{X}|\, |\nabla \hat{X}|^2 \hat{\varphi} \, dw.$$

I.e. by (3.19) again

$$\int_B |\nabla^2 X(t)|^2 \, dw \leq c \int_B (|\partial_t X(t)|^2 + |\nabla X(t)|^4) \, dw.$$

(3.9) and our assumption $\varepsilon(R) \leq \varepsilon_3$ now imply that

$$\int_B |\nabla^2 X(t)|^2 \, dw \leq c \int_B |\partial_t X(t)|^2 \, dw + cR^{-2},$$

and (3.25) yields the estimate

$$\int_B |\nabla^2 X(t)|^2 \, dw \leq c(T, \tau, R)$$

for all $t \in [\tau, T], \tau > 0$, resp. the global bound

$$\int_B |\nabla^2 X(t)|^2 \, dw \leq c(T, R, \|X_0\|_{H^{2,2}(B,\mathbf{R}^3)}) \tag{3.26}$$

for regular initial data.

By Sobolev's embedding theorem $H^{2,2}(B) \hookrightarrow C^0(\bar{B})$ and $X(t)$ is uniformly continuous locally on $]0, T]$, resp. on $[0, T]$ for regular X_0. In particular, $\hat{D}$ contains a uniform neighborhood of $\bar{B} \times]0, T]$, resp. of $\bar{B} \times [0, T]$, and we can proceed to derive estimates for X from (3.18) and the local regularity theory for linear parabolic equations with right hand side bounded in L^q for all $q<\infty$ locally on $]0,T]$, resp. globally on $[0,T]$ for $X_0\in\mathscr{C}_2(S)$.

The contended Hölder continuity thus is a consequence of [11, Theorem 3.10.1].

For weak solutions $X\in V(B^T)$ the time derivative has to be replaced by difference quotients. The remainder of the proof stays the same. $\square$

Remark 3.13. Once Hölder continuity and a priori Hölder bounds have been established for X higher regularity and a priori bounds for derivatives of X may be obtained in a standard manner by locally mapping a neighborhood of S around a boundary point of X onto a plane e.g. by introducing normal coordinates around S and reflecting the transformed surface $\check{X}$ in this plane to obtain a solution of a parabolic system

$$\partial_t\check{X}-\Delta\check{X}=\check{\Gamma}(\check{X})(\nabla\check{X},\nabla\check{X})$$

with coefficients $\check{\Gamma}\in C^{m-3}$ if $S\in C^m, m\geq 4$. The standard regularity theory for such systems (cp. [11]) together with our Hölder bounds now implies that X really is as smooth as the data permit, i.e. of class C^{m-2} in space and of class C^{m-3} in time for any $m\geq 4$ with all derivatives up to this order Hölder continuous on $\bar{B}\times]0,T]$.

If X_0 is sufficiently smooth we obtain regularity and a priori bounds up to $t=0$.

Remark 3.14. As in [24, Lemmata 3.7′, 3.10′, Remark 3.11′] we can also prove local regularity and a priori bounds for solutions $X\in\bigcap_{T<\bar{T}}V(B^T)$ on any subset $D'\subset\bar{B}^{\bar{T}}$ with the property that for some $R>0$

$$\sup_{(w,t)\in D'}D(X(t);B_R(w)\cap B)\leq\varepsilon_4$$

where $\varepsilon_4>0$ is a suitable constant depending only on S and H.

Uniqueness.

LEMMA 3.15. *Let $H\in\mathbf{R}, X_0\in\mathscr{C}(S)$ and suppose that $X^{(1)},X^{(2)}\in V(B^T)$ are weak solutions to* (3.1)–(3.4) *with $X^{(1)}(0)=X^{(2)}(0)=X_0$. Then $X^{(1)}\equiv X^{(2)}$.*

Proof. Let $\hat{X}^{(j)}$ be the extension of $X^{(j)}$ defined in Lemma 3.10, $\hat{\varphi}^{(j)}$ the associated truncation function, $j=1,2$. Define $\hat{\varphi}=\min\{\hat{\varphi}^{(1)},\hat{\varphi}^{(2)}\}$. Subtracting equations (3.18) for $\hat{X}^{(1)},\hat{X}^{(2)}$ and testing with $(\hat{X}^{(1)}-\hat{X}^{(2)})\hat{\varphi}^2$ we obtain the following estimates for $\hat{Y}=\hat{X}^{(1)}-\hat{X}^{(2)}$:

$$|\partial_t \hat{Y} - a(w)\,\Delta \hat{Y}| \leqslant c|\nabla \hat{X}||\nabla \hat{Y}| + c|\nabla \hat{X}|^2|\hat{Y}|,$$

$$\int_{(\mathbf{R}^2)^T} \partial_t\left(\frac{|\hat{Y}|^2\hat{\varphi}^2}{2}\right) + |\nabla \hat{Y}|^2\hat{\varphi}^2\,dw\,dt \leqslant c\int_{(\mathbf{R}^2)^T} \{|\hat{Y}|^2|\partial_t\hat{\varphi}|\hat{\varphi} + |\nabla \hat{Y}|\,|\hat{Y}|\,|\nabla\hat{\varphi}|\hat{\varphi} + |\nabla \hat{Y}|\,|\hat{Y}|\hat{\varphi}^2$$
$$+|\nabla \hat{X}|\,|\nabla \hat{Y}|\,|\hat{Y}|\hat{\varphi}^2 + |\nabla \hat{X}|^2|\hat{Y}|^2\hat{\varphi}^2\}\,dw\,dt. \tag{3.27'}$$

For brevity we have denoted $|\nabla \hat{X}^{(1)}| + |\nabla \hat{X}^{(2)}| =: |\nabla \hat{X}|$, etc. By (3.21)

$$|\partial_t\hat{\varphi}| \leqslant c|\partial_t \hat{X}|, \quad |\nabla\hat{\varphi}| \leqslant c|\nabla \hat{X}|,$$

and we may bound the right hand side of (3.27′) by

$$c\int_{B^T} |Y|^2(|\partial_t X| + |\nabla X|^2) + |\nabla Y|\,|Y|(1 + |\nabla X|)\,dw\,dt. \tag{3.27''}$$

Again we have used the fact that

$$\hat{Y}(w,t) = (\hat{X}^{(1)} - \hat{X}^{(2)})(w,t) = (R(X^{(1)}) - R(X^{(2)}))\left(\frac{w}{|w|^2}, t\right)$$
$$= \int_0^1 \nabla R(X^{(2)} + \vartheta(X^{(1)} - X^{(2)}))\cdot(X^{(1)} - X^{(2)})\,d\vartheta\Big|_{(w/(|w|^2),\,t)}$$

whence

$$|\hat{Y}(w,t)| \leqslant c\left|Y\left(\frac{w}{|w|^2}, t\right)\right|$$

for all $w \notin B$, where $Y = X^{(1)} - X^{(2)}$, and integral estimates for $\hat{Y}$ can be obtained from estimates for Y on B^T. A similar statement applies to $\nabla \hat{Y}$.

