H^p SPACES OF SEVERAL VARIABLES

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Introduction

The classical theory of H^p spaces could be considered as a chapter of complex function theory—although a fundamental one, with many intimate connections to Fourier analysis.(1) From our present-day perspective we can see that its heavy dependence on such special tools as Blaschke products, conformal mappings, etc. was not an insurmountable obstacle barring its extension in several directions. Thus the more recent n-dimensional theory (begun in [24], but with many roots in earlier work) succeeded in some measure

(1) See Zygmund [28], Chapter III in particular.

because it was able to exploit and generalize in a decisive way the circle of ideas centering around conjugate harmonic functions, harmonic majorization, etc. (1)

The purpose of the present paper (2) is to develop a new viewpoint about the H^p spaces which pushes these ideas even further into the background, but which brings to light the real variable meaning of H^p. Besides the substantial clarification that this offers, it allows us to resolve several questions that could not be attacked by other means. We shall now sketch the requisite background.

Background

We can isolate three main ideas that have made their appearance in the last several years and which can be said to be at the root of the present development. We list them in order of occurrence.

(i). The realization that the results of boundedness of certain singular integral operators could be extended from the L^p spaces, $1<p<\infty$, to the H^p spaces, $p\leqslant 1$. (3) But those results had two draw-backs, first, an esthetic one, since the proofs often depended on more complicated auxilliary functions, (e.g. the g_λ^* functions and the H^p inequalities for the S function of Calderón [3] and Segovia [16]) which in reality did not help to clarify matters. A more basic objection was that with those methods, operators such as the more strongly singular integrals (corresponding to $\theta>0$, in the definition in § 1), could not be treated at all.

(ii). The theorem of Burkholder, Gundy and Silverstein [2], that in the classical situation of an analytic function $F=u+iv$, the property $F\in H^p$, $0<p<\infty$, is equivalent with the non-tangential max.function of u belonging to L^p. This striking theorem (proved, incidentally, by Brownian motion) raised, however, many questions. Explicitly: how could the results be extended to n-dimensions; and implicitly: what was the rôle, fundamental or merely incidental, of the Poisson kernel in these matters?

(iii). The third idea (which is part of the subject matter of the present paper) is the identification of the dual of H^1 with the space BMO, the space of functions of bounded mean oscillation. The latter space had previously been introduced in a different context by John and Nirenberg [13]; it had since been noted that in several instances BMO served as a substitute for L^∞. The duality ties together these facts. But more significantly, it leads to new ways of approaching various problems about H^1, and sometimes also gives us hints about possible extensions to H^p, $p<1$.

(1) Compare [25, Chapter VI]. Among other significant generalizations of the classical theory are H^p spaces of several complex variables, and analogues in the context of Banach algebras.

(2) Some of the results of the present paper were announced in two seperate notes [10] and [22].

(3) See [21, Chapter VII].

Main results

Our results are of two kinds: those valid for all H^p, $0<p<\infty$, and sharper and more far-reaching ones for H^1. We sketch first the results for H^1.

Our first main result is that the dual of H^1 is BMO. At the same time that we prove this we also give several equivalent characterizations of BMO, among which is one in terms of Poisson integrals (Theorems 2 and 3 in § 2) as a rather immediate consequence of the duality we obtain boundedness on H^1 of a variety of singular integral transformations, and of certain maximal functions. The duality, in fact, allows us to obtain these results under "sharp" hypotheses.

Further applications require the introduction of another idea, namely the function $f^\#$. It is defined by

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy.$$

Observe that $f\in\text{BMO}$ is equivalent with $f^\#\in L^\infty$. The result we prove is that $f^\#\in L^p$ implies $f\in L^p$, if $p<\infty$; this may be viewed as the inverse of the corresponding inequality for the maximal function. Its significance for us is that it provides the link between BMO and L^p, and so allows us to interpolate in the complex sense between H^1 and L^p. The resulting interpolation theorem (Corollaries 1 and 2 in § 5) is the key step in the new estimates we make for L^p multipliers. Examples of our results are as follows. First, if σ is the uniform measure distributed on the unit sphere of $\mathbf{R}^n$, then the operator

$$f \to \left(\frac{\partial}{\partial x}\right)^\alpha \sigma * f$$

is bounded on L^p, whenever $\left|\frac{1}{2}-\frac{1}{p}\right| \leqslant \frac{1}{2} - \frac{|\alpha|}{n-1}$.

Secondly, we also obtain sharp L^p estimates, $1<p<\infty$, for multipliers of the form $m(\xi)=\exp(i|\xi|^a)/|\xi|^\beta$, $|\xi|$ large, where $0<a<1$, $\beta>0$, with m locally smooth.

The results we have sketched for H^1 make up parts II and III of our paper. In parts IV and V we carry out the general theory, valid for H^p, $0<p<\infty$.

We say, by definition, that a harmonic function $u(x,t)$ on $\mathbf{R}^{n+1}_+$ is in H^p if it and a requisite number of conjugates satisfy an appropriate boundedness condition on L^p. (See the definition in § 8). For such harmonic functions we may speak of their boundary values, $\lim_{t\to 0} u(\cdot,t)=f(\cdot)$, taken in the sense of distributions. Then $u(\cdot,t)=P_t * f$. Conversely, for tempered distributions f on $\mathbf{R}^n$, we may ask when these arise as H^p boundary values. The answer is contained in the equivalence of the following four properties:

(1) $f = \lim_{t\to 0} u(\cdot, t)$, $u \in H^p$.

(2) $\sup_{t>0} |(f * \varphi_t)(x)| \in L^p$, with $\varphi_t(x) = t^{-n}\varphi\left(\frac{x}{t}\right)$, for all sufficiently "regular" φ (say $\varphi \in \mathcal{S}$).

(3) That (2) holds for *one* such φ, with $\int_{\mathbf{R}^n} \varphi\, dx \neq 0$.

(4) $\sup_{|x-y|<t} |u(y,t)| \in L^p$.

The equivalence of (1) and (4) is the generalization of the Burkholder–Gundy–Silverstein theorem, which we prove by means of the corresponding results for the Lusin S-function. But also of capital importance is the equivalence of these two with (2) and (3), which gives the real-variable interpretation of the classes H^p. It shows that the H^p spaces are utterly intrinsic and arise as soon as we ask simple questions about regularizing distributions with approximate identities. There is thus no need, when formulating certain basic properties, to have recourse to analytic functions, conjugate harmonic functions, Poisson integrals, etc.

That these ideas can be applied to Fourier analysis on H^p spaces and in particular to singular integrals, can be understood as follows. In making the usual L^1 estimates (for singular integral operators) it is, in effect, the Hardy-Littlewood maximal function which is controlling, and is at the bottom of the weak-type estimates that occur in this context. However because of property (2) on H^p, appropriate substitutes of the maximal function are bounded in L^p, and this leads to H^p results for various operators. In brief: our equivalences (1) to (4) make it almost routine to carry over the main ideas of the usual Calderón–Zygmund techniques to H^p.

Further remarks

To some extent the different sections of this paper may be read independently. For instance, the reader mainly interested in the duality of H^1 with BMO, and its applications to L^p multipliers need only look at parts II and III; while for the reader principally interested in the generalities about H^p, parts IV and V would suffice. On the other hand, anyone who wants to understand the various interrelations that exist and the larger picture we are sketching, must resign himself to read the whole paper.

Our results suggest the possibility of several generalizations and also raise a variety of problems. Some examples are:

(1) It seems likely that the theory given here goes over to any smooth compact manifold, in place of $\mathbf{R}^n$. In particular it is indicated that Theorem 11 should have a complete analogue in that context, and thus lead to intrinsically defined H^p classes.

(2) Another problem is the extension of these results to products of half-spaces, and more particularly to the H^p theory of polydiscs or tube domains over octants.[1]

(3) We may ask what is the meaning of the conjugacy conditions (in § 8), depending on p, in the definition of H^p, particularly in view of the fact that the other characterizations mentioned above do not vary with p. For instance, are the exponents p_k the best possible, as for as the definition of H^p is concerned?

We have obtained partial results for the first two of these problems, and hope to return to these matters at another time.

With great pleasure we thank A. Zygmund who sparked the research of parts II and III by his incisive questions about BMO. We owe a real debt to D. Burkholder and R. Gundy for several illuminating discussions on H^p spaces and for many helpful ideas contained in their earlier work. Finally, we thank R. Wheeden for several useful observations concerning the sawtooth region in section 7.

II. Duality of H^1 and BMO

1. Functions of bounded mean oscillation: preliminaries

Let f be a locally integrable function on $\mathbf{R}^n$. Then f is of *bounded mean oscillation* (abbreviated as BMO) if

$$\sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx = \|f\|_* < \infty \tag{1.1}$$

where the supremum ranges over all finite cubes Q in $\mathbf{R}^n$, $|Q|$ is the Lebesgue measure of Q, and f_Q denote the mean value of f over Q, namely $f_Q = (1/|Q| \int_Q f(x)dx)$. The class of functions of bounded mean oscillation, modulo constants, is a Banach space with the norm $\|\cdot\|_*$ defined above.

We note first that a consequence of (1.1) is the seemingly stronger condition

$$\sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|^2\,dx \leqslant A\,\|f\|_*^2 < \infty \tag{1.1'}$$

which is itself an immediate corollary of an inequality of John and Nirenberg about functions of bounded mean oscillation. Their inequality is as follows (see [13])

$$|\{x\in Q: |f(x)-f_Q|>\alpha\}| \leqslant e^{-c\alpha/\|f\|_*}|Q|, \quad \text{for every } \alpha>0.$$

We observe next that if f is BMO then $\int_{\mathbf{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}\,dx < \infty$, and more precisely

(1) For H^p theory in this context see [28], Chapter 17, and [25], Chapter III.

$$\int_{\mathbf{R}^n} \frac{|f(x) - f_Q|}{1+|x|^{n+1}}\, dx \leqslant A\,\|f\|_*, \tag{1.2}$$

where Q is the cube whose sides have length 1, and is centered at the origin. (¹)

Let us prove (1.2). Let Q_{2^k} be the cube with the same center as Q but whose sides have common length 2^k. Then of course $|\int_{Q_{2^{k-1}}}[f(x)-f_{Q_{2^k}}]dx| \leqslant \int_{Q_{2^k}}|f(x)-f_{Q_{2^k}}|dx \leqslant 2^{nk}\|f\|_*$, and therefore $|f_{Q_{2^{k-1}}}-f_{Q_{2^k}}| \leqslant 2^n\|f\|_*$. Adding these inequalities gives $|f_{Q_{2^k}}-f_Q| \leqslant 2^n k\|f\|_*$, and finally

$$\int_{Q_{2^k}} |f(x)-f_Q|\, dx \leqslant 2^{nk}[1+2^n k]\,\|f\|_*.$$

A last addition in k then gives (1.2). Now the mapping $f(x)\to f(\delta^{-1}x)$, $\delta>0$, is clearly a Banach space isometry of BMO to itself. Thus, by making the indicated change of variables (1.2) leads to the following extension of itself

$$\int_{\mathbf{R}^n} \frac{|f(x)-f_{Q_\delta}|\,dx}{\delta^{n+1}+|x|^{n+1}} \leqslant \frac{A}{\delta}\,\|f\|_*, \quad \delta>0 \tag{1.2'}$$

where Q_δ is the cube whose sides have length δ, and is centered at 0.

We show next that the class BMO arises as the image of L^∞ under a variety of "singular integral" transformations. Let K be an integrable function on $\mathbf{R}^n$, and suppose θ is a fixed parameter with $0\leqslant\theta<1$. If $\theta=0$ we shall assume that

$$\int_{|x|\geqslant 2|y|} |K(x-y)-K(x)|\,dx \leqslant B, \quad \text{all } y \neq 0$$

and
$$|\hat{K}(\xi)| \leqslant B.$$

When $0<\theta<1$, we assume K vanishes when $|x|\geqslant 1$, and if $|y|<1$,

$$\int_{|x|\geqslant 2|y|^{1-\theta}} |K(x-y)-K(x)|\,dx \leqslant B; \text{ also } |\hat{K}(\xi)| \leqslant B(1+|\xi|)^{-n\theta/2}.$$

THEOREM 1. *The mapping $f\to T_0(f)=K*f$ is bounded from L^∞ to* BMO, *with a bound that can be taken to depend only on B (and not the L^1 norm of K).*

This theorem is known in the case $\theta=0$; (see [17], and [20]). We shall assume therefore that $0<\theta<1$. The latter class of transformation corresponds to the "weakly-strongly" singular integrals of [9].

