

NON-EXISTENCE, NON-UNIQUENESS AND IRREGULARITY OF SOLUTIONS TO THE MINIMAL SURFACE SYSTEM

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Table of contents

| | |
|---|----|
| § 1. Introduction | 1 |
| § 2. The minimal surface system | 2 |
| § 3. A brief summary of known results in codimension one | 4 |
| § 4. The existence of solutions of dimension two | 5 |
| § 5. The non-uniqueness and non-stability of solutions of dimension two | 6 |
| § 6. The non-existence of solutions in dimensions ≥ 4 | 11 |
| § 7. The existence of non-parametric minimal cones | 13 |

§1. Introduction

The study of real-valued functions whose graphs are minimal surfaces has a long and rich history, and by now we have a fairly profound understanding of the subject. In contrast, almost nothing is known about vector-valued functions whose graphs are minimal surfaces, and this paper should explain, at least in part, why this is so. It will be shown that many of the deep and beautiful results for non-parametric minimal surfaces in codimension one fail utterly in higher codimensions.

We shall be concerned primarily with the Dirichlet problem for the minimal surface system on a bounded, convex domain Ω in \mathbf{R}^n . Using an old argument of Rado, we shall show that for $n=2$, the Dirichlet problem is solvable for arbitrary continuous boundary data. However, we then construct examples to show that these solutions are *not unique* in general. Moreover, we shall show that such surfaces *need not even be stable* in contrast with the fact that in codimension one, non-parametric minimal surfaces are absolutely area minimizing.

We shall then show that for $n \geq 4$, the Dirichlet problem is *not even solvable* in general.

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In fact, for each $n \geq 4$ it is possible to find a C^ω function $f: S^{n-1} \rightarrow \mathbf{R}^k$ for some integer k , $2 \leq k \leq n-2$, with the property that there are no Lipschitz solutions F to the minimal surface system in D^n such that $F|_{S^{n-1}} = f$. Moreover, the same statement holds for all f in a large C^1 neighborhood of f . Such a family of examples can be generated for each non-trivial element of $\pi_{n-1}(S^{k-1})$.

Finally, by examining certain specific cases in detail we are able to show the existence of non-parametric minimal cones. In particular, this gives examples of Lipschitz solutions to the minimal surface system which are not C^1 , thereby making sharp the basic regularity result of Morrey which states that any C^1 solution is real analytic.

We would like to thank Bill Allard for several informative conversations relating to this work.

§2. The minimal surface system

Let Ω be an open set in \mathbf{R}^n and consider a C^2 immersion $F: \Omega \rightarrow \mathbf{R}^{n+k}$. Then F is a minimal immersion if and only if F satisfies the system

$$\sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial F}{\partial x^j} \right) = 0, \quad (2.1)$$

where $g = \det((g_{ij}))$, $((g^{ij})) = ((g_{ij}))^{-1}$ and $g_{ij} = \langle \partial F / \partial x^i, \partial F / \partial x^j \rangle$. This is equivalent to the requirement that $F(\Omega)$ have mean curvature identically zero.

The immersion F is said to be non-parametric if it has the form $F(x) = (x, f(x))$ for some function $f: \Omega \rightarrow \mathbf{R}^k$. In this case the system (2.1) has the form:

$$\begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij}) = 0; & j = 1, \dots, n \\ \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) = 0, \end{cases} \quad (2.2)$$

where g and $((g^{ij}))$ are defined as above, and in this case

$$g_{ij} = \delta_{ij} + \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle.$$

This is clearly equivalent to the system:

$$\begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij}) = 0; & j = 1, \dots, n \\ \sum_{i,j=1}^n g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} = 0. \end{cases} \quad (2.2)$$

For functions of class C^2 , this system can be replaced by the smaller one:

$$\sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) = 0. \quad (2.3)$$

To see this we note that (2.3) simply expresses the fact that the vertical projection of the mean curvature vector to the graph of f is identically zero.

By a somewhat more subtle argument, it is shown in [30] that the system (2.2) (or (2.2)') for C^2 functions is also equivalent to:

$$\sum_{i,j=1}^n g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} = 0. \quad (2.4)$$

Note that the system (2.2) (or (2.2)') is defined in the weak sense for any locally Lipschitz function f on Ω . In this case (2.2) is equivalent to the condition that the first variation of area of the graph of f with respect to smooth, compactly supported deformations of $\Omega \times \mathbf{R}^k$, is zero. In other words, the graph of f is a stationary integral varifold in the sense of Almgren (cf. [3]). This leads naturally to the following.

CONJECTURE 2.1. The systems (2.2) and (2.3) are equivalent for any locally Lipschitz function f in Ω .

Suppose now that Ω is bounded and strictly convex and that $d\Omega$ is of class C^r for $r \geq 2$. For the remainder of the paper we shall be concerned with the following.

DIRICHLET PROBLEM. Given a function $\phi: d\Omega \rightarrow \mathbf{R}^k$ of class C^s , $0 \leq s \leq r$, find a function $f \in C^0(\bar{\Omega}) \cap \text{Lip}(\Omega)$ such that f satisfies the minimal surface system (2.2) in Ω and $f|_{d\Omega} = \phi$.

When $s \geq 1$, we further require that the area of the graph of f be finite.