From Hölder's inequality and (3.27) we now obtain

$$\int_B |Y(T)|^2\,dw + \int_{B^T} |\nabla Y|^2\,dw\,dt \leqslant c\left(\int_{B^T} |\partial_t X|^2 + |\nabla X|^4\,dw\,dt\right)^{1/2}\cdot\left(\int_{B^T} |Y|^4\,dw\,dt\right)^{1/2}$$
$$+\frac{1}{2}\int_{B^T} |\nabla Y|^2\,dw\,dt + c\int_{B^T} |Y|^2\,dw\,dt.$$

With no loss of generality we may assume that $T>0$ is chosen such that

$$\int_B |Y(T)|^2\,dw = \sup_{0\leqslant t\leqslant T}\int_B |Y(t)|^2\,dw.$$

Estimating the L^4-norm of Y by (3.9′) we conclude that

$$\begin{aligned}
&\sup_{0\le t\le T}\int_B |Y(t)|^2\,dw+\int_{B^T}|\nabla Y|^2\,dw\\
&\quad\le c\left(\int_{B^T}|\partial_t X|^2+|\nabla X|^4\,dw\,dt\right)^{1/2}\left(\sup_{0\le t\le T}\int_B |Y(t)|^2\,dw+\int_{B^T}|\nabla Y|^2\,dw\,dt\right)\\
&\qquad+cT\sup_{0\le t\le T}\int_B |Y(t)|^2\,dw,
\end{aligned}$$

and by absolute continuity of the Lebesgue integral the right-hand-side can be made

$$\le\frac{1}{2}\left(\sup_{0\le t\le T}\int_B |Y(t)|^2\,dw+\int_{B^T}|\nabla Y|^2\,dw\,dt\right)$$

for $T>0$ sufficiently small. Hence $Y\equiv 0$ on B^T for small $T>0$. By iteration $Y\equiv 0$ on B^T for any $T>0$ such that $X^{(1)}, X^{(2)}$ are defined. □

Local existence. Local existence of solutions to (3.1)–(3.4) will be obtained by a fixed point method.

LEMMA 3.16. *For any $H\in\mathbf{R}$, any $X_0\in\mathscr{C}(S)$ there exists a number $\bar T>0$ depending only on H, S and the number*

$$\bar R=\sup\left\{R\in\,]0,1]\,\middle|\,\sup_{w\in B}D(X_0;B_R(w)\cap B)\le\frac{\bar\varepsilon}{4}\right\}$$

such that (3.1)–(3.4) *admits a solution $X\in V(B^{\bar T})$. Here, $\bar\varepsilon<0$ is the least of the constants ε_j appearing in Lemmata* 3.7, 3.8, 3.12.

If $|H|<1/L$, then $\|X\|_{L^\infty(B^{\bar T})}\le L$.

Proof. First consider smooth initial data $X_0\in\mathscr{C}_2(S)$. By reflection in S we may extend X_0 to some ball $B_\varrho(0)$, $\varrho>1$. Denote

$$\hat X_0(w)=\begin{cases} X_0(w), & |w|\le 1,\\ R\left(X_0\left(\dfrac{w}{|w|^2}\right)\right), & |w|>1,\end{cases}$$

and for $T>0$ let

$$(B_\varrho)^T=B_\varrho(0)\times[0,T].$$

For sufficiently small $\sigma>0, T>0$ and a suitable number $q>4$ to be determined later consider the set

$$\Xi = \left\{ \hat{X} \in V((B_\varrho)^T) | \hat{X}(0) = \hat{X}_0, \operatorname*{ess\,sup}_{0\le t\le T} \int_{(B_\varrho)^T} |\nabla \hat{X}(t) - \nabla \hat{X}_0|^q \, dw \le \sigma \right\}.$$

Endowed with the topology of $V((B_\varrho)^T)$, Ξ is closed and convex.

To $\hat{X} \in \Xi$ we now associate the unique solution $\hat{Y} =: f(\hat{X}) \in V((B_\varrho)^T)$ of the Cauchy-Dirichlet problem

$$\partial_t \hat{Y} - a(w)\Delta \hat{Y} + \hat{\Gamma}(w, \hat{X})(\nabla \hat{X}, \nabla \hat{X}) = 0 \quad \text{in} \quad (B_\varrho)^T \tag{3.28}$$

with initial and boundary data

$$\hat{Y}(0) = \hat{X}_0, \tag{3.29}$$

$$\hat{Y} = \hat{X} \quad \text{on} \quad \partial B_\varrho \times [0, T]. \tag{3.30}$$

From (3.28) and [11, Theorem 4.10.1 and (10.12), p. 355] if $X_0 \in C^2$ we have uniform estimates

$$\int_{B^T} |\partial_t \hat{Y}|^{q/2} + |\nabla^2 \hat{Y}|^{q/2} \, dw \, dt \le c(X_0)$$

Hence it is possible to a priori bound $\hat{Y}$ on the cylinder surface $\partial B_{1/\varrho}(0) \times [0, T]$ in a trace space $W^{l, l/2}_{q/2}$ with $l>3/2$, cp. [11, Lemma 2.3.4]. In particular, for sufficiently small $\varrho>1, T>0, \sigma>0$, the points $\hat{Y}(w/|w|^2, t)$, $|w| = \varrho$, $0 \le t \le T$, will all lie in a δ-neighborhood of S and their reflection in S will be defined. We may therefore define a map $F: \Xi \to \Xi$ by letting $\hat{Z} = F(\hat{X})$ be the unique solution to the problem

$$\partial_t \hat{Z} - a(w)\Delta \hat{Z} + \hat{\Gamma}(w, \hat{X})(\nabla \hat{X}, \nabla \hat{X}) = 0 \quad \text{in} \quad (B_\varrho)^T \tag{3.31}$$

with initial and boundary data

$$\hat{Z}(0) = \hat{X}_0, \tag{3.32}$$

$$\hat{Z}(w, t) = R\left(\hat{Y}\left(\frac{w}{|w|^2}, t\right)\right) \quad \text{on} \quad \partial B_\varrho \times [0, T]. \tag{3.33}$$

Note that if $q>4$ is chosen sufficiently large from [11, Theorem 4.9.1] and our above observation about the regularity of $\hat{Y}$ on $\partial B_{1/\varrho}(0) \times [0, T]$ we obtain uniform Hölder estimates for $\nabla\hat{Z}$ in space and time. In particular, for small enough $T>0$ we obtain $\hat{Z}\in\Xi$ and $F:\Xi\to\Xi$. Moreover, $F(\Xi)$ is bounded in $V((B_\varrho)^T)$.