(¹) Here and below the constant A may vary from inequality to inequality. A is always independent of the function f, the cube Q, etc. but may depend on the dimension n or other explicitly indicated parameters.

Let Q be any cube of diameter δ, which for simplicity we may assume is centered at the origin. Take first $\delta \leqslant 1$. Write $f = f_1 + f_2$, where $f_1 = f$ in the ball $|x| \leqslant 2\delta^{1-\theta}$, and $f_2 = f$ when $|x| > 2\delta^{1-\theta}$. We write also $u = T_0(f)$, and $u_j = T_0(f_j)$, $j = 1, 2$. In terms of the Fourier transform $\hat{u}_1(\xi) = \hat{K}(\xi)\hat{f}_1(\xi) = |\xi|^{n\theta/2}|\hat{K}(\xi)|\xi|^{-n\theta/2}\hat{f}_1(\xi)$. According to our assumptions, the factor $|\xi|^{n\theta/2}\hat{K}(\xi)$ is bounded by B, while by the Hardy–Littlewood–Sobolev theorem of fractional integration $|\xi|^{-n\theta/2}\hat{K}(\xi)|\xi|^{n\theta/2}\hat{f}_1(\xi)$ is the Fourier transform of an L^p function whose L^p norm does not exceed $A\|\hat{K}(\xi)|\xi|^{n\theta/2}\hat{f}_1(\xi)\|_2 \leqslant AB\|\hat{f}_1(\xi)\|_2 = AB\|f_1\|_2$, with $1/p = 1/2 - \theta/2$; see e.g. [21, 116–120]. Thus $\int_Q |u_1|^p dx \leqslant \int_{\mathbf{R}^n} |u_1|^p dx \leqslant A^p B^p \|f_1\|_2^p \leqslant A^p B^p \delta^{n(1-\theta)p/2} \|f\|_\infty^p$.

Summarizing, we get

$$\frac{1}{|Q|}\int_Q |u_1(x)|\,dx \leqslant AB\|f\|_\infty. \tag{1.3}$$

Now let $a_Q = \int K(-y) f_2(y)\,dy$. Since $u_2(x) - a_Q = \int [K(x-y) - K(y)] f_2(y)\,dy$, if $|x| \leqslant \delta$, (which is certainly the case if $x \in Q$), then $|u_2(x) - a_Q| \leqslant \int_{|x| \geqslant 2|y|^{1-\theta}} |K(x-y) - K(-y)|\,dy$ $\|f\|_\infty \leqslant B\|f\|_\infty$. When this is combined with (1.3) it gives

$$\frac{1}{|Q|}\int_Q |u(x) - a_Q|\,dx \leqslant A'B\|f\|_\infty \tag{1.4}$$

from which it follows immediately that

$$\frac{1}{|Q|}\int_Q |u(x) - u_Q|\,dx \leqslant 2A'B\|f\|_\infty. \tag{1.5}$$

This disposes of the case when the diameter δ of Q is not greater than 1. Suppose now $\delta > 1$. Let c be a positive constant, sufficiently small so that $c\delta^{1-\theta} \leqslant \delta + 1$, for all $\delta \geqslant 1$. Let $f_1(x) = f(x)$ if $|x| \leqslant c\delta^{1-\theta}$, and $f_2(x) = f(x)$, if $|x| > c\delta^{1-\theta}$. Then for u_1 we get, as before, the estimate (1.3). However when $|x| \leqslant \delta$, $u_2(x) = \int_{|y|<1} K(y) f_2(x-y)\,dy = 0$, since then $x - y$ ranges outside the support of f_2. This leads, as above to (1.4) (with $a_Q = 0$) and hence to (1.5). The proof of the theorem is therefore complete.(1)

The theorem we have just proved will be extended below to show that the operator T_0, in effect, maps BMO to itself.

We show now by an example how Theorem 1 can be applied to the standard singular integrals (other applications are below). Let $K(x) = \Omega(x)/|x|^n = c_n x_j/|x|^{n+1}$, $j = 1, \ldots, n$, be the kernels of the Riesz transforms. For every $\varepsilon > 0$ consider their truncation K_ε, defined by $K_\varepsilon(x) = K(x)$ if $|x| > \varepsilon$, and $K_\varepsilon(x) = 0$, if $|x| \leqslant \varepsilon$. If f is a bounded function define $R_j(f)$ by

(1) An extension of this result to a general class of pseudo-differential operators has been obtained by one of us in [11].

$$R_j(f)(x) = \lim_{\varepsilon \to 0} \int_{\mathbf{R}^n} [K_\varepsilon(x-y) - K_1(-y)] f(y)\, dy = u(x). \tag{1.6}$$

We observe that for each fixed ε and x the integral converges absolutely; it is also well known that the limit exists almost everywhere in x and, say, in the L^2 norm on each finite cube. We claim that with our assumption that $f \in L^\infty$ the $R_j(f)$ belong to BMO. To see this write $u_\varepsilon(x) = \int_{\mathbf{R}^n}[K_\varepsilon(x-y) - K_1(-y)]f(y)\,dy$, and $c_\varepsilon = \int_{\mathbf{R}^n}[K_\varepsilon(-y) - K_1(-y)]f\,dy$. Note also that $u_\varepsilon - u_N = K_{\varepsilon N} * f$, where $K_{\varepsilon N} = K_\varepsilon - K_N$, and the integrable kernels $K_{\varepsilon N}$ satisfy the conditions for Theorem 1 (with $\theta = 0$), uniformly in ε and N. Thus

$$\frac{1}{|Q|} \int_Q |u_\varepsilon(x) - u_N(x) - u_{\varepsilon Q} + u_{NQ}|\, dx \leqslant A\, \|f\|_\infty \tag{1.7}$$

with A independent of ε, N and Q. We remark that $u_N(x) - c_N = \int [K_N(x-y) - K_N(-y)] f(y)\, dy \to 0$ uniformly as $N \to \infty$, if $c_N = \int [K_N(-y) - K_1(-y)] f(y)\, dy$. Since in (1.7) we can replace u_N by $u_N - c_N$ without changing the inequality, we get, upon letting $N \to \infty$ and then $\varepsilon \to 0$

$$\frac{1}{|Q|} \int_Q |u(x) - u_Q|\, dx \leqslant A\, \|f\|_\infty,$$

which is our desired conclusion. We remark that everything we have said extends to the case when $K(x) = \Omega(x)/|x|^n$ is any Calderón–Zygmund kernel, i.e. where Ω is homogeneous of degree zero, satisfies a Dini condition, and has mean-value zero on the unit sphere.

2. Duality of H^1 and BMO

We shall be studying the equivalence of several definitions of the H^p spaces later. For the present, however, it will be useful to adopt the following definition when $p = 1$. H^1 consists of that class of L^1 functions f, so that there exists L^1 functions $f_1, f_2, \dots, f_n$ with the property that $\hat{f}_j(\xi) = (i\xi_j/|\xi|)\hat{f}(\xi)$. We write $f_j = R_j(f)$. To define the H^1 norm (see also [21, p. 221]) we set

$$\|f\|_{H^1} = \|f\|_1 + \sum_{j=1}^{n} \|R_j f\|_1.^{(1)}$$

It will be technically useful to use a certain dense subspace H_0^1 of H^1, ([21, p. 225]). If $f \in H_0^1$, then among other things, it is bounded and rapidly decreasing at infinity. In particular, this shows that $H^1 \cap L^2$ is dense in H^1. With these matters out of the way we come to one of our key results.

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(1) The definition of R_j we have used here differs from (1.6) by an additive constant. Since in the latter context we view the range of the R as BMO, the additive constant here does not lead to any real ambiguity.

THEOREM 2. *The dual of H^1 is* BMO, *in the following sense.*

(a) *Suppose $\varphi \in$ BMO. Then the linear functional $f \to \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx$, initially defined for $f \in H^1_0$, has a bounded extension to H^1.*

(b) *Conversely, every continuous linear functional on H^1 arises as in (a) with a unique element φ of* BMO.

The norm of φ as a linear functional on H^1 is equivalent with the BMO *norm.*

The proof of Theorem 2 requires certain other equivalent characterizations of the class BMO. In order to state them we observe that if φ is any function that satisfies $\int_{\mathbf{R}^n} |\varphi(x)|/(1+|x|^{n+1})\,dx < \infty$, then its Poisson integral $\varphi(x,t)$ is well defined as

$$\varphi(x,t) = \int_{\mathbf{R}^n} P_t(x-y)\,\varphi(y)\,dy, \ t>0, \qquad \text{where } P_t(x) = \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}.$$

THEOREM 3. *The following three conditions on φ are equivalent:*

(i) *φ is* BMO.

(ii) $\varphi = \varphi_0 + \sum_{j=1}^{n} R_j(\varphi_j)$, *where* $\varphi_0, \varphi_1, \ldots, \varphi_n \in L^\infty$

(iii) $\int_{\mathbf{R}^n} \frac{|\varphi(x)|\,dx}{1+|x|^{n+1}} < \infty$, *and* $\sup_{x^0 \in \mathbf{R}^n} \int_{T(x^0,h)} t\,|\nabla\varphi|^2\,dx\,dt \leqslant Ah^n$, $0<h<\infty$ *where* $|\nabla\varphi|^2 = \left|\frac{\partial\varphi}{\partial t}\right|^2 + \sum_{j=1}^{n}\left|\frac{\partial\varphi}{\partial x_j}\right|^2$, *and* $T(x^0,h) = \{(x,t): 0<t<h,\ |x-x^0|<h\}$, *with* $\varphi(x,t)$ *the Poisson integral of φ.*

The various implications in Theorems 2 and 3 will be proved in the order which we schematize as follows: $2 \Rightarrow 3(\text{ii}) \Rightarrow 3(\text{i}) \Rightarrow 3(\text{iii}) \Rightarrow 2$.

Let B be the Banach space which consists of the direct sum of $n+1$ copies of $L^1(\mathbf{R}^n)$. That is, $B = \{(f_0, f_1, \ldots, f_n),\ f_j \in L^1(\mathbf{R}^n)\}$. We define a norm on B by setting $\|(f_0, f_1, \ldots, f_n)\| = \sum_{j=0}^n \|f_j\|_1$. Let S be the subspace of B for which $f_j = R_j(f_0)$, $j=1, \ldots n$. Clearly S is a closed subspace of B, and the mapping $f_0 \to (f_0, R_1 f_0, \ldots, R_n f_0)$ is a Banach space isometry of H^1 to S. Any continuous linear functional on H^1 can be identified with a corresponding functional defined on S, and hence by the Hahn–Banach theorem, it extends to a continuous linear functional on B. Now $B = L^1 \oplus L^1 \ldots \oplus L^1$ and thus the dual to B is equivalent to $L^\infty \oplus L^\infty \ldots \oplus L^\infty$. Restricting attention to S (and hence H^1) we get the following conclusion. Suppose l is a continuous linear functional on H^1, then there exists $\varphi_0, \varphi_1 \ldots \varphi_n \in L^\infty$, so that

$$l(f) = \sum_{j=0}^{n} \int_{\mathbf{R}^n} f_j \varphi_j\,dx, \quad \text{where } f = f_0, \text{ and } f_j = R\ (f),\ j=1, \ldots n. \tag{2.1}$$

Now the "anti-Hermitian" character of the Riesz transforms gives us

$$\int_{\mathbf{R}^n} R_j(f)\,\varphi_j\,dx = -\int_{\mathbf{R}^n} f R_j(\varphi_j)\,dx, \quad \text{if } f\in H_0^1,\ \varphi_j \in L^\infty.$$

(This is obvious by the Fourier transform if both f_0 and φ_j are in L^2; the case where we assume that $\varphi_j \in L^\infty$ and $f \in H_0^1$ follows from this by a standard limiting argument whose details may be left to the reader).