The study of the Dirichlet problem usually falls into two distinct parts, those of the existence and the regularity of solutions. It is an unusual fact that for the non-parametric minimal surface system more is known about regularity than existence.

THEOREM 2.2. (Interior regularity; C. B. Morrey [22], [23]). Any C^1 function f which satisfies the system (2.2) is real analytic.

THEOREM 2.3. (Boundary regularity; W. Allard [2]). Suppose that ϕ is of class $C^{s,\alpha}$ for $2 \leq s \leq \infty$ or $s = \omega$, and let f be any solution to the Dirichlet problem for ϕ in Ω . Then there is a neighborhood U of $d\Omega$ such that $f \in C^{s,\alpha}(U \cap \bar{\Omega})$.

Note. Theorem 2.3 is deduced from the work in [2] as follows. Since Ω is strictly convex and ϕ is class C^2 , the arguments in 5.2 of [2] show that the graph Γ_f of f in $\mathbf{R}^n \times \mathbf{R}^k$, at each point of its boundary, has a tangent cone consisting of a finite number

of half-planes (of dimension n), each of which projects non-singularly into \mathbf{R}^n . Since each such cone is the limit of a sequence of dilations of Γ_r , it is easy to see that the cone consists, in fact, of a single half-plane. It follows that the density $\Theta(\|\Gamma_r\|, x) = \frac{1}{2}$ at each boundary point x . Theorem 2.3 is then an immediate corollary of the Regularity theorem in [2, § 4].

Remark. Perhaps it should be pointed out that if Ω is not convex, the Dirichlet problem is not necessarily solvable in general even for $n=2$ and $r=s=\omega$. In fact when $n=2$, convexity is necessary and sufficient for a solution to exist corresponding to arbitrary C^s boundary values [12]. When $n>2$, convexity is still sufficient, but the precise necessary and sufficient condition is that the mean curvature of the boundary with respect to the interior normal be everywhere non-negative [17].

§ 3. A brief summary of known results in codimension one

For comparison with our later results in higher codimension we list here some of the facts known about non-parametric minimal surfaces in codimension one.

THEOREM 3.1. *For $k=1$, the Dirichlet problem is solvable for arbitrary continuous boundary data. Furthermore:*

- (a) *The solution is unique.*
- (b) *The solution is real analytic.*
- (c) *The solution is absolutely area minimizing, i.e., its graph is the unique integral current of least mass in \mathbf{R}^{n+k} for the given boundary (the graph of ϕ).*

The existence for C^2 boundary data follows from Jenkins and Serrin [17], and for general continuous boundary data from the *a priori* estimates in [5]. The regularity of Lipschitz solutions follows from de Giorgi [7]. (See [37] or [25].) The remainder of the theorem is classic. (See [21, pp. 156–7] for part (c).)

For the purpose of completeness we mention the following strong removable singularities result.

THEOREM 3.2. *Let K be a compact subset of Ω with Hausdorff $(n-1)$ -measure zero. Then any (smooth) function $f: (\Omega - K) \rightarrow \mathbf{R}$ which satisfies the minimal surface equation (2.2)' in $\Omega - K$ extends to a (smooth) solution in all of Ω .*

This was first proved by Bers [4] for $n=2$ and $K = \{\text{point}\}$. The general 2-dimensional case was proved by Nitsche [27]. The result in arbitrary dimensions is the work of de Giorgi and Stampacchia [8].

Note. It has been known for some time that this last result is not true in general

codimensions. In fact, in [28] an example is given of a bounded function $f: \{(x, y) \in \mathbf{R}^2: 0 < x^2 + y^2 \leq 1\} \rightarrow \mathbf{R}^2$, whose graph is a minimal surface in \mathbf{R}^4 , but which does not extend continuously across 0. However, under certain (necessary) restrictive hypotheses, removable singularity results can be proved in higher codimension (cf. [14], [31]).

§4. The existence of solutions of dimension two

The following result is not new. It is an immediate consequence of old techniques of T. Rado and has certainly been known to the authors for some time. (See, for example [29, Thm. 7.2].) We include a proof here because the arguments involved will be useful in the next chapter.

THEOREM 4.1. *For $n=2$ and any $k \geq 1$, there exist solutions $f \in C^\omega(\Omega) \cap C^0(\bar{\Omega})$ to the Dirichlet problem for any given continuous boundary function $\phi: d\Omega \rightarrow \mathbf{R}^k$.*

Proof. We begin by recalling the notion of a *generalized parametric minimal surface* in \mathbf{R}^n having a given Jordan curve γ as boundary. This is a map $\psi: \Delta \rightarrow \mathbf{R}^n$, where $\Delta = \{z = x + iy \in \mathbf{C}: |z| \leq 1\}$, with the following properties:

- (a) $\psi \in C(\Delta) \cap C^\omega(\Delta^\circ)$,
- (b) $|\psi_x|^2 = |\psi_y|^2$ and $\langle \psi_x, \psi_y \rangle = 0$,
- (c) $\Delta\psi = 0$,
- (d) $\psi|_{d\Delta}: d\Delta \rightarrow \gamma$ is a homeomorphism.

