FIBER SPACES OVER TEICHMÜLLER SPACES

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Introduction

We first summarize the results of this paper for the simplest and most important special case: the Teichmüller spaces $T(p, n)$ of surfaces of type (p, n), i.e. closed Riemann surfaces of genus p with n punctures. The points of $T(p, n)$ are equivalence classes of orientation preserving homeomorphisms of a fixed surface S_0 of type (p, n) onto other such surfaces; two mappings, f_1 and f_2, are considered equivalent if there is a conformal mapping h such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. Homotopy classes of orientation preserving automorphisms of S_0 form the modular group Mod (p, n) which acts naturally on $T(p, n)$, and $X(p, n) = T(p, n)/\text{Mod}\,(p, n)$ is the space of moduli (conformal equivalence classes) of surfaces of type (p, n). We assume throughout that $3p-3+n \geqslant 0$. The space $T(p, n)$ has a canonical structure of a complex $(3p-3+n)$-dimensional manifold, the action of Mod (p, n) on $T(p, n)$ is holomorphic and properly discontinuous, and $X(p, n)$ is a normal complex space.

A central result in Teichmüller space theory asserts that $T(p, n)$ admits an essentially canonical representation as a bounded domain in $\mathbf{C}^{3p-3+n}$. In proving this result [7] one attaches to every $\tau \in T(p, n)$ a Jordan domain $D(\tau)$ and a quasi-Fuchsian group G^τ, both depending holomorphically on τ, such that τ is the equivalence class of mappings of S_0 onto $D(\tau)/G^\tau$. The fiber space $F(p, n)$ over $T(p, n)$ is the set of pairs (τ, z), with $\tau \in T(p, n)$, $z \in D(\tau)$.

We shall show that the group Mod (p, n) can be extended to a group mod (p, n) which acts holomorphically and properly discontinuously on $F(p, n)$. The quotient $Y(p, n) = F(p, n)/\text{mod}\,(p, n)$ is a normal complex space and a fiber space over $X(p, n)$ with the

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property: the fiber over a point $x \in X(p, n)$ representing a Riemann surface S is isomorphic to $S/\mathrm{Aut}\,(S)$ where Aut (S) is the group of all conformal automorphisms of S. For $n=0$, the existence of such a fiber space was asserted, without proof, by Teichmüller [16] and proved, in a completely different way, by Baily [5].

We shall also establish an isomorphism between $T(p, n+1)$ and $F(p, n)$ which conjugates mod (p, n) into a subgroup of index $n+1$ of Mod $(p, n+1)$. This implies that $X(p, n+1)$ is a ramified $(n+1)$-sheeted covering space of $Y(p, n)$.

In the last section the isomorphism theorem is used to represent the space $T(p, n)$ for $p=0, 1, 2$ as Bergman domains, that is as sets of r-tuples of complex numbers $(z_1, \ldots, z_r)$ determined by the requirement: z_j lies in a Jordan domain depending holomorphically on the variables z_k, $k<j$. It is hoped that this representation will prove useful.

In the body of the paper we deal with a more general case, Teichmüller spaces $T(G)$ and modular groups Mod (G) of arbitrary Fuchsian groups G. (If G does not have a fundamental domain of finite non-Euclidean area, $T(G)$ is a domain in an infinitely dimensional complex Banach space and Mod (G) need not act discontinuously.) The construction of the fiber space $F(G)$ and of the extended modular group mod (G) goes through in all cases. The isomorphism theorem can be stated and proved whenever G has no elements of finite order. We give two proofs of this theorem, one relies on a topological result of D. B. Epstein [12], the other is self-contained.

The results of this paper have been announced without proof in the survey article [9]; this article also contains all needed definitions and results and an extensive bibliography.

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§ 1. Teichmüller spaces

In this section we fix our notations and recall some basic definitions and facts.

Let U denote the *upper half-plane* of the complex plane; we denote by Q the group of all *quasiconformal automorphisms* of U, and, for every $\omega \in Q$, we denote by $K(\omega)$ the *dilatation* of ω. The elements $\omega \in Q$ with $K(\omega)=1$ form the subgroup Q_{conf} of conformal automorphisms of U; it can be identified with the *real Möbius group*. It is known that every $\omega \in Q$ can be extended, by continuity, to an automorphism of the closure of U in the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$; by abuse of language, we shall denote this extension by the same letter ω. The elements $w \in Q$ *normalized* by the conditions $w(0)=0$, $w(1)=1$, $w(\infty)=\infty$ form the subgroup Q_{norm}. Clearly,

$$Q = Q_{\text{norm}} Q_{\text{conf}} = Q_{\text{conf}} Q_{\text{norm}}, \quad Q_{\text{conf}} \cap Q_{\text{norm}} = 1,$$

where 1 denotes the trivial group $1 = \{\text{id}\}$.

The set Q_{norm} is made into a *complete metric space* (but not into a topological group) by defining the Teichmüller distance δ between two elements w and $\hat{w}$ as

$$\delta(w, \hat{w}) = \log K(\hat{w} \circ w^{-1}).$$

By a *Fuchsian group* we mean, in this paper, a discrete subgroup G of Q_{conf}. The region of discontinuity $\Omega(G)$ of G, i.e. the largest open subset of $\hat{\mathbf{C}}$ on which G acts properly discontinuously, is either the union of U and the lower halfplane L or a domain containing $U \cup L$; the set $\Lambda(G) = \hat{\mathbf{C}} - \Omega(G)$ is called the *limit set* of G and is the set of accumulation points of orbits of G. The group G is called of the *first* or of the *second kind* according to whether $\Lambda(G)$ coincides with the extended real axis $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ or not. For instance, the trivial group 1 is of the second kind.

Let G be a Fuchsian group (this notation will be kept throughout this paper). A quasiconformal automorphism $\omega \in Q$ is called *compatible with* G if $\omega G \omega^{-1} \subset Q_{\text{conf}}$. If so, $\omega G \omega^{-1}$ is a Fuchsian group and is of the first kind if and only if G is, and the mapping $g \mapsto \omega \circ g \circ \omega^{-1}$ is a (type preserving) isomorphism. The set of elements $\omega \in Q$ compatible with G will be denoted by $Q(G)$; thus $Q = Q(1)$. We set $Q_{\text{norm}}(G) = Q(G) \cap Q_{\text{norm}}$.

Two elements, ω and $\hat{\omega}$, of Q will be called *equivalent* if $\omega | \mathbf{R} = \hat{\omega} | \mathbf{R}$. The elements equivalent to the identity form a normal subgroup $Q_0 \subset Q_{\text{norm}}$. If ω and $\hat{\omega}$ are compatible with a Fuchsian group G of the first kind, then they are equivalent if and only if the isomorphisms $g \mapsto \omega \circ g \circ \omega^{-1}$ and $g \mapsto \hat{\omega} \circ g \circ \hat{\omega}^{-1}$ coincide. The *equivalence class* of $\omega \in Q$ will be denoted by $[\omega]$.

The *Teichmüller space* $T(G)$ of a Fuchsian group G is the set of equivalence classes $[w]$ of elements $\omega \in Q_{\text{norm}}(G)$. The canonical surjection $Q(G) \mapsto T(G)$ defines a Teichmüller distance function δ_G on $T(G)$ and makes $T(G)$ into a *complete metric space*. In particular, the *universal Teichmüller space* $T(1)$ is the factor group Q_{norm}/Q_0; it is *not*, however, a topological group.

It is clear that if G and $G_1 \subset G$ are two Fuchsian groups, then $T(G) \subset T(G_1)$. It turns out that $T(G)$ is *closed* in $T(G_1)$ and the embedding $T(G) \hookrightarrow T(G_1)$ is a *homeomorphism*. Also, $\delta_{G_1} | T(G) \times T(G) \leqslant \delta_G$; it is not known whether the equality sign holds in general. In particular, every Teichmüller space $T(G)$ is a closed subset of the universal Teichmüller space $T(1)$.

Let $L_\infty(U)$ be the usual complex Banach space of (equivalence classes of) bounded measurable functions, let $L_\infty(U)_1$ be the open unit ball in $L_\infty(U)$, and, for a Fuchsian group G, let $L_\infty(U, G)$ be the closed linear subspace of $L_\infty(U)$ consisting of elements μ satisfying

$$\mu(g(z))\,\overline{g'(z)}/g'(z) = \mu(z) \quad \text{for } g \in G.$$

Also, let $L_\infty(U, G)_1 = L_\infty(U)_1 \cap L_\infty(U, G)$.

Every $\omega \in Q$ has a.e. partial derivatives and a *Beltrami coefficient* $\mu = (\partial\omega/(\partial\bar{z})/(\partial\omega/\partial z)$ which belongs to $L_\infty(U)_1$; also

$$K(\omega) = \frac{1+\|\mu\|}{1-\|\mu\|}$$

where $\|\ \|$ is the L_∞ norm. Every $\mu \in L_\infty(U)_1$ is the Beltrami coefficient of a unique normalized automorphism $w \in Q_{\text{norm}}$; we write

$$w = w_\mu$$

and note that $w_\mu \in Q(G)$ if and only if $\mu \in L_\infty(U, G)$.

Thus there is a canonical bijection $L_\infty(U, G)_1 \to Q_{\text{norm}}(G)$, which is a homeomorphism, and there is a continuous surjection $\mu \mapsto [w_\mu]$ of $L_\infty(U, G)_1$ onto $T(G)$ which defines in $T(G)$ a *complex structure* (of a ringed space). This is the same structure as the one given by the embedding $T(G) \subset T(1)$, for $L(U)$ has a continuous projection on $L_\infty(U, G)$.

The Teichmüller space $T(G)$, with its complex structure, can be realized, canonically, as a *bounded domain* of a complex Banach space $B_2(L, G)$ defined as follows: the elements of $B_2(L, G)$ are holomorphic functions $\varphi(z)$ defined in the lower halfplane L, satisfying the functional equation of quadratic differentials:

$$\varphi(g(z))\,g'(z)^2 = \varphi(z) \quad \text{for } g \in G,$$

and having a finite norm

$$\|\varphi\|_B = \sup y^2 |\varphi(z)| \quad (z = x + iy \in L).$$

It is clear that $B_2(L, G)$ is a closed linear subspace of $B_2(L, G_1)$ for every Fuchsian group $G_1 \subset G$, and it is known that $\dim B_2(L, G) < \infty$ if and only if G is finitely generated and of the first kind.

For every $\mu \in L_\infty(U)_1$ there is a unique quasiconformal automorphism w of $\hat{\mathbf{C}}$ with $w(0)=0$, $w(1)=1$, $w(\infty)=\infty$, such that $w\,|\,U$ has the Beltrami coefficient μ and $w\,|\,L$ is conformal. We write

$$w = w^\mu$$

and denote by $\varphi^\mu(z)$ the *Schwarzian derivative* of $w^\mu(z)$ in L:

$$\varphi^\mu = u' - \tfrac{1}{2}u^2 \quad \text{where } u(z) = \frac{d}{dz} \log \frac{d}{dz}(w^\mu\,|\,L).$$

By Nehari's theorem [14], $\varphi^\mu \in B_2(L, 1)$ and $\|\varphi^\mu\|_B \leqslant \frac{3}{2}$. It has been shown that $w^\mu|L$, $w^\mu|\mathbf{R}$ and φ^μ depend only on $[w_\mu]$, that $\varphi^\mu \in B_2(L, G)$ if $\mu \in L_\infty(U, G)$, and that *the mapping* $[w_\mu] \mapsto \varphi^\mu$ *is a biholomorphic bijection of* $T(G)$ *onto a holomorphically convex domain in* $B_2(L, G)$ containing the open ball of radius $\frac{1}{2}$. From now on we shall identify $T(G)$ with its canonical image in $B_2(L, G)$.

§ 2. Fiber spaces over Teichmüller spaces

To every point τ of the universal Teichmüller space $T(1)$ there is associated an *unbounded Jordan domain* $D_u(\tau)$ defined as follows. Let $\tau = [w_\mu]$; since $w^\mu(L)$ depends only on $[w_\mu]$ and not on the particular choice of μ, so does $w^\mu(U)$, the complement of the closure of $w_\mu(L)$. We set $D_u(\tau) = w^\mu(U)$. The boundary of $D_u(\tau)$ is the directed Jordan curve $w^\mu(\hat{\mathbf{R}})$; it admits the parametric representation $\zeta = w^\mu(x)$, $-\infty \leqslant x < \infty$. For every fixed $x \in \hat{\mathbf{R}}$, ζ depends holomorphically on $\mu \in L_\infty(U)$, as follows from the results of [3], and since $w^\mu(x)$ depends only on $\tau = [w_\mu]$, ζ is, for a fixed x, a holomorphic function of $\tau \in T(G)$. In this sense $D_u(\tau)$ *depends holomorphically on* τ.

We also define a *bounded Jordan domain* $D_b(\tau)$, by the following construction. Let $\eta_1(z)$ and $\eta_2(z)$ be two linearly independent solutions of the ordinary differential equation

$$2\eta''(z) + \varphi^\mu(z)\eta(z) = 0, \quad z \in L \tag{2.1}$$

normalized by the initial conditions

$$\eta_1(-i) = \eta_2'(-i) = 1, \quad \eta_1'(-i) = \eta_2(-i) = 0, \tag{2.2}$$

and set

$$W^\mu(z) = \eta_1(z)/\eta_2(z). \tag{2.3}$$

Then, as is well-known, W^μ has the Schwarzian derivative φ^μ, so that there is a complex Möbius transformation β_μ such that

$$W^\mu = \beta_\mu \circ w^\mu \tag{2.4}$$

in L; we use this relation to define $W^\mu(z)$ for all $z \in \hat{\mathbf{C}}$, and we set $D_b(\tau) = W^\mu(U)$. The definition is legitimate since $W^\mu(U)$ is the complement of the closure of $W^\mu(L)$ and $W^\mu|L$ depends only on $[w_\mu]$.

We show now that β_μ depends only on $[w_\mu]$, and that this dependence is holomorphic. Near $z = -i$ we have that $w^\mu(z) = a + b(z+i) + c(z+i)^2 + \dots$ where $a \neq 0$, $b \neq 0$; here a, b, c depend holomorphically on μ, and depend only on $[w_\mu]$, since $w^\mu|L$ depends only on $[w_\mu]$. On the other hand, by (2.1), (2.2) and (2.3) we have, near $z = -i$, that $W^\mu(z) = (z+i)^{-1} + \hat{b}(z+i) + \hat{c}(z+i)^2 + \dots$. Hence $\beta_\mu(t) = [ct + (b^2 - ac)]/(bt - ab)$. This proves the assertion. We conclude, as before that $D_b(\tau)$ *depends holomorphically on* τ.

Now let G be a Fuchsian group. The *unbounded fiber space* $F_u(G)$ over $T(G)$ is the set of pairs (τ, z) with $\tau \in T(G)$, $z \in D_u(\tau)$; the *bounded fiber space* $F_b(G)$ is the set of pairs (τ, z) with $\tau \in T(G)$, $z \in D_b(\tau)$; the terminology will be justified below. Both fiber spaces are subsets of $B_2(L, G) \oplus \mathbf{C}$ and restrictions to $T(G)$ of the *universal fiber spaces* $F_u(1)$, $F_b(1)$. There is a canonical bijection $F_u(G) \to F_b(G)$ which takes a point $(\tau, z) \in F_u(G)$, with $\tau = [w_\mu]$, into the point $(\tau, \beta_\mu(z))$ of $F_b(G)$.

THEOREM 1. *The fiber spaces $F_b(G)$ and $F_u(G)$ are domains, the canonical bijection $F_u(G) \to F_b(G)$ is biholomorphic, and $F_b(G)$ is bounded.*

Proof. For every $\mu \in L_\infty(U)_1$, the function

$$\chi(\zeta) = \left[-2iW^\mu\left(\frac{i\zeta + i}{\zeta - 1}\right)\right]^{-1}, \quad |\zeta| < 1$$

is holomorphic and univalent, and $\chi(0) = 0$, $\chi'(0) = 1$. By Koebe's one-quarter theorem, $|\chi(\zeta)| \geqslant \frac{1}{4}$ for $|\zeta| = 1$. Therefore $|W^\mu(x)| \leqslant 2$ for $x \in \hat{\mathbf{R}}$. Hence $D_b([w_\mu]) = W^\mu(U)$ lies in the disc $|z| \leqslant 2$. Thus $F_b(1)$ is a bounded set in $B_2(L, 1) \oplus \mathbf{C}$.