Finally, F is compact. To see this consider a bounded subset of Ξ. By weak compactness of $V((B_\varrho)^T)$ and uniform boundedness of $\nabla\hat{X}(t)$ in $L^q(B_\varrho(0))$ for any $t\in[0,T]$ and any $X\in\Xi$ this subset is compact in $L^2([0,T];H^{1,4}(B_\varrho;\mathbf{R}^3))$. Moreover, the associated set of traces $\hat{Y}|_{\partial B_{1/\varrho}\times[0,T]}$ is compact in the trace space $W_2^{3/2,3/4}$. From (3.31)–(3.33) and [11, Theorem 4.9.1] it now follows that F is compact in the $V((B_\varrho)^T)$-topology. By the Schauder fixed point theorem F has a fixed point $\hat{X}=\hat{Z}$. Necessarily

$$\hat{Y}(w,t)=\hat{X}(w,t)=\hat{Z}(w,t)=R\left(\hat{Y}\left(\frac{w}{|w|^2},t\right)\right)\quad\text{on}\quad \partial B_\varrho\times[0,T].$$

I.e. $\hat{Y}$ is also a solution to (3.31)–(3.33). Hence $\hat{Y}=\hat{Z}=\hat{X}$ and $\hat{X}$ is a solution to (3.18). But by construction also $R(\hat{X}(w/|w|^2,t))$ is a solution to (3.18) in $[B_\varrho(0)\setminus B_{1/\varrho}(0)\times[0,T]]$ with the same initial and boundary data as $\hat{X}$. Our proof of uniqueness for (3.1)–(3.4), cp. Lemma 3.15, conveys to this situation and we infer that $\hat{X}(w,t)\equiv R(\hat{X}(w/|w|^2,t))$. In particular, $X\in\mathcal{C}(S)$ and (3.3) is satisfied. (3.4) also holds—otherwise $\nabla^2\hat{X}\notin L^2((B_\varrho)^T)$. Finally, by Lemma 3.5 if $|H|<1/L$ estimate (3.6) will be satisfied.

This proves local existence for smooth initial data $X_0\in\mathcal{C}(S)\cap C^2(B;\mathbf{R}^3)$. To obtain local existence for arbitrary $X_0\in\mathcal{C}_0(S)$ we approximate X_0 by smooth data X_0^m and let $X^m\in V(B^{T_m})$ be the corresponding solution. Note that by Lemmata 3.7, 3.8, 3.12 each X^m persists as a regular solution to (3.1)–(3.4) for at least a time $\bar{T}=\bar{\varepsilon}\bar{R}^2/2c_2>0$. In fact, Lemma 3.7 guarantees the estimate $\varepsilon(\bar{R})\leq\bar{\varepsilon}$ for all X^m on $[0,\bar{T}]$, and Lemmata 3.8, 3.12 apply. Moreover, we have a uniform bound

$$\int_{(B_\varrho)^{\bar{T}}}|\partial_t X^m|^2+|\nabla^2X^m|^2+|\nabla X^m|^4\,dw\,dt+\sup_{0\leq t\leq\bar{T}}D(X^m(t))\leq c(\bar{R}).$$

and we may extract a subsequence that converges weakly to a solution of (3.1)–(3.4).

If $|H|<1/L$, estimate (3.6) for X^m implies the same estimate for X. □

Asymptotics. Let us now investigate the global behavior of the solutions to (3.1)–(3.4).

LEMMA 3.17. *Let $H \in \mathbf{R}$. Suppose $X \in \bigcap_{T<\infty} V(B^T)$ is a solution to* (3.1)–(3.4), *and suppose that for $\bar{T}=\infty$ condition* (3.7) *is satisfied while for some $R>0$ there holds*

$$\sup_{(w,t)\in B^\infty} D(X(t); B_R(w)\cap B) \leq \bar{\varepsilon}.$$

Then X is globally regular and there exists a sequence $t_m \to \infty$ such that $X(t_m) \in \mathscr{C}_2(S)$ and

$$X(t_m) \to \bar{X} \quad \text{in} \quad H^{2,2}(B; \mathbf{R}^3)$$

where $\bar{X}$ is a solution to (1.1)–(1.4).

Proof. Our assumptions and Lemma 3.7 imply that for $m \in \mathbf{N}$:

$$\int_{B_m^{m+1}} |\partial_t X|^2 \, dw \, dt \to 0 \quad (m \to \infty)$$

$$\int_{B_m^{m+1}} |\nabla^2 X|^2 \, dw \, dt \leq c \quad \text{uniformly in } m.$$

By Fubini's theorem we may thus choose a sequence $t_m \to \infty$ such that $X_m = X(t_m) \in \mathscr{C}_2(S)$ satisfies

$$\partial_t X(t_m) = \Delta X_m - 2H X_{m_u} \wedge X_{m_v} \to 0 \quad \text{in } L^2(B)$$

$$\partial_n X_m(w) \perp T_{X_m(w)} S, \quad \text{a.e.} \quad \text{on } \partial B$$

$$\sup_{w\in B} D(X_m, B_R(w)\cap B) \leq \bar{\varepsilon}, \quad \text{for all } m.$$

The claim now follows from Theorem 3.19 below. □

Singularities. Likewise we may analyse singularities created by concentration of Dirichlet's integral.

LEMMA 3.18. *Let $H \in \mathbf{R}$, $|H| < 1/L$. Suppose that for some $\bar{T} \leq \infty$, $X \in \bigcap_{T<\bar{T}} V(B^T)$ is a solution to* (3.1)–(3.4) *satisfying the modulus estimate* (3.6). *Assume that condition* (3.7) *is satisfied while for all $R>0$:*

$$\lim_{T \nearrow \bar{T}} \sup_{(w,t)\in B^T} D(X(t); B_R(w)\cap B) \geq \bar{\varepsilon}.$$

Then there exist sequences $t_m \nearrow \bar{T}$, $w_m \in B$, $R_m \searrow 0$ *and a surface* $\bar{X} \in H^{2,2}_{\mathrm{loc}}(\mathbf{R}^2_+; \mathbf{R}^3)$ *such that* $X(t_m) \in \mathscr{C}_2(S)$ *and the rescaled functions*

$$X_m(w) \equiv X(w_m + wR_m, t_m) \to \bar{X} \quad \text{in} \quad H^{2,2}_{\mathrm{loc}}(\mathbf{R}^2_+; \mathbf{R}^3)$$

after a possible rotation of coordinates. Moreover, $D(\bar{X}) < \infty$, *and* $\bar{X}$ *is conformal to some non-constant regular solution to* (1.1)–(1.4).

Proof. For a sequence $R_m \to 0$ let $t_m \leq \bar{T}$ be maximal with the property that for some $w_m \in B$

$$D(X(t_m), B_{R_m}(w_m) \cap B) = \sup_{(w,t) \in B^{t_m}} D(X(t); B_{R_m}(w) \cap B) = \bar{\varepsilon}.$$

Clearly $t_m \nearrow \bar{T}$ as $m \to \infty$.