Therefore

$$l(f) = \int_{\mathbf{R}^n} f\left\{\varphi_0 - \sum_{j=1}^{n} R_j(\varphi_j)\right\} dx.$$

Thus we have proved that when restricted to H_0^1, every continuous linear functional arises from φ which can be written as $\varphi = \varphi_0 - \sum_{j=1}^n R_j(\varphi_j)$, with $\varphi_0, \dots \varphi_n \in L^\infty$. This proves the implication $2 \Rightarrow 3$(ii). The implication 3(ii)$\Rightarrow$3(i) is immediate, since we have seen earlier that the Riesz transforms of an L^∞ function are BMO. We consider therefore next the implication 3(i)$\Rightarrow$3(iii).

Observe that we have already proved (see (1.2)) that if $\varphi \in \text{BMO}$ then $\int |\varphi(x)|/(1+|x|^{n+1})\,dx < \infty$. In proving the second inequality of (iii) let us assume that $x^0 = 0$. We let $Q_{4h} \subset \mathbf{R}^n$ be the cube whose sides have length $4h$, with center the origin. We write χ for the characteristic function of this cube, and $\tilde{\chi}$ for the characteristic function of the complement. With $\varphi_{Q_{4h}}$ denoting the mean-value of φ over Q_{4h}, we have

$$\varphi = \varphi_{Q_{4h}} + (\varphi - \varphi_{Q_{4h}})\,\chi + (\varphi - \varphi_{Q_{4h}})\,\tilde{\chi} = \varphi_1 + \varphi_2 + \varphi_3.$$

We also have $\varphi(x,t) = \varphi_1(x,t) + \varphi_2(x,t) + \varphi_3(x,t)$, for the corresponding Poisson integrals. In the integral with the gradient square, φ_1 contributes nothing since it is constant. Now

$$\int_{T(0,h)} t\,|\nabla\varphi_2|^2\,dx\,dt \leqslant \int_{\mathbf{R}_+^{n+1}} t\,|\nabla\varphi_2|^2\,dx\,dt = \|g(\varphi_2)\|_2^2 = \tfrac{1}{2}\,\|\varphi_2\|_2^2 = \tfrac{1}{2}\int_{Q_{4h}} |\varphi - \varphi_{Q_{4h}}|^2\,dx,$$

(by [21, p. 83]).

The last quantity does not exceed $Ah^n\,\|\varphi\|_*^2$, by (1.1′). Altogether then

$$\int_{T(0,h)} t\,|\nabla\varphi_2|^2\,dx\,dt \leqslant Ah^n\,\|\varphi\|_*^2.$$

However

$$|\nabla\varphi_3(x,t)| \leqslant \int |\nabla P_t(x-y)|\,|\varphi_3(y)|\,dy \leqslant A\int_{\complement Q_{4h}} \left[\frac{1}{t+|x-y|}\right]^{n+1} |\varphi(y) - \varphi_{Q_{4h}}|\,dy.$$

So if $(x,t) \in T(0,h)$ then

$$\left(\frac{1}{t+|x-y|}\right)^{n+1} \leqslant A\frac{1}{h^{n+1}+|y|^{n+1}},$$

when $y \in \complement Q_{4h}$, and therefore by (1.2′) $|\nabla\varphi_3(x,t)| \leqslant Ah^{-1}\|\varphi\|_*$.

From this it is obvious that $\int_{T(0,h)} t|\nabla\varphi_3(x,t)|^2 dx\,dt \leqslant Ah^n\|\varphi\|_*^2$, and a combination of the above remarks shows that

$$\int_{T(0,h)} t|\nabla\varphi(x,t)|^2\,dx\,dt \leqslant Ah^n\,\|\varphi\|_*^2$$

which concludes the proof of the implication 3(i) ⇒ 3(iii).

The last implication, and the deepest, is that whenever φ satisfies the condition 3(iii), then it gives rise to a continuous linear functional $f \to \int f\varphi\,dx$ on H^1.

To see this we begin by showing that for appropriate $f \in H^1$

$$\left|\int_{\mathbf{R}_+^{n+1}} t(\nabla f(x,t))\,(\nabla\varphi(x,t))\,dx\,dt\right| \leqslant A\,\|f\|_{H^1}. \tag{2.2}$$

The f we deal with have the following properties: there exists $F=(u_0, u_1, ..., u_n)$, so that the function $u_j(x,t)$ satisfy Cauchy–Riemann equations in $\mathbf{R}_+^{n+1}$; F is continuous and rapidly decreasing at infinity in $\overline{\mathbf{R}}_+^{n+1}$; $|F|>0$ and $\Delta|F| = O(|x|+t+1)^{-n-\delta}$ in $\mathbf{R}_+^{n+1}$; and finally $u_0(x,0)=f(x)$. The fact that $f\in H^1$ means then that $\int_{\mathbf{R}^n}\sup_{t>0}|F(x,t)|\,dx \leqslant A\|f\|_{H^1}$. By a simple limiting argument it suffices to prove the inequalities for such f (see [21, 225–227]).

Now the quantity on the left of (2.2) is clearly majorized by

$$\int_{\mathbf{R}_+^{n+1}} t|\nabla f(x,t)|\,|\nabla\varphi(x,t)|\,dx\,dt \leqslant \int_{\mathbf{R}_+^{n+1}} t|\nabla F(x,t)|\,|\nabla\varphi(x,t)|\,dx\,dt$$
$$\leqslant \left(\int_{\mathbf{R}_+^{n+1}} t|\nabla\varphi(x,t)|^2|F(x,t)|\,dx\,dt\right)^{\frac12}\left(\int_{\mathbf{R}_+^{n+1}} t|F(x,t)|^{-1}|\nabla F(x,t)|^2\,dx\,dt\right)^{\frac12}.$$

Let $q=(n-1)/n$, $g(x)=|F(x,0)|^q$, and $g(x,t)$ the Poission integral of g. Then $|F(x,t)| \leqslant (g(x,t))^p$, with $p=1/q>1$, $g\in L^p(\mathbf{R}^n)$, and $\|g\|_p^p = \int_{\mathbf{R}^n}|F(x,0)|\,dx \leqslant A\|f\|_{H^1}$; (see [21, 222–223]). From this we get that

$$\int_{\mathbf{R}_+^{n+1}} t|\nabla\varphi(x,t)|^2|F(x,t)|\,\mathrm{d}x\,dt \leqslant \int_{\mathbf{R}_+^{n+1}} t|\nabla\varphi(x,t)|^2(g(x,t))^p\,dx\,dt. \tag{2.3}$$

The last integral can be estimated by observing that the condition

$$\sup_{x^0}\int_{T(x^0,h)} t|\nabla\varphi|^2\,dx\,dt \leqslant Ah^n,$$

is precisely the property that the measure $t|\nabla\varphi|^2 dx\,dt$ on $\mathbf{R}^{n+1}_+$ satisfies the hypothesis of an inequality of Carleson (for which see [15] and [21, p. 236]). The result is that the second integral in (2.3) is majorized by $A\|g\|_p^p \leqslant A'\|f\|_{H^1}$.

Next we invoke the inequality $|F|^{-1}|\nabla F|^2 \leqslant (n+1)\,\Delta(|F|)$ ([21, p. 217]) and hence

$$\int_{\mathbf{R}^{n+1}_+} t|F|^{-1}|\nabla F|^2 dx\,dt \leqslant (n+1)\int_{\mathbf{R}^{n+1}_+} t\Delta(|F(x,t)|)\,dx\,dt = (n+1)\int_{\mathbf{R}^n} |F(x,0)|\,dx \leqslant A\,\|f\|_{H^1}.$$

The next-to-the-last inequality holds by a simple argument involving Green's theorem.

Thus a combination of the last few inequalities proves (2.2). To conclude the proof of the implication $3(\mathrm{iii}) \Rightarrow 2$ we observe that

$$\int_{\mathbf{R}^n} f(x)\,\varphi(x)\,dx = 2\int_{\mathbf{R}^{n+1}_+} t(\nabla f(x,t))\,(\nabla\varphi(x,t))\,dx\,dt$$

whenever, say, both f and φ are in $L^2(R^n)$, (see [21, 83, 85]). The extension of this identity to the case when $f \in H^1_0$, and $\varphi \in \mathrm{BMO}$ is then routine. This shows that

$$\left|\int_{\mathbf{R}^n} f\varphi\,dx\right| \leqslant A\,\|f\|_{H^1}, \quad \text{whenever } f \in H^1_0,$$

and hence $f \to \int f\varphi\,dx$ extends to a continuous linear functional on H^1. Since the series of implications $2 \Rightarrow 3(\mathrm{ii}) \Rightarrow 3(\mathrm{i}) \Rightarrow 3(\mathrm{iii}) \Rightarrow 2$ have all been proved, the fact that the norm of φ as a linear functional on H^1 is equivalent with its BMO norm follows either by a priori grounds (the closed graph theorem), or when one keeps track of the various constants that arise in the proof just given. QED

Remarks. (a) We sketch an alternate proof of the main step $(3\mathrm{iii}) \Rightarrow \varphi \in (H^1)^*$ above. Given $u(x,t)$ harmonic and $h \in [0,\infty]$, define $S_h(u)(x) = (\iint_{|x-y|<t<h} t^{1-n}|\nabla u(y,t|^2 dy\,dt)^{\frac{1}{2}}$. This auxilliary function is intimately connected both with H^1 and BMO. For any $f \in H^1$ with Poisson integral $f(x,t)$, we know that $\|S_\infty(f)(x)\|_1 \leqslant C\|f\|_{H^1}$ (see Calderón [3], Segovia [16] and Theorem 9 Corollary 1 below). In addition, any φ whose Poisson integral $\varphi(x,t)$ satisfies (3iii) must also satisfy $\int_{B(y,h)} (S_h(\varphi)(x))^2 dx \leqslant Ch^n$ for all y, h, which implies:

Let $h(x) = \sup\,\{h \geqslant O\,|\,S_h(\varphi)\,(x) \leqslant 1\,000\,\mathrm{C}\}$. (Thus, $S_{h(x)}(\varphi)\,(x) \leqslant 1\,000\,\mathrm{C}$ automatically). (2.4)

Then $|\{x \in B(y,h)\,|\,h(x) > h\}| \geqslant ch^n$.

Now to show that $\varphi \in (H^1)^*$, we write

$$\left|\int_{\mathbf{R}^n} f(x)\,\varphi(x)\,dx\right| \leqslant 2\iint_{\mathbf{R}^{n+1}_+} t\,|\nabla f(y,t)|\,|\nabla\varphi(y,t)|\,dy\,dt$$

$$\leqslant C\int_{\mathbf{R}^n}\iint_{|x-y|<t<h(x)} t^{1-n}\,|\nabla f(y,t)|\,|\nabla\varphi(y,t)|\,dy\,dt\,dx$$

(as follows from (2.4) and a change in the order of the triple integral)

$$\leqslant C\int_{\mathbf{R}^n}\left(\iint_{|x-y|<t<h(x)} t^{1-n}\,|\nabla f(y,t)|^2\,dy\,dt\right)^{\frac{1}{2}}\left(\iint_{|x-y|<t<h(x)} t^{1-n}\,|\nabla\varphi(y,t)|^2\,dy\,dt\right)^{\frac{1}{2}}dx$$

$$\leqslant C\int_{\mathbf{R}^n} S_\infty(f)\,(x)\,S_{h(x)}\,(x)\,dx \leqslant C\,\|f\|_{H^1},$$

since $S_{h(x)}(\varphi)\,(x) \leqslant 1\,000\,\mathrm{C}$ and $\|S_\infty(f)\|_1 \leqslant C\,\|f\|_{H^1}$. The proof is complete.

b) Theorems 2 and 3 have natural analogues in martingale theory. Let $\mathfrak{F}_0\subseteq\mathfrak{F}_1\subseteq\mathfrak{F}_2\subseteq\dots$ $\dots\subseteq\mathfrak{F}_\infty$ be an increasing sequence of Borel fields. For each $\mathfrak{F}_\infty$-measurable function f, form the conditional expectation $f_n = E(f\,|\,\mathfrak{F}_n)$. We say that $f\in H^1$ if the maximal function $f^*(x) = \sup_{n\geqslant 0}|f_n(x)|$ belongs to L^1. Let BMO denote the space of f's for which

(α) $\|f_n - f_{n+1}\|_\infty \leqslant C$ for all n and

(β) $\|E(|f-f_n|\,|\,\mathfrak{F}_n)\|_\infty \leqslant C$ for all n.