To show that $F_b(G)$ is a domain it is enough to show that it is open, and this requires to demonstrate the following. For every $\mu_0 \in L_\infty(U, G_1)_1$ and every $z_0 \in D_b([w_{\mu_0}])$ there are positive numbers ε_1 and ε_2 such that, for every $\nu \in L_\infty(U, G)$ with $\|\nu\| = 1$, and for every $t \in \mathbf{C}$ with $|t| < \varepsilon_1$, the disc $|z - z_0| < \varepsilon_2$ lies in $D_b([w_{\mu_0} + t\nu])$. Let $\varepsilon_2 > 0$ be so small that the disc $|z - z_0| < 2\varepsilon_2$ lies in $D_b([w_{\mu_0}])$. It is enough to find an ε_1 such that, for all ν and t as above, and for all $x \in \mathbf{R}$, one has $|W_{\mu_0}(x) - W_{\mu_0 + t\nu}(x)| < \varepsilon_2$. We choose an $\varepsilon > 0$ so small that for ν as above and for $|t| < \varepsilon$, $\mu_0 + t\nu \in L_\infty(U, G)_1$. For every $x \in \mathbf{R}$ the holomorphic function of t, $W_{\mu_0 + t\nu}(x) - W_{\mu_0}(x)$, vanishes for $t = 0$ and has a modulus not exceeding 4 for $|t| < \varepsilon$. By Schwarz' lemma the desired inequality holds with $\varepsilon_1 = \varepsilon\varepsilon_2/4$.

The other assertion of the theorem follows from the fact, established above, that β_μ in equation (2.4) depends holomorphically on $[w_\mu] \in T(G)$.

Often there is no need to distinguish between the two isomorphic fiber spaces, and one writes simply $F(G)$.

§ 3. The universal modular group

Every $\omega \in Q$ induces an automorphism ω_* of Q_{norm} defined as follows: if $w \in Q_{\text{norm}}$, $\omega_*(w)$ is the unique element of Q_{norm} which can be written as

$$\omega_*(w) = \alpha \circ w \circ \omega^{-1} \quad (\alpha \in Q_{\text{conf}}). \tag{3.1}$$

Note that α depends on w. Each ω_* is an isometry and a holomorphic mapping of Q_{norm}, and since $[\omega_*(w)]$ depends only on $[\omega]$ and on $[w]$, ω_* may be considered as an (isometric and holomorphic) automorphism $[\omega]_*$ of $T(1)$, which depends only on $[\omega]$ and maps $[w] \in T(1)$ into $[\omega_*(w)]$. The group of all these automorphisms is called the *universal modular group* and is denoted by Mod (1); it can be identified with the factor group Q/Q_0. It turns out that the action of Mod (1) on $T(1)$ can be extended to an action on the fiber space $F(1)$ over $T(1)$ which respects the fiber space structure.

THEOREM 2. *The group* Mod (1) *operates on* $F_u(1)$, *as a group of holomorphic automorphisms, according to the rule: if* $\omega \in Q$, $w = w_\mu$ *for some* $\mu \in L_\infty(U)_1$, *and* $z \in D_u([w_\mu]) = w^\mu(U)$, *then*

$$[\omega]_*([w_\mu], z) = ([w_\nu], \hat{z}) \tag{3.2}$$

where

$$\nu \in L_\infty(U)_1 \text{ with } \omega_*(w_\mu) = w_\nu \tag{3.3}$$

and

$$\hat{z} = w^\nu \circ \omega \circ (w^\mu)^{-1}(z). \tag{3.4}$$

(The action of Mod (1) on $F_b(1)$ is defined similarly.)

The proof is somewhat long and will be broken up into several lemmas.

LEMMA 3.1. *The mapping* $z \mapsto \hat{z}$ *defined by (3.4) is a conformal bijection of* $D_u([w_\mu])$ *onto* $D_u([w_\nu])$ *and depends only on the equivalence classes* $[w_\mu]$, *and* $[\omega]$ (*and not on the particular choices of* μ *and* ω).

Proof. Let h_μ be defined by

$$w_\mu = h_\mu \circ w^\mu | U; \tag{3.5}$$

then h_μ: $D_u([w_\mu]) \to U$ is a conformal bijection since w_μ and $w^\mu | U$ have the same Beltrami coefficient μ. This bijection keeps 0, 1, ∞ fixed, hence it depends only on $[w_\mu]$. Similarly, $w_\nu = h_\nu \circ w^\nu | U$, and the conformal bijection h_ν: $D_u([w_\nu]) \to U$ depends only on $[w_\nu] = [\omega_*(w_\mu)]$ and thus only on $[w_\mu]$ and $[\omega]$. Now, by (3.4) and (3.3),

$$\hat{z} = h_\nu^{-1} \circ w_\nu \circ \omega \circ w_\mu^{-1} \circ h_\mu = h_\nu^{-1} \circ (\alpha \circ w_\mu \circ w^{-1}) \circ \omega \circ w_\mu^{-1} \circ h_\mu = h_\nu^{-1} \circ \alpha \circ h_\mu.$$

Since $\alpha \in Q_{\text{conf}}$, and α clearly depends only on $[w_\nu]$ and $[\omega]$, the assertion follows.

Lemma 3.1 implies that the right side of (3.2) depends only on $[\omega]$, $[w_\nu]$ and z.

LEMMA 3.2. *For* μ *and* z *as in Theorem 3, and for* $\omega_1, \omega_2 \in Q$, *we have*

$$[\omega_1 \circ \omega_2]_*([w_\mu], z) = [\omega_1]_* \circ [\omega_2]_*([w_\mu], z).$$

The proof is by a direct verification and is left to the reader. The lemma implies that Mod (1) is a group of bijections of $F_u(1)$.

LEMMA 3.3. *Every* $[\omega]_*$ *is a continuous self-mapping of* $F_u(1)$.

Proof. Since it is known that the restriction of $[w]_*$ on the "first coordinate" $[w_\mu]$ is an isometric self-mapping of $T(1)$, it suffices to prove the following. Let ω be fixed and let μ, μ_j, $j=1, 2, \ldots$, be elements in $L_\infty(U)_1$ such that $\lim \delta_1([w_{\mu_j}], [w_\mu])=0$. For every j, and for $z \in w^\mu(U)$, set $\hat{z}_j = w^{\nu_j} \circ \omega \circ (w^{\mu_j})^{-1}(z)$, where $w_{\nu_j}=\omega_*(w_{\mu_j})$. Then $\lim \hat{z}_j=\hat{z}$, uniformly on compact subsets of $w^\mu(U)$.

Since $\hat{z}_j$ depends only on the equivalence class $[w_{\mu_j}]$ we loose no generality in assuming that $\lim \delta(w_{\mu_j}, w_\mu)=0$; then also $\lim \delta(w_{\nu_j}, w_\nu)=0$. Using standard properties of quasiconformal mappings one verifies that w^{μ_j}, $(w^{\mu_j})^{-1}$ and w^{ν_j} converge, uniformly on compact sets, to w^μ, $(w^\mu)^{-1}$ and w^ν, respectively. Whence the assertion.

LEMMA 3.4. *Let $t \mapsto \sigma_t$ be a holomorphic mapping of the disc $|t|<\varepsilon$ in* $\mathbf{C}$ *into $L_\infty(U)_1$. There is an ε_0, $0<\varepsilon_0<\varepsilon$, and a holomorphic mapping $t\mapsto \mu_t$ of $|t|<\varepsilon_0$ into $L_\infty(U)_1$ such that* (i) *$\mu_t(z)$, $z\in U$ is, for $|t|<\varepsilon_0$, a real analytic function of $x=\operatorname{Re} z$, $y=\operatorname{Im} z$, and, for every compact set $\Delta\subset U$ and every integer $n>0$, the moduli of μ_t and its partial x and y derivatives up to the order n are bounded on Δ by a constant depending only on Δ and n, and* (ii) *$[\mu_t]=[\sigma_t]$ for $|t|<\varepsilon_0$.*

Proof. There are constants θ_0 and θ such that $0<\theta_0<1$, $0<\theta<\frac{1}{2}$, and if $\tau\in L_\infty(U)$ and $\|\tau\|<\theta_0$, then $\|\varphi^\tau\|_B<\theta$ (here $\|\ \|$ is the L_∞ norm, $\|\ \|_B$ the norm defined in § 1, and φ^τ the Schwarzian derivative of $w^\tau|L$).

Write the given w_{σ_0} in the form $w_{\sigma_0}=w_{\varrho_1}\circ\ldots\circ w_{\varrho_r}$ where $\varrho_j\in L_\infty(U)_1$ and $\|\varrho_j\|<\theta_0$, $j=1, \ldots, r$. This is easily done; observe, for instance, that for every $\sigma\in L_\infty(U)$ with $\|\sigma\|<k<1$, we have that $w_\sigma=w_\tau\circ w_{\sigma/2}$ where $\|\tau\|\leqslant k_1\|\sigma\|$ with k_1, $0<k_1<1$, depending only on k.

Next, set $\tau_j(z)=-2y^2\varphi^{\varrho_j}(\bar{z})$. Then $\tau_j\in L_\infty(U)_1$, $\|\tau_j\|<\theta$, and by the *Ahlfors-Weill lemma* (cf. [4]), $\varphi^{\tau_j}=\varphi^{\varrho_j}$, so that $[w_{\tau_j}]=[w_{\varrho_j}]$. We define μ_0 by the condition $w_{\mu_0}=w_{\tau_1}\circ\ldots\circ w_{\tau_r}$. Then $[w_{\mu_0}]=[w_{\sigma_0}]$.

Now let ε_0, $0<\varepsilon_0<\varepsilon$ be so small that, for $|t|<\varepsilon_0$, if we define $\eta_t\in L_\infty(U)_1$ by the requirement: $w_{\eta_t}=w_{\sigma_t}\circ w_{\sigma_0}^{-1}$, then $\|\eta_t\|<\theta_0$ so that $\|\varphi^{\eta_t}\|<\theta$. Set $\zeta_t(z)=-2y^2\varphi^{\eta_t}(\bar{z})$, and define μ_t by the requirement: $w_{\mu_t}=w_{\zeta_t}\circ w_{\mu_0}$. Then the mapping $t\mapsto\eta_t$ is holomorphic and so are the mappings $t\mapsto\zeta_t$ and $t\mapsto\mu_t$. Also, $[w_{\zeta_t}]=[w_{\eta_t}]$, by the Ahlfors-Weill lemma, so that $[w_{\mu_t}]=[w_{\eta_t}\circ w_{\mu_0}]=[w_{\eta_t}\circ w_{\sigma_0}]=[w_{\sigma_t}]$.

Thus $t\mapsto\mu_t$ satisfies condition (ii). Noting that the numbers $\|\varphi^{\varrho_1}\|, \ldots, \|\varphi^{\varrho_r}\|$ and $\|\varphi^{\eta_t}\|$ are all bounded by the same constant $\theta<\frac{1}{2}$, one verifies that condition (i) is also satisfied.

LEMMA 3.5. *For a fixed $\omega\in Q$ and for a fixed (relevant) z, the number $\hat{z}$ defined by* (3.4) *depends holomorphically on $\mu\in L_\infty(U)_1$.*

Proof. It is required to prove the following. Let $t \mapsto \mu_t$ be a holomorphic mapping of the disc $|t| < \varepsilon$ in $\mathbf{C}$ into $L_\infty(U)_1$. Let $z_0 \in w^{\mu_0}(U)$ and let ε_0, $0 < \varepsilon_0 < \varepsilon$, be so small that $z_0 \in w^{\mu_t}(U)$ for $|t| < \varepsilon_0$. Let $w_{\nu_t} = \omega_*(w_{\mu_t})$ and, for $z \in w^{\mu_t}(U)$, set

$$\hat{z} = g_t(z) = w^{\nu_t} \circ \omega \circ (w^{\mu_t})^{-1}(z). \tag{3.6}$$

Then $t \mapsto g_t(z_0)$ is a holomorphic function of t near $t = 0$.

In view of Lemmas 3.1 and 3.4 we may assume that $t \mapsto \mu_t$ has property (i) of Lemma 3.4. The Beltrami equation $\partial w^{\mu_t}/\partial \bar{z} = \mu_t(z)(\partial w^{\mu_t}/\partial z)$ is an elliptic system of two first order partial differential equations for the real and imaginary parts of w^{μ_t}. This system is uniformly elliptic as long as $\|\mu_t\|$ is bounded away from 1. Under this hypothesis the solutions $w^{\mu_t}(z)$ are uniformly bounded on compact sets in U. Using standard theory of elliptic partial differential equations we conclude from property (i) that, for $[t] < \varepsilon_0$, $w^{\mu_t}(z)$ is real analytic in x and y and the partial derivatives of $w^{\mu_t}(z)$, up to any given order, are uniformly bounded on compact subsets of U. It is known (cf. [3]) that, for any fixed $z \in U$, the number $w^{\mu_t}(z)$ depends holomorphically on μ_t and hence on t. In view of the remark made above, the same is true of the partial derivatives of $w^{\mu_t}(z)$. For instance,

$$\frac{\partial w^{\mu_t}(z)}{\partial x} = \lim_{\mathbf{R} \ni h \to 0} \frac{w^{\mu_t}(z+h) - w^{\mu_t}(z)}{h}$$

is a holomorphic function of t, since the limit is attained uniformly in t.

Next, the Beltrami coefficient ν_t of $\omega_*(w_{\mu_t})$ depends holomorphically on μ_t (as is known and easy to check) and hence on t. Therefore $w^{\nu_t}(z)$ is, for every fixed $z \in U$, a holomorphic function of t; it is uniformly bounded for $|t| < \varepsilon_0$ and z restricted to a compact set in U.

Finally, let $\zeta_0 = (w^{\mu_0})^{-1}(z_0)$, let r be a number with $0 < r < \operatorname{Im} \zeta_0$, and let C_t be the image under w^{μ_t} of the circle $\zeta = \zeta_0 + re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. Then C_t is a smooth Jordan curve, and if $|t|$ is small enough, z_0 lies in the domain interior to C_t. We restrict ourselves to such t and apply Cauchy's formula to the holomorphic function $z \mapsto g_t(z)$, cf. Lemma 3.1. Noting (3.6) we have that

$$\begin{aligned} g_t(z_0) &= \frac{1}{2\pi i} \int_{C_t} \frac{g_t(z)\, dz}{z - z_0} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{g_t \circ w^{\mu_t}(\zeta_0 + re^{i\theta})}{w^{\mu_t}(\zeta_0 + re^{i\theta}) - z_0} \, \frac{\partial w^{\mu_t}(\zeta_0 + re^{i\theta})}{\partial \theta} \, d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{w^{\nu_t} \circ \omega(\zeta_0 + re^{i\theta})}{w^{\mu_t}(\zeta_0 + re^{i\theta}) - z_0} \, \frac{\partial w^{\mu_t}(\zeta_0 + re^{i\theta})}{\partial \theta} \, d\theta. \end{aligned}$$

This formula exhibits $g_t(z_0)$ as a holomorphic function of t.

This lemma implies that the mapping $[\omega]_*$: $F_u(1) \to F_u(1)$ is holomorphic. Theorem 2 is now proved. The verification of the following theorem is trivial.

THEOREM 3. *Let G be a Fuchsian group and let $\omega \in Q(G)$. Then $[\omega]_* \mid F(G)$ is an isomorphism of $F(G)$ onto $F(\omega G \omega^{-1})$.*

Here F may be interpreted as either F_u or F_b. By an isomorphism we mean a biholomorphic bijection which respects the fiber space structure. In particular $[\omega]_*$ maps $T(G)$ holomorphically onto $T(\omega G \omega^{-1})$. It is easy to check that this latter mapping is a Teichmüller isometry in the following sense: the δ_G distance is taken into the δ_{G_1} distance, $G_1 = \omega G \omega^{-1}$.