The fundamental work of Douglas and Rado asserts the existence of such a minimal surface for any given Jordan curve $\gamma \subset \mathbf{R}^N$ (cf. [9], [32] or [6]).

Theorem 4.1 will be an immediate consequence of the following result.

THEOREM 4.2. *Let γ be a Jordan curve in \mathbf{R}^{2+k} and suppose that γ can be expressed as the graph of a continuous function $\phi: d\Omega \rightarrow \mathbf{R}^k$ where Ω is a bounded, convex domain in \mathbf{R}^2 . Then every generalized parametric minimal surface with boundary γ has a one-to-one, non-singular projection onto Ω , i.e., every such surface can be expressed as the graph of a function $f: \bar{\Omega} \rightarrow \mathbf{R}^k$ where $f \in C^0(\bar{\Omega}) \cap C^\omega(\Omega)$ and f satisfies (2.2) in Ω .*

To prove this we shall need the following basic result of T. Rado [33]. (Rado states the theorem for $N=3$, however the proof works in general, cf. [19], [29].)

PROPOSITION 4.3. *Suppose $\psi: \Delta \rightarrow \mathbf{R}^N$ is a generalized parametric minimal surface with boundary γ , where γ is an arbitrary Jordan curve in \mathbf{R}^N . Let $\lambda: \mathbf{R}^N \rightarrow \mathbf{R}$ be a linear function*

and suppose that at a point $(x_0, y_0) \in \Delta^0$, the function $\lambda \circ \psi$ has a zero of order k , i.e.,

$$\left. \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} (\lambda \circ \psi) \right|_{(x_0, y_0)} = 0 \quad \text{for } i + j \leq k.$$

Then the hyperplane $H = \{X \in \mathbf{R}^N: \lambda(X) = 0\}$ meets γ in at least $(2k + 2)$ connected components.

Roughly, this proposition follows from the maximum principle and the observation that the zeros of $\lambda \circ \psi$ in a neighborhood of a zero of order k have the structure of the zeros of a harmonic homogeneous polynomial of degree $k + 1$.

Proof of Theorem 4.2. Let $\pi: \mathbf{R}^2 \times \mathbf{R}^k \rightarrow \mathbf{R}^2$ be projection and consider the map $\pi \circ \psi: \Delta \rightarrow \bar{\Omega}$. (Note that $\pi \circ \psi(\Delta^0) \subset \Omega$ by the strict maximum principle.) Suppose that at some point $(x_0, y_0) \in \Delta^0$ we had $\text{rank}(d(\pi \circ \psi)) \leq 1$. Then for a non-trivial linear function of the form $\lambda(x, y, z_1, \dots, z_n) = \lambda(x, y) = ax + by$, we would have that $\lambda \circ \psi$ has a critical point of order ≥ 1 at (x_0, y_0) . It then follows from Proposition 4.3 that the hyperplane H defined by $\lambda = \lambda(x_0, y_0)$ must intersect γ in at least 4 components. However, by the convexity of Ω , $H \cap \gamma$ has exactly two components. It follows then that $\pi \circ \psi: \Delta^0 \rightarrow \Omega$ is a local diffeomorphism which extends to a homeomorphism of the boundaries. It follows easily that $\pi \circ \psi$ is one-to-one. This completes the proof.

We now state another immediate consequence of the arguments above. Given a linear function λ on \mathbf{R}^N and $c \in \mathbf{R}$, let $H_c(\lambda) = \{X \in \mathbf{R}^N: \lambda(X) = c\}$.

PROPOSITION 4.4. *Let γ be a Jordan curve in \mathbf{R}^N and suppose $\lambda: \mathbf{R}^N \rightarrow \mathbf{R}$ is a linear function such that $H_c(\lambda) \cap \gamma$ has at most five components for each $c \in \mathbf{R}$. Then for any generalized parametric minimal surface $\psi: \Delta \rightarrow \mathbf{R}^N$ with γ as boundary, the function $\lambda \circ \psi$ has only non-degenerate critical points in Δ^0 , and those critical points have index 1.*

§5. The non-uniqueness and non-stability of solutions of dimension two

In this section we shall prove the following result. Let

$$D = \{(x, y) \in \mathbf{R}^2: x^2 + y^2 \leq 1\}.$$

THEOREM 5.1. *There exists a real analytic function $\phi: dD \rightarrow \mathbf{R}^2$ with the property that there exist at least three distinct solutions (each of class C^ω) to the Dirichlet problem for ϕ in D . Moreover, one of these solutions represents an unstable minimal surface; that is, the area of the graph can be decreased by arbitrarily small deformations which fix the boundary.*

Proof. Let γ denote the graph of ϕ in \mathbf{R}^4 . It will suffice to show that there exist two geometrically distinct parametric minimal surfaces with boundary γ , each of which has

the following additional property:

The surface minimizes the area and Dirichlet integrals among all piecewise differentiable maps of Δ into Euclidean space which carry $d\Delta$ homeomorphically onto γ . (5.1)

If we succeed in proving this, then by the theorems of Morse-Tompkins [24] and Shiffman [35], there is a third, *unstable* parametric minimal surface with γ as boundary. By Theorem 4.2 every generalized parametric minimal surface with boundary γ is, in fact, a non-parametric surface over D and we will have proved the theorem.