Then $(H^1)^* = \mathrm{BMO}$. We shall not give the proof, but only remark that the reasoning in (a) above goes over to the martingale setting. A. Garsia and C. Herz have since discovered other proofs of the martingale version of $(H^1)^* = \mathrm{BMO}$.

3. Some applications to H^1

Our purpose here is to give those applications of the duality of H^1 and BMO which are rather straight-forward consequences of this relation. Further applications will be found in part III below.

COROLLARY 1. *Let $T_0(f) = K * f$ be as in Theorem 1. Then T_0 is a bounded operator of H^1 to itself, with a bound that can be taken to depend only on B.*

If we considered T_0 as bounded mapping from H^1 to L^1 only, the corresponding conclusion would follow immediately from Theorems 1 and 2. However the proof of the full assertion requires a further idea.

Let $\tilde{K}(x) = K(-x)$, and $\tilde{T}(f) = \tilde{K} * f$ be the "dual" to T_0. Then clearly

$$\int_{\mathbf{R}^n} T_0(f)\,\varphi\,dx = \int_{\mathbf{R}^n} f\tilde{T}(\varphi)\,dx \tag{3.1}$$

whenever $f \in L'$ and $\varphi \in L^\infty$. Assume now that $f \in H^1$, and let φ range over the unit ball of L^∞. It then follows from (3.1) and Theorem 2 that $\|T_0(f)\|_1 \leqslant A\|f\|_{H^1}\{\sup_{\|\varphi\|_\infty \leqslant 1}\|\tilde{T}(\varphi)\|_*\}$. However $\tilde{T}$ satisfies the same conditions as T, and thus by Theorem 1 $\|\tilde{T}(\varphi)\|_* \leqslant A\|\varphi\|_\infty$. Altogether then

$$\|T_0(f)\|_1 \leqslant A\|f\|_{H^1} \tag{3.2}$$

where A depends only on the bound B (and not the L^1 norm of K).

We now invoke the fact (which is trivial only when $n=1$), that the Riesz transforms are bounded on H^1. That is, we have the inequality

$$\|R_j f\|_{H^1} \leqslant A\|f\|_{H^1}, \quad j=1, \ldots, n, \; f \in H^1 \tag{3.3}$$

where $(R_j f)\hat{}\,(\xi) = (i\xi_j/|\xi|)\hat{f}(\xi)$; or equivalently

$$\|R_j R_k f\|_1 \leqslant A\left\{\|f\|_1 + \sum_{j=1}^{n}\|R_j(f)\|_1\right\}, \quad f \in H^1, \; 1 \leqslant j, k \leqslant n.$$

(See [21, p. 232]). (1)

Now $T_0 R_j = R_j T_0$, as bounded operators on H^1. (At this stage we use the trivial result that T_0 is bounded on H^1, but with a norm that may depend on the L^1 norm of K.) Combining (3.2) and (3.3) then gives

$$\|T_0(f)\|_{H^1} = \|T_0(f)\|_1 + \sum_{j=1}^{n}\|R_j T_0 f\|_1 \leqslant A\{\|f\|_{H^1} + \sum\|R_j f\|_{H^1}\} \leqslant A'\|f\|_{H^1},$$

with A' that depends only on B. The proof of the corollary is therefore complete.

The corollary allows us to prove the boundedness of singular integrals on H^1 in various new circumstances. For the singular integrals corresponding to $0<\theta<1$, no results were known previously concerning boundedness on H^1. In the case of the "standard" singular integrals (corresponding to $\theta=0$) where boundedness results were originally developed, the above technique leads to those results but with sharper conditions. We state two theorems of this kind.

First let $m(\xi)$ be a function satisfying $|m(\xi)| \leqslant B$, and

$$\sup_{0<R<\infty} R^{2|\alpha|-n}\int_{R<|\xi|<2R}|m^{(\alpha)}(\xi)|^2 d\xi \leqslant B, \quad \text{for } 0 \leqslant |\alpha| \leqslant k,$$

where k is the smallest integer $>n/2$. Then m is a multiplier on H^1. These conditions on m are exactly Hörmander's hypothesis for the Mihlin multiplier theorem. Secondly it can be shown that the singular integrals with the Calderón–Zygmund kernels $\Omega(x)/|x|^n$ (where Ω

(1) The last inequality could also be proved by appealing to (3.2).

is homogeneous of degree zero, satisfies Dini's condition and has mean-value zero) give bounded operators on H^1. Both these results, which could be proved from Corollary 1 by what are now rather standard arguments, sharpen the corresponding theorems in [21].

Let T be the dual of the operator $\tilde{T}$, where the latter is considered as an operator on H^1. Then T is an extension of T_0. From Corollary 1 (applied to $\tilde{K}$ in place of K) and Theorem 2 one then obtains immediately

COROLLARY 2. *T is a bounded operator from* BMO *to itself, with a bound that can be taken to depend only on B.*

We show next how one can obtain sharp results for maximal functions on H^1. Maximal functions of this kind, but studied by different methods, are at the heart of the "real-variable" theory of H^p described in part V. Let φ be a function on $\mathbf{R}^n$ which satisfies either condition (A) or (B) below; for that matter various other conditions of the same kind would also do.

(A) φ has compact support and φ satisfies a Dini condition, i.e. if

$$\omega(\delta)=\sup_{|x-y|\leqslant\delta}|\varphi(x)-\varphi(y)|, \quad \text{then} \int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta<\infty.$$

Alternatively,

(B) for some $\varepsilon>0$,

$$|\varphi(x)|\leqslant A(1+|x|)^{-n-\varepsilon} \quad\text{and}\quad |\varphi(x-y)-\varphi(x)|\leqslant\frac{A|y|^{\varepsilon}}{(1+|x|)^{n+2\varepsilon}}, \text{ for } 2|y|\leqslant|x|.$$

For such φ, let $\varphi_t(x)=t^{-n}\varphi(x/t)$, $t>0$.

THEOREM 4. *Suppose φ satisfies either* (A) *or* (B) *above. Then whenever $f\in H^1$*

$$\sup_{t>0}|(f*\varphi_t)(x)|\in L^1,$$

and

$$\int_{\mathbf{R}^n}|\sup_{t>0}(f*\varphi_t)(x)|\,dx\leqslant A\,\|f\|_{H^1}. \tag{3.4}$$

The same conclusion holds for the "non-tangential" version, that is, where $|\sup_{t>0}(f*\varphi)(x)|$ is replaced by

$$\sup_{|x-y|<\alpha t}|f*\varphi_t(y)|.$$

We shall prove the theorem for the case $\sup_{t>0}|(f*\varphi_t)(x)|$, under alternative (A). The other variants are proved in the same way, and the details may be left to the interested reader.

It is no loss of generality to assume that the support of φ is inside the unit ball. Let $x\to t(x)$ be any positive, measurable function on $\mathbf{R}^n$ which is bounded and bounded away from zero. It will suffice to show that

$$\int_{\mathbf{R}^n} |(f * \varphi_{t(x)})(x)|\, dx \leqslant A\, \|f\|_{H^1} \tag{3.4}$$

where the constant A above does not depend on the particular function $t(x)$ that is used. We dualize this inequality, and see by Theorem 3 that the problem is reduced to the following: consider the mapping

$$\Psi \to \int_{\mathbf{R}^n} (t(x))^{-n} \varphi\left(\frac{x-y}{t(x)}\right) \Psi(x)\, dx = \Phi(y).$$

Then this is a bounded mapping from L^∞ to BMO, with a bound independent of $t(x)$. The proof of this last assertion follows the same lines as the proof of Theorem 1. Fix a cube $Q=Q_h$ whose sides have length h and whose center is y^0, and let Q_{2h} be the cube with the same center as Q, whose sides have length $2h$. We shall estimate Φ in Q by writing $\Phi=\Phi_1+\Phi_2$, where Φ_j arises from Ψ_j, $\Psi=\Psi_1+\Psi_2$, $\Psi_1=\chi\Psi$, $\Psi_2=(1-\chi)\Psi$, and χ is the characteristic function of Q_{2h}. Now

$$\int_Q |\Phi_1(y)|\, dy \leqslant \int_{\mathbf{R}^n} |\Phi_1(y)|\, dy \leqslant \|\varphi\|_1 \|\Psi_1\|_\infty \leqslant A\,|Q|\, \|\Psi\|_\infty. \tag{3.5}$$

Write
$$a_Q = \int_{\complement Q_{2h}} (t(x))^{-n} \varphi\left(\frac{x-y^0}{t(x)}\right) \Psi(x)\, dx.$$

Then
$$\Phi_2(y) - a_Q = \int_{\complement Q_{2h}} (t(x))^{-n} \left[\varphi\left(\frac{x-y}{t(x)}\right) - \varphi\left(\frac{x-y^0}{t(x)}\right)\right] \Psi(x)\, dx$$

and in view of the support condition on φ we may assume either $t(x) \geqslant |x-y|$ or $t(x) \geqslant |x-y^0|$. Since if $y\in Q$ and $x \in \complement Q_{2h}$ then $|x-y| \approx |x-y^0|$, we obtain that in either case $t(x) \geqslant c\,|x-y^0|$ for some small positive c. The Dini condition applied to φ therefore gives

$$|\Phi_2(y) - a_Q| \leqslant A \int_{|x-y^0|>c_1 h} |x-y^0|^{-n}\, \omega\left(\frac{y-y^0}{c\,|x-y^0|}\right) dx\, \|\Psi\|_\infty \leqslant A' \|\Psi\|_\infty,$$

as long as $y \in Q_h$. Combining this with (3.5) gives

$$\int_Q |\Phi(y) - a_Q|\, dy \leqslant A\,|Q|\, \|\Psi\|_\infty$$

where A is independent of Q or $t(x)$, which is the desired result.

We remark that the Dini type condition imposed on φ cannot be essentially relaxed Thus in one dimension the conclusion of Theorem 4 would be false if we took for φ the characteristic function of the unit interval. To see this consider the (limiting) case when $t(x)=|x|$; φ is the characteristic function of $[-1, 1]$. Let Ψ be the characteristic function of $[0, 1]$. Then

$$\Phi(y)=\int_{-\infty}^{\infty} t(x)^{-1}\varphi\left(\frac{x-y}{t(x)}\right)\Psi(x)\,dx=\int_0^1 x^{-1}\varphi\left(\frac{x-y}{x}\right)dx=\log\frac{2}{y}, \quad \text{if } 0<y<2,$$

and $\Phi(y)=0$ otherwise. But it is easy to see that Φ is not BMO. Therefore by the duality, the mapping $f\to(f*\varphi_{t(x)})(x)$ cannot be bounded from H^1 to L^1, and hence the property (3.4) does not hold for this φ.

III. Applications to L^p boundedness

4. The function $f^\#$

In order to apply the above duality to the boundedness in L^p of various operators, we shall need to introduce a device that mediates between BMO and the L^p spaces. This device is the function $f^\#$ defined as follows. Whenever f is locally integrable on $\mathbf{R}^n$ we set

$$f^\#(x)=\sup_{x\in Q}\left\{\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy\right\}. \tag{4.1}$$

Of course, $f\in\text{BMO}$ is identical with the statement $f^\#\in L^\infty$; the interest of $f^\#$ is the fact that $f^\#\in L^p$, $p<\infty$ implies $f\in L^p$. Define the maximal function Mf by

$$(Mf)(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

As is well known, if $f\in L^p$, then $Mf\in L^p$, and $\|Mf\|_p\leqslant A_p\|f\|_p$, when $1<p\leqslant\infty$. Obviously $Mf\geqslant|f|$. The precise statement concerning $f^\#$ is as follows.