The mappings $[\omega]_* \mid T(G)$ and $[\omega]_* \mid F(G)$, for $\omega \in Q(G)$, are called *allowable isomorphisms*.

§ 4. Modular groups

Let $N(G)$ and $N_{\text{conf}}(G)$ be the normalizers of G in Q and in Q_{conf}, respectively. The *extended modular group* mod (G) of a Fuchsian group G is defined as the subgroup of the universal modular group Mod (1) induced by $N(G)$; thus mod (G) can be identified with the quotient $N(G)/(N(G) \cap Q_0)$. In view of Theorem 3 the elements of mod (G) induce allowable automorphisms of $F_u(G)$, and also of $F_b(G)$.

Let $\alpha \in N_{\text{conf}}(G)$. Then the coset $[\alpha]$ contains no elements of Q_{conf} distinct from α. We may therefore, by abuse of language, identify $[\alpha]$ and $[\alpha]_*$ with α. Hence G may be considered as a subgroup of mod (G), a normal subgroup, of course. The *modular group* Mod (G) of G is defined as the factor group

$$\operatorname{Mod}(G) = \operatorname{mod}(G)/G \simeq (N(G)/(N(G) \cap Q_0))/G \tag{4.1}$$

The element of Mod (G) induced by $\omega \in N(G)$ will be denoted by $\langle \omega \rangle$.

One verifies easily that, for $g \in G$ and $w \in Q_{\text{norm}}(G)$, one has $g_*(w) = w$. Hence $[g]_* \mid T(G) = \text{id}$, and, by Theorem 3, every element $\langle \omega \rangle$ of Mod (G) induces an allowable automorphism $[\omega]_* \mid T(G)$ of $T(G)$. However, the action of Mod (G) on $T(G)$ need not be effective; a non-neutral element of Mod (G) may induce the identity mapping on $T(G)$.

THEOREM 5. *Let $\omega \in Q(G)$ and $\hat{G} = \omega G \omega^{-1}$. The allowable isomorphism $T(G) \to T(\hat{G})$, $F(G) \to F(\hat{G})$ conjugate the actions of* Mod (G) *and of* mod (G) *into those of* Mod $(\hat{G})$ *and of* mod $(\hat{G})$.

The proof is clear.

THEOREM 6. *The action of* Mod (G) *on $T(G)$ is effective if G is of the first kind and its* *signature is not* $(0, 3; \nu_1, \nu_2, \nu_3)$ *with at least two of the* ν_j *equal,* $(1, 1; \nu)$, $(1, 2, \nu, \nu)$ *or* $(2, 0)$. *The action of* mod (G) *on* $F(G)$ *is always effective.*

Before proving this theorem we recall how one defines the signature

$$(p, n; \nu_1, \nu_2, \ldots, \nu_n)$$

of a finitely generated Fuchsian group of the first kind. The number p is the genus of U/G, n is the number of non-conjugate maximal elliptic or parabolic subgroups of G, and $\nu_1, \ldots, \nu_n$ are the orders of these subgroups, arranged in ascending order. If G has no torsion, the only possible value for a ν_j is ∞ and the signature is determined by the *type*

$$(p, n)$$

of G. We recall that $\dim T(G) = 3p - 3 + n$. The only restrictions on the signature of a Fuchsian group are: $p \geqslant 0$, $n \geqslant 0$, $2 \leqslant \nu_j \leqslant \infty$ and $2p - 2 + n - (1/\nu) - \ldots - (1/\nu_n) > 0$.

Proof of Theorem 5. Assume that the action of Mod (G) on $T(G)$ is not effective. This means that there is an $\omega_0 \in Q(G)$ such that

$$[\alpha \circ w \circ \omega_0^{-1}] = [w] \quad \text{for all } w \in Q_{\text{norm}}(G),\ \alpha \in Q_{\text{conf}} \text{ depending on } w, \tag{4.2}$$

$$[\omega_0] \neq [g] \quad \text{for all } g \in G. \tag{4.3}$$

Applying (4.2) to $w = \text{id}$ we obtain that $[\alpha \circ \omega^{-1}] = [\text{id}]$. Hence we may assume that $\omega_0 = \alpha$, so that

$$\omega_0 \in N_{\text{conf}}(G) - G. \tag{4.4}$$

One sees by (4.4) that ω_0 induces a conformal self-mapping $\theta \neq \text{id}$ of U/G such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \omega_0\ } & U \\ \downarrow & & \downarrow \\ U/G & \xrightarrow{\ \theta\ } & U/G \end{array} \tag{4.5}$$

is commutative. Here (and hereafter) unmarked vertical arrows denote natural projections. One sees from (4.5) that θ moves a point over which the covering $U \to U/G$ is ramified of order ν into another such point. This implies already that if the type of G is $(0, 3)$ or $(1, 2)$, two of the orders ν must be equal.

Now consider some $w \in T(G)$, and let α be determined from (4.2). One checks at once that $\alpha \in N(wGw^{-1})$. Also $\alpha \notin wGw^{-1}$, for otherwise there would be a $g_0 \in G$ with $\alpha = w \circ g_0 \circ w^{-1}$ and (4.2) would imply that $[\omega_0] = [g_0]$, contradicting (4.3). As before, α induces a conformal self-mapping of U/wGw^{-1} distinct from the identity and preserving the orders of ramification of the covering $U \to U/wGw^{-1}$. But if $0 < \dim T(G) < \infty$ and the type of G is not (0, 3), (1, 1), (1, 2) or (2, 0), then one can find a $w \in T(G)$ so that U/wGw^{-1} admits no such conformal automorphisms. One sees this by noticing that a "general" compact Riemann surface of genus $p > 2$, a torus with $n > 2$ "general" punctures, and a sphere punctured at $n > 3$ "general" points admit no non-trivial conformal automorphisms.

Assume now that (4.2) and (4.3) holds. To complete the proof we must show that for μ and ν in $L_\infty(U, G)_1$, connected by the relation $w_\nu = \omega_*(w_\mu)$, the equality

$$\hat{z} = z \tag{4.6}$$

cannot hold for all $z \in w^\mu(U)$; here $\hat{z}$ is given by equation (3.4). But we have $[w_\mu] = [w_\nu]$ by (4.2) and therefore $w^\mu = w^\nu$. If (4.6) holds for all $z \in w^\mu(U)$, then $w^\mu | U = w^\mu \circ \omega_0 | U$ or $\omega_0 = \mathrm{id}$ which contradicts (4.3).

Remark. Theorem 5 could be strengthened if the following statement were true. Every Riemann surface which is not the three times punctured sphere, a once punctured torus, or a closed surface of genus 2, and which does not admit a continuous group of conformal automorphisms, is quasiconformally equivalent to a Riemann surface which admits no non-trivial conformal automorphisms whatsoever.

This sounds quite reasonable, but I know of no proof.

THEOREM 7. *If* $\dim T(G) < \infty$, *then the groups* Mod (G) *and* mod (G) *act properly discontinuously on* $T(G)$ *and* $F(G)$, *respectively.*

Proof. We may assume that $\dim T(G) > 0$, otherwise $T(G)$ is a point, $F_u(G) = U$ and mod (G) a Fuchsian group. It is enough to prove that the groups in question are discrete.

The discreteness of Mod (G) is a classical result of Fricke. Nevertheless we sketch a proof, for the convenience of the reader.

Since $\dim T(G) < \infty$, the group G has in U a fundamental domain which is compact except for finitely many "parabolic cusps". Using this one shows easily that for every $A > 0$ there are only finitely many non-conjugate hyperbolic elements $g \in G$ such that the non-Euclidean distance between some point $z \in U$ and the point $g(z)$ is less than A. This implies that the set

$$\{t \,|\, t = (\text{trace } (g))^2,\ g \in G\} \text{ is discrete.} \tag{4.7}$$

Let $g_1, g_2, \ldots, g_r$ be a set of generators for G chosen so that g_1 and g_2 are hyperbolic, the fixed points of g_1 being separated by those of g_2. Such generators exist since $\dim T(G) > 0$. One can find a finite set Γ of words $\gamma_1, \gamma_2, \ldots, \gamma_s$ in $g_1, \ldots, g_r$ such that the sequence $\{(\text{trace } (\gamma_j))^2\}$ determines the sequence $\{g_j\}$, except for a conjugation in Q_{conf}.

Assume now that $\{\omega_n\} \subset N(G)$ is such that $\lim \langle\omega_n\rangle([w]) = \lim [\omega_{n*}(w)] = [w]$ for every $w \in T(G)$. In particular $\lim [\omega_{n*}(\mathrm{id})] = [\mathrm{id}]$, so that there are elements $\alpha_n \in Q_{\mathrm{conf}}$ with $\alpha_n \circ \omega^{-1} \in Q_{\mathrm{norm}}$ and $\lim [\alpha_n \circ \omega^{-1}] = [\mathrm{id}]$. Hence, for every $g \in G$,

$$\lim (\mathrm{trace}\,(\alpha_n \circ \omega_n^{-1} \circ g \circ \omega_n \circ \alpha_n^{-1}))^2 = (\mathrm{trace}\,(g))^2.$$

In view of (4.7) we have, for n large and $g \in \Gamma$,

$$(\mathrm{trace}\,(\omega_n^{-1} \circ g \circ \omega_n))^2 = (\mathrm{trace}\,(g))^2.$$

Recalling how Γ has been chosen we see that there exist elements $\beta_n \in Q_{\mathrm{conf}}$ such that, for n large and $g \in G$,

$$\omega_n^{-1} \circ g \circ \omega_n = \beta_n \circ g \circ \beta_n^{-1}.$$

This implies that $[\omega_n^{-1}] = [\beta_n]$. Without changing $[\omega_n]$ we may assume that $\omega_n = \beta_n^{-1}$, so that $\omega_n \in N_{\mathrm{conf}}(G)$. Since $N_{\mathrm{conf}}(G)/G$ is known to be finite (this follows from the hypothesis that $\dim T(G) < \infty$) we conclude that $\langle\omega_n\rangle([w]) = [w]$ for all $w \in T(G)$ and all large n. Thus Mod (G) is discrete.

Now let $\{\omega_n\} \subset N(G)$ be such that

$$\lim [\omega_n]_*(\tau, z) = (\tau, z) \quad \text{for } (\tau, z) \in F_u(G). \tag{4.8}$$

Let $\mu \in L_\infty(U, G)_1$, let $\nu_n \in L_\infty(U, G)_1$ be such that $w_{\nu_n} = \omega_{n*}(w_\mu)$, and let $\hat{z}_n$ be determined by the relation

$$\hat{z}_n = w^{\nu_n} \circ \omega_n \circ (w^\mu)^{-1}(z), \quad z \in w^\mu(U),$$

cf. equation (3.4). Now relation (4.8), for $\tau = [w_\mu]$, reads

$$\lim [w_{\nu_n}] = [w_\mu], \quad \lim \hat{z}_n = z. \tag{4.9}$$

By the previous argument we know that $[\omega_n] = [\gamma_n]$, $\gamma_n \in N_{\mathrm{conf}}(G)$ and $[w_{\nu_n}] = [w_\mu]$ for large n. For such n we may assume, without changing $\hat{z}_n$, that $\omega_n = \gamma_n$, and $w_{\nu_n} = w_\mu$, so that $w^{\nu_n} = w^\mu$, $\hat{z}_n = w^\mu \circ \gamma_n \circ (w^\mu)^{-1}(z)$. Thus the second equation (4.9) implies that $\lim \gamma_n(z) = z$ for $z \in U$. Since $N_{\mathrm{conf}}(G)$ is known to be a Fuchsian group we conclude that $g_n = \mathrm{id}$, $\hat{z}_n = z$ for large n.

Thus $[\omega_n]_*(\tau, z) = (\tau, z)$ for large n. This shows that mod (G) is discrete.

§ 5. Fiber spaces over moduli spaces

In this section we consider only finitely generated Fuchsian groups G of the first kind. Given two such groups, G and $\hat{G}$, the existence of a $\omega \in Q(G)$ with $\omega G \omega^{-1} = \hat{G}$ is equivalent to the condition that G and $\hat{G}$ have the same signature. If this condition is satisfied, then there exists a $w \in Q_{\mathrm{norm}}(G)$ such that wGw^{-1} is conjugate to $\hat{G}$ (in the group Q_{conf}). Furthermore, if w and $\hat{w}$ are two elements of $Q_{\mathrm{norm}}(G)$, then the groups wGw^{-1} and $\hat{w}G\hat{w}^{-1}$ are conjugate if and only if there is a $\omega \in Q(G)$ with $\omega_*(w) = \hat{w}$.

It follows from Theorem 3 that $T(G)$ and $F(G)$ are determined up to isomorphisms by the signature $(p, n; \nu_1, ..., \nu_n)$ of G. So are, by Theorem 5, the groups Mod (G) and mod (G). The notations $T(p, n; \nu_1, ..., \nu_n)$, $F(p, n; \nu_1, ..., \nu_n)$, Mod $(p, n; \nu_1, ..., \nu_n)$, mod $(p, n; \nu_1, ..., \nu_n)$ are therefore legitimate. We may also denote the quotients

$$X(G) = T(G)/\mathrm{Mod}\,(G), \quad Y(G) = F(G)/\mathrm{mod}\,(G)$$

by $X(p, n; \nu_1, \dots, \nu_n)$ and $Y(p, n; \nu_1, \dots, \nu_n)$, respectively. Observe that $X(p, n; \nu_1, \dots, \nu_n)$ is the *space of moduli* (conjugacy classes) of Fuchsian groups of signature $(p, n; \nu_1, ..., \nu_n)$.

By Theorem 7, and by a general theorem of H. Cartan [11], $X(G)$ and $Y(G)$ are normal complex spaces. Recalling the definition of the action of mod (G) on $F(G)$, we see that there is a natural holomorphic surjection $Y(G) \to X(Y)$, induced by the mapping $(\tau, z) \mapsto z$ of $F(G)$ onto $T(G)$.

LEMMA 5.1. *Let $\mu \in L_\infty(U, G)_1$. The inverse image Σ_μ of the point $[w_\mu]$ under the mapping $Y(G) \to X(G)$ is isomorphic to the quotient $U/N_{\mathrm{conf}}(w_\mu G w_\mu^{-1})$.*

Proof. We interpret $Y(G)$ as $F_u(G)/\mathrm{mod}\,(G)$. In view of Theorems 3 and 5 we loose no generality in assuming that $[w_\mu] = [\mathrm{id}]$ or even that $\mu = 0$. We must therefore consider the subgroup Γ of mod (G) which keeps the fiber $([\mathrm{id}], z)$, $z \in U$ of $F_u(G)$ over the point $[\mathrm{id}] \in T(G)$ fixed and determine the quotient $\Sigma_0 = D_u([\mathrm{id}])/\Gamma = U/\Gamma$. Now, an element $\omega \in N(G)$ induces an element $[\omega]_* \in \Gamma$ if and only if $[\omega]_*([\mathrm{id}]) = [\mathrm{id}]$, i.e., if and only if there is an $\alpha \in Q_{\mathrm{conf}}$ such that $[\alpha \circ \omega^{-1}] = [\mathrm{id}]$, i.e., if and only if $[\omega]_* = [\alpha]_*$, $\alpha \in N_{\mathrm{conf}}(G)$. Hence $\Sigma_0 = U/N_{\mathrm{conf}}(G)$, as asserted.