For clarity of exposition we shall begin with an example in codimension 3. We shall then indicate how similar arguments will produce examples in codimension 2.

Consider the regular, C^∞ Jordan curve γ_0 in \mathbf{R}^3 pictured in Figure 1. This curve lies in the union of two pairs of parallel planes, each pair a distance ε apart where $\varepsilon \ll 1$. The curve is assumed to be invariant under the symmetry $\sigma_0(x, y, z) = (-x, -z, y)$. Note that $\sigma_0|_{\gamma_0}$ preserves the orientation of γ_0 .

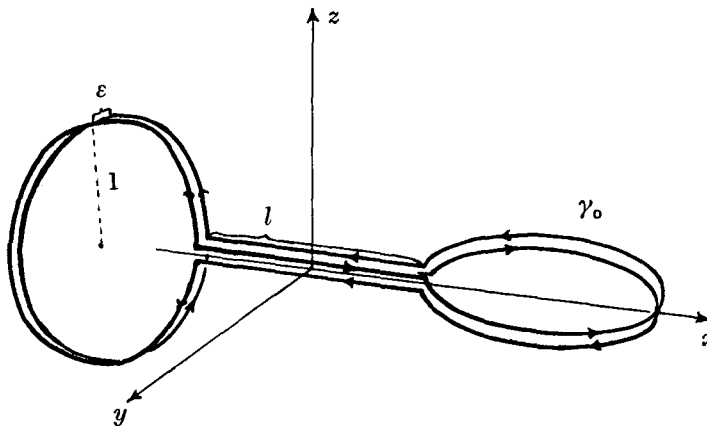


Figure 1

Consider now the mapping $\phi_0: dD \rightarrow \gamma_0$ given by (the appropriate multiple of) arc-length, and set $\phi = R\phi_0$ for some $R \gg 1$. We then define γ to be the graph of ϕ in \mathbf{R}^5 . Note that since σ_0 is an isometry and ϕ is an arc-length parametrization, we have

$$\sigma_0(\phi(u, v)) = \phi(-v, u)$$

Hence γ is invariant under the symmetry

$$\sigma(u, v, x, y, z) = (-v, u, -x, -z, y).$$

Up to a scalar change, γ represents an ever-so-slight deformation of γ_0 into five-space. Our first major claim concerning the curve γ is the following.

PROPOSITION 5.2. *Let $\psi: \Delta \rightarrow \mathbf{R}^5$ be any generalized parametric minimal surface with boundary γ , and let $\lambda: \mathbf{R}^5 \rightarrow \mathbf{R}$ be the coordinate function $\lambda = x$. Then $\lambda \circ \psi$ has exactly one critical point in Δ and that critical point is non-degenerate.*

Proof. Each hyperplane $H_c(\lambda) = \{X \in \mathbf{R}^5: \lambda(X) = c\}$ clearly meets γ in at most four points. Hence, by Proposition 4.4 every critical point of $\lambda \circ \psi$ in Δ^0 is non-degenerate and of index 1.

We now consider the corresponding non-parametric surface $f: D \rightarrow \mathbf{R}^3$ guaranteed by Theorem 4.2. If $\pi: \mathbf{R}^5 \rightarrow \mathbf{R}^3$ denotes projection onto the first two coordinates, then $h = \pi \circ \psi: \Delta \rightarrow D$ is a homeomorphism and we have that

$$\psi(z) = (h(z), f(h(z)))$$

for all $z \in \Delta$. Since h is a diffeomorphism on the interior of Δ and since $\lambda \circ f \circ h = \lambda \circ \psi$, we see that the function $\lambda \circ f$ has the property that each of its critical points in D^0 is non-degenerate and of index 1.

When restricted to the boundary, the function $\lambda \circ f|_{\partial D} = \lambda \circ \phi$ has exactly four critical points, two non-degenerate maxima and two non-degenerate minima.

We are now in a position to apply elementary Morse theory to $\lambda \circ f$. For each $c \in \mathbf{R}$ we consider the sublevel set $D_c = \{(x, y) \in D: \lambda \circ f(x, y) \leq c\}$. For c slightly larger than $c_0 = \inf_D \{\lambda \circ f\}$, D_c consists of two components, each homeomorphic to a disk. As c increases, we add a one-handle to this manifold for each critical point in D^0 . Since $D_\infty = D$, we conclude that there must be exactly one critical point of $\lambda \circ f$ in D^0 . It follows that $\lambda \circ \psi$ has exactly one critical point in Δ^0 . This completes the proof of the proposition.