THEOREM 5. *Suppose $f\in L^{p_0}(\mathbf{R}^n)$, for some p_0. Assume that $1<p<\infty$, $1\leqslant p_0\leqslant p$, and that $f^\#\in L^p(\mathbf{R}^n)$. Then $Mf\in L^p(\mathbf{R}^n)$, and we have the a priori inequality*

$$\|Mf\|_p\leqslant A_p\|f^\#\|_p. \tag{4.2}$$

Proof. We apply the Calderón–Zygmund lemma to $|f|$. For fixed $\alpha>0$, we divide $\mathbf{R}^n$ into a mesh of equal cubes so that $(1/|Q|)\int_Q|f|\,dx\leqslant\alpha$ for every cube in this mesh. (This can be done since $f\in L^{p_0}$, $p_0<\infty$.) Next by repeated bisection, obtain a disjoint family of cubes $\{Q_j^\alpha\}$, so that

$$\alpha < \frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha}|f|\,dx \leqslant 2^n\alpha \tag{4.3}$$

and $|f(x)| \leqslant \alpha$ if $x \notin \bigcup_j Q_j^\alpha$. (For details see Calderón and Zygmund [5].)

This decomposition can be carried out simultaneously for all values of α. It is then convenient to restrict attention to a fixed family of meshes—the "dyadic" ones. Note that if $\alpha_1 > \alpha_2$, the cubes in $\{Q_j^{\alpha_1}\}$ are then sub-cubes of the cubes in $\{Q_j^{\alpha_2}\}$. Let us denote by $\mu(\alpha) = \sum_j |Q_j^\alpha|$. The main estimate will be the inequality:

$$\mu(\alpha) \leqslant \left|\left\{x: f^\# > \frac{\alpha}{A}\right\}\right| + \frac{2}{A}\,\mu(2^{-n-1}\alpha) \tag{4.4}$$

which is to hold for all positive α and A.

Fix a cube $Q_{j_0}^{\alpha 2^{-n-1}} = Q_0$, and look at all the cubes $Q_j^\alpha \subset Q_{j_0}^{\alpha 2^{-n-1}}$. We divide consideration into two cases.

Case 1. $Q_0 \subset \{x: f^\# > \alpha/A\}$. Then trivially,

$$\sum_{Q_j^\alpha \subset Q_0} |Q_j^\alpha| \leqslant |\{x: f^\# > \alpha/A\} \cap Q_0|. \tag{4.5}$$

Case 2. $Q_0 \not\subset \{x: f^\# > \alpha/A\}$.

Then obviously
$$\frac{1}{|Q_0|}\int_{Q_0}|f(x) - f_{Q_0}|\,dx \leqslant \alpha/A. \tag{4.6}$$

However by (4.3) $|f_{Q_0}| \leqslant 2^n(2^{-n-1}\alpha) = \alpha/2$, and $|f|_{Q_j^\alpha} > \alpha$. Hence

$$\int_{Q_j^\alpha}|f(x) - f_{Q_0}|\,dx \geqslant \frac{\alpha}{2}|Q_j^\alpha|,$$

where Q_j^α is any cube $\subset Q_0$. Summing over all such cubes and comparing with (4.6) gives us

$$\sum_{Q_j^\alpha \subset Q_0} |Q_j^\alpha| \leqslant \left(\frac{2}{A}\right)|Q_0|. \tag{4.7}$$

Finally we sum over all the cubes Q_0 (these are the cubes in $\{Q_j^{\alpha 2^{-n-1}}\}$), taking into account the estimates for cases 1 and 2 in (4.5) and (4.7). This proves (4.4).

Let $\lambda(\alpha) = |\{x: Mf(x) > a\}|$ be the distribution function of the maximal function Mf. When we compare λ with μ we obtain two estimates

$$\begin{cases}\mu(\alpha) \leqslant \lambda(\alpha)\\ \lambda(\alpha) \leqslant c_1\mu(c_2\alpha), \text{ where } c_1 \text{ and } c_2 > 0.\end{cases} \tag{4.8}$$

Observe that $|Q_j^\alpha|^{-1}\int_{Q_j^\alpha}|f(x)|\,dx>\alpha$, so that $(Mf)(x)>\alpha$ whenever $x\in Q_j^\alpha$, and therefore $\{x: Mf(x)>\alpha\}\supset\bigcup_j Q_j^\alpha$, which gives the first inequality.

Next let $\tilde{Q}_j$ be the cube with the same centers as Q_j^α but expanded by a factor of 2. Suppose $x\notin\bigcup_j\tilde{Q}_j$, and let Q be any cube such that $x\in Q$. Consider

$$\int_Q |f(y)|\,dy=\int_{Q\cap\{\bigcup_j Q_j^\alpha\}}|f(y)|\,dy+\int_{Q\cap\{\complement\bigcup_j Q_j^\alpha\}}|f(y)|\,dy.$$

In the second integral on the right the value of $|f|$ does not exceed α. Thus this integral is majorized by $\alpha|Q|$. For the first integral we use the simple geometric observation: if $Q\cap Q_j^\alpha\neq 0$, and $Q\not\subset\tilde{Q}_j^\alpha$ (because $x\in Q$), then $Q_j^\alpha\subset\tilde{\tilde{Q}}$. Here $\tilde{\tilde{Q}}$ is the cube with the same center as Q but expanded by a factor of 4. Therefore for the first integral we have

$$\int_{Q\cap(\bigcup_j Q_j^\alpha)}|f(y)|\,dy\leqslant\sum_{Q_j^\alpha\subset\tilde{\tilde{Q}}}\int_{Q_j^\alpha}|f(y)|dy\leqslant\sum_{Q_j^\alpha\subset\tilde{\tilde{Q}}}2^n\alpha|Q_j^\alpha|\leqslant 2^n\cdot 4^n\alpha|Q|.$$

Altogether then, for such Q, $\int_Q|f(y)|\,dy\leqslant(1+2^n4^n)\,\alpha|Q|$; this gives $Mf(x)\leqslant(1+2^n4^n)\,\alpha$. Therefore $\{x: Mf(x)>(1+2^n4^n)\,\alpha\}\subset\bigcup_j\tilde{Q}_j^\alpha$, which means that $\lambda((1+2^n4^n)\,\alpha)\leqslant 2^n\mu(\alpha)$, proving the second inequality in (4.8).

Let us now consider the quantity I_N defined by $I_N=p\int_0^N\alpha^{p-1}\mu(\alpha)\,d\alpha$. By (4.8) $I_N\leqslant p\int_0^N\alpha^{p-1}\lambda(\alpha)\,d\alpha$, and we know that $p_0\int_0^\infty\alpha^{p_0-1}\lambda(\alpha)\,d\alpha=\|Mf\|_{p_0}^{p_0}<\infty$, if $1<p_0<\infty$ or else $\lambda(\alpha)=O(\alpha^{-1})$ if $p_0=1$. In either case, then, $I_N<\infty$, since $p_0\leqslant p$, and $1<p$, according to our hypotheses. Let us carry out the corresponding integration of both sides of (4.4) over the interval $0<\alpha<N$. We then get

$$I_N\leqslant p\int_0^\infty\left|\left\{x: f^\#>\frac{\alpha}{A}\right\}\right|\alpha^{p-1}\,d\alpha+\frac{2}{A}\,p\int_0^N\alpha^{p-1}\mu(2^{-n-1}\alpha)\,d\alpha.$$

Clearly $$p\int_0^N\alpha^{p-1}\mu(2^{-n-1}\alpha)\,d\alpha=p\cdot 2^{(n+1)p}\int_0^{N2^{-n-1}}\alpha^{p-1}\mu(\alpha)\,d\alpha\leqslant p\,2^{(n+1)p}\,I_N.$$

Hence, $$I_N\leqslant A^p\,\|f^\#\|_p^p+\frac{2\cdot 2^{(n+1)p}}{A}\,I_N.$$

Choose now $A=4\cdot 2^{(n+1)p}$. The resulting inequality is therefore $I_N\leqslant 2\cdot 4^p\cdot 2^{(n+1)p^2}\|f^\#\|_p^p$. When we let $N\to\infty$ we see that by (4.8) we have

$$\|M(f)\|_p^p=p\int_0^\infty\alpha^{p-1}\lambda(\alpha)\,d\alpha\leqslant c_1c_2^{-p}p\int_0^N\alpha^{p-1}\mu(\alpha)\,d\alpha=c_1c_2^{-p}\lim_{N\to\infty}I_N\leqslant c_1c_2^{-p}\cdot 2\cdot 4^p\cdot 2^{(n+1)p^2}\|f^\#\|_p^p.$$

Thus $Mf\in L^p$, and the inequality (4.2) is also proved, with $A_p\leqslant C^p$. In particular, note that A_p remains bounded as $p\to 1$.

5. Intermediate spaces

The properties of the function $f^{\#}$, together with the duality, allow us to determine the intermediate spaces between H^1 and L^p, and also between BMO and L^p, in terms of the complex method of interpolation. To be specific, let $[.,.]_\theta$ be the complex method of interpolation as described in Calderón [4]. Then we can state the following identities, with $1<p\leqslant\infty$, $p'^{-1}+p^{-1}=1$, $q'^{-1}+q^{-1}=1$, $0<\theta<1$, and $q^{-1}=1-\theta+\theta p^{-1}$:

$$[H^1, L^p]_\theta = L^q, \quad \text{and} \quad [\text{BMO}, L^{p'}]_\theta = L^{q'}. \tag{5.1}$$

In order not to get side-tracked in various details that are not relevent to the central subject of this paper, we shall not give a proof of this result here. Instead we shall formulate and prove two corollaries of it, which are of the form most useful for applications. In addition, these corollaries already contain the essential ideas of the general identity (5.1).

We shall deal with a mapping $z\to T_z$ from the closed strip $0\leqslant R(z)\leqslant 1$ to bounded operators on $L^2(\mathbf{R}^n)$. We shall assume that this mapping is analytic in the interior of the strip, and strongly continuous and uniformly bounded in the closed strip.

COROLLARY 1. *Suppose*

$$\sup_{-\infty<y<\infty} \|T_{iy}(f)\|_1 \leqslant M_0 \|f\|_{H^1}, \quad \textit{for } f\in L^2\cap H^1, \tag{5.2}$$

and

$$\sup_{-\infty<y<\infty} \|T_{1+iy}(f)\|_2 \leqslant M_1 \|f\|_2, \quad \textit{for } f\in L^2. \tag{5.3}$$

Then $\|T_t(f)\|_p\leqslant M_t\|f\|_p$, *for* $f\in L^2\cap L^p$ *whenever* $0<t<1$, $p^{-1}=1-\frac{1}{2}t$, *and* M_t *depends on* M_0, M, *and* t *only.*

COROLLARY 2. *Suppose*

$$\sup_{-\infty<y<\infty} \|T_{iy}(f)\|_* \leqslant M_0 \|f\|_\infty, \quad \textit{for } f\in L^2\cap L^\infty, \tag{5.2$'$}$$

and

$$\sup_{-\infty<y<\infty} \|T_{1+iy}(f)\|_2 \leqslant M_1 \|f\|_2, \quad \textit{for } f\in L^2. \tag{5.3$'$}$$

Then $\|T_t(f)\|_{p'}\leqslant M_t\|f\|_{p'}$, *for* $f\in L^2\cap L^{p'}$, *where* $p^{-1}+p'^{-1}=1$ *and with the notations of Corollary 1.*

Proof. Our proof of Corollary 1 will at the same time give a proof for Corollary 2. The proof of Corollary 2 will not, however, require the duality of H^1 and BMO. For each z in the strip let S_z denote the (Hilbert space) adjoint of $T_{\bar z}$. Thus

$$\int_{\mathbf{R}^n} T_z(f)\,\bar g\,dx = \int_{\mathbf{R}^n} f\,\overline{S_{\bar z}(g)}\,dx, \quad f,g\in L^2(\mathbf{R}^n). \tag{5.4}$$

Observe that as a result $z \to S_z$ is analytic in the interior, and strongly continuous and uniformly bounded in the closure of the strip. Moreover if $z \to T_z$ satisfies the conditions of the type (5.2) and (5.3), then $z \to S_z$ satisfies conditions of the type (5.2)′ and (5.3)′. Let us prove this for (5.2)′. By (5.4), whenever $g \in L^2 \cap L^\infty$, then the mapping $f \to \int_{\mathbf{R}^n} f S_{iy}(g)\,dx$ is the restriction to $L^2 \cap H^1$ of a bounded linear functional on H^1 whose norm does not exceed $M_0\|g\|_\infty$. Therefore by Theorem 2, $S_{iy}(g) \in \mathrm{BMO}$, and $\|S_{iy}(g)\|_* \leqslant \mathrm{A}M_0\|g\|_\infty$. This proves the condition of the type (5.2)′ for S_z, but with a possibly larger constant. (The increase in the size of the bound is immaterial in what follows.) The condition of type (5.3)′ follows immediatly from the self-duality of L^2.