By Lemma 3.8 there exists a constant $c_3 = \bar{\varepsilon}/2c_2$ such that for all $t \in [t_m - c_3 R_m^2, t_m]$

$$D(X(t), B_{2R_m}(w_m) \cap B) \geq \frac{\bar{\varepsilon}}{4} > 0.$$

Moreover, by Lemma 3.7 and (3.7)

$$\int_{B^{t_m}_{t_m - c_3 R_m^2}} |\nabla^2 X|^2 \, dw \, dt \leq c$$

uniformly for all m while again by assumption (3.7) and absolute continuity of the Lebesgue integral

$$\int_{B^{t_m}_{t_m - c_3 R_m^2}} |\partial_t X|^2 \, dw \, dt \to 0 \quad (m \to \infty).$$

Finally, by (3.7) also

$$\operatorname*{ess\,sup}_{t_m - c_3 R_m^2 \leq t \leq t_m} D(X(t)) \leq c < \infty$$

uniformly for all m.

Hence if we rescale

$$X^m(w, t) = X(w_m + R_m w, t_m + R_m^2 t)$$

and let

$$B^m = \{w \in \mathbf{R}^2 | w_m + R_m w \in B\}$$

our new functions $X^m \in V((B^m)_{-c_3})$ will satisfy (3.1)–(3.3) on $(B^m)_{-c_3}$ together with the estimates

$$\int_{(B^m)_{-c_3}} |\partial_t X^m|^2 \, dw \, dt \to 0 \quad (m \to \infty),$$

$$\int_{(B^m)_{-c_3}} |\nabla^2 X^m|^2 \, dw \, dt \leq c,$$

$$\operatorname*{ess\,inf}_{-c_3 \leq t \leq 0} D(X^m(t); B_2(0) \cap B^m) \geq \frac{\bar{\varepsilon}}{4} > 0$$

$$\operatorname*{ess\,sup}_{(w,t) \in (B^m)_{-c_3}} D(X^m(t); B_1(w) \cap B^m) \leq \bar{\varepsilon},$$

$$\operatorname*{ess\,sup}_{-c_3 \leq t \leq 0} D(X^m(t)) \leq c < \infty,$$

uniformly in m.

Choosing $\tau_m \in [-c_3, 0]$ suitably we can achieve that $X_m = X^m(\tau_m)$ satisfies $X_m \in H^{2,2}(B^m; \mathbf{R}^3)$ and

$$\partial_t X^m(\tau_m) = \Delta X_m - 2H X_{m_u} \wedge X_{m_v} \to 0 \quad \text{in} \quad L^2(B^m)$$

while

$$\partial_n X_m(w) \perp T_{X_m(w)} S, \quad \text{a.e.} \quad \text{on } \partial B^m$$

and

$$0 < c \leq D(X_m; B_2(0) \cap B^m) \leq D(X_m) \leq c' < \infty,$$

$$\sup_{w \in B^m} D(X_m; B_1(w) \cap B^m) \leq \bar{\varepsilon}$$

uniformly in m. Shifting time we may assume $\tau_m = 0$. Again Theorem 3.19 may now be invoked. It follows that either $B^m \to \mathbf{R}^2$ or B^m (after rotation) exhaust a hemi-space $\mathbf{R}^2_+$.

In the first case moreover

$$X^m \to \bar{X} \quad \text{locally in} \quad H^{2,2}(\mathbf{R}^2; \mathbf{R}^3)$$

where $\bar{X}$ is a conformal branched covering of a sphere of radius $1/|H| > L$. But this is impossible by (3.6).

This leaves as the only possibility that

$$X_m \to \bar{X} \quad \text{locally in} \quad H^{2,2}(\mathbf{R}^2_+; \mathbf{R}^3)$$

where $\bar{X}$ is conformal to a non-constant solution of (1.1)–(1.4) □

A local Palais-Smale condition. The following compactness result may be interpreted as a local Palais-Smale condition for the functional E_H on the dense subset $\mathscr{C}_2(S) \subset \mathscr{C}(S)$, cp. [24, Proposition 5.1].

For reason of exposition we have scaled domains to achieve a uniform control of the densities $D(X_m; B_1(w) \cap B^m)$. In spirit, however, the result below is of the same nature as [3], [4], [15], [17], [22], [23, Proposition 3.7], [24, Proposition 5.1], [27] where "compactness modulo separation of spheres" is observed.

THEOREM 3.19. *Suppose $B^m \subset \mathbf{R}^2$ is a sequence of balls, $X_m \in H^{2,2}(B^m; \mathbf{R}^3)$ a sequence of surfaces such that*

$$X_m(\partial B^m) \subset S \tag{3.34}$$

$$\partial_n X_m(w) \perp T_{X_{m(w)}} S, \quad \text{a.e.} \quad \text{on } \partial B^m \tag{3.35}$$

$$D(X_m) \leq c < \infty, \quad \text{uniformly in } m \tag{3.36}$$

$$\sup_{w \in B^m} D(X_m; B_1(w) \cap B^m) \leq \bar{\varepsilon}, \quad \text{uniformly in } m \tag{3.37}$$

while for some $H \in \mathbf{R}$

$$\int_{B^m} |\Delta X_m - 2H X_{m_u} \wedge X_{m_v}|^2 \, dw \to 0. \tag{3.38}$$

Then there exists a limiting domain $B^\infty \subset \mathbf{R}^2$ and a surface $\bar{X} \in H^{2,2}_{\mathrm{loc}}(\bar{B}^\infty; \mathbf{R}^3)$ such that for a subsequence $m \to \infty$

$$B^m \to B^\infty$$

$$X_m \to \bar{X} \quad \text{in} \quad H^{2,2}_{\mathrm{loc}}(\bar{B}^\infty; \mathbf{R}^3),$$

and $\bar{X}$ solves (1.1)–(1.4) *on B^∞*. I.e. *$\bar{X}$ satisfies*

$$\Delta\bar{X} = 2H\bar{X}_u \wedge \bar{X}_v \quad \text{in } B^\infty, \tag{3.39}$$

$$|\bar{X}_u|^2 - |\bar{X}_v|^2 = 0 = \bar{X}_u \cdot \bar{X}_v \quad \text{in } B^\infty, \tag{3.40}$$

$$\bar{X}(\partial B^\infty) \subset S \tag{3.41}$$

$$\partial_n \bar{X}(w) \perp T_{\bar{X}(w)} S \quad \text{on } \partial B^\infty. \tag{3.42}$$

Moreover,

$$D(\bar{X}) < \infty.$$

In particular, if $B^m = B^\infty = B$, the sequence X_m accumulates (in $H^{2,2}$) at a regular solution $\bar{X}$ of (1.1)–(1.4) *with curvature H.*

If $B^\infty = \mathbf{R}^2_+$ the surface X_m accumulates at a surface $\bar{X}$ which is conformal to a regular solution of (1.1)–(1.4) *with curvature H.*

Finally, if $B^\infty = \mathbf{R}^2$, X_m accumulates at a branched conformal covering of a sphere of radius $1/|H|$.