We can reformulate the result as

THEOREM 8. *There is a normal complex fiber space over the space of moduli of Fuchsian groups with signature $(p, n; \nu_1, ..., \nu_n)$, the fiber over a point representing the conjugacy class of a group G being isomorphic to $U/N_{\mathrm{conf}}(G)$.*

If G has no torsion, we write $T(p, n), ..., Y(p, n)$ instead of $T(p, n; \nu_1, ..., \nu_n)$ etc. It is known that $T(p, n)$, Mod (p, n) and $X(p, n)$ are the Teichmüller space, the modular group and the space of moduli, respectively, of compact Riemann surfaces of genus p with n punctures, as defined in the introduction. It is also known ([10], cf. also [13]) that there are canonical isomorphisms

$$T(p, n; \nu_1, ..., \nu_n) \to T(p, n)$$

which induce isomorphisms between $\operatorname{Mod}(p, n; \nu_1, \ldots, \nu_n)$ and subgroups of $\operatorname{Mod}(p, n)$. It follows from classical uniformization theory that there are canonical bijections

$$X(p, n; \nu_1, \ldots, \nu_n) \to X(p, n; \hat{\nu}_1, \ldots, \hat{\nu}_n) \tag{5.1}$$

whenever the signatures are such that $\nu_j = \nu_{j+1}$ if and only if $\hat{\nu}_j = \hat{\nu}_{j+1}$. It is also easy to see that there are canonical surjections

$$X(p, n; \nu_1, \ldots, \nu_n) \to X(p, n). \tag{5.2}$$

It is not difficult to show that (5.1) is biholomorphic and (5.2) is a finitely many sheeted ramified holomorphic covering.

We note that Theorem 8 has the following

COROLLARY. *There exists a normal complex fiber space $Y(p, n)$ over the space $X(p, n)$ of moduli of compact Riemann surfaces of genus p with n punctures such that the fiber over a point of $X(p, n)$ representing the conformal equivalence class of a Riemann surface S is isomorphic to $S/\operatorname{Aut}(S)$ where $\operatorname{Aut}(S)$ is the group of all conformal self-mappings of S.*

For $n=0$ this was conjectured by Teichmüller [16] and proved, in an entirely different way, by Baily [5].

§ 6. The isomorphism theorem

In this and the following sections we consider only *torsion free* Fuchsian groups. We shall show that for such a group G the fiber space $F(G)$ is isomorphic to a Teichmüller space $T(\dot{G})$ for another Fuchsian group $\dot{G}$.

THEOREM 9. *Let G be a torsion free Fuchsian group, a a point in U, $\hat{a}$ the image of a under the natural projection $U \to U/G$. Let $\dot{G}$ be another torsion free Fuchsian group and $u\colon U/\dot{G} \to (U/G) - \{\hat{a}\}$ a conformal bijection. Then there is a canonical isomorphism (biholomorphic bijection) $T(\dot{G}) \to F(G)$.*

Before proving the theorem we make some remarks which will be used also in the following section.

Remark 1. We recall that a *puncture* P of a Riemann surface S is defined by a domain $D \subset S$ and a conformal bijection θ of D onto the unit disc punctured at the origin (the domain $0 < |\zeta| < 1$) such that a sequence $\{P_j\} \subset S$, with $\lim \theta(P_j) = 0$, diverges on S. Any such sequence is said to converge to P. There is an obvious equivalence relation and a natural way of topologizing the union of S and the set of its punctures, and of making this union into a Riemann surface.

Now assume that $S=U/\Gamma$, Γ a Fuchsian group. Then there is a natural one-to-one correspondence between the punctures P of S and the conjugacy classes of maximal parabolic subgroups Γ_0 of Γ, defined as follows. Let Δ be a fundamental region of Γ_0 in U, and let z_0 be the fixed point of the parabolic generator of Γ_0. The subgroup Γ_0 belongs to P if, given any sequence $\{z_j\}\subset\Delta$ with $\lim z_j=z_0$, the image of this sequence under the natural projection $U\to U/\Gamma$ converges to P.

Under the hypotheses of Theorem 7, there is a *distinguished puncture* of $U/\dot{G}$ which may be denoted by $u^{-1}(\hat{a})$.

Remark 2. We recall that an open arc of an *ideal boundary curve* of a Riemann surface S is defined by a domain $D\subset S$ and a conformal bijection θ of D onto the unit disc with the property: any sequence $\{P_j\}\subset D$ for which $\zeta=\lim\theta(P_j)$ exists, diverges on S if and only if $\operatorname{Im}\zeta>0$, $|\zeta|=1$; such a sequence is said to converge to a point on an ideal boundary curve. There is an obvious equivalence relation and a natural way of topologizing the union of S and its ideal boundary curves.

Now assume that $S=U/\Gamma$, Γ a Fuchsian group. Then the ideal boundary curves of S can be identified with the components of $(\hat{\mathbf{R}}-\Lambda(\Gamma))/\Gamma$, and the projection $U\to U/\Gamma$ extends, by continuity, to a mapping $(U\cup\hat{\mathbf{R}})-\Lambda(\Gamma)\to[(U\cup\hat{\mathbf{R}})-\Lambda(\Gamma)]/\Gamma$. Note that Γ is of the first kind if and only if U/Γ has no ideal boundary curves.

Remark 3. Under the hypotheses of Theorem 7, if G and $\hat{a}$ are given, the point a is determined but for an action of G. We denote the set of all points $g(a)$, $g\in G$, by A. This set is determined by G and $\hat{a}$; it is infinite except if $G=1$. The choice of G and $\hat{a}$ determines $\dot{G}$ up to a conjugation in the group Q_{conf} of real Möbius transformations. To find $\dot{G}$, choose a holomorphic universal covering v: $U\to U-A$ and let $\dot{G}$ consist of all $\gamma\in Q_{\text{conf}}$ for which $v(z)$ and $v\circ\gamma(z)$ are always G equivalent. Then there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad v \quad} & U-A \\ \downarrow & & \downarrow \\ U/\dot{G} & \xrightarrow{\quad u \quad} & (U-A)/G=(U/G)-\{\hat{a}\} \end{array} \tag{6.1}$$

where unmarked vertical arrows denote natural projections. Clearly, u is a conformal bijection. There is an exact sequence

$$1\to V\hookrightarrow\dot{G}\xrightarrow{\chi}G\to 1 \tag{6.2}$$

where V is the covering group of v, and

$$v \circ \gamma = \chi(\gamma) \circ v \quad \text{for } \gamma \in \dot{G}. \tag{6.3}$$

Conversely, given G, a, $\dot{G}$ and u, there is exactly one v satisfying the above conditions. Indeed, since u can be lifted to a conformal bijection between the universal covering surfaces of $U/\dot{G}$ and of $(U-A)/G$, and since the covering $U-A \to (U-A)/G$ is subordinated to a universal covering, we have a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\alpha} & U & \xrightarrow{v_0} & (U-A) \\ \downarrow & & & \searrow^{v_1} & \downarrow \\ U/\dot{G} & & \xrightarrow{u} & & (U-A)/G \end{array} \tag{6.4}$$

where $\alpha \in Q_{\text{conf}}$ and v_0, v_1 are holomorphic universal coverings. Now set $v = v_0 \circ \alpha$; then (6.4) becomes (6.1).

In proving Theorem 7 we shall work with the unbounded fiber space $F_u(G)$. The proof will be given in a sequence of lemmas.

LEMMA 6.1. *If $G=1$, $V=\dot{G}$ is a cyclic group with a parabolic generator. If $G \neq 1$, V contains infinitely many hyperbolic elements with distinct fixed points.*

Proof. If $G=1$, then $U-A=U-\{a\}$. This implies the first statement. If $G \neq 1$, the fundamental group of $U-A$, which is isomorphic to $V \subset \dot{G}$ is infinitely generated. This implies the second statement.

LEMMA 6.2. *The groups G, $\dot{G}$ and V are either all three of the first kind or all three of the second kind. In all cases, $\Lambda(V)=\Lambda(\dot{G})$. If G is of the second kind the diagram* (6.1) *extends by continuity to the diagram*

$$\begin{array}{ccc} U \cup (\hat{\mathbf{R}} - \Lambda(\dot{G})) & \xrightarrow{v} & (U-A) \cup (\hat{\mathbf{R}} - \Lambda(G)) \\ \downarrow & & \downarrow \\ [U \cup (\hat{\mathbf{R}} - \Lambda(\dot{G}))]/\dot{G} & \xrightarrow{u} & [(U-A) \cup (\hat{\mathbf{R}} - \Lambda(G))]/G \end{array} \tag{6.5}$$

(by abuse of language we do not distinguish between u, v and their continuous extensions).

The assertion of the lemma follows from Remark 2 above and from the following two observations. The ideal boundary curves of $(U/G)-\{\hat{a}\}$ can be identified with those of $U/\dot{G}$, by means of the mapping u. The ideal boundary curves of $U-A$ are the components of $\hat{\mathbf{R}} - \Lambda(G)$, since $\Lambda(G)$ is the set of accumulation points of A.

LEMMA 6.3. *The mapping* R: $L_\infty(U, G)_1 \to F_u(G)$ *which sends a* $\nu \in L_\infty(U, G)_1$ *into* $([w_\nu], w^\nu(a)) \in F_u(G)$ *is a holomorphic surjection. The complex structure of* $F_u(G)$ *could have been defined by means of this surjection.*

Proof. Let $\nu \in L_\infty(U, G)_1$ be given and let Δ be a simply connected fundamental domain for the group $G^\nu = w^\nu G(w^\nu)^{-1}$ in $D_u([w_\nu]) = w^\nu(U)$ with a smooth boundary and containing $w^\nu(a)$ in its interior. One can deform the mapping w^ν in the interior of Δ, and then, correspondingly, in all fundamental regions $w^\nu \circ g \circ (w^\nu)^{-1}(\Delta)$, $g \in G$, which are equivalent to Δ, so that the new mapping remains quasiconformal and has a Beltrami coefficient $\sigma \in L_\infty(U, G)_1$ with $[w_\sigma] = [w_\nu]$, and $w^\sigma(a)$ is equal to a given point in Δ. Using this remark one verifies easily that R is a surjection.

The other statements of the lemma follow from the fact that $w^\nu(a)$ is a holomorphic function of $\nu \in L_\infty(U, G)_1$ and that the differential of $L_\infty(U, G)_2 \to T(G)$ is, at each point, surjective, and its kernel has a complementary subspace.

LEMMA 6.4. *The mapping* $\hat{R}$: $L_\infty(U, G)_1 \to T(G) \times U$ *which sends a* $\nu \in L_\infty(U, G)_1$ *into the pair* $([w_\nu], w_\nu(a))$ *is a real analytic surjection and there is a real analytic bijection,*

$$l\colon F_u(G) \to T(G) \times U \text{ with } \hat{R} = l \circ R.$$

Proof. We have that $w_\nu = h_\nu \circ w^\nu | U$ where h_ν is the conformal mapping of $D_u([w_\nu]) = w^\nu(U)$ onto U normalized by the conditions $h_\nu(0) = 0$, $h_\nu(1) = 1$, $h_\nu(\infty) = \infty$, cf. the proof of Lemma 3.1. Since h_ν depends only on $[w_\nu]$,

$$F_u(G) \ni ([w_\nu], z) \overset{l}{\mapsto} ([w_\nu], h_\nu(z)) \in T(G) \times U$$

is a bijection and $l \circ R = \hat{R}$. The rest at the proof is left to the reader. Note that $w_\nu(a)$ is known to depend real analytically (though not holomorphically) on $\nu \in L_\infty(U, G)_1$.

LEMMA 6.5. *There is a linear isometric bijection* ϱ: $L_\infty(U, \dot{G}) \to L_\infty(U, G)$ *defined by the condition:* $\nu = \varrho(\mu)$ *if and only if*

$$\nu(v(z))\, \overline{v'(z)} / v'(z) = \mu(z), \tag{6.6}$$

or equivalently, if and only if there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad w_\mu \quad} & U \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle v_\mu} \\ U - A & \xrightarrow{w_\nu = w_{\varrho(\mu)}} & w_\nu(U - A) \end{array} \tag{6.7}$$

where v_μ *is a holomorphic universal covering. If so, then the covering group of* v_μ *is* $w_\mu V w_\mu^{-1}$. (By abuse of language we do not distinguish between w_ν and $w_\nu | U - A$.)

Proof. If $\nu \in L_\infty(U)$, μ in (6.6) is well defined. If $\mu \in L_\infty(U)$ is given, ν is also well defined, since if $v(\hat{z}) = v(z)$, then $\hat{z} = \gamma(z)$ for some $\gamma \in V$, and hence $v'(z) = v'(\gamma(z))\gamma'(z)$ so that

$$\mu(\hat{z})\, v'(\hat{z}) / \overline{v'(\hat{z})} = \mu(\gamma(z))\, v'(\gamma(z)) / \overline{v'(\gamma(z))}$$
$$= \mu(\gamma(z))\,[v'(z)/\overline{v'(z)}]\,\overline{\gamma'(z)}/\gamma'(z) = \mu(z)\, v'(z)/\overline{v'(z)}.$$

Let $v \in L_\infty(U, G)$, and assume that (6.6) holds. We have, by (6.2) and (6.3), that to every $\gamma \in \dot{G}$ there is a $g \in G$ with $v(\gamma(z)) = g(v(z))$, $v'(\gamma(z))\gamma'(z) = g'(v(z))v'(z)$. Hence, by (6.6),

$$\mu(\gamma(z))\,\overline{\gamma'(z)}/\gamma'(z) = \nu(g(v(z)))\overline{g'(v(z))\, v'(z)}/g'(v(z))\, v'(z) = \nu(v(z))\,\overline{v'(z)}/v'(z) = \mu(z),$$

so that $\mu \in L_\infty(U, \dot{G})$. One computes similarly that if (6.6) holds, and $\mu \in L_\infty(U, \dot{G})$, then $\nu \in L_\infty(U, G)$. It is clear that ϱ is linear and norm preserving.

For a given $\mu \in L_\infty(U, \dot{G})_1$, define ν by (6.6) and lift the mapping $w_\nu \mid U - A$ to universal covering surfaces. This yields a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad \omega \quad} & U \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle \hat{v}} \\ U - A & \xrightarrow{\quad w_\nu \quad} & w_\nu(U - A) \end{array} \tag{6.8}$$

where $\hat{v}$ is a holomorphic universal covering. Here we may replace $\hat{v}$ by $\hat{v} \circ \alpha$ and ω by $\alpha^{-1} \circ \omega$, where $\alpha \in Q_{\mathrm{conf}}$. One computes that ω is quasiconformal and that $\omega_{\bar{z}}/\omega_z = \mu$. Hence there is a unique α such that $\alpha^{-1} \circ \omega = w_\mu$. With this α, set $v_\mu = \hat{v} \circ \alpha$; then (6.8) becomes (6.7).

The existence of the commutative diagram (6.7), with a holomorphic v_μ implies relation (6.6); the proof is a calculation.

From (6.7) we see that a $\gamma \in Q_{\mathrm{conf}}$ satisfies $v_\mu \circ \gamma = v_\mu$ if and only if $w_\nu \circ v \circ w_\mu^{-1} \circ \gamma = w_\nu \circ v \circ w_\mu^{-1}$, that is, if and only if $v \circ w_\mu^{-1} \circ \gamma \circ w_\mu = v$. This is so if and only if $w_\mu^{-1} \circ \gamma \circ w_\mu \in V$. Hence $w_\mu V w_\mu^{-1}$ is the covering group of v_μ.

LEMMA 6.6. *Let* μ_0 *and* μ_1 *be elements of* $L_\infty(U, \dot{G})_1$ *with* $[w_{\mu_0}] = [w_{\mu_1}]$, *and set* $\nu_0 = \varrho(\mu_0)$, $\nu_1 = \varrho(\mu_1)$. *Then* $\hat{R}(\nu_0) = \hat{R}(\nu_1)$.

Proof. Since $[w_{\mu_0}] = [w_{\nu_0}]$, $w_{\mu_0} \circ \gamma \circ w_{\mu_0}^{-1} = w_{\mu_1} \circ \gamma \circ w_{\mu_1}^{-1}$ for all $\gamma \in \dot{G}$. In particular, the groups $w_{\mu_0} V w_{\mu_0}^{-1}$ and $w_{\mu_1} V w_{\mu_1}^{-1}$ coincide, so that $w_{\nu_0}(U - A)$ and $w_{\nu_1}(U - A)$ are conformally equivalent (cf. the preceding lemma). We have therefore the commutative diagram

$$\begin{array}{ccccccc} U & & \xrightarrow{\quad w_{\mu_0} \quad} & U & \xleftarrow{\quad w_{\mu_1} \quad} & & U \\ {\scriptstyle v}\big\downarrow & & {\scriptstyle v_{\mu_0}}\swarrow & & \searrow{\scriptstyle v_{\mu_1}} & & \big\downarrow{\scriptstyle v} \\ (U - A) & \xrightarrow{w_{\nu_0}} & w_{\nu_0}(U - A) & \xrightarrow{\alpha} & w_{\nu_1}(U - A) & \xleftarrow{w_{\nu_1}} & (U - A) \end{array} \tag{6.9}$$

where α is a conformal bijection. Since A is discrete in U, so are $w_{\nu_0}(A)$ and $w_{\nu_1}(A)$, and we conclude, by the theorem on removable singularities, that α is (the restriction of) an element of Q_{conf}.