We may therefore assume that S satisfies the conditions of Corollary 2, and we shall prove that, as a result, S satisfies the conclusions of that corollary. Write $F = S_z(f)$, and let $x \to Q(x)$ be any measurable function from $\mathbf{R}^n$ to cubes in $\mathbf{R}^n$ with the property that $x \in Q(x)$, and let $\eta(x, y)$ be any measurable function on $\mathbf{R}^n \times \mathbf{R}^n$ such that $|\eta(x, y)| = 1$. Define the operator $f \to U_z(f)$ by

$$U_z(f)(x) = \frac{1}{|Q(x)|}\int_{Q(x)} [F(y) - F_{Q(x)}]\,\eta(x,y)\,dy, \quad F = S_z(f).$$

Observe that $|U_z(f)(x)| \leqslant F^\#(x)$, and conversely $\sup |U_z(f)(x)| = F^\#(x)$, if the supremum is taken over all possible functions $x \to Q(x)$ and $(x, y) \to \eta(x, y)$ described above. Since $\|F^\#\|_2 \leqslant 2\|M(F)\|_2 \leqslant C\|F\|_2$, it is easy to see that the function $z \to \int_{\mathbf{R}^n} U_z(f) g\,dx$, is analytic in the strip $0 < R(z) < 1$, and continuous and bounded in the closure of that strip. Also

$$\|U_{iy}(f)\|_\infty \leqslant \|F^\#\|_\infty = \|F\|_* = \|S_{iy}(f)\|_* \leqslant AM_0\|f\|_\infty, \quad \text{if } f \in L^2 \cap L^\infty.$$

Similarly $\quad \|U_{1+iy}(f)\|_2 \leqslant \|F^\#\|_2 \leqslant C\|F\|_2 = C\|S_{1+iy}(f)\|_2 \leqslant CM_1\|f\|_2 \quad \text{for } f \in L^2.$

We may therefore apply to the analytic family of operators $z \to U_z$ a known interpolation theorem (see e.g. [25, chapter V]) and conclude that

$$\|U_t(f)\|_{p'} \leqslant (AM_0)^{1-t}(CM)^t\|f\|_{p'} \tag{5.5}$$

whenever $f \in L^2 \cap L^{p'}$, with $p'^{-1} = \frac{1}{2}t$, and $0 < t < 1$. Observe that bound $(AM_0)^{1-t}(CM_1)^t$ does not depend on the particular choice of the function $x \to Q(x)$ and $\eta(x, y)$. We take therefore the supremum over all such functions, and then (5.5) yields

$$\|F^\#\|_{p'} = \|(S_t(f))^\#\|_{p'} \leqslant (AM_0)^{1-t}(CM_1)^t\|f\|_{p'}, \quad f \in L^2 \cap L^{p'}.$$

If $f \in L^2$, of course $S_t(f) \in L^2$, and obviously $2 \leqslant p' < \infty$. We may then invoke Theorem 5 (see (4.2)), and conclude that

$$\|S_t(f)\|_{p'} \leq \|M[S_t(f)]\|_{p'} \leq A_{p'}\|(S_t(f))^{\#}\|_{p'} \leq A_{p'}(AM_0)^{1-t}(CM_1)^t\|f\|_{p'}$$

whenever $f \in L^2 \cap L^{p'}$. Thus the Corollary 2 is proved. Going back to T_z via the duality (5.4) also proves Corollary 1 (with $M_t = A_{p'}(AM_0)^{1-t}(CM_1)^t$).

Remarks. As is the case in theorems of this kind, estimates of the type (5.2) can be replaced by weaker bounds, such as $\|T_{iy}(f)\|_1 \leq M_0(y)\|f\|_{H^1}$, where $M_0(y) = O(e^{e^{a|y|}})$, $a < \pi/2$. Similarly for (5.3), (5.2)′, and (5.3′). In practice, however, all that is needed is the case where $M_i(y) = O(1+|y|)^N$, for some N. This case can be deduced directly from the corollaries as stated by taking $e^{z^2}T_z$ instead of T_z.

In the classical case ($n=1$) Corollary 1 had been known previously, even in a more extended form which applied to all H^p spaces; see Stein and Weiss [23], Zygmund [28, Chapter XII], and the earlier literature cited there. However, when $n=1$ one used complex methods (e.g. Blaschke products, etc.), and these of course are unavailable in the general context treated here.

6. L^p boundedness of certain convolution operators

We begin by dealing with operators of the type arising in Theorem 1 (in § 1). Here we shall assume that $0<\theta<1$, because there is no analogue of the result below when $\theta=0$. We shall also change the hypotheses slightly. K will be a distribution of compact support, which is integrable away from the origin. Its Fourier transform $\hat{K}$ is of course a function. We make the following assumptions

$$\begin{cases} \displaystyle\int_{|x|\geq 2|y|^{1-\theta}} |K(x-y)-K(x)|\,dx \leq B, \quad 0<|y|\leq 1 \\ |\hat{K}(\xi)| \leq B(1+|\xi|)^{-n\theta/2} \end{cases} \tag{6.1}$$

THEOREM 6. *Suppose K satisfies the above assumptions. Then $|\xi|^\gamma \hat{K}(\xi)$ is a bounded multiplier for $(L^p(\mathbf{R}^n), L^p(\mathbf{R}^n))$, if $|\frac{1}{2}-p^{-1}| \leq \frac{1}{2}-\gamma/n\theta$, $1<p<\infty$, and $\gamma \geq 0$.*

The main interest of the theorem is the case when $|\frac{1}{2}-p^{-1}| = \frac{1}{2}-\gamma/n\theta$; the full assertion is then a consequence of this, and anyway the result when $|\frac{1}{2}-p^{-1}| < \frac{1}{2}-\gamma/n\theta$ was known previously.

Let φ be a fixed C^∞ function on $\mathbf{R}^n$ with compact support and which is normalized i.e. $\int_{\mathbf{R}^n}\varphi\,dx = 1$. Let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $\varepsilon>0$, and write $K_\varepsilon = K * \varphi_\varepsilon$. It is not difficult to show (see [9]) that when $0<\varepsilon\leq 1$ the (C^∞) functions K_ε satisfy the condition (6.1) uniformly in ε, and of course have their support in a fixed bounded set. Moreover $\hat{K}_\varepsilon(\xi) = \hat{K}(\xi)\hat{\varphi}(\varepsilon\xi) \to \hat{K}(\xi)$, as $\varepsilon \to 0$. It is easy to see, then, that it suffices to prove the following: For each such ε,

$|\xi|^\gamma \hat{K}_\varepsilon(\xi)$ is a bounded multiplier on (L^p, L^p), (where $|\frac{1}{2}-p^{-1}|=\frac{1}{2}-\gamma/n\theta$), with a bound independent of ε.

For this purpose write T_z as the multiplier operator on L^2 defined by

$$(T_z f)\hat{\ }(\xi) = |\xi|^{n\theta z/2} \hat{K}_\varepsilon(\xi) \hat{f}(\xi),$$

whenever $f \in L^2(\mathbf{R}^n)$. In view of fact that $\hat{K}_\varepsilon(\xi) = O(|\xi|^{-n\theta/2})$, it follows that $z \to T_z$ is analytic in the strip $0 < R(z) < 1$, strongly continuous and uniformly bounded in the closure of that strip. Clearly, also

$$\sup_{-\infty<y<\infty} \|T_{1+iy} f\|_2 \leqslant A\|f\|_2 \tag{6.2}$$

where A does not depend on ε.

Next assume $f \in L^2 \cap H^1$. Then we can write

$$T_{iy} f = I_{iy}(K_\varepsilon * f)$$

where $(I_{iy}(F))\hat{\ }(\xi) = |\xi|^{in\theta y/2} \hat{F}(\xi)$.

By Corollary 1 (in § 3), the operators $f \to K_\varepsilon * f$ are bounded on H^1, with norms that are uniformly bounded in ε. Also by Corollary 1 the fractional integration operators of purely imaginary order, I_{iy}, are bounded on H^1 with norms that do not exceed $A(1+|y|)^{n+1}$. (This was known before). Thus

$$\|T_{iy}(f)\|_1 = \|I_{iy}(K_\varepsilon * f)\|_1 \leqslant A(1+|y|)^{n+1} \|K_\varepsilon * f\|_{H^1} \leqslant A(1+|y|)^{n+1} \|f\|_{H^1}. \tag{6.3}$$

Finally, consider $e^{z^2} T_z$ instead of T_z. The former then satisfies all the conditions of Corollary 1 (in § 5), where the bounds M_0 and M_1 are independent of ε. The corollary then gives the desired result when $1 < p < 2$, $p^{-1} - \frac{1}{2} = \frac{1}{2} - \gamma/n\theta$, where $p^{-1} = 1 - \frac{1}{2}t$, and $\gamma = n\theta t/2$. The corresponding statement for $\frac{1}{2} - p^{-1} = \frac{1}{2} - \gamma/n\theta$ is just the dual of the preceeding, and this concludes the proof of Theorem 6.

The argument we just gave clearly generalizes to yield also the proof of the following theorem of wider scope.

THEOREM 7. *Suppose m is a bounded multiplier on (H^1, H^1). Assume also that $|m(\xi)| \leqslant A|\xi|^{-\delta}$, $\delta > 0$. Then $|\xi|^\gamma m(\xi)$ is a bounded multiplier for (L^p, L^p) if $1 < p < \infty$ and $|\frac{1}{2} - p^{-1}| \leqslant \frac{1}{2} - \gamma/2\delta$, and $\gamma \geqslant 0$.*

We define the operator T_z by $T_z(f)\hat{\ }(\xi) = |\xi|^{\delta z} m(\xi) \hat{f}(\xi)$. The argument is then the same as for Theorem 6. An immediate corollary is

COROLLARY. *Suppose $d\mu$ is a finite Borel measure on $\mathbf{R}^n$ and assume its Fourier transform $\hat{\mu}(\xi)$ is $O(|\xi|^{-\delta})$, as $|\xi| \to \infty$, $\delta \geqslant 0$. Then $|\xi|^\gamma \hat{\mu}(\xi)$ is a multiplier for (L^p, L^p) if $1 \leqslant p \leqslant \infty$, $|\frac{1}{2} - p^{-1}| \leqslant \frac{1}{2} - \gamma/2\delta$, and $\gamma \geqslant 0$.*

Obviously $\|f * d\mu\|_{H^1} \leq \|d\mu\| \, \|f\|_{H^1}$, and $(f * d\mu)^\wedge = \hat{\mu}(\xi)\hat{f}(\xi)$, so $\hat{\mu}(\xi)$ is a multiplier on H^1, and the corollary follows from Theorem 7.

We shall now give some representative examples of the applications of Theorems 6 and 7.

Example 1. Let ψ be a fixed C^∞ function on $\mathbf{R}^n$ which vanishes near the origin, and is $=1$ for all sufficiently large ξ. Suppose $0 < a < 1$. Then

$$\psi(\xi)\, e^{i|\xi|^a} |\xi|^{-b} = m_{a,b}(\xi) \tag{6.4}$$

is a multiplier on (L^p, L^p), whenever $1 < p < \infty$, and

$$\left|\frac{1}{2} - \frac{1}{p}\right| \leq \left(\frac{b}{n}\right) \frac{(\frac{1}{2}n + \lambda)}{(b + \lambda)}, \quad \text{with } \lambda = \left(\frac{na}{2} - b\right) \Big/ (1-a), \ 0 \leq b < \frac{na}{2}. \tag{6.5}$$

In fact the analysis of Wainger [27, p. 41–53] shows that whenever $n + \lambda > 0$, then $m_{a,b}(\xi) = \hat{K}(\xi)$, where $K = K_1 + K_2$ and K_1 is a distribution of compact support, while K_2 is an L^1 function. Moreover $\hat{K}_2(\xi) = O(|\xi|^{-N})$ as $|\xi| \to \infty$ for all N, and thus $\hat{K}_1(\xi) \sim \psi(\xi)^{i|\xi|^a} |\xi|^{-b}$. In addition it one can show that $K_1(x)$ is a function away from the origin with

$$K_1(x) \sim c_1 |x|^{-n-\lambda} e^{ic_2|x|^{a'}}, \quad \text{as } x \to 0$$

where $a^{-1} + a'^{-1} = 1$. Also $|\nabla K_1(x)| \leq |x|^{-n-1+a'}$.