Proof. Testing with ΔX_m we obtain

$$\int_{B^m} |\Delta X_m|^2\, dw \leq H \int_{B^m} |\nabla X_m|^2 |\Delta X_m|\, dw + o(1) \left(\int_{B^m} |\Delta X_m|^2\, dw \right)^{1/2}$$

where $o(1) \to 0$ $(m \to \infty)$. By (3.9′) this gives

$$\begin{aligned} \int_{B^m} |\Delta X_m|^2\, dw &\leq c \int_{B^m} |\nabla X_m|^4\, dw + o(1) \\ &\leq c\bar{\varepsilon} \int_{B^m} |\nabla^2 X_m|^2\, dw + c + o(1). \end{aligned}$$

Similarly, testing with $R_m^2(\partial^2/\partial\phi^2)X_m$—where (r, ϕ) denote polar coordinates on B^m and R_m^{-1} denotes the radius of B^m—yields that

$$R_m^2 \int_{B^m} \left|\nabla \frac{\partial}{\partial\phi} X_m\right|^2 dw \leq c\bar{\varepsilon} \int_{B^m} |\nabla^2 X_m|^2\, dw + c$$

$$+R_m^2\int_{\partial B^m}|\partial_n X_m\cdot G(X_m)\left[\frac{\partial}{\partial\phi}X_m\cdot\nabla G(X_m)\frac{\partial}{\partial\phi}X_m\right]d\phi+o(1)$$
$$\leqslant c\bar{\varepsilon}\int_{B^m}|\nabla^2X_m|^2\,dw+c+o(1),$$

cp. the proof of Lemma 3.7.

Together, these estimates allow us to conclude that

$$\int_{B^m}|\nabla^2X_m|^2\,dw\leqslant c\quad\text{uniformly.}\tag{3.44}$$

By Sobolev's embedding theorem, in particular the X_m are equicontinuous and hence may be extended to a ϱ-neighborhood $\hat{B}^m$ of B^m by reflection in S, with $\varrho>0$ independent of m. (3.34) and (3.35) guarantee that these extensions $\hat{X}_m\in H^{2,2}(\hat{B}^m;\mathbf{R}^3)$ and satisfy (3.44) on $\hat{B}^m$, cp. (3.19). Moreover, as in the proof of Lemma 3.10, for the extensions $\hat{X}_m$ of X_m from (3.38) we derive that

$$\int_{\hat{B}^m}|a(w)\Delta\hat{X}_m-\hat{\Gamma}(w,\hat{X}_m)(\nabla\hat{X}_m,\nabla\hat{X}_m)|^2\,dw\to 0\tag{3.45}$$

for all m with uniformly continuous coefficients $a\geqslant 1$.

Suppose (as we may) that $B^m\to B^\infty$. By (3.44) we may select a subsequence $\hat{X}_m$ such that for any bounded $\Omega\subset\mathbf{R}^2$

$$\hat{X}_m\to\bar{X}\quad\text{weakly in}\quad H^{2,2}(B^\infty\cap\Omega;\mathbf{R}^3).$$

Note that for any such Ω we always have $\hat{B}^m\supset B^\infty\cap\Omega$ if m is sufficiently large.

By Rellich's theorem therefore also

$$\hat{X}_m\to\bar{X}\quad\text{strongly in}\quad H^{1,q}(B^\infty\cap\Omega;\mathbf{R}^3)$$

and

$$\hat{X}_m\to\bar{X}\quad\text{strongly in}\quad H^{1,q}(\partial B^\infty\cap\Omega;\mathbf{R}^3)$$

for any $q<\infty$. Hence we may pass to the limit $m\to\infty$ in (3.34), (3.35) and (3.38) and find that $\bar{X}$ solves (3.39), (3.41) and (3.42) as claimed.

Moreover, letting $\hat{\varphi}_m$ be the cutoff function associated with $\hat{X}_m$ by (3.20),

$\hat{\varphi}=\min\{\hat{\varphi}_m, \hat{\varphi}_n\}$, $\zeta\in\mathscr{C}_0^\infty(\Omega)$, upon testing the difference of equations (3.44) for $m, n\in\mathbf{N}$ with the function $\Delta(\hat{X}_m-\hat{X}_n)\hat{\varphi}\zeta$ we obtain:

$$\begin{aligned}\int_{\mathbf{R}^2}|\nabla^2(\hat{X}_m-\hat{X}_n)|^2\hat{\varphi}\zeta\,dw &= \int_{\mathbf{R}^2}|\Delta(\hat{X}_m-\hat{X}_n)|^2\hat{\varphi}\zeta\,dw+o(1)\\ &\le c\int_{\mathbf{R}^2}[\hat{\Gamma}(w,\hat{X}_m)(\nabla\hat{X}_m,\nabla\hat{X}_m)-\hat{\Gamma}(w,\hat{X}_n)(\nabla\hat{X}_n,\nabla\hat{X}_n)]\\ &\quad\times\Delta(\hat{X}_m-\hat{X}_n)\hat{\varphi}\zeta\,dw+o(1)\to 0 \quad\text{as}\quad m\to\infty,\end{aligned}$$

and $\hat{X}_m\to\bar{X}$ strongly in $H^{2,2}(B^\infty\cap\Omega;\mathbf{R}^3)$ for any bounded $\Omega\subset\mathbf{R}^2$.

The remaining assumptions of the theorem are now easily verified.

Note that by (3.39) the complex valued function of $w=u+iv\in B^\infty\subset\mathbf{R}^2\cong\mathbf{C}$

$$\Phi(w)\equiv|\bar{X}_u|^2-|\bar{X}_v|^2-2i\bar{X}_u\cdot\bar{X}_v$$

is holomorphic and by (3.43) is integrable over B^∞.

In case $B^\infty=\mathbf{R}^2$ from the mean value theorem for harmonic functions

$$\Phi(w)=\frac{1}{2\pi R}\int_{\partial B_R(w)}\Phi(w')\,dw'$$

upon letting $R\to\infty$ suitably we obtain at once that $\Phi\equiv0$, i.e. that $\bar{X}$ is conformal.

Similarly, if $B^\infty=\mathbf{R}^2_+$, by (3.41) and (3.42) Φ is real on $\partial B^\infty=\{(u,v)|v=0\}$. By reflection the imaginary part of Φ may be extended to a harmonic function $\in L^1(\mathbf{R}^2)$, hence it must vanish identically by the preceding argument. The Cauchy-Riemann equations now imply that $\Phi\equiv$const. But $\Phi\in L^1(\mathbf{R}^2_+)$, thus $\Phi\equiv0$, and $\bar{X}$ is conformal.