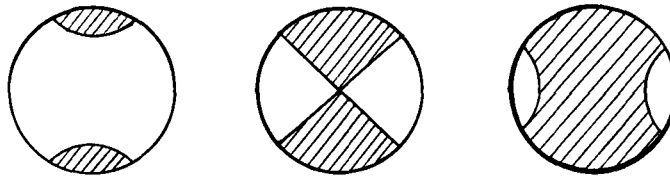


Figure 2

From the work of Douglas [9] we know that there exists at least one parametric minimal surface $\psi: \Delta \rightarrow \mathbf{R}^5$ having boundary γ and satisfying condition (5.1). Let us suppose that there are no other such surfaces which are geometrically distinct from ψ (i.e., which

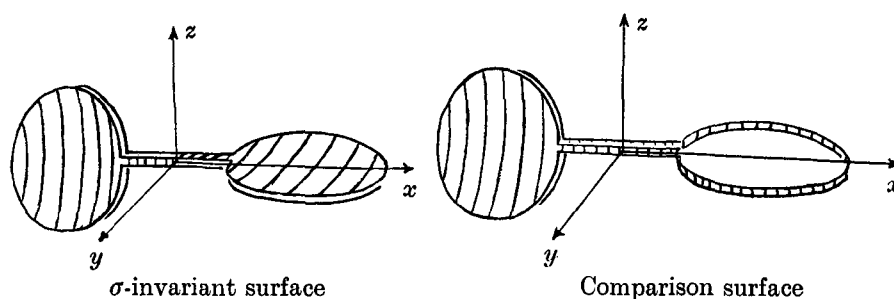


Figure 3

are not just a reparameterization of the same surface). Then since $\sigma: \mathbf{R}^5 \rightarrow \mathbf{R}^5$ is an isometry which preserves γ , we see that $\sigma \circ \psi$ is again a parametric minimal surface having boundary γ and satisfying condition (5.1). Hence, ψ and $\sigma \circ \psi$ must have the same image surface, say Σ , in \mathbf{R}^5 ; that is, $\sigma(\Sigma) = \Sigma$. By Proposition 5.2 the linear function $\lambda = x$ has exactly one critical point p when restricted to Σ^0 . It follows that $\sigma(p) = p$, and so $p = 0$.

We will now show that the surface ψ cannot satisfy condition 5.1 for R sufficiently large. In fact we claim that the area of Σ satisfies the inequality

$$A(\Sigma) \geq 4\pi R^2 \quad (5.2)$$

whereas one can easily construct a surface Σ' with boundary γ such that

$$A(\Sigma') \leq [2\pi + 2(\pi + l)\varepsilon]R^2 + O(R) + \pi \quad (5.3)$$

where l is the length of the four parallel arcs of γ (cf. Fig. 1). To construct this surface one first chooses a parametrization $\psi_0: D \rightarrow \mathbf{R}^3$ of the comparison surface pictured in Figure 3, with $\psi_0|_{\partial D} = \phi_0$. The area of the comparison surface is bounded above by $2\pi + 2(\pi + l)$. The surface Σ' is then defined to be the graph of the map $\psi = R\psi_0$. It is easy to verify the inequality (5.3).

To deduce (5.2) we consider the hyperplane $H = \{X \in \mathbf{R}^5: \lambda(X) = 0\}$ and observe that $\Sigma - H$ decomposes into four connected components, (each parameterized by ψ restricted to one of the four wedge-shaped regions in the middle picture of Figure 2). Denote these components by $\Sigma_1, \dots, \Sigma_4$, and observe that the boundary of each Σ_i maps by orthogonal projection onto the boundary of a key-shaped region R_i in either the (x, y) -plane or the (x, z) -plane. Consequently, each Σ_i maps by orthogonal projection onto R_i . Since $A(R_i) \geq \pi R^2$, it follows that $A(\Sigma_i) \geq \pi R^2$ for each i and (5.2) is established.

We have shown that there must be two geometrically distinct parametric minimal surfaces with boundary γ , which satisfy condition (5.1). This establishes an example of the type claimed in the theorem with the exception that the curve is class C^∞ and the codimension is three.

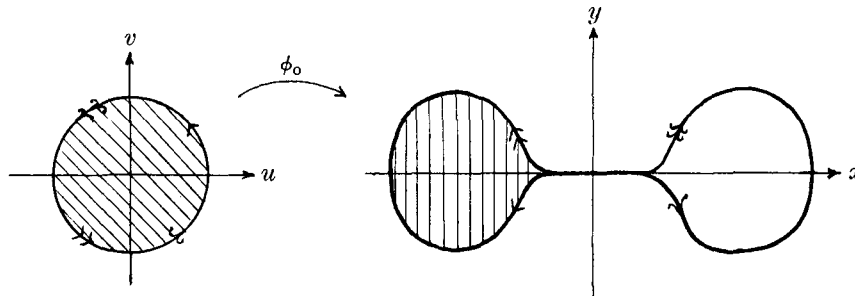


Figure 4

We now observe that throughout the above argument it is possible to replace the curve γ with a real analytic approximation. This is done by replacing ϕ with a sufficiently large piece of its Fourier expansion. Since the Fourier polynomials preserve such properties as

$$F\left(\theta + \frac{\pi}{2}\right) = e^{i(\pi/2)} F(\theta)$$

and

$$F\left(\theta + \frac{\pi}{2}\right) = -F(\theta)$$

for (complex-valued) functions F on the circle, we have that the analytic approximation will also be σ -invariant.

We shall now indicate how to construct an example of codimension 2. The arguments are all essentially the same except that one begins with a map $\phi = R\phi_0$ where $R \gg 1$ and $\phi_0: dD \rightarrow \mathbf{R}^2$ is a suitably parametrized double tracing of the curve pictured in Figure 4, where each loop is traversed once in each direction, and where the parameter is a multiple of arc length.