It follows from Lemma 6.2 that if G is of the second kind the above diagram extends by continuity to the following

$$\begin{array}{ccccccc} U \cup (\hat{\mathbf{R}} - \Lambda(\dot{G})) & \xrightarrow{w_{\mu_0}} & & U \cup (\hat{\mathbf{R}} - \Lambda(w_{\mu_0}\dot{G}w_{\mu_0}^{-1})) & & \xleftarrow{w_{\mu_1}} & U \cup (\mathbf{R} - \Lambda(\dot{G})) \\ \downarrow v & & \swarrow v_{\mu_0} & & v_{\mu_1} \searrow & & \downarrow v \\ U^{\#} & \xrightarrow{w_{\nu_0}} & w_{\nu_0}(U^{\#}) & \xrightarrow{\alpha} & w_{\nu_1}(U^{\#}) & \xleftarrow{w_{\nu_1}} & U^{\#} \end{array} \tag{6.10}$$

where $$U^{\#} = (U \cup \hat{\mathbf{R}}) - (A \cup \Lambda(G)).$$

By hypothesis, $w_{\mu_1}^{-1} \circ w_{\mu_0}$ commutes with all elements of $\dot{G}$. By (6.9) and (6.3) it follows that $w_{\nu_1}^{-1} \circ \alpha \circ w_{\nu_0}$ commutes with all elements of G, hence leaves all fixed points of hyperbolic elements of G fixed, hence leaves all points of $\Lambda(G)$ fixed. By hypothesis, $w_{\mu_1}^{-1} \circ w_{\mu_0}$ leaves every point of $\hat{\mathbf{R}}$ fixed, and it follows from (6.10) that $w_{\nu_1}^{-1} \circ \alpha \circ w_{\nu_0}$ leaves every point of $\hat{\mathbf{R}} - \Lambda(G)$ fixed. Thus $w_{\nu_0}^{-1} \circ \alpha \circ w_{\nu_0} | \hat{\mathbf{R}} = \text{id}$. Since w_{ν_0} and w_{ν_1} leave 0, 1, ∞ fixed so does α. Hence $\alpha = \text{id}$ and we conclude that $[w_{\nu_0}] = [w_{\nu_1}]$.

Observe now that the point a may be considered as a puncture on $U - A = v(U)$, and let Γ_0 be a maximal parabolic subgroup of V belonging to this puncture (cf. Remark 1 above). Let γ_0 be a generator of Γ_0. To simplify writing, assume that $\gamma_0(z) = z + 1$ (this can be achieved conjugating $\dot{G}$ in Q_{conf}). Let Δ denote the region $0 \leqslant \operatorname{Re} z < 1$, $\operatorname{Im} z > 0$. If $\{z_j\}$ is a sequence in Δ with $\lim |z_j| = \infty$, then $\lim v(z_j) = a$. Set $\Delta' = w_{\mu_1}^{-1} \circ w_{\mu_0}(\Delta)$. Since $w_{\mu_1}^{-1} \circ w_{\mu_0}$ is an automorphism of $U \cup \mathbf{R}$ which commutes with all elements of $\dot{G}$, Δ' is also a fundamental region for Γ_0 in U. For every sequence $\{z'_j\} \subset \Delta'$ with $\lim |z'| = \infty$, we have that $\lim v(z'_j) = a$. Noting that $\alpha = \text{id}$, we obtain from (6.9), for $n = 1, 2, \ldots$

$$v \circ w_{\mu_1}^{-1} \circ w_{\mu_0}(in) = w_{\nu_1}^{-1} \circ w_{\nu_0} \circ v(in).$$

For $n \to \infty$ this yields $a = w_{\nu_1}^{-1} \circ w_{\nu_0}(a)$ on $w_{\nu_1}(a) = w_{\nu_0}(a)$. This completes the proof of Lemma 6.6.

LEMMA 6.7. *There is a holomorphic surjection $T(\dot{G}) \to F_u(G)$ which, for every $\mu \in L_\infty(U, \dot{G})$ takes $[w_\mu]$ into $R(\varrho(\mu))$. If this surjection is injective, it is biholomorphic.*

This follows at once from Lemmas 6.3, 6.4, 6.5, and 6.6.

We now prove Theorem 7 for the special case $G = 1$.

LEMMA 6.8. *If $G=1$ the surjection $T(\dot{G})\to F_u(G)$ is injective.*

Proof. To simplify writing we assume (without loss of generality, cf. Lemma 6.1) that $a=i$ and that v: $U\to U-\{i\}$ is chosen as

$$v(z)=i\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}}. \tag{6.11}$$

Let $\mu\in L_\infty(U,G)_1$ and $\nu=\varrho(\mu)$. Then v_μ: $U\to U-\{w_\nu(i)\}$ must be of the form $v_\mu=\alpha\circ v$ with $\alpha\in Q_{\text{conf}}$ and $\alpha(i)=w_\nu(i)$. The group $V=\dot{G}$ is generated by $z\mapsto z+1$ and w_μ satisfies $w_\mu(z+1)=w_\mu(z)+c$, with some $c>0$. Setting $z=0$ we see that $c=1$. We use the continuous extension of the commutative diagram (6.7), noting that $\Lambda(\dot{G})=\{\infty\}$. For $0<x<1$ we obtain that $w_\nu(-\cot\pi x)=\alpha(-\cot\pi w_\mu(x))$. Letting $x\to 0$ we obtain $\alpha(\infty)=\infty$, so that

$$\alpha(z)=v(z)\operatorname{Im} w_\nu(i)+\operatorname{Re} w_\nu(i). \tag{6.12}$$

For $0<x<1$ we obtain from (6.7), (6.11) and (6.12) that

$$w_\nu(-\cot\pi x)=-(\cot\pi w_\mu(x))\operatorname{Im} w_\nu(i)+\operatorname{Re} w_\mu(i).$$

Thus the knowledge of $\hat{R}(\nu)=([w_\nu],\,w_\nu(i))$ determines $w_\mu(x)$ for $0<x<1$ and hence for all $x\in\mathbf{R}$.

The rest of the proof of Theorem 9 will be based on a topological theorem by D. B. Epstein which asserts that an automorphism of an orientable surface *with base point* which is homotopic to the identity is isotopic to the identity ([12], p. 101). Actually we need only a weak corollary of this result which we state as

LEMMA 6.9. *Let S be a Riemann surface which is not homeomorphic to the sphere, the plane, the punctured plane or to a torus. Let $\hat{a}\in S$ and let θ be a topological orientation-preserving automorphism of S with $\theta(\hat{a})=\hat{a}$. Assume that θ is homotopic to the identity by a homotopy which leaves $\hat{a}$ fixed. Then $\theta\,|\,S-\{\hat{a}\}$ is homotopic to the identity.*

In §§ 8 and 9 we shall give a proof of Theorem 9 which does not assume Lemma 6.9.

LEMMA 6.10. *The surjection $T(\dot{G})\to F_u(G)$ is injective.*

Proof. We assume that $G\neq 1$, otherwise there is nothing to prove (cf. Lemmas 6.1 and 6.8). Since $G\neq 1$, we loose no generality in assuming that $0, 1, \infty$ are among the fixed points of elements of $\dot{G}$; this can be achieved by conjugating $\dot{G}$ in Q_{conf} (cf. Lemma 6.2).

Now let μ_0 and μ_1 be two elements of $L_\infty(U,\dot{G})_1$ and set $\nu_0=\varrho(\mu_0)$, $\nu_1=\varrho(\mu_1)$. Assume that $\hat{R}(\nu_0)=\hat{R}(\nu_1)$. We must show that $[w_{\mu_0}]=[w_{\mu_1}]$.

Our hypothesis implies that $w_{\nu_0} \circ g \circ w_{\nu_0}^{-1} = w_{\nu_1} \circ g \circ w_{\nu_1}^{-1}$ for all $g \in G$, so that the two groups, $w_{\nu_0} G w_{\nu_0}^{-1}$ and $w_{\nu_1} G w_{\nu_1}^{-1}$, are the same; we denote this group by G_ν. The hypothesis also implies that $w_{\nu_0}(a) = w_{\nu_1}(a)$, $w_{\nu_0}(A) = w_{\nu_1}(A)$; we denote this point and this set by a_ν and A_ν, respectively. Finally, the hypothesis implies that $w_{\nu_1} \circ w_{\nu_1}^{-1}$ commutes with all elements of G_ν. Let $\hat{a}_\nu$ denote the image of a_ν under the projection $U \to U/G_\nu$.

Now we construct the *Ahlfors homotopy* $\hat{\omega}_t$ between $w_{\nu_0} \circ w_{\nu_1}^{-1}$ and the identity: for every t, $0 \leqslant t \leqslant 1$, and for every $z \in U$, $\hat{\omega}_t(z)$ is that point on the non-Euclidean segment joining $w_{\nu_0} \circ w_{\nu_1}^{-1}(z)$ to z which divides the non-Euclidean length of this segment in the ratio $t/(1-t)$. Then $\hat{\omega}_t(z)$ depends continuously on (t, z) and, for every $g \in G$, $\hat{\omega}_t(g(z)) = g(\hat{\omega}_t(z))$. Hence there is, for every t, a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\hat{\omega}_t} & U \\ \downarrow & & \downarrow \\ U/G_\nu & \xrightarrow{\omega_t} & U/G_\nu \end{array} \tag{6.13}$$

which, for $t = 0$, becomes

$$\begin{array}{ccc} U & \xrightarrow{w_{\nu_0} \circ w_{\nu_1}^{-1}} & U \\ \downarrow & & \downarrow \\ U/G_\nu & \xrightarrow{\omega_0} & U/G_\nu \end{array} \tag{6.14}$$

where ω_0 is a homeomorphism onto. For $t = 1$ we have $\omega_1 = \text{id}$, $\hat{\omega}_1 = \text{id}$. Now $\hat{\omega}_t$ is a homotopy of $w_{\nu_0} \circ w_{\nu_1}^{-1}$ into the identity keeping a_ν fixed, so that ω_t is a homotopy of ω_0 into the identity keeping $\hat{a}_\nu$ fixed. By Lemma 6.9 there is a homotopy Ω_t of $\omega_0 | (U/G_\nu) - \{a_\nu\}$ into the identity. Using Lemma 6.5 (cf. diagram (6.7)) and restricting diagram (6.13) we construct the commutative diagram

$$\begin{array}{ccccc} U & \xleftarrow{w_{\mu_1}} & U & \xrightarrow{w_{\mu_0}} & U \\ \downarrow v_{\mu_1} & & \downarrow v & & \downarrow v_{\mu_0} \\ U - A_\nu & \xleftarrow{w_{\nu_1} | U - A} & U - A & \xrightarrow{w_{\nu_0} | U - A} & U - A_\nu \\ \downarrow & & & & \downarrow \\ (U - A_\nu)/G_\nu & & \xrightarrow{\omega_0 | (U - A)/G_\nu} & & (U - A_\nu)/G_\nu \end{array} \tag{6.15}$$

Using the outer square of this diagram we lift the homotopy Ω_t and obtain, for each t the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad\hat{\Omega}_t\quad} & U \\ {\scriptstyle v_{\mu_1}}\downarrow & & \downarrow{\scriptstyle v_{\mu_0}} \\ U-A_\nu & & U-A_\nu \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ (U-A_\nu)/G_\nu & \xrightarrow{\quad\Omega_t\quad} & (U-A_\nu)/G_\nu \end{array} \tag{6.16}$$

where $\hat{\Omega}_t(z)$ depends continuously on (t, z), $\hat{\Omega}_0 = w_{\mu_0} \circ w_{\mu_1}^{-1}$, $\Omega_0 = \omega_0 \,|\, (U-A_\nu)/G_\nu$, $\Omega_1 = \text{id}$, and π denotes the natural projection.

The mappings $\pi \circ v_{\mu_0}$ and $\pi \circ v_{\mu_1}$ are universal coverings, and v_{μ_0}, v_{μ_1} are universal coverings of the same domain $U - A_\nu$. Hence there is an $\eta \in Q_{\text{conf}}$ with $v_{\mu_1} = v_{\mu_0} \circ \eta$. It follows easily from Lemma 6.5 and from (6.1), (6.2), (6.3) that the covering groups of $\pi \circ v_{\mu_0}$ and $\pi \circ v_{\mu_1}$ are $\dot{G}_0 = w_{\mu_0} \dot{G} w_{\mu_0}^{-1}$ and $\dot{G}_1 = w_{\mu_1} \dot{G} w_{\mu_1}^{-1}$, respectively. We conclude from (6.16) that for every $\gamma \in \dot{G}_1$, every t and ever $z \in U$, there is a unique $\hat{\gamma} \in \dot{G}_0$ such that $\hat{\Omega}_t \circ \gamma(z) = \hat{\gamma} \circ \hat{\Omega}_t(z)$. Since this $\hat{\gamma}$ must depend continuously on z and on t, and $\dot{G}_0$ is discrete, $\hat{\gamma}$ depends only on γ. Since $\Omega_1 = \text{id}$ we have $\pi \circ v_{\mu_0} \circ \hat{\Omega}_1 = \pi \circ v_{\mu_1} = \pi \circ v_{\mu_0} \circ \eta$. Hence $\hat{\Omega}_1 \circ \eta^{-1} \in \dot{G}_0$ and $\hat{\Omega}_1 \in Q_{\text{conf}}$. Now, for every $\gamma \in \dot{G}_1$, we have $w_{\mu_0} \circ w_{\mu_1}^{-1} \circ \gamma \circ w_{\mu_1} \circ w_{\mu_0}^{-1} = \hat{\Omega}_0 \circ \gamma \circ \hat{\Omega}_0^{-1} = \hat{\Omega}_1 \circ \gamma \circ \hat{\Omega}_1^{-1}$. Hence $w_{\mu_0} \circ w_{\mu_0}^{-1} \,|\, \Lambda(\dot{G}_1) = \hat{\Omega}_1 \,|\, \Lambda(\dot{G}_1)$. By hypothesis, $\Lambda(\dot{G})$ contains the points 0, 1, ∞; so does $\Lambda(\dot{G}_1)$ since $\Lambda(\dot{G}_1) = w_{\mu_1}(\Lambda(\dot{G}))$. But $w_{\mu_0} \circ w_{\mu_1}^{-1}$ leaves 0, 1, ∞ fixed. So does therefore $\hat{\Omega}_1$, and since $\hat{\Omega}_1$ is a Möbius transformation, $\hat{\Omega}_1 = \text{id}$. Hence

$$w_{\mu_0} \circ w_{\mu_1}^{-1} \,|\, \Lambda(\dot{G}_0) = \text{id}. \tag{6.17}$$

Since $\hat{\Omega}_1 = \text{id}$, we have that $\eta^{-1} \in \dot{G}_0$. Hence $v_{\mu_1} = v_{\mu_0} \circ \eta = v_{\mu_0}$. Also, $\dot{G}_0 = \dot{G}_1$ and $w_{\mu_0} V w_{\mu_0}^{-1} = w_{\mu_1} V w_{\mu_1}^{-1}$. This latter group will be denoted by V_μ; it is the covering group of v_{μ_0}.