By conformal equivalence of $B\cong\mathbf{R}^2_+$ and (conformal invariance of (3.39)–(3.43)) this argument also proves conformality of $\bar{X}$ in the case $B^\infty=B$ and concludes the proof in this case.

If $B^\infty=\mathbf{R}^2_+$, by conformal equivalence $B\cong\mathbf{R}^2_+$ again, the map $\bar{X}$ will be conformal to a surface $\bar{X}\in\mathscr{C}(S)$ satisfying (1.1)–(1.4) in a weak (distribution) sense. By the regularity result of [6] $\bar{X}$ is regular and the theorem is also verified in this case.

Finally, the characterization of solutions to (3.39) on $B^\infty=\mathbf{R}^2$ with finite Dirichlet integral follows e.g. from [3]. □

Proof of Theorem 3.1. Existence, uniqueness and regularity of solutions $X\in V(B^T)$ to (3.1)–(3.4) for small $T>0$ follow from Lemmata 3.12, 3.15, 3.16 and Remark 3.13. Also (3.6) will be satisfied whenever $|H|<1/L$.

By iteration, local solutions may be continued either globally—and their asymptotic behavior is given by Lemma 3.17—or until a singularity is encountered. In this case under assumption (3.7) the desired conclusion follows from Lemma 3.18.

Proof of Theorem 3.2. If $H=0$ we have $E_H=D$ and Lemma 3.6 implies the uniform a priori bound

$$\int_{B^T} |\partial_t X|^2 \, dw \, dt + D(X(T)) \leq D(X_0) \tag{3.7}$$

for all solutions to (3.1)–(3.4) with $X_0 \in \mathscr{C}(S)$. Theorem 3.1 now guarantees the existence of a unique regular solution X to (3.1)–(3.4) (for $H\equiv 0$) in a time interval $[0, T_1[$ where T_1 is characterized by (3.5).

By Remark 3.14 and arguments in the proof of [24, Theorem 4.2] we infer that X remains regular up to T_1 with exception of (finitely many) points $w_1^{(1)}, \ldots, w_{l_1}^{(1)}$ where non-constant minimal surfaces solving (1.1)–(1.4) for $H=0$ separate (in the sense of Lemma 3.18). Moreover, by (3.7) for some $t_m \nearrow T_1$: $X(t_m) \to X_1$ weakly in $\mathscr{C}(S)$, and finiteness of the number of singularities follows from (3.7), Lemma 3.8 and the estimate

$$\begin{aligned} 0 \leq D(X_1) &\leq D(X_0) - \liminf_{R \to 0} \sum_{l=1}^{l_1} \liminf_{t \to T_1} D(X(t); B_{2R}(w_l^{(1)}) \cap B) \\ &\leq D(X_0) - \frac{l_1}{c_2} \bar{\varepsilon}. \end{aligned}$$

By Theorem 3.1 X may now be continued to a (weak) solution of (3.1)–(3.4) on a larger interval $[0, T_2[$ by solving the initial value problem (3.1)–(3.3) with $X(T_1)=X_1$. Moreover, X will be a regular solution to (3.1)–(3.4) on $\bar{B} \times]0, T_2[$ with exception of the points $(w_l^{(1)}, T_1)$. By induction we obtain a solution X to (3.1)–(3.4) on intervals $[0, T_k[$ where at each T_k finitely many (l_k) non-constant solutions to (1.1)–(1.4) for $H=0$ separate. Moreover, letting $X_k = \text{w-lim}_{t \to T_k} X(t)$

$$D(X_k) \leq D(X_0) - \sum_{j=1}^{k} l_j \frac{\bar{\varepsilon}}{c_2}$$

and there can be at most a finite number of singular points $(w_l^{(k)}, T_k)$ in space-time. In particular, for some $k \in \mathbb{N}$ we must have $T_k = \infty$. This proves Theorem 3.2. □

4. Proof of Theorem 1.1

To illustrate the general principle first consider *the case* $H=0$. Suppose by contradiction that there is no non-constant solution to (1.1)–(1.4) for $H=0$. Theorem 3.2 and Lemma 3.17 then guarantee the existence of a global solution $X(t;X_0)$ through any initial data $X_0\in\mathscr{C}(S)$, and $X(t,X_0)$ converges to a solution of (1.1)–(1.4) as $t\to\infty$.

Choose some path $p_0\in P$. By Lemma 3.5 and continuous dependence of X on the initial data for all $t<\infty$ then also

$$p_t(s)\equiv X(t;p_0(s))\in P$$

and therefore by Lemma 2.4 for all t

$$\sup_{0\leq s\leq 1} D(p_t(s))\geq\beta_0>0. \tag{4.1}$$

Let $t_m\to\infty$ and suppose the supremum in (4.1) is achieved at $s=s_m$ for each $t=t_m$. By compactness $s_m\to s$, and since $D(X(t))$ is non-increasing for any solution X to (3.1)–(3.4) there results

$$D(X(t_m);p_0(s))=\lim_{n\to\infty} D(X(t_m);p_0(s_n))\geq\lim_{n\to\infty} D(X(t_n);p_0(s_n))\geq\beta_0>0$$

and $X(t_m;p_0(s))$ for a suitable sequence $t_m\to\infty$ converges to a non-constant solution of (1.1)–(1.4) for $H=0$.

Next consider *the case* $0<H<\bar{H}\leq 1/L$. Again assume that (1.1)–(1.4) possesses only the trivial (constant) solutions.

Note the identity

$$E_H(X)=\frac{\bar{H}-H}{\bar{H}}D(X)+\frac{H}{\bar{H}}E_{\bar{H}}(X) \tag{4.2}$$

which holds for all $X\in\mathscr{C}(S)$, provided $E_H(X)$ and $E_{\bar{H}}(X)$ are defined by means of the same extension operator, cp. Section 2.

Let $p\in P$. For convenience we normalize

$$E_H(p(0))=E_{\bar{H}}(p(0))=0>E_H(p(1)>E_{\bar{H}}(p(1)). \tag{4.3}$$

(If necessary, p may be reparametrized via $s\mapsto 1-s$ to achieve (4.3).)