The graph of ϕ is invariant by the symmetries $\sigma_k: \mathbf{R}^4 \rightarrow \mathbf{R}^4$, $k=1, 2$ where

$$\sigma_1(u, v, x, y) = (-u, -v, x, -y)$$

$$\sigma_2(u, v, x, y) = (-u, v, -x, y).$$

Let γ be the graph of ϕ . If we again assume that there are not two geometrically distinct minimal surfaces with boundary γ satisfying condition (5.1), then the surface Σ given by the Douglas solution to the Plateau problem for γ must be invariant under σ_1 and σ_2 . The linear function $\lambda = x$ has exactly one critical point p on Σ . This follows from an argument entirely similar to the proof of Proposition 5.2. Consequently, $\sigma_1(p) = p$, $\sigma_2(p) = p$, and so $p = 0$. The rest of the argument proceeds as above. The appropriate comparison

surface is given by a map $\Psi = R\Psi_0$ where Ψ_0 is chosen to be linear on the lines $u + v = c$. For $|c| \leq 1$, Ψ_0 will be constant on these lines. The image of Ψ_0 will be the curve together with the region bounded by the loop on the left (covered twice). This completes the proof of Theorem 5.1.

§ 6. The non-existence of solutions in dimensions ≥ 4

The main result of this section is the following. Let $D^n = \{x \in \mathbf{R}^n: \|x\| \leq 1\}$ and set $S^n = dD^{n+1}$.

THEOREM 6.1. *Let $\phi: S^{n+k} \rightarrow S^n \subset \mathbf{R}^{n+1}$ be any C^2 mapping which is not homotopic to zero as a map into S^n , and suppose $k > 0$. Then there is an R_ϕ such that for each $R \geq R_\phi$ there is no solution to the Dirichlet Problem for the boundary function $\phi_R = R \cdot \phi$.*

Remark 6.2. It follows from the Implicit function theorem (cf. [26]) that the Dirichlet problem is always solvable for sufficiently small boundary data. Hence, given ϕ as in Theorem 6.1, there is an $r_\phi > 0$ such that for all r , $|r| \leq r_\phi$, the Dirichlet problem is solvable for boundary data $\phi_r = r \cdot \phi$.

Proof. The proof of this theorem rests on the following two results. In the statements, the term *minimal variety* means any integral current T with compact support such that the first variation of mass is zero with respect to any smooth deformation supported away from the boundary of T . In our applications we will only need to consider currents which are given as the oriented graphs of Lipschitz functions.

THEOREM A. *Let V be any p -dimensional minimal variety in \mathbf{R}^N which is regular at the boundary (i.e., in a neighborhood of $\text{supp } dV$), V is given by an oriented, C^1 submanifold with boundary). Then the mass of V is given by the formula*

$$\mathbf{M}(V) = \frac{1}{p} \int_{dV} \langle v, x \rangle * \mathbf{1} \quad (6.1)$$

where x is the position vector with respect to any euclidean coordinate system and where v is the unit exterior normal field to dV along dV .

Note. The mass of V is the “weighted volume”. If V is a Lipschitz submanifold, $\mathbf{M}(V)$ is just the Hausdorff p -measure of that submanifold.

We shall only need this result in the special case that dV lies in a sphere about the origin in \mathbf{R}^N . Under this assumption the proof is quite easy, and the reader is referred to [13] or [20] for details.

THEOREM B. *Let V be a p -dimensional minimal variety in \mathbf{R}^N . Suppose that $x_0 \in \mathbf{R}^N$ lies on V and that $B(x_0, R) = \{x \in \mathbf{R}^N: \|x - x_0\| < R\}$ does not meet the boundary of V . Then*

$$\mathbf{M}(V) \geq c_p R^p \quad (6.2)$$

where c_p is the volume of the unit ball in \mathbf{R}^p .

This is a direct consequence of the well known fact that $\phi(r) \equiv \mathbf{M}(V \cap B(x_0, r))/c_p r^p$ is monotone increasing and $\lim_{r \rightarrow 0} \phi(r) \geq 1$. (See [10], [13] or [20].)

Let us now suppose that for fixed R , there is a solution f_R to the Dirichlet problem for ϕ_R . Let V_R be the minimal variety given by the graph of f_R in \mathbf{R}^N where $N = 2(n+1) + k$.

By Theorem 2.3 V_R is regular at the boundary. We clearly have that $dV_R = \{(x, R\phi(x)): \|x\| = 1\}$ is contained in the sphere of radius $\sqrt{1+R^2}$ about the origin in \mathbf{R}^N . Hence, it follows from (6.1) that

$$\mathbf{M}(V_R) \leq \frac{\sqrt{1+R^2}}{n+k+1} \mathbf{M}(dV_R). \quad (6.3)$$

We now observe that f_R represents a homotopy of ϕ_R to zero in \mathbf{R}^{n+1} . Since ϕ_R is not homotopic to zero as a map into S^n and since $\mathbf{R}^{n+1} - \{0\}$ has the homotopy type of S^n , we conclude that $f_R(D^{n+k+1})$ cannot be contained in $\mathbf{R}^{n+1} - \{0\}$. Hence, there must be a point $x_0 \in D^{n+k+1}$ such that $f_R(x_0) = 0$, i.e., there is a point of the form $(x_0, 0)$ on V_R .