The upper half of the commutative diagram (6.15) extends, by continuity, to

$$\begin{array}{ccc} (U \cup \hat{\mathbf{R}}) - \Lambda(\dot{G}_0) & \xrightarrow{\quad w_{\mu_0} \circ w_{\mu_1}^{-1}\quad} & (U \cup \hat{\mathbf{R}}) - \Lambda(\dot{G}_0) \\ {\scriptstyle v_{\mu_0}}\downarrow & & \downarrow{\scriptstyle v_{\mu_0}} \\ (U \cup \hat{\mathbf{R}}) - (A \cup \Lambda(G_\nu)) & \xrightarrow{\quad w_{\nu_0} \circ w_{\nu_0}^{-1}\quad} & (U \cup \hat{\mathbf{R}}) - (A \cup \Lambda(G_\nu)) \end{array}$$

(by abuse of language we use the old names for the various mappings involved). For x in some component I of $\hat{\mathbf{R}} - \Lambda(\dot{G}_0)$ we have $v_{\mu_0} \circ w_{\mu_0} \circ w_{\mu_1}^{-1}(x) = w_{\nu_0} \circ w_{\nu_1}^{-1} \circ v_{\mu_0}(x) = v_{\mu_0}(x)$. Hence there is a (unique) element $\gamma \in V_\mu$ with $w_{\mu_0} \circ w_{\mu_1}^{-1}(x) = \gamma(x)$. Since γ depends continuously on x and V_μ is discrete, γ is the same for all x in I. The endpoints of the interval I belong to $\Lambda(\dot{G}_0)$; we conclude from (6.17) that γ leaves these endpoints fixed, so that $\gamma(I)=I$. Let Γ denote the maximal subgroup of V_μ which leaves I fixed, so that $\gamma \in \Gamma$. Since Γ is isomorphic to the fundamental group of $v_{\mu_0}(I)$, a component of $\hat{\mathbf{R}}-\Lambda(G_\nu)$, and all components of $\hat{\mathbf{R}}-\Lambda(G_\nu)$ are homeomorphic to $\mathbf{R}$, $\Gamma=1$. Thus $\gamma=\text{id}$ and we conclude that

$$w_{\mu_0}\circ w_{\mu_1}^{-1}|\hat{\mathbf{R}}-\Lambda(\dot{G}_0)=\text{id}. \tag{6.18}$$

Together with (6.17) this shows that $w_{\mu_0}|\mathbf{R}=w_{\mu_1}|\mathbf{R}$.

Thus Lemma 6.10 is proved, and so is Theorem 9. We note the

COROLLARY. *There is a canonical isomorphism* $T(p, n+1)\to F(p, n)$.

§ 7. A relation between modular groups

We recall that by Theorem 6 the extended modular group mod (G) acts effectively on $F(G)$. By the same theorem, and under the hypotheses of Theorem 9, the modular group Mod $(\dot{G})$ acts effectively on $T(\dot{G})$ provided $\dim T(G)<\infty$ and G is not of type (1.1). Indeed, $U/\dot{G}$ cannot be an (unpunctured) closed surface of genus 2 (since it has at least one puncture) or a once punctured torus, or a thrice punctured sphere (since U/G can be neither an unpunctured torus nor a twice punctured sphere).

THEOREM 10. *Under the hypotheses of Theorem 9, the isomorphism* $T(\dot{G})\to F(G)$ *induces an isomorphism between (the action of) a subgroup* $\text{Mod}_0(\dot{G})$ *of* Mod $(\dot{G})$ *and the group* mod (G).

If U/G *is compact, then* $\text{Mod}_0(\dot{G})=\text{Mod}(\dot{G})$. *If* U/G *has precisely* n *punctures, the index of* $\text{Mod}_0(\dot{G})$ *in* Mod $(\dot{G})$ *is* $n+1$.

Note that U/G has precisely n punctures if and only if G has precisely n conjugacy classes of maximal parabolic subgroups.

Before proving the theorem we note two immediate consequences.

COROLLARY 1. *The isomorphism* $T(p, n+1)\to F(p, n)$ *induces an isomorphism between* mod (p, n) *and a subgroup of index* $n+1$ *of* Mod $(p, n+1)$.

COROLLARY 2. *There is a holomorphic surjection*

$$Y(p, n)\to X(p, n+1)$$

which is an isomorphism for $n=0$, *an* $(n+1)$*-sheeted ramified covering for* $n>0$.

Of course, the statement of Corollary 2 is almost self-evident, since a generic point of $Y(p, n)$ is represented by a Riemann surface of genus p with n punctures and one distinguished point.

Proof of Theorem 10. Every element $\Omega \in N(\dot{G})$ induces a quasiconformal automorphism f: $U/\dot{G} \to U/\dot{G}$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \Omega\ } & U \\ \downarrow & & \downarrow \\ U/\dot{G} & \xrightarrow{\ f\ } & U/\dot{G} \end{array} \tag{7.1}$$

commutes. All quasiconformal automorphism can be so induced. Two elements, Ω_1 and Ω_2 induce the same f if and only if $\Omega_1 \circ \Omega_2^{-1} \in \dot{G}$. If $[\Omega_1] = [\Omega_2]$, then the induced mappings, f_1 and f_2 are homotopic by a homotopy which leaves the ideal boundary curves of $U/\dot{G}$ pointwise fixed. (Construct an Ahlfors homotopy between Ω_1 and Ω_2, cf. the proof of Lemma 6.10 in § 6, project in onto $U/\dot{G}$ and note that the projection extends to $[(U \cup \hat{\mathbf{R}}) - \Lambda(\dot{G})]/\dot{G}$.)

Conversely, if f_1 and f_2 are homotopic by a homotopy which leaves the ideal boundary curves of $U/\dot{G}$ pointwise fixed, they can be induced by equivalent elements of $N(\dot{G})$ (as is seen by lifting the homotopy to $U \cup \hat{\mathbf{R}} - \Lambda(\dot{G})$ via the covering $U \to U/\dot{G}$).

Let $N_0(\dot{G})$ consist of those $\Omega \in N(\dot{G})$ which induce mappings f leaving the distinguished puncture $u^{-1}(\dot{a})$ fixed. One sees at once that $N_0(G)$ is a group, and can be characterized by the condition: if Γ_0 is a maximal parabolic subgroup of $\dot{G}$, belonging to the distinguished puncture, so is $\Omega \Gamma_0 \Omega^{-1}$. Let $\text{Mod}_0\,(\dot{G})$ be the subgroup of $\text{Mod}\,(G)$ induced by $N_0(\dot{G})$. It is clear that if U/G is compact, $N_0(\dot{G}) = \dot{G}$ and $\text{Mod}_0\,(G) = \text{Mod}\,(G)$, and that if U/G has precisely n punctures, $U/\dot{G}$ has precisely $n+1$ and $[\text{Mod}\,(\dot{G})\colon \text{Mod}_0\,(\dot{G})] = n+1$.

Since A is discrete in U, the fundamental group $\pi_1(U-A)$ is generated by elements ζ_j corresponding to loops running once around a point a_j of A and 0 times around every point $a_k \neq a_j$ of A. The covering v: $U \to U - A$ induces an isomorphism between the covering group V and $\pi_1(U-A)$; under this isomorphism ζ_j corresponds to a generator γ_j of a maximal parabolic subgroup $\Gamma_j \subset V$ belonging to the puncture a_j of $U - A$. Viewed as subgroups of $\dot{G}$ all Γ_j belong to the puncture $u^{-1}(\dot{a})$. Thus, if $\Omega \in N_0(\dot{G})$, then $\Omega V \Omega^{-1} \subset V$, that is, $N_0(\dot{G}) \subset N(V)$.

It follows that every $\Omega \in N_0(\dot{G})$ induces a quasiconformal automorphism ω such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\Omega} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\ U-A & \xrightarrow{\omega} & U-A \end{array} \tag{7.2}$$

This ω can be extended, by continuity, to an automorphism of U; by abuse of language, we denote the extension again by ω. For $\gamma \in \dot{G}$, set $\chi(\gamma)=g$, $\Omega \circ \gamma \circ \Omega^{-1}=\hat{\gamma}$, $\chi(\hat{\gamma})=\hat{g}$, cf. relation (6.3). We have that $v \circ \Omega \circ \gamma = \omega \circ v \circ \gamma = \omega \circ g \circ v$ and also $v \circ \Omega \circ \gamma = v \circ \hat{\gamma} \circ \Omega = \hat{g} \circ v \circ \Omega = \hat{g} \circ \omega \circ v$. Thus $\omega \circ g \circ v = \hat{g} \circ \omega \circ v$ and since v is onto A, $\omega \circ g = \hat{g} \circ \omega$ or $\omega \in N(G)$. Furthermore, $\omega(a) \subset A$.

Conversely, let $\omega \in N(G)$ be given, with $\omega(a) \subset A$. Then the restriction $\omega | U-A$ can be lifted to U via v. We obtain the commutative diagram (7.2) and conclude that $\Omega \in N(\dot{G})$. Indeed, with $\gamma \in \dot{G}$, $g=\chi(g)$, $\omega \circ g \circ \omega^{-1}=\hat{g}$ and $\hat{\gamma} \in \dot{G}$ such that $\chi(\hat{\gamma})=\hat{g}$, we have that $v \circ \Omega \circ \gamma = \omega \circ v \circ \gamma = \omega \circ g \circ v = \hat{g} \circ \omega \circ v = \hat{g} \circ v \circ \Omega = v \circ \hat{\gamma} \circ \Omega$ so that $\Omega \circ \gamma \circ \Omega^{-1} \circ \hat{\gamma}^{-1} \in V$. Also, if Γ_1 is a maximal parabolic subgroup of V belonging to the puncture a, $\Omega \Gamma_1 \Omega^{-1} \subset V$ belongs to the puncture $\omega(a) \subset A$. Thus $\Gamma_1 \subset \dot{G}$ and $\Omega \Gamma_1 \Omega^{-1} \subset \dot{G}$ both belong to $u^{-1}(a)$, and $\Omega \in N_0(\dot{G})$.

Let $N_A(G)$ denote the subgroup of all $\omega \in N(G)$ satisfying $\omega(a) \subset A$. We observe now that *if* $\Omega \in N_0(\dot{G})$ *and* $\omega \in N_A(G)$ *are connected by diagram* (7.2), *then the isomorphism* $T(\dot{G}) \to F_u(G)$ *of Theorem 9 transforms the element* $\langle \Omega \rangle$ *of* Mod $(\dot{G})$ *induced by* Ω *into the element* $[\omega]_*$ *of* mod (G) *induced by* ω.

Indeed, let $\mu \in L_\infty(U, \dot{G})_1$ and $\nu = \varrho(\mu)$, cf. Lemma 6.5, so that we have the commutative diagram (6.7). Then $([w_\nu], w^\nu(a))$ is the image of $[w_\nu]$ under $T(\dot{G}) \to F_u(G)$. Let α and β be elements of Q_{conf} such that $\alpha \circ w_\nu \circ \omega^{-1}$ and $\beta \circ w_\mu \circ \Omega^{-1}$ belong to Q_{norm}, and define $\hat{\mu} \in L_\infty(U, \dot{G})_1$, $\hat{\nu} \in L_\infty(U, G)_1$ by the requirements that $w_{\hat{\mu}} = \beta \circ w_\mu \circ \Omega^{-1}$ and $w_{\hat{\nu}} = \alpha \circ w_\nu \circ \omega^{-1}$. Then $\langle \Omega \rangle$ sends $[w_\mu]$ into $[w_{\hat{\mu}}]$ and $[\omega]_*$ sends $([w_\nu], w^\nu(a))$ into $([w_{\hat{\nu}}], w^{\hat{\nu}}(a))$. Our assertion will be proved once we show that $\hat{\nu} = \varrho(\hat{\mu})$, that is, once we establish a commutative diagram

$$\begin{array}{ccccccc} U & \xleftarrow{\Omega} & U & \xrightarrow{w_\mu} & U & \xrightarrow{\beta} & U \\ {\scriptstyle v}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle v_\mu}\downarrow & & \downarrow{\scriptstyle v_{\hat{\mu}}} \\ U-A & \xleftarrow{\omega} & U-A & \xrightarrow{w_\nu} & U-w_\nu(A) & \xrightarrow{\alpha} & U-\alpha \circ w_\nu(A) \end{array} \tag{7.3}$$

with $v_{\hat{\mu}}$ a holomorphic universal covering (by abuse of language we write w_ν instead of $w_\nu | U-A$, etc.). But this is easy, since the first and second square are simply (7.2) and (6.7). To get the third square let $\hat{v}$: $U \to \alpha \circ w_\nu(A)$ be some holomorphic universal covering and lift the mapping $\alpha | U - w_\nu(A)$ to obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\hat{\alpha}} & U \\ {\scriptstyle v_\mu}\downarrow & & \downarrow{\scriptstyle \hat{v}} \\ U - w_\nu(A) & \xrightarrow{\alpha} & U - \alpha \circ w_\nu(A) \end{array}$$

where $\hat{\alpha} \in Q_{\text{conf}}$. Now set $v_{\hat{\mu}} = \hat{v} \circ \hat{\alpha} \circ \beta^{-1}$, to obtain (7.3).

It remains to show that for every $\omega \in N(G)$, $[\omega]_* \in \operatorname{mod}(G)$ can be writen as $[\hat{\omega}]_*$ with $\hat{\omega} \in N_A(G)$. For a given ω, let Δ be a fundamental polygon for G in U containing $\omega(a)$ as an interior point and containing no points of A on its boundary. Then Δ contains a unique $g_0(a) \subset A$ as an interor point, where $g_0 \in G$. Let Δ_0 be a relatively compact subdomain of Δ containing $\omega(a)$ and $g_0(a)$. There is a $\omega_0 \in N(G)$ such that $\omega_0 \circ \omega(a) = g(a)$ and $\omega_0 | \Delta - \Delta_0 = \text{id}$. Set $\hat{\omega} = \omega_0 \circ \omega$. Then $\hat{\omega} \in N_A(G)$, $[\hat{\omega}] = [\omega]$ and hence $[\hat{\omega}]_* = [\omega]_*$.

§ 8. Standard coordinates

In this section we give a *local description* of the mapping $T(\dot{G}) \to F(G)$ of Theorem 9 for the case when G is a finitely generated Fuchsian group of the first kind. This description will be used, in the next section, to give a direct proof of the isomorphism theorem. Throughout the present section we assume that

$$\dim T(G) = r < \infty. \tag{8.1}$$

We denote by $B_2(U, G)$ the space of bounded holomorphic quadratic differentials for G in U. The definition is the same as of the space $B_2(L, G)$, cf. § 1, in particular relation (1.1), except that the functions considered are defined in U. Also, in view of (8.1), the two conditions

$$\sup |y^2 \varphi(z)| < \infty \tag{8.2}$$

and

$$\iint_{U/G} |\varphi(z)|\, dx\, dy < \infty \tag{8.3}$$

are equivalent (for holomorphic solutions of (1.1); note that the integral in (8.3) is meaningful since $|\varphi| dx dy$ is G invariant). Of course, $\dim B_2(U, G) = r$.

The following is a well-known result, sometimes called Teichmüller's lemma (cf. [6]).

LEMMA 8.1. *Let $\sigma \in L_\infty(U, G)$. Then for $\varepsilon \in \mathbf{C}$, $|\varepsilon|$ small, the condition that the Teichmüller distance between $[w_{\varepsilon\sigma}]$ and* [id] *be $o(\varepsilon)$, that is that*

$$\log K_\varepsilon = o(\varepsilon),\ \varepsilon \to 0 \text{ where } K_\varepsilon = \inf K(w),\ w \in [w_{\varepsilon\sigma}] \tag{8.4}$$

is equivalent to the condition

$$\iint_{U/G} \sigma(z)\,\varphi(z)\,dx\,dy = 0 \quad \textit{for all } \varphi \in B_2(U, G). \tag{8.5}$$

Note that the integrand $\sigma\varphi dxdy$ in (8.5) is G invariant.

If σ satisfies (8.4) and (8.5), it is called *locally trivial*. (Lemma 8.1 is true, but harder to prove, also if $\dim T(G) = \infty$.)