Let $X=X(t;p(s))$ denote the unique local solution to (3.1)–(3.4) through $X_0=p(s)$,

$0 \leqslant s \leqslant 1$, guaranteed by Theorem 3.1. Note that Lemma 3.6 and (4.2) imply for any such X and any $T>0$ in the domain of X the estimate:

$$\begin{aligned} L\int_{B^T} |\partial_t X|^2\, dw\, dt + L^2(\bar{H}-H)\, D(X(T)) + \frac{1}{\bar{H}} E_{\bar{H}}(X(T)) \\ \leqslant \frac{1}{H}\left(\int_{B^T} |\partial_t X|^2\, dw\, dt + \frac{\bar{H}-H}{H} D(X(T) + \frac{H}{\bar{H}} E_{\bar{H}}(X(T))\right) \\ = \frac{1}{H}\left(\int_{B^T} |\partial_t X|^2\, dw\, dt\right) + \frac{1}{H} E_H(X(T)) \leqslant \frac{1}{H} E_H(X(0)). \end{aligned} \tag{4.4}$$

In particular, if for some $\varepsilon>0$ we let $p \in P$ satisfy (4.3) and

$$\sup_s E_H(p(s)) \leqslant \beta_H - H \operatorname{vol}(S) + H\varepsilon,$$

which is possible by (2.9) and (4.3), from (4.4) we obtain

$$L\int_{B^T} |\partial_t X|^2\, dw\, dt + L^2(\bar{H}-H)\, D(X(T)) + \frac{1}{\bar{H}} E_{\bar{H}}(X(T)) \leqslant \frac{\beta_H}{H} - \operatorname{vol}(S) + \varepsilon, \tag{4.5}$$

for any $X = X(t; p(s))$, and any $T>0$ such that $X \in V(B^T)$ is defined.

Let $T(s) \leqslant \infty$ be maximal such that $X(\cdot, p(s))$ is defined on $[0, T(s)[$ and consider an arbitrary continuous path

$$p'(s) = X(t(s); p(s)), \quad 0 \leqslant s \leqslant 1, 0 \leqslant t(s) < T(s),$$

connecting $p'(0) = X(t(0); p(0)) = p(0)$ with $p'(1) = X(t(1); p(1)) = p(1)$. Note that by (3.6) $\|p'(s)\|_{L^\infty} \leqslant L$. Hence $p' \in P$ and by (2.9)

$$\sup_s E_{\bar{H}}(p'(s)) \geqslant \beta_{\bar{H}} - \bar{H} \operatorname{vol}(S). \tag{4.6}$$

Inserting (4.6) into (4.5) we infer that for some $s \in [0,1]$, $X = X(\cdot; p(s))$, $T = t(s)$:

$$L\int_{B^T} |\partial_t X|^2\, dw\, dt + L^2(\bar{H}-H)\, D(X(T)) + \frac{\beta_{\bar{H}}}{\bar{H}} \leqslant \frac{\beta_H}{H} + \varepsilon, \tag{4.7}$$

where $X(T)$ maximizes $E_{\bar{H}}$ on p'. Since $\varepsilon>0$ is arbitrary this proves

LEMMA 4.1. *The function $H \mapsto \beta_H / H$ is non-increasing in* $]0, 1/L[$.

But any function of bounded variation, in particular any monotone function is a.e. differentiable. Now let

$$\mathscr{H}_+ = \left\{ H \in \left]0, \frac{1}{L}\right[\;\middle|\; H \to \frac{\beta_H}{H} \text{ is differentiable at } H \right\}.$$

Then $\bar{\mathscr{H}}_+ = [0, 1/L]$ and for any $H \in \mathscr{H}_+$ there exists $c>0$ such that

$$\limsup_{\bar{H} \searrow H} \frac{1}{\bar{H}-H} \left(\frac{\beta_H}{H} - \frac{\beta_{\bar{H}}}{\bar{H}} \right) \leq c. \tag{4.8}$$

Fix such H and for $\bar{H}>H$ close to H choose $\varepsilon = c(\bar{H}-H)$ in (4.7). For any p', $X=X(\cdot\,;p(s))$, $T=t(s)$ as above then there holds

$$\int_{B^T} |\partial_t X|^2 \, dw \, dt \leq c(\bar{H}-H) \tag{4.9}$$

$$D(X(T)) \leq 2c \tag{4.10}$$

uniformly in $\bar{H}$ and p'.

(4.9) and (4.10) almost give us the criterion (3.7) which we need to invoke Theorem 3.1. In order to obtain a uniform bound on D on a time interval of length $\geq cR^2$, where $\varepsilon(R) \leq \bar{\varepsilon}$, we now argue as follows:

For a sequence $H_m \to H$, $H_m > H$, let $p = p_m \in P$ be chosen corresponding to $\varepsilon = \varepsilon_m = c(H_m - H)$. The index m will be implicit in the following. Let $t(s) = \min\{T(s), 1\}$, $t(0) = t(1) = 1$, and consider the simply connected set

$$V = \{(s,t) \in \mathbf{R}^2 \mid 0 \leq s \leq 1, t(s) \leq t \leq 1\}.$$

LEMMA 4.2. $E_{\bar{H}}(X(t;p(s))) \to -\infty$ if $(s,t) \to (s_0, t_0) \in \partial V$, $t_0 < 1$.

Proof. Otherwise (4.5) implies (3.7) while $X(\cdot\,, p(s_0))$ becomes singular at $T \leq t_0$. By Theorem 3.1 a non-constant solution to (1.1)–(1.4) separates from X at $\bar{T}$, contrary to our assumption. □

Perturbing the boundary of V slightly for any m we obtain a path $p' \in P$, $p'(s) = X(t'(s); p(s))$, $t'(s) < T(s)$ with the property that $t'(s) = 1$ if $E_{H_m}(p'(s)) \geq 0$. In particular, (4.9) and (4.10) will hold for suitable solutions $X = X(\cdot\,; p(s))$ with $T=1$. I.e. we have proved:

LEMMA 4.3. *For any $H \in \mathscr{H}_+$ there exists a constant c_0 and a sequence of solutions $X_m \in V(B^1)$ to* (3.1), (3.3) *and* (3.4) *such that $X_m(1)$ maximizes E_{H_m} on $p'_m \in P$ and*

$$\int_{B^1} |\partial_t X_m|^2 \, dw \, dt \to 0 \quad (m \to \infty)$$

while

$$D(X_m(1)) \leq c_0 \quad \textit{uniformly.}$$

Now let $t_m<1$ be maximal with the property that $D(X_m(t_m)) \geq 2c_0$, $t_m=0$ if no such number exists. Let

$$\varepsilon_m(R) = \sup_{(w,t)\in B^1_{t_m}} D(X_m(t); B_R(w) \cap B),$$

and let $R_m>0$ be maximal with the property that

$$\varepsilon_m(R_m) \leq \bar{\varepsilon}.$$

LEMMA 4.4. *$R_m^2/(1-t_m) \leq c < \infty$ uniformly in m.*

Proof. By Lemma 3.7 for $t \leq t_m + R_m^2$, $t \leq 1$:

$$\int_{B^t_{t_m}} |\nabla^2 X_m|^2 \, dw \, dt \leq 2c_0 c_1 \left(1 + \frac{t-t_m}{R_m^2}\right) \leq c.$$