Now set $\theta = a$, and $\lambda = 0$ (i.e. $b = na/2$). Then it follows from what we have just said that the distribution K_1 satisfies the conditions (6.1), for Theorem 6, ($\hat{K}_1(\xi) \sim \psi(\xi)^{i|\xi|^a} |\xi|^{-\frac{na}{2}}$ in this case). By taking $\gamma = \frac{1}{2}na - b$ we obtain the desired result.

Example 2. Define T_λ by

$$T_\lambda(f) = \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| \leq 1} f(x-y) \left(e^{i|y|^{a'}} / |y|^{n+\lambda}\right) dy, \quad f \in C_0^\infty \tag{6.6}$$

where $a' < 0$, $\lambda \geq 0$.

Then $\|T_\lambda(f)\|_p \leq A_p \|f\|_p$, where p, a and λ are related by (6.5), and $a^{-1} + a'^{-1} = 1$. This example is very closely related to the previous one and can be obtained by the same analysis.

The results in examples 1 and 2, when there is strict inequality in (6.5), are due to Hirschmann [14] and Wainger [27], who also show that the L^p inequalities do not hold when $|\frac{1}{2} - p^{-1}| > (b/n)[\frac{1}{2}n + \lambda/(b+\lambda)]$. The sharp result, i.e. the case of equality in (6.5), is new.

Example 3. Let $d\sigma$ be the uniform mass distributed on the unit sphere in $\mathbf{R}^n$. Consider the distribution $(\partial/\partial x)^\alpha \sigma$, where $\alpha=(\alpha_1, ..., \alpha_n)$ and $|\alpha| = \alpha_1+\alpha_2, \ldots, +\alpha_n$. Then if $n \geqslant 3$ (the cases $n=1$ and $n=2$ are vacuous),

$$\left[\left(\frac{\partial}{\partial x}\right)^\alpha \sigma\right]^\wedge \text{ is a multiplier for } (L^p, L^p), \text{ when } \left|\frac{1}{2}-\frac{1}{p}\right| \leqslant \frac{1}{2}-\frac{|\alpha|}{n-1}. \tag{6.7}$$

In fact
$$\left[\left(\frac{\partial}{\partial x}\right)^\alpha \sigma\right]^\wedge = (-2\pi i\xi)^\alpha \hat{\sigma}(\xi) = \frac{(-2\pi i\xi)^\alpha}{|\xi|^{|\alpha|}} |\xi|^{|\alpha|} \hat{\sigma}(\xi).$$

It is very well known that $\hat{\sigma}(\xi) = O(|\xi|^{-(n-1)/2})$, and thus $|\xi|^{|\alpha|}\hat{\sigma}(\xi)$ is a multiplier on (L^p, L^p) when $|\frac{1}{2}-p^{-1}| \leqslant \frac{1}{2}-|\alpha|/(n-1)$, according to the corollary. In addition since $(-2\pi i\xi)^\alpha/|\xi|^{|\alpha|}$ is homogeneous of degree 0 and smooth on the unit sphere, it is a multiplier on (L^p, L^p), for $1<p<\infty$; (see e.g. [21, 96]).

(6.7) answers several questions raised in the study of A_p algebras; see Eymard [8]. Results of this kind may also be used to give new estimates for solutions of the wave equation.

IV. Characterization of H^p in terms of boundary properties of harmonic functions

7. Area integral and non-tangential max.functions

This part is organized as follows. Section 7 contains the basic result relating the L^p norms, $0<p$, of the non-tangential max.function and the area integral of any harmonic function. The consequences for the H^p spaces are then set down in Section 8. These are, among others, the extension to n-dimensions of the theorem of Burkholder–Gundy–Silverstein [2] characterizing H^p. Section 9 consists of a series of lemmas about harmonic functions that are used in sections 7 and 8 and also in later parts. Section 10 presents the second main result of this part, namely that max.functions formed with "arbitrary" approximate identities work as well for H^p as the one formed with the Poisson kernel. This result was anticipated, in the context of H^1, in § 3; for the case of general H^p it is taken up again as the main theme of Part V.

We begin with the first theorem, and fix the notation: $u=u(x,t)$ will be a harmonic function in $\mathbf{R}^{n+1}_+$; p will be an exponent so that $0<p<\infty$; and Γ_1 and Γ_2 will be a pair of cones whose vertices are the origin, i.e. $\Gamma_i = \{(x,t): |x|<c_i t\}$, $i=1, 2$. We denote by $\Gamma_i(x)$, $(x \in \mathbf{R}^n)$, the translate of Γ_i so that its vertex is x.

THEOREM 8. *With the notation above, set*

$$u^*(x) = \sup_{(x',t)\in\Gamma_1(x)} |u(x',t)|, \quad \textit{and} \quad (Su)(x) = \left(\int_{\Gamma_2(x)} |\nabla u(x',t)|^2 t^{1-n}\,dx'\,dt\right)^{\frac{1}{2}}.$$

Then the following two conditions are equivalent

(a) $u^* \in L^p$

(b) $u(x, t) \to 0$, *as* $t \to \infty$, *and* $S(u) \in L^p$.

Moreover, $\|u^*\|_p \approx \|S(u)\|_p$, $0 < p < \infty$. (1)

This result is well-known when $1 < p < \infty$; see [19]. Thus in proving the theorem we shall assume that $p < 2$. (2) The proof will be an adaptation of the proof of the earlier known "local analogue" of this theorem, namely that the sets where $u^* < \infty$ and where $S(u) < \infty$ are equivalent. We shall therefore follow the main lines of the proof of the local analogue (see [21, chapter VII]), but we will need to find the right quantitative estimates in place of certain qualitative statements.

The implication (a) $\Rightarrow$ (b): $\|S(u)\|_p \leqslant C\|u^*\|_p$.

It is convenient to make certain additional assumptions that will be removed at the end of the proof. We assume: u is the Poisson integral of an L^2 function; and the cone defining S is strictly contained in the cone defining u^*, i.e. $c_2 < c_1$.

We let E be the closed set $E = \{x \in \mathbf{R}^n\colon u^*(x) \leqslant \alpha\}$ and B its complement. So, if λ_{u^*} is the distribution function of u^*, then $\lambda_{u^*}(\alpha) = |B|$. Write $\mathcal{R} = \bigcup_{x \in E} \Gamma_2(x)$. By a simple argument, if $|u(x, t)| \leqslant \alpha$ in $\bigcup_{x \in E} \Gamma_1(x)$, we have $|t \nabla u(x, t)| \leqslant C\alpha$, for $(x, t) \in \mathcal{R}$. Now

$$\int_E (Su(x))^2 dx = \iint_R |\nabla u(y,t)|^2 |\{x \in E\colon (y,t) \in \Gamma_2(x)\}| t^{1-n} dy\, dt \leqslant C \iint_R t|\nabla u(y,t)|^2 dy\, dt.$$

An estimate for the last integral is obtained by replacing the region $\mathcal{R}$ by an approximating family of sub-regions, $\mathcal{R}_\varepsilon$; and then transforming the resulting integrals by Green's theorem. We can choose the $\mathcal{R}_\varepsilon$ so that their boundaries, $\mathcal{B}_\varepsilon$, are given as hypersurfaces $t = c_2^{-1}\delta_\varepsilon(x)$, with $\delta_\varepsilon(x)$ smooth and $|\partial\delta_\varepsilon/\partial x_j| \leqslant 1$, $j = 1, \ldots, n$. See [21; 206].

In applying Green's theorem the point corresponding to the boundary "at infinity" will vanish in view of the assumption that u is the Poisson integral of an L^2 function, as the reader may easily verify. Since $\Delta|u|^2 = 2|\nabla u|^2$ we have

$$\int_E (Su(x))^2 dx \leqslant \lim_{\varepsilon \to 0} \left\{ \int_{B_\varepsilon} t \frac{\partial |u|^2}{\partial n} d\sigma - \int_{B_\varepsilon} |u|^2 \frac{\partial t}{\partial n} d\sigma \right\}. \text{(3)} \tag{7.1}$$

(1) In his thesis at Washington University, Jia-Arng Chao has obtained an analogue of this theorem and a partial analogue of Theorem 9 in the context of p-adic fields.

(2) When $n = 1$, see Burkholder, Gundy and Silverstein [2].

(3) We have $\iint_R (A\Delta B - B\Delta A)\, dx\, dt = \int_{\partial R} (A(\partial B/\partial n) - B(\partial A/\partial n))\, d\sigma$, and take $A = t$, $B = \frac{1}{2}|u|^2$.

We now divide the boundary $\mathcal{B}_\varepsilon$ into parts $\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon^E \cup \mathcal{B}_\varepsilon^B$, where $\mathcal{B}_\varepsilon^E$ is that part above the set E, and $\mathcal{B}_\varepsilon^B$ that part lying above the set B. However, $\sup_{t>0}|u(x,t)|$ and $\sup_{t>0} t|\nabla u(x,t)|$ are both in L^2, and $\lim_{t\to 0} t|\nabla u(x,t)|=0$ almost everywhere. Thus

$$\int_{B_\varepsilon^E} t\frac{\partial |u|^2}{\partial n}\, d\sigma \to 0,$$

as $\varepsilon \to 0$, since $d\sigma \approx dx$. We have already observed that $|t\nabla u(x,t)| \leqslant C\alpha$ in $\mathcal{R}$; also $|u(x,t)| \leqslant \alpha$ in $\mathcal{R}$, since $\mathcal{R} = \bigcup_{x\in E}\Gamma_2(x)$, and $u^*(x) \leqslant \alpha$ for $x \in E$. Therefore

$$\left|\int_{B_\varepsilon^B} t\frac{\partial |u|^2}{\partial n}\, d\sigma\right| \leqslant C\alpha^2 \int_{B_\varepsilon^B} d\sigma \leqslant C\alpha^2 |B| = C\alpha^2 \lambda_{u^*}(\alpha).$$

Next
$$\int_{B_\varepsilon} |u|^2 \frac{\partial t}{\partial n}\, d\sigma = \int_{B_\varepsilon^E} + \int_{B_\varepsilon^B}.$$

The first integral is dominated by $\int_E (u^*)^2 d\sigma \leqslant C\int_E (u^*)^2 dx \leqslant C\int_0^\alpha t\lambda_{u^*}(t)\,dt$, since $u^* \leqslant \alpha$ on E. For the second integral we have as majorant $\int_{B_\varepsilon^B} |u|^2 d\sigma \leqslant \alpha^2 \int_B d\sigma \leqslant C\alpha^2|B| = C\alpha^2\lambda_{u^*}(\alpha)$. Altogether then by (7.1)

$$\int_E (Su(x))^2\, dx \leqslant C\left\{\alpha^2\lambda_{u^*}(\alpha) + \int_0^\alpha t\lambda_{u^*}(t)\,dt\right\}.$$

From this, and the fact that $|\complement E| = |B| = \lambda_{u^*}(\alpha)$, it follows that

$$\lambda_{S(u)}(\alpha) \leqslant C\left\{\lambda_{u^*}(\alpha) + \alpha^{-2}\int_0^\alpha t\lambda_{u^*}(t)\,dt\right\}. \tag{7.2}$$

Integrating with respect to α then gives,

$$\|S(u)\|_p^p = p\int_0^\infty \alpha^{p-1}\lambda_{S(u)}(\alpha)\,d\alpha \leqslant C\left\{p\int_0^\infty \alpha^{p-1}\lambda_{u^*}(\alpha)\,d\alpha + p\int_0^\infty t\lambda_{u^*}(t)\left\{\int_t^\infty \alpha^{p-3}\,d\alpha\right\}dt\right\}$$
$$\leqslant C\|u^*\|_p^p, \quad \text{if } 0<p<2.$$

The converse (b) $\Rightarrow$ (a): $\|u^*\|_p \leqslant C\|S(x)\|_p$.