Now let $\nu \in L_\infty(U, G)_1$ and set $G_\nu = w_\nu G w_\nu^{-1}$; clearly, $\dim B_2(U, G_\nu) = r$. Let $\varphi_1, \ldots, \varphi_r$ be a basis in $B_2(U, G_\nu)$ which is orthonormal with respect to the Petersson scalar product:

$$\iint_{U/G_\nu} y^2 \varphi_j(z)\,\overline{\varphi_k(z)}\,dx\,dy = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases} \tag{8.6}$$

If $\zeta_1, \ldots, \zeta_r$ are complex numbers, then

$$\sigma(z) = y^2[\zeta_1 \overline{\varphi_1(z)} + \ldots + \zeta_r \overline{\varphi_r(z)}] \tag{8.7}$$

belongs to $L_\infty(U, G_\nu)$, to $L_\infty(U, G_\nu)_1$ if $|\zeta_1|^2 + \ldots + |\zeta_r|^2$ is small. The mapping

$$(\zeta_1, \ldots, \zeta_r) \mapsto [w_\sigma \circ w_\nu] \tag{8.8}$$

is a holomorphic mapping of a neighborhood of the origin in $\mathbf{C}^r$ into $T(G)$ which sends the origin into $[w_\nu]$. The rank of this mapping at the origin equals to the rank at the origin of the mapping $(\zeta_1, \ldots, \zeta_r) \mapsto [w_\sigma]$ of $\mathbf{C}^r$ into $T(G_\nu)$. Applying Lemma 8.1 to the group G_ν we see that σ cannot be locally trivial without vanishing identically. Hence the rank considered is r. Actually more is true; the mapping (8.8) is a bijection of its whole domain of definition, as is seen from the Ahlfors–Weill lemma, cf. the proof of 6.10 in § 6.

At any rate, $\zeta_1, \ldots, \zeta_r$ can be used as complex coordinates in a neighborhood of $[w_\nu]$ in $T(G)$. (These coordinates have been introduced in [6] and studied by Ahlfors in [1].) We call $(\zeta_1, \ldots, \zeta_r)$ *standard coordinates* in $T(G)$ about $[w_\nu]$. The construction implies

LEMMA 8.2. *The standard coordinates about a point in* $T(G)$ *are determined uniquely, except for a unitary transformation.*

From now on we assume that G is *torsion free*. Let a be a point in U; we shall determine certain standard complex coordinates in $F_u(G)$ near the point $([w_\nu], w^\nu(a))$. Let A_ν be the G_ν orbit of $w_\nu(a)$. Then $A_\nu = w_\nu(A)$ where A is the G orbit of a, and A_ν is discrete in U. Let $\lambda(z)|dz|$ be the *Poincaré metric* in $U - A_\nu$, that is, the unique complete conformal Riemannian metric in $U - A_\nu$ with constant curvature -1. We have that

$$\lambda(g(z))\,|g'(z)| = \lambda(z) \quad \text{for } g \in G_\nu,$$

$$\lambda(z) > \frac{1}{y}, \tag{8.9 a}$$

and

$$\lambda(z) \sim -\frac{1}{|z-z_0| \log |z-z_0|} \quad \text{for } z \to z_0 \in A_\nu. \tag{8.9 b}$$

Let $\Psi_1, \ldots, \Psi_r$ be a basis in $B_2(U, G_\nu)$ orthonormal with respect to a scalar product based on λ; this means that

$$\iint_{U/G_\nu} \lambda(z)^{-2} \Psi_j(z) \overline{\Psi_k(z)}\, dx\, dy = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases} \tag{8.10}$$

The integrals are meaningful since $\lambda^{-2}\Psi_j \overline{\Psi_k} dx\, dy$ is G_ν invariant; they converge because of (8.9).

If $t_1, \ldots, t_r$ are complex numbers and

$$s(z) = \lambda(z)^{-2}[t_1 \overline{\Psi_1(z)} + \ldots + t_r \overline{\Psi_r(z)}] \tag{8.11}$$

then $s \in L_\infty(U, G_\nu)$ and s is locally trivial if and only if $t_1 = \ldots = t_r = 0$. We conclude as before that the rank of the holomorphic mapping

$$(t_1, \ldots, t_r) \mapsto [w_s \circ w_r]$$

at the origin is r, so that $t_1, \ldots, t_r$ could be used as complex coordinates in $T(G)$ near $[w_\nu]$. We call $t_1, \ldots, t_r$ the *semi-standard coordinates* about $[w_\nu]$ in $T(G)$ belonging to the point $w_\nu(a)$. Our construction implies

LEMMA 8.3. *The semi-standard coordinates about a point in $T(G)$ belonging to a point in U, are determined uniquely except for a unitary transformation.*

Next, set $D = D_u([w_\nu]) = w^\nu(U)$, $A^\nu = w^\nu(A)$, $G^\nu = w^\nu G(w^\nu)^{-1}$, so that G^ν is a discrete group of complex Möbius transormations mapping D onto itself, and A^ν is the G^ν orbit of $w^\nu(a)$. For $z \in D - A^\nu$, define

$$\Psi^0(z) = -\frac{1}{\pi} \sum_{g \in G^\nu} \frac{g'(z)^2}{g(z)[g(z)-1][g(z)-w^\nu(a)]}. \tag{8.12}$$

Since

$$\iint_D \frac{dx\, dy}{z(z-1)[z-w^\nu(a)]} < \infty,$$

the Poincaré series in (8.12) converges uniformly and absolutely on compact subsets of $D - A^\nu$, Ψ^0 is holomorphic in $D - A^\nu$, Ψ^0 has simple poles at all points of A^ν,

$$\Psi^0(g(z))g'(z)^2 = \Psi^0(z) \quad \text{for } g \in G^\nu, \tag{8.13}$$

and

$$\iint_{D/G^\nu} |\Psi^0(z)|\, dx\, dy < \infty.$$

Using the conformal mapping

$$h_\nu = w_\nu \circ (w^\nu)^{-1}: D \to U \tag{8.14}$$

(cf. the proof of Lemma 3.1) we define a function $\Psi_0(z)$, $z \in U - A_\nu$, by the requirement

$$\Psi_0(h(z))\, h_\nu'(z)^2 = \Psi^0(z);$$

in view of (8.13) the definition is legitimate. This function is holomorphic in $U - A_\nu$, has simple poles at all points of A_ν, and, since $h_\nu G^\nu h_\nu^{-1} = G_\nu$, we have

$$\Psi_0(g(z))g'(z)^2 = \Psi_0(z) \quad \text{for } g \in G_\nu, \tag{8.15}$$

$$\iint_{U/G_\nu} |\Psi_0(z)|\, dx\, dy < \infty. \tag{8.16}$$

From (8.15) and (8.16) one concludes in the usual way (using Fourier series for Ψ_0 near parabolic fixed points of G_ν) that in every fundamental domain for G_ν in U the function $|y^{-2}\Psi_0(z)|$ is uniformly bounded except near a point of A_ν. This implies that

$$\iint_{U/G_\nu} \lambda(z)^{-2} |\Psi_0(z)|^2\, dx\, dy < \infty. \tag{8.17}$$

Hence we can find numbers $c_1, c_2, \ldots, c_r$ and $c \neq 0$, such that the function

$$\Psi_{r+1}(z) = c\Psi_0(z) + cc_1\Psi_1(z) + \ldots + cc_r\Psi_r(z) \tag{8.18}$$

satisfies the relations

$$\iint_{U/G_\nu} \lambda(z)^{-2}\Psi_{r+1}(z)\overline{\Psi_j(z)}\, dx\, dy = 0, \quad j = 1, 2, \ldots, r, \tag{8.19}$$

and

$$\iint_{U/G} \lambda(z)^{-2} |\Psi_{r+1}(z)|^2\, dx\, dy = 1. \tag{8.20}$$

If $(\zeta_1, \ldots, \zeta_{r+1}) \in \mathbf{C}^{r+1}$, set

$$\sigma(z) = \lambda(z)^{-2}[\zeta_1\overline{\Psi_1(z)} + \ldots + \zeta_r\overline{\Psi_r(z)} + \zeta_{r+1}\overline{\Psi_{r+1}(z)}] \tag{8.21}$$

and note that $\sigma \in L_\infty(U, G_\nu)$. We define an element $\tau \in L_\infty(U, G)_1$, for sufficiently small values of $|\zeta_1|^2 + \ldots + |\zeta_{r+1}|^2$, by the requirement that

$$w_\tau = w_\sigma \circ w_\nu \tag{8.22}$$

and consider the mapping

$$(\zeta_1, \ldots, \zeta_{r+1}) \mapsto ([w_\tau], w^\tau(a)) \tag{8.23}$$

of a neighborhood of the origin in $\mathbf{C}^{r+1}$ into $F_u(G)$; the image of the origin under this mapping is $([w_\nu], w^\nu(a))$. Since σ depends holomorphically on $(\zeta_1, \ldots, \zeta_{r+1})$ so do τ and $w^\tau(a)$. Thus the mapping (8.23) is holomorphic.

LEMMA 8.4. *The rank of the mapping* (8.23) *at* $\zeta_1 = \ldots = \zeta_{r+1} = 0$ *is* $r+1$.

Proof. Let $(t_1, \ldots, t_r)$ be the semi-standard coordinates in $T(G)$ about the point $[w_\nu]$, belonging to the point $w_\nu(a)$, and let t_{r+1} be a complex number restricted to the neighborhood of $w^\nu(a)$. The mapping (8.23) defines $t_1, \ldots, t_r$ and $t_{r+1} = w^\tau(a)$ as holomorphic functions of $\zeta_1, \ldots, \zeta_{r+1}$ defined near the origin; we must show that

$$\frac{\partial(t_1, \ldots, t_{r+1})}{\partial(\zeta_1, \ldots, \zeta_{r+1})} \neq 0 \quad \text{at } \zeta_1 = \ldots = \zeta_{r+1} = 0.$$

By construction

$$\frac{\partial t_j}{\partial \zeta_k} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \quad \text{for } 1 \leqslant j, k \leqslant r \text{ at } \zeta_1 = \ldots = \zeta_{r+1} = 0,$$

and it will suffice to show that

$$\frac{\partial t_j}{\partial \zeta_{r+1}} = 0 \quad \text{for } j = 1, \ldots, r; \ \zeta_1 = \ldots = \zeta_{r+1} = 0 \tag{8.24}$$

and

$$\frac{\partial t_{r+1}}{\partial \zeta_{r+1}} \neq 0 \quad \text{for } \zeta_1 = \ldots = \zeta_{r+1} = 0. \tag{8.25}$$

Define $\sigma_0 \in L_\infty(U, G_\nu)$ by

$$\sigma_0(z) = \lambda(z)^{-2} \overline{\Psi_{r+1}(z)}. \tag{8.26}$$

Condition (8.24) is equivalent to the requirement that σ_0 be locally trivial. This is so, in view of Lemma 8.1 and (8.19).

Now define, for $\varepsilon \in \mathbf{C}$, $|\varepsilon|$ sufficiently small, the element $\tau_\varepsilon \in L_\infty(U, G)$ by the requirement

$$w_{\tau_\varepsilon} = w_{\varepsilon\sigma_0} \circ w_\nu.$$

A well-known (formal but legitimate) calculation of the Beltrami coefficient τ_ε yields

$$\tau_\varepsilon = \frac{\varepsilon \hat{\sigma}_0 + \nu}{1 + \varepsilon \hat{\sigma}_0 \bar{\nu}} \quad \text{where } \hat{\sigma}_0(z) = \sigma_0 \circ w_\nu(z) \frac{\overline{\partial w_\nu(z)/\partial z}}{\partial w_\nu(z)/\partial z}. \tag{8.27}$$

Condition (8.25) is equivalent to the following:

$$\frac{\partial w^{\tau_\varepsilon}(a)}{\partial \varepsilon} \neq 0 \quad \text{for } \varepsilon = 0,$$

or, setting
$$f_\varepsilon = w^{\tau\varepsilon}\circ (w^\nu)^{-1},$$
to
$$\frac{\partial f_\varepsilon(w^\nu(a))}{\partial \varepsilon} \neq 0 \quad \text{for } \varepsilon = 0. \tag{8.28}$$

Now f_ε is a quasiconformal automorphism of $\hat{\mathbf{C}}$ which leaves $0, 1, \infty$ fixed. Let θ be the Beltrami coefficient of f_ε,
$$\theta = \frac{\partial f_\varepsilon/\partial \bar{z}}{\partial f_\varepsilon/\partial z};$$
then
$$\theta \,|\, \mathbf{C} - D = 0$$
and one computes, using (8.14) and (8.27), that for $z \in D$,
$$\theta(z) = \varepsilon\theta_0(z), \quad \theta_0(z) = \sigma_0 \circ h_\nu(z)\, \overline{h_\nu'(z)}/h_\nu'(z).$$
By a known formula (cf. [18], p. 133 or [2], p. 104)
$$\begin{aligned}\left.\frac{\partial f_\varepsilon(w^\nu(a))}{\partial \varepsilon}\right|_{\varepsilon=0} &= -\frac{1}{\pi}\iint_D \frac{\theta_0(z)\,dx\,dy}{z(z-1)\,[z-w^\nu(z)]}\\ &= -\frac{1}{\pi}\iint_D \frac{\lambda(h_\nu(z))^{-2}\,\overline{\Psi_{r+1}(h_\nu(z))}\,\overline{h_\nu'(z)}/h_\nu'(z)\,dx\,dy}{z(z-1)\,[z-w^\nu(z)]}\\ &= -\frac{1}{\pi}\sum_{g\in G^\nu}\iint_{g(\Delta)} \frac{\theta_0(z)\,dx\,dy}{z(z-1)\,[z-w^\nu(z)]}\end{aligned}$$
where Δ is a smoothly bounded fundamental domain for G^ν in D. Since $\theta_0(g(z))\,\overline{(g'(z)}/g'(z) = \theta_0(z)$ for $g \in G^\nu$, a simple calculation yields
$$\begin{aligned}\left.\frac{\partial f_\varepsilon(w^\nu(a))}{\partial \varepsilon}\right|_{\varepsilon+0} &= \iint_\Delta \theta_0(z)\,\Psi^0(z)\,dx\,dy = \iint_\Delta \sigma_0(h_\nu(z))\,\Psi_0(h_\nu(z))\,|h_\nu'(z)|^2\,dx\,dy\\ &= \iint_{U/G_\nu} \sigma_0(z)\,\Psi_0(z)\,dx\,dy = \iint_{U/G_\nu} \lambda(z)^{-2}\,\overline{\Psi_{r+1}(z)}\,\Psi_0(z)\,dx\,dy = \frac{1}{c} \neq 0\end{aligned}$$
where we used (8.26), (8.18), (8.19) and (8.20). Thus (8.28) is proved and so is Lemma 8.4.

The lemma implies that $\zeta_1, \ldots, \zeta_{r+1}$ can be used as complex coordinates in neighborhood of the point $([w_\nu], w^\nu(a))$ in $F_u(G)$. We call them *standard coordinates* about this point.

LEMMA 8.5. *The standard coordinates about a point in $F_u(G)$ are uniquely determined, except that the first r coordinates may be subject to a unitary transformation and the $(r+1)$-st may be multiplied by a complex number of absolute value* 1.

This follows from the construction.

Now we are in a position to state

LEMMA 8.6. *Under the hypotheses of Theorem 9, and for* $\dim T(G) = r < \infty$, *the surjection* $T(\dot G) \to F_u(G)$ *of Lemma 6.7 takes (appropriately chosen) standard coordinates about any point in* $T(\dot G)$ *into standard coordinates about the corresponding point in* $F_u(G)$.