Hence for such t

$$\begin{aligned} D(X_m(t)) &= D(X_m(t_m)) + \int_{t_m}^{t} \frac{d}{dt} D(X_m(t)) \, dt \\ &= 2c_0 - \int_{B^t_{t_m}} \partial_t X_m \cdot \Delta X_m \, dw \, dt \\ &\geq 2c_0 - \left(\int_{B^t_{t_m}} |\partial_t X_m|^2 \, dw \, dt\right)^{1/2} \left(\int_{B^t_{t_m}} |\nabla^2 X_m|^2 \, dw \, dt\right)^{1/2} \\ &> c_0 \end{aligned}$$

for sufficiently large m. It follows that $t_m + R_m^2 \leq 1$. □

The proofs of either Lemma 3.17 or 3.18 (depending on whether $R_m \geq R > 0$ or $R_m \to 0$) now show that a sequence of surfaces $X_m(t'_m)$, $t'_m \in [t_m, 1]$, either converges in

$H^{2,2}(B;\mathbf{R}^3)$ to a solution $\bar{X}$ of (1.1)–(1.4) or (after rescaling) converges locally to some $\bar{X}\in H^{2,2}_{\mathrm{loc}}(\mathbf{R}^2_+;\mathbf{R}^3)$ which is conformal to a non-constant, regular solution to (1.1)–(1.4).

In the second event the proof is complete. In this first case it might happen that $\bar{X}\equiv$const. But then $D(X_m(t'_m))\to 0$ and by Lemma 3.8 also $D(X_m(1))\to 0$.

Inspection of the proof of Lemma 2.4 now shows that E_{H_m} cannot achieve its supremum on $p'=p'_m$ at $X_m(1)$ for large m. The contradiction shows that $\bar{X}\not\equiv$const., and the proof is complete in the case $H\geq 0$.

The case $H\leq 0$ follows by reversal of orientation. □

Appendix

LEMMA A1. *Let $Y\in H^{1,2}(B)$ be harmonic in B with $Y(\partial B)\subset S$ in $L^2(\partial B;\mathbf{R}^3)$. Then*

$$\overline{Y(B)}\subset(Y(B)\cup S);$$

in particular, for any harmonic function $Y\in H^{1,2}(B;\mathbf{R}^3)$ with boundary on S there exists a point P lying interior to S such that $P\notin\overline{Y(B)}$.

Proof. By local continuity of Y it suffices to show that for any $\varepsilon>0$ there exists $r<1$ such that the image $Y(w)$ of any point w, $|w|>r$, lies in a ε-neighborhood of S.

Let $w_0\in\partial B$ and for $\varrho>0$ denote

$$B'_\varrho=B_\varrho(w_0)\cap B,$$

$$C'_\varrho=\partial B_\varrho(w_0)\cap B,$$

and let s measure arc length along C'_ϱ. Note that by Fubini's theorem $\partial Y/\partial s\in L^2(C'_\varrho;\mathbf{R}^3)$ for a.e. ϱ and $Y(w_1)$, $Y(w_2)\in S$, where w_1, w_2 denote the endpoints of C'_ϱ. Then analogous to the well-known Courant-Lebesgue-lemma for any $\delta>0$ we have:

$$\int_{B'_\delta}|\nabla Y|^2\,dx\geq\int_0^\delta\int_{C'_\varrho}\left|\frac{\partial}{\partial s}Y\right|^2 ds\,d\varrho$$

$$\geq\frac{\delta}{2}\operatorname*{ess\,inf}_{\varrho\in[\delta/2,\delta]}\left(\int_{C'_\varrho}\left|\frac{\partial}{\partial s}Y\right|^2 ds\right).$$

Choosing $\varrho\in[\delta/2,\delta]$ such that

$$\int_{C'_\varrho}\left|\frac{\partial}{\partial s}Y\right|^2 ds\leq\frac{4}{\delta}\int_{B'_\delta}|\nabla Y|^2\,dw,$$

by Hölder's inequality we may estimate for $w \in C'_\varrho$

$$[\operatorname{dist}(Y(w), S)]^2 \leqslant \left(\int_{C'_\varrho} \left|\frac{\partial}{\partial s} Y\right| ds\right)^2 \leqslant \pi\varrho \int_{C'_\varrho} \left|\frac{\partial}{\partial s} Y\right|^2 ds \leqslant 4\pi \int_{B'_\delta} |\nabla Y|^2 dw; \quad \text{(A.1)}$$

moreover, by absolute continuity of the Lebesgue integral the right hand side is smaller than $\varepsilon^2/4$, provided $\delta>0$ is sufficiently small.

Now given any $w' \in B'_\varrho$ there is a conformal map τ of B'_ϱ onto B such that $\tau(w')=0$. Denote $\tilde{Y}=Y \circ \tau^{-1}$. By the mean value property of harmonic functions

$$Y(w') = \tilde{Y}(0) = (2\pi)^{-1} \int_{\partial B} \tilde{Y}\, dw.$$

Hence by Hölders inequality and (A.1)

$$\begin{aligned}
\operatorname{dist}(Y(w'), S) &= \operatorname{dist}(\tilde{Y}(0), S) \\
&\leqslant \operatorname*{ess\,inf}_{\tilde{w} \in \partial B} |\tilde{Y}(0) - \tilde{Y}(\tilde{w})| + \operatorname*{ess\,sup}_{\tilde{w} \in \partial B} \operatorname{dist}(\tilde{Y}(\tilde{w}), S) \\
&\leqslant (2\pi)^{-2} \int_{\partial B} \int_{\partial B} |\tilde{Y}(w) - \tilde{Y}(\tilde{w})|\, dw\, d\tilde{w} + \frac{\varepsilon}{2} \qquad \text{(A.2)} \\
&\leqslant c \left(\int_{\partial B} \int_{\partial B} \frac{|\tilde{Y}(w) - \tilde{Y}(\tilde{w})|^2}{|w-\tilde{w}|^2}\, dw\, d\tilde{w}\right)^{1/2} + \frac{\varepsilon}{2}
\end{aligned}$$

and the latter double integral is equivalent to Douglas' integral which in turn is equivalent to Dirichlet's integral

$$c^{-1} \int_{\partial B} \int_{\partial B} \frac{|\tilde{Y}(w) - \tilde{Y}(\tilde{w})|^2}{|w-\tilde{w}|^2}\, dw\, d\tilde{w} \leqslant \int_B |\nabla \tilde{Y}|^2 dw = \int_{B'_\varrho} |\nabla Y|^2 dw \leqslant \int_{B'_\delta} |\nabla Y|^2 dw. \quad \text{(A.3)}$$

In conclusion, if $\delta>0$ is sufficiently small we may cover the annulus

$$A = \{w \in B \mid |w| > 1-\delta/4\}$$

by balls $B'_\varrho, \varrho \in [\delta/2, \delta]$, where for any point $w' \in B'_\varrho$

$$\operatorname{dist}(Y(w'), S) < \varepsilon.$$

Now we may let $r=1-\delta/4$ to achieve our claim. □

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