We now have that

$$\text{dist}[(x_0, 0), dV_R] = \min_{\|x\|=1} \sqrt{\|x - x_0\|^2 + \|R\phi(x)\|^2} > R.$$

Hence, we may apply Theorem B to V_R in the ball of radius R about $(x_0, 0)$ and conclude that

$$\mathbf{M}(V_R) \geq c_{n+k+1} R^{n+k+1}. \quad (6.4)$$

One can easily see that there is a constant c such that $\mathbf{M}(dV_R) \leq cR^n$ for all $R > 0$. Combining this with the inequalities (6.3) and (6.4) gives the estimates

$$c_{n+k+1} R^{n+k+1} \leq \mathbf{M}(V_R) \leq \frac{\sqrt{1+R^2}}{n+k+1} cR^n.$$

Since $k > 0$, the theorem follows immediately.

Example 6.3. The simplest example of a map satisfying the hypotheses of Theorem 6.1 is the Hopf map $\eta: S^3 \rightarrow S^2$ given by

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 \bar{z}_2) \quad (6.5)$$

where S^3 is considered as the unit sphere in $\mathbb{C}^2 = \mathbb{R}^4$ and S^2 as the unit sphere in $\mathbb{R} \times \mathbb{C} = \mathbb{R}^3$. In this case we can make a fairly explicit calculation of R_η .

Observe that the group $SU(2)$ acts naturally on \mathbb{C}^2 and on \mathbb{R}^3 (as $SO(3) = SU(2)/\mathbb{Z}_2$). The map η is equivariant with respect to these actions. This is most easily seen by recalling that $S^2 = \mathbb{P}^1(\mathbb{C})$ and $SO(3) = PU(2)$, the group of isometries of the projective line. It follows that the graph of each $\eta_R = R\eta$, $R > 0$, is invariant under the joint action of $SU(2)$ on \mathbb{R}^7 . Since $SU(2)$ is transitive on S^3 , each such graph is an orbit of this action.

We now wish to compute the volume of the graph of η_R . By the homogeneity it will suffice to compute the volume element at a single point. At any point $x \in S^3$ we can choose an orthonormal basis e_1, e_2, e_3 of $T_x(S^3)$ such that $(\eta_R)_*(e_3) = 0$ and $(\eta_R)_*(e_j)$, $j = 1, 2$, are perpendicular and of length $2R$. Hence the metric induced by the graphing immersion $x' \mapsto (x', \eta_R(x'))$ at x has matrix

$$\begin{pmatrix} 1 + 4R^2 & & 0 \\ & 1 + 4R^2 & \\ 0 & & 1 \end{pmatrix}$$

with respect to the basis e_1, e_2, e_3 . It follows that the volume of the graph Γ_R of η_R is

$$\text{vol}(\Gamma_R) = \int_{S^3} (1 + 4R^2) *1 = (1 + 4R^2) 2\pi^2. \quad (6.6)$$

Therefore, combining the inequalities (6.3) and (6.4) in this case we get

$$\frac{\pi^2}{2} R^4 \leq \mathbf{M}(f_R) \leq \frac{\pi^2}{2} \sqrt{1 + R^2} (1 + 4R^2).$$

It follows that *there are no solutions to the Dirichlet problem for η_R in D^4 whenever $R \geq 4.2$.*

It is interesting to note that while there is no non-parametric minimal variety with boundary Γ_R for $R \geq 4.2$, there must be some minimal variety with this boundary by the basic work of Federer and Fleming [11]. In fact there must be such a variety which is $SU(2)$ -invariant [18]. This symmetric solution corresponds to a geodesic arc in the orbit space (cf. Hsiang and Lawson [16]). It can be described topologically as (the closure of) the graph of the rational map $\varrho: D^4 \rightarrow \mathbb{P}^1(\mathbb{C})$ given by $\varrho(Z_1, X_2) = [Z_1/Z_2]$. It is diffeomorphic to the oriented 2-disk bundle over S^2 of Chern class 1.

§7. The existence of non-parametric minimal cones

The existence of solutions to the Dirichlet problem for boundary data $\phi_R = R \cdot \phi$ when R is small, and the non-existence when R is large, lead one to suspect that there should be a critical value R_0 for which there exists some sort of singular non-parametric

solution. In the special case of the Hopf map, the symmetries indicate that this singular solution should simply be the cone over the boundary values. We shall show that this is indeed the case.

Recall that for any submanifold $M^p \subset S^n$, the cone $C(M^p) = \{tx \in \mathbf{R}^{n+1}: x \in M^p \text{ and } t > 0\}$ is a minimal variety in \mathbf{R}^{n+1} if and only if M^p is a minimal submanifold of S^n . For each α , $0 < \alpha \leq 1$ we consider the embedding $i_\alpha: S^3 \rightarrow S^6$ given by

$$i_\alpha(x) = (\alpha x, \sqrt{1 - \alpha^2} \eta(x)), \quad (7.1)$$

where $\eta: S^3 \rightarrow S^2$ is the map given by (6.5). Each submanifold $i_\alpha(S^3)$ is an orbit of the action of $SU(2)$ on $\mathbf{R}^7 = \mathbf{C}^2 \times \mathbf{R}^3$ given by the natural diagonal map $SU(2) \hookrightarrow SU(2) \times SO(3)$ (as we saw in the last section). In fact, these orbits are principal orbits since $SU(2)$ acts freely on them. Now by a basic result of Wu-Yi Hsiang [15] the principal orbits of maximal volume are minimal submanifolds of S^6 . Therefore, we want to compute the volume function on the orbit space $X = S^6/SU(2)$.