Proof. Let $\mu \in L_\infty(U, \dot G)_1$ and $\nu = \varrho(\mu)$, ϱ being the mapping defined in § 6.5. Let $\Psi_1, \ldots, \Psi_{r+1}$ be the functions used to define the standard coordinates in $F_u(G)$ about $([w_\nu], w^\nu(a))$ and let v_μ: $U \to U - A_\nu = w_\nu(U - A)$ be the covering map in diagram (6.7). Then we have that

$$\lambda(v_\mu(z))\,|v_\mu'(z)| = \frac{1}{y}.$$

Next, define the functions

$$\hat\Psi_j(z) = \Psi_j(v_\mu(z))\, v_\mu'(z)^2, \quad j = 1, \ldots, r+1.$$

One verifies by a computation that the $\hat\Psi_j$ belong to $B_2(U, \dot G_\mu)$ where $\dot G_\mu = w_\mu \dot G w_\mu^{-1}$, and that

$$\iint_{U/\dot G_\mu} y^2 \Psi_j(z)\, \overline{\Psi_k(z)}\, dx\, dy = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Let $\zeta_1, \ldots, \zeta_{r+1}$ be complex numbers with small absolute values, let σ and τ be defined by (8.21) and by (8.22), respectively, and define $\hat\sigma \in L_\infty(U, \dot G_\mu)_1$ and $\hat\tau \in L_\infty(U, \dot G)_1$ by the relations:

$$\hat\sigma(z) = y^2[\overline{\zeta_1 \hat\Psi_1(z)} + \ldots + \overline{\zeta_{r+1} \hat\Psi_{r+1}(z)}],$$

and

$$w_{\hat\tau} = w_{\hat\sigma} \circ w_\mu.$$

There is a universal covering $\hat v$: $U \to w_\sigma \circ w_\nu(U - A)$ and a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\;w_{\hat\sigma}\;} & U \\ {\scriptstyle v_\mu}\downarrow & & \downarrow{\scriptstyle \hat v} \\ w_\nu(U - A) & \xrightarrow{\;w_\sigma\;} & w_\sigma \circ w_\nu(U - A) \end{array} \tag{8.29}$$

this is established in the same way as diagram (6.7) in the proof of Lemma 6.5. Diagram (8.29) and (6.7) yield the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\;w_{\hat\tau}\;} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \hat v} \\ U - A & \xrightarrow{\;w_\tau\;} & w_\tau(U - A) \end{array}$$

which shows that

$$\hat\tau = \varrho(\tau).$$

This proves the assertion.

Lemma 8.6 contains the desired local description.

§ 9. Direct proof of the isomorphism theorem

We shall now establish Theorem 9 *without* assuming previous knowledge of the topological result stated as Lemma 6.9. The proof will also yield this lemma, for the case when the fundamental group of the surface S is finitely generated.

We begin by proving, as before, Lemmas 6.1–6.8. Next we consider the case $\dim T(G) < \infty$. Let $\nu \in L_\infty(U, G)_1$, and note that the preimages of the point $([w_\nu]\, w^\nu(a)) \in F_u(G)$ under the mapping $T(\dot{G}) \to F_u(G)$ form a *discrete set*. Indeed, if μ, $\hat{\mu} \in L_\infty(U, \dot{G})_1$ are such that $\varrho(\mu) = \varrho(\hat{\mu}) = \nu$, cf. Lemma 6.5, then the Riemann surfaces $U/w_\mu G w_\mu^{-1}$ and $U/w_{\hat{\mu}} G w_{\hat{\mu}}$ are both conformally equivalent to $w_\nu(U - A)/w_\nu G w_\nu^{-1}$, so that $w_\mu G w_\mu^{-1}$ and $w_{\hat{\mu}} G w_{\hat{\mu}}^{-1}$ are conjugate in Q_{conf}. Thus $[w_\mu]$ and $[w_\mu]$ are equivalent under the group Mod $(\dot{G})$, which acts discontinuously on $T(\dot{G})$. The discreteness of every fiber of $T(\dot{G}) \to F_u(G)$, together with Lemma 8.6 imply that $T(\dot{G}) \to F_n(G)$ is an *unbounded unramified covering*.

Now, $F_u(G)$ is homeomorphic to $T(G) \times U$, by Lemma 6.4. Since $\dim T(G) < \infty$, $T(G)$ is a cell, by a classical theorem of Fricke. In our case, G without torsion, this also follows from the theory of extremal quasiconformal mapping (cf., for instances [6]). We conclude that $F_u(G)$ is a cell, so that $T(\dot{G}) \to F_n(G)$ is an isomorphism.

Now we can prove Lemma 6.9 assuming that S *is compact, except perhaps for finitely many punctures*. There is no loss of generality in assuming that the automorphism θ of S, which leaves the point $\hat{a}$ fixed, is quasiconformal. Indeed, in view of our hypothesis on S, $\theta \,|\, S - \{\hat{a}\}$ is homotopic to a quasiconformal automorphism. We may also assume that $S = U/G$, G a torsion free Fuchsian group, and $\hat{a}$ is the image of some $a \in U$ under $U \to U/G$. In view of our assumption on S, $\dim T(G) < \infty$. There is an $\omega \in N(G)$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \omega\ } & U \\ \downarrow & & \downarrow \\ U/G & \xrightarrow{\ \theta\ } & U/G \end{array} \tag{9.1}$$

commutes. Let Θ_t, $0 \leqslant t \leqslant 1$ be the homotopy which takes θ into the identity leaving $\hat{a}$ fixed. It lifts, for each t, to a mapping Ω_t such that we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \Omega_t\ } & U \\ \downarrow & & \downarrow \\ U/G & \xrightarrow{\ \Theta_t\ } & U/G \end{array} \tag{9.2}$$

with $\Theta_0 = \theta$, $\Omega_0 = \omega$, $\Theta_1 = \text{id}$,

$$\Omega_t \circ g \circ \Omega_t^{-1} \in G \tag{9.3}$$

and
$$\Omega_t(a) \subset A \tag{9.4}$$

where A is, as in § 6, the G orbit of a. For $t=1$, the commutativity of (9.2) implies that $\Omega_1 \in G$. But the commutativity of (9.1) defines ω only up to premultiplication by an element of G. Hence we may replace ω by $\Omega_1^{-1} \circ \omega$, and Ω_t by $\Omega_1^{-1} \circ \Omega_t$, i.e., we may assume that

$$\Omega_1 = \mathrm{id}. \tag{9.5}$$

Observe that $\Omega_t(z)$ depends continuously on $(t, z) \in [0, 1] \times U$. Since the group G is discrete, $\Omega_t \circ g \circ \Omega_t^{-1}$ depends only on $g \in G$, and not on t. By (9.5), we have that $\omega \circ g \circ \omega^{-1} = g$ for all $g \in G$. Since G is of the first kind, we conclude that $[\omega] = [\mathrm{id}]$. Hence $\omega \in Q_{\mathrm{norm}}$ and there is a $\nu \in L_\infty(U, G)_1$ with $\omega = [w_\nu]$, $[w_\nu] = [\mathrm{id}]$.

Since A is discrete, $\Omega_t(a)$ in (9.4) does not depend on t; by (9.5) we have that $\Omega_t(a) = a$; for $t=0$, this yields $w_\nu(a) = a$. Since $[w_\nu] = [\mathrm{id}]$ implies that $w_\nu = w^\nu$ we conclude that $w^\nu(a) = a$. Thus, for $\dot{G}$ as in Theorem 9, we have that

$$T(\dot{G}) \to F_u(G) \quad \text{takes } [\mathrm{id}] \text{ into } ([w_\nu], w^\nu(a)). \tag{9.6}$$

Now let
$$\mu = \varrho^{-1}(\nu) \in L_\infty(U, \dot{G})_1, \tag{9.7}$$

cf. Lemma 6.5. Then we have the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad w_\mu \quad} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v_\mu} \\ U-A & \xrightarrow{\quad w_\nu | U-A \quad} & U-A \\ \downarrow & & \downarrow \\ (U-A)/G & \xrightarrow{\quad \theta|(U-A)/G \quad} & (U-A)/G \end{array} \tag{9.8}$$

obtained by combining (6.7) and (9.1), noting that $\omega = w_\nu$, $w_\nu(A) = A$, and restricting the latter diagram.

Since by (9.7) and Lemma 6.7, $T(\dot{G}) \to F_u(G)$ takes $[w_\mu]$ into $([w_\nu], w^\nu(a))$ and the mapping is known to be a bijection, (9.6) implies that $[w_\mu] = [\mathrm{id}]$. Hence there is an Ahlfors homotopy (cf. the proof of Lemma 6.10) of w_μ into the identity. Using (9.8) the Ahlfors homotopy can be projected into a homotopy of $\theta | (U-A)/G = \theta | (U/G) - \{\hat{a}\}$ into the identity. This concludes the argument.

Since Lemma 6.9 is purely topological, we have actually established it whenever S has a finitely generated fundamental group, since every such surface is homeomorphic to a compact Riemann surface with at most finitely many punctures. We can now repeat the proof of Lemma 6.10 and establish Theorem 9 for all *finitely generated* groups G.

Now let G be infinitely generated. There is a sequence of finitely generated subgroups $G_1, G_2, \ldots$, of G with

$$1 \neq G_1 \subset G_2 \subset G_3 \subset \ldots, \quad G = G_1 \cup G_2 \cup G_3 \cup \ldots.$$

We are given the universal covering $v\colon U \to U - A$ with covering group $V \subset \dot{G}$. Let A_j denote the G_j orbit of a, so that

$$A_1 \subset A_2 \subset A_3 \subset \ldots, \quad A = A_1 \cup A_2 \cup A_3 \cup \ldots.$$

Let $v_j\colon U \to U - A_j$ be the holomorphic universal covering determined by the conditions:

$$v_j(i) = v(i), \; v_j'(i)\overline{v'(i)} > 0.$$

We can repeat the construction of the mapping ϱ given in § 6, cf., in particular, Lemma 6.5. There are uniquely determined torsion free Fuchsian groups V_j, $\dot{G}_j$, conformal bijections $u_j\colon U/\dot{G}_j \to (U - A_j)/G_j$ and exact sequences

$$1 \to V_j \hookrightarrow \dot{G}_j \xrightarrow{\chi_j} G_j \to 1,$$

where V_j is the covering group of v_j, and

$$h_j \circ \gamma = \chi_j(\gamma) \circ h_j \quad \text{for } \gamma \in \dot{G}_j.$$

For every j there is a bijection

$$\varrho_j\colon L_\infty(U, \dot{G}_j)_1 \to L_\infty(U, G_j)_1$$

such that $\nu = \varrho_j(\mu)$ if and only if

$$\nu(v_j(z))\overline{v_j'(z)}/v_j'(z) = \mu(z). \tag{9.9}$$

Observe that $L_\infty(U, G) \subset L_\infty(U, G_j)$ for all j.

LEMMA 9.1. *Let* $\nu \in L_\infty(U, G)_1$ *and set* $\mu_j = \varrho_j^{-1}(\nu)$ $j = 1, 2, \ldots$. *Then*

$$\lim_{j \to \infty} w_{\mu_j}(z) = w_\mu(z) \;\; \textit{for} \;\; z \in U \cup \mathbf{R}. \tag{9.10}$$

Proof. A standard function theoretical argument shows that

$$\lim_{j \to \infty} v_j(z) = v(z) \;\; \text{for} \;\; z \in U$$

uniformly on compact subsets (cf., for instance, the proof of Lemma 15 in [10]). By (9.9) and (6.6) we have that

$$\lim_{j\to\infty} \mu_j(z) = \mu(z), \quad z \in U.$$

Since also $\|\mu_j\| = \|\nu\| < 1$, the conclusion (9.10) follows (cf., for instance, [3]).

Now we can prove Theorem 9 for G. Let μ, $\hat{\mu} \in L_\infty(U, \dot{G})_1$ and assume that $\nu = \varrho(\mu)$ and $\hat{\nu} = \varrho(\hat{\mu})$ satisfy $[w_\nu] = [w_{\hat{\nu}}]$, $w^\nu(a) = w^{\hat{\nu}}(a)$. We must show that $[w_\mu] = [w_{\hat{\mu}}]$. Define $\mu_j = \varrho_j^{-1}(\nu)$ $\hat{\mu}_j = \varrho_j^{-1}(\hat{\nu})$ where ϱ_j is the mapping defined above. Since G_j is finitely generated we know that $[w_{\mu_j}] = [w_{\hat{\mu}_j}]$, for all j. The desired conclusion follows from Lemma 9.1.

§ 10. Teichmüller spaces of low genus

Let M be a complex manifold and, for every $m \in M$, let $\Delta(m)$ be a Jordan domain in $\hat{\mathbf{C}}$. We say that $\Delta(m)$ *depends holomorphically on* m of there is a continuous mapping $(m, t) \mapsto Z_m(t)$ of $M \times \hat{\mathbf{R}}$ into $\hat{\mathbf{C}}$ such that, for every fixed t, $Z_m(t)$ depends holomorphically on m and, for every fixed m, $t \mapsto Z_m(t)$ is an orientation preserving homeomorphism of $\hat{\mathbf{R}}$ onto the boundary of $\Delta(m)$. (We orient the boundary of a Jordan domain so that the domain is to the left, and we consider $\hat{\mathbf{R}}$ as the boundary of U.)

Example. Let G be a Fuchsian group with dim $T(G) < \infty$. Then the Jordan domains $D_u([w_\mu])$ and $D_b([w_\mu])$, cf. § 2, depend holomorphically on $\tau = [w_\mu] \in T(G)$. In the first case, we can set $Z_\tau(t) = w^\mu(t)$, in the second, $Z_\tau(t) = W^\mu(t)$.

A domain $A \subset \mathbf{C}^r$ will be called a *Bergman domain* if either $r = 0$, or $r > 0$ and there is a Bergman domain $M \subset \mathbf{C}^{r-1}$ and, for every $m \in M$, a Jordan domain $\Delta(m) \subset \hat{\mathbf{C}}$ depending holomorphically on m, such that A consists of all pairs (m, z) with $m \in M$ and $z \in \Delta(m)$.

THEOREM 11. *The Teichmüller spaces $T(p, n)$ with $p = 0$, $n \geq 3$, with $p = 1$, $n \geq 1$, and with $p = 2$, $n \geq 0$ can be represented as bounded Bergman domains.*

Proof. Let G be a torsion free Fuchsian group such that $T(G)$ can be identified with $T(p, n)$. Let $\dot{G}$ be related to G as in Theorem 9, so that $T(\dot{G})$ can be identified with $T(p, n+1)$. Since $F_u(G)$ and $F_b(G)$ are isomorphic to $T(\dot{G})$, we conclude from the example above that $T(p, n+1)$ is a (bounded) Bergman domain if $T(p, n)$ is.

Now $T(0, 3)$ is a point and $T(1, 1)$ is a Jordan domain. Hence the statement of the theorem is true for $p = 0$ and $p = 1$. It is also true for $p = 2$ in view of the known

LEMMA 10.1. *$T(0, 4)$ is isomorphic to $T(1, 1)$, $T(1, 2)$ to $T(0, 5)$ and $T(0, 6)$ to $T(2, 0)$.*

We sketch a proof, for the sake of completeness. If G is a Fuchsian group with dim $B_2(L, G) < \infty$, then $T(G) = T(1) \cap B_2(L, G)$. It follows that if $\hat{G}$ and $G \subset \hat{G}$ are Fuchsian groups with dim $T(G) =$ dim $T(\hat{G}) < \infty$, then $T(G) = T(\hat{G})$.

Now let G be a Fuchsian group of signature $(1, 1; \infty)$ [or $(1, 1, \infty, \infty)$, or $(2, 0)$] so that $T(G)$ may be identified with $T(1, 1)$ [or with $T(1, 2)$, or with $T(2, 0)$]. Then U/G admits a conformal involution J which leaves precisely 3 [or precisely 4, or precisely 6] points fixed. Lifting J to U we obtain a $\gamma \in Q_{\text{conf}}$ such that γ and G generate a Fuchsian group $\hat{G}$ of signature $(0, 4; 2, 2, 2, \infty)$ [or $(0, 5; 2, 2, 2, 2, \infty)$, or $(0, 6; \infty, \infty, \infty, \infty, \infty, \infty)$]. By the result of [10], $T(\hat{G})$ may be identified with $T(0, 4)$ [or with $T(0, 5)$, or with $T(0, 6)$]. In all cases, dim $T(G) =$ dim $T(\hat{G})$.

Recently Patterson [15] proved that Lemma 10.1 exhausts *all* isomorphisms between distinct spaces $T(p, n)$.

Question: *Are Teichmüller spaces $T(p, n)$ with $p > 2$ representable as Bergman domains?*

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