We first observe that the orbit space itself is highly symmetric. Recall that $SU(2) \cong Sp(1) \cong S^3$ is just the group of unit quaternions, and the above representation on $\mathbf{C}^2 \cong \mathbf{H}$ is just quaternion multiplication on the left. The group $SU(2)$ also acts on $\mathbf{C}^2 \cong \mathbf{H}$ by quaternion multiplication on the right. Let ρ denote this representation, and extend it trivially to $\tilde{\rho} = \rho \oplus \mathbf{1}d$ on \mathbf{R}^7 . It is clear that $\tilde{\rho}$ commutes with the action above (since left and right multiplication commute). Hence, $\tilde{\rho}$ descends to an action on the orbit space X . An easy computation shows that the generic orbits of $\tilde{\rho}$ on S are two-dimensional, and that $SU(2)/X$ is diffeomorphic to a closed interval. The endpoints of the interval correspond to the two singular orbits:

$$S^3 = \{(x, 0) \in \mathbf{R}^4 \times \mathbf{R}^3: \|x\| = 1\} \quad \text{and} \quad S^2 = \{(0, y) \in \mathbf{R}^4 \times \mathbf{R}^3: \|y\| = 1\}.$$

The family of mappings $i_\alpha: S^3 \rightarrow S^6$, $0 \leq \alpha \leq 1$, represents a curve in the orbit space between the two singular orbits, and therefore all the isometry classes of orbits are represented by this family. Hence, to find an orbit of maximal volume in S^6 we need only find an orbit of maximal volume in this family.

Let $v(\alpha)$ denote the volume of $i_\alpha(S^3)$. Then a computation similar to the one in § 6 or a direct interpretation of formula (6.6) shows that

$$v(\alpha) = 2\pi^2\alpha(4 - 3\alpha^2).$$

This function reaches its maximum in $[0, 1]$ at $\alpha = 2/3$. We conclude that

$$i_{2/3}(S^3) = \left\{ \left(\frac{2}{3}x, \frac{\sqrt{5}}{3}\eta(x) \right) \in \mathbf{R}^4 \times \mathbf{R}^3: \|x\| = 1 \right\}$$

is a minimal submanifold of S^6 .

It follows that the cone over this submanifold is a minimal variety in \mathbf{R}^7 . Therefore, we have proved

THEOREM 7.1. *The Lipschitz function $f: \mathbf{R}^4 \rightarrow \mathbf{R}^3$ given by*

$$f(x) = \frac{\sqrt{5}}{2} \|x\| \eta \left(\frac{x}{\|x\|} \right) \quad \text{for } x \neq 0,$$

where η is the Hopf map (6.5), is a solution to the minimal surface system (2.2).

In particular, this shows that there exist Lipschitz solutions to (2.2) which are not of class C^1 .

Note. This theorem has been verified by direct computation.⁽¹⁾

Analogous examples can be constructed using the Hopf maps

$$\eta': S^7 \rightarrow S^4 \quad \text{and} \quad \eta'': S^{15} \rightarrow S^8.$$

The corresponding volume functions are

$$v'(\alpha) = \frac{\pi^4}{3} \alpha^3 (4 - 3\alpha^2)^2 \quad \text{and} \quad v''(\alpha) = \frac{2\pi^8}{7!} \alpha^7 (4 - 3\alpha^2)^4.$$

Observe that these submanifolds are all examples of compact minimal varieties in the sphere whose normal planes are at a constant acute angle with respect to a fixed plane in \mathbf{R}^N . By a result of de Giorgi this is not possible in codimension one unless the submanifold is a totally geodesic subsphere. A result of Simons [36], refined by Reilly [34], states that if M is a compact minimal submanifold of codimension- k in S^{N-1} with normal plane field ν satisfying

$$\langle \nu, \nu_0 \rangle > \sqrt{\frac{2k-2}{3k-2}}$$

for some fixed k -plane ν_0 in \mathbf{R}^N , then M is a totally geodesic subsphere. The examples above show that there is a positive lower bound for the "best" constant possible in this theorem. In fact, for the Hopf maps η , η' , and η'' above, there are planes ν_0 such that $\langle \nu, \nu_0 \rangle$ is constant and equal to $1/9$, $(1/8\sqrt{7})$ and $(7^4 \cdot 2^{-11} \cdot 3^{-5} \sqrt{7/5})$ respectively.

Remark. It follows from the work of J. L. M. Barbosa ("An extrinsic rigidity theorem for minimal immersions from S^2 into S^n ," to appear) that there are no non-parametric

⁽¹⁾ We should like to thank the MACSYMA program at M.I.T. and Bill Gosper of the Stanford University Artificial Intelligence Laboratory.

minimal cones of dimension three or less. Moreover, using this fact one can prove that every Lipschitz solution of the minimal surface system in three (or fewer) independent variables is real analytic. Of course, by the examples above such a statement is false for the system in four or more variables.

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