We assume, as above, that u is the Poisson integral of an L^2 function; in addition we suppose that the cone defining u^* is strictly contained in the cone defining S, i.e. $c_1 < c_2$.

We let E be the closed set $\{x \in \mathbf{R}^n\colon S(u)(x) \leqslant \alpha\}$, and B its complement. Thus $\lambda_{S(u)}(\alpha) = |B|$. Now let E_0 be those points at which E has relative density at least $\frac{1}{2}$; more precisely set $E_0 = \{x \in \mathbf{R}^n$: for every cube Q, such that $x \in Q$, $|E \cap Q| \geqslant \frac{1}{2}|Q|\}$. Observe that since E is closed, $E_0 \subset E$; clearly E_0 is also closed. If χ is the characteristic function of

$B = \complement E$, then $\complement E_0 = B^* = \{x: M(\chi) > \frac{1}{2}\}$, where M is the maximal function. Thus $|B^*| \leqslant C|B| = C\lambda_{S(u)}(\alpha)$.

We now form the region $\mathcal{R} = \bigcup_{x \in E_0} \Gamma_1(x)$, with the corresponding aproximating regions $\mathcal{R}_\varepsilon$, and try to estimate the integral of $|u|^2$ on $\partial \mathcal{R}_\varepsilon = \mathcal{B}_\varepsilon$. We have

$$\int_E (Su(x))^2 dx = \int_{\bigcup\limits_{x \in E} \Gamma_2(x)} |\nabla u(y,t)|^2 |\{x \in E: (y,t) \in \Gamma_2(x)\}| t^{1-n} dy\, dt.$$

In the second integral we restrict integration over $(y,t) \in \mathcal{R}$. Then $(y,t) \in \mathcal{R} \Leftrightarrow$ for some $\bar{x} \in E_0$, $(y,t) \in \Gamma_1(\bar{x})$, i.e. $|\bar{x} - y| < c_1 t$. But then $(y,t) \in \Gamma_2(x)$, whenever $|x - \bar{x}| < (c_2 - c_1)t$. Thus $|\{x \in E: (y,t) \in \Gamma_2(x)\}| \geqslant |E \cap B|$, where B is the ball of center $\bar{x} \in E_0$, and radius $(c_2 - c_1)t$. In view of the definition of E_0, the latter quantity exceeds ct^n, and so

$$\int_E (Su(x))^2 dx \geqslant C \iint_R t |\nabla u(y,t)|^2 dy\, dt \geqslant C \iint_{R_\varepsilon} t |\nabla u(y,t)|^2 dy\, dt.$$

We transform the last integral by Green's theorem, obtaining as above

$$\int_E (Su(x))^2 dx \geqslant C_1 \int_{B_\varepsilon} |u(y,t)|^2 d\sigma - C_2 \int_{B_\varepsilon} |u(y,t)|\, t\, |\nabla u(y,t)|\, d\sigma \tag{7.3}$$

where C_1 and C_2 are two positive constants (independent of ε).

Let $\mathcal{I}_\varepsilon = (\int_{B_\varepsilon} |u(y,t)|^2 d\sigma)^{\frac{1}{2}}$. We have $\int_{B_\varepsilon} |u(y,t)|^2 d\sigma \leqslant \int_{\mathbf{R}^n} |u^*|^2 d\sigma \leqslant C \int_{\mathbf{R}^n} |u^*|^2 dx < \infty$, in view of the assumption that u is the Poisson integral of an L^2 function. Hence $\mathcal{I}_\varepsilon$ is finite for every ε. Next

$$\int_{B_\varepsilon} |u(y,t)|\, t\, |\nabla u(y,t)|\, d\sigma = \int_{B_\varepsilon^{B^*}} + \int_{B_\varepsilon^{E_0}}.$$

We know that $|t \nabla u(y,t) \leqslant| C\alpha$ in $\mathcal{R}$, since $S(u)(x) \leqslant \alpha$ for $x \in E$. By Schwarz's inequality, we get therefore that

$$\int_{B_\varepsilon^{B^*}} \leqslant \mathcal{I}_\varepsilon \alpha \sigma(\mathcal{B}_\varepsilon^{B^*})^{\frac{1}{2}} \leqslant \mathcal{I}_\varepsilon \alpha C |B^*|^{\frac{1}{2}} \leqslant C \mathcal{I}_\varepsilon (\alpha^2 \lambda_{S(u)}(\alpha))^{\frac{1}{2}}.$$

Also, as we have seen before, $I_\varepsilon = \int_{B_\varepsilon^{E_0}} \to 0$, with $\varepsilon \to 0$. Hence (7.3) gives $\mathcal{I}_\varepsilon^2 \leqslant C \int_E (Su(x))^2 dx + C\mathcal{I}_\varepsilon (\alpha^2 \lambda_{Su}(\alpha))^{\frac{1}{2}} + CI_\varepsilon$ and therefore

$$\mathcal{I}_\varepsilon^2 = \int_{B_\varepsilon} |u(y,t)|^2 d\sigma \leqslant C \left\{ \int_E (Su(x))^2 dx + \alpha^2 \lambda_{S(u)}(\alpha) \right\} \tag{7.4}$$

if ε is small enough.

Next for each $\varepsilon>0$, define a function f^ε on $\mathbf{R}^n$ by setting $f^\varepsilon(x)=C|u(x,\varphi_\varepsilon(x))|+C\alpha\chi_{B^*}(x)$ where χ_{B^*} is the characteristic function of B^*, and $t=\varphi_\varepsilon(x)$ is the equation of the hypersurface $\mathcal{B}_\varepsilon=\partial\mathcal{R}_\varepsilon$. Let $U_\varepsilon(x,t)$ be the Poisson integral of the f^ε, which because of (7.4) are obviously L^2 functions. Then as in [21; 211] we have the majorization $|u(x,t)|\leqslant U_\varepsilon(x,t)$ on $\mathcal{B}_\varepsilon$, whence on $\mathcal{R}_\varepsilon$.

We select then a subsequence of the f^ε which converges weakly to $f\in L^2$. Notice that because of (7.4) we then have

$$\int_{\mathbf{R}^n}|f|^2\,dx\leqslant C\left\{\int_E(Su(x))^2\,dx+\alpha^2\lambda_{S(u)}(\alpha)\right\}. \tag{7.5}$$

Passing to the limit we obtain, that

$$|u(x,t)|\leqslant U(x,t)\quad (x,t)\in\mathcal{R}$$

where U is the Poisson integral of f, and therefore $u^*(x)\leqslant U^*(x)$, for $x\in E_0$. So of course,

$$\int_{E_0}(u^*(x))^2\,dx\leqslant\int_{E_0}(U^*(x))^2\,dx\leqslant C\int_{\mathbf{R}^n}(f(x))^2\,dx.$$

Thus $|\{x\in E_0: u^*(x)>\alpha\}|\leqslant C\{\lambda_{S(u)}(\alpha)+\alpha^{-2}\int_0^t t\lambda_{S(u)}(t)\,dt\}$ and $|\{x\in \complement E_0\}|=|B^*|\leqslant C\{\lambda_{S(u)}(\alpha)\}$. Altogether then

$$\lambda_{u^*}(\alpha)\leqslant C\left\{\lambda_{S(u)}(\alpha)+\alpha^{-2}\int_0^\alpha t\lambda_{S(u)}(t)\,dt\right\}. \tag{7.6}$$

This inequality is the same kind as (7.2), but with u^* and $S(u)$ interchanged. Carrying out a similar integration gives

$$\|u^*\|_p\leqslant C\|S(u)\|_p,\quad \text{if } 0<p<2.$$

We need now only to remove the restrictions on u and the cones Γ_1 and Γ_2 to conclude the proof of Theorem 8. Assume therefore $u^*\in L^p$, $p\leqslant 2$. Then if $u_\varepsilon(x,t)=u(x,t+\varepsilon)$, it follows by Lemma 3 in § 9 below, that $\sup_{t>0}\int_{\mathbf{R}^n}|u_\varepsilon(x,t)|^2dx<\infty$, and hence u_ε is the Poisson integral of an L^2 function. Therefore by what we have proved

$$\|S(u_\varepsilon)\|_p\leqslant C\|u_\varepsilon^*\|_p$$

and a simple limiting argument, involving the monotone convergence theorem then shows that $\|S(u)\|_p\leqslant C\|u^*\|_p$.

The argument for the converse is only slightly more complicated. Let $u_{\varepsilon N}(x,t)=u(x,t+\varepsilon)-u(x,t+N)$. Then

$$|u_{\varepsilon N}(x,t)| \leqslant \left|\int_{t+\varepsilon}^{t+N} \frac{\partial u}{\partial s}(x,s)\,ds\right| \leqslant \left(\int_0^\infty s\left|\frac{\partial u}{\partial s}(x,s)\right|^2 ds\right)^{\frac{1}{2}} \left(\int_{t+\varepsilon}^{t+N} s^{-1} ds\right)^{\frac{1}{2}}$$

$$\leqslant g_1(u)\left(\log \frac{N}{\varepsilon}\right)^{\frac{1}{2}} \leqslant C_{\varepsilon,N} S(u)(x),$$

in view of the known majorization $g(u)(x) \leqslant CS(u)(x)$, (see [21; 90]). Thus $\sup_{t>0} \int_{\mathbf{R}^n} |u_{\varepsilon N}(x,t)|^p dx < \infty$, and again by Lemma 3 of § 9 we see that $u_{\varepsilon N}$ is the Poisson integra lof an L^2 function.

Thus by what is already proved

$$\|u^*_{\varepsilon N}\|_p \leqslant C\|S(u_{\varepsilon N})\|_p.$$

However $S(u_{\varepsilon N})(x) \leqslant S(u_\varepsilon)(x) + S(u_N)(x) \leqslant CS(u)(x)$, as is easily verified, and moreover

$$\limsup_{\substack{\varepsilon\to 0\\ N\to\infty}} \|u^*_{\varepsilon N}\|_p \geqslant \|u^*\|_p,$$

by Fatou's lemma, since $u_N \to 0$ as $N\to\infty$. The result is then $\|u^*\|_p \leqslant C\|S(u)\|_p$.

Finally the restriction on the cones (we supposed that Γ_2 is strictly contained in Γ_1 to prove (a) ⇒ (b), and the reverse to prove (b) ⇒ (a)), is removed by the following lemma which will also be useful later.

LEMMA 1. *Let $u(x,t)$ be any continuous function on $\mathbf{R}^{n+1}_+$. If the non-tangential maximal function $u^*(x) = \sup_{|x-y|<t} |u(y,t)| \in L^p (0<p<\infty)$, then $u^*_N(x) = \sup_{|x-y|<Nt} |u(y,t)|$ is also in L^p; in fact the "tangential" maximal function*

$$u^{**}(x) = \sup_{(y,t)\in R^{n+1}_+} |u(y,t)| \left(\frac{t}{|x-y|+t}\right)^M \in L^p, \quad \text{for } M > n/p.$$

Moreover, $$\|u^{**}\|_p \leqslant C_M \|u^*\|_p.$$

Proof. Let $E_\alpha = \{u^*(x) > \alpha\}$, and $E^*_\alpha = \{M(\chi_{E_\alpha})(x) > C/N^n\}$. Then we have $|E^*_\alpha| \leqslant CN^n |E_a|$ by the maximal theorem. On the other hand, $u^*_N(x) \leqslant \alpha$ for $x \notin E^*_\alpha$. For, pick any (y,t) with $|x-y| < Nt$. The ball $B(y;t)$ cannot be contained in E_α since if it were $M(\chi_{E_\alpha})(x) \geqslant |B(x;Nt)|^{-1} |B(y;t)| \geqslant c/N^n$. Therefore $u^*(z) \leqslant \alpha$ for some $z \in B(y;t)$, which implies $|u(y,t)| \leqslant u^*(z) \leqslant \alpha$. Thus, $u^*_N \leqslant \alpha$ except on E^*_α. So

$$\int_{\mathbf{R}} (u^*_N(x))^p dx = c\int_0^\infty \alpha^{p-1} |\{u^*_N(\cdot) > \alpha\}|\, d\alpha \leqslant C\int_0^\infty \alpha^{p-1} |E^*_\alpha|\, d\alpha$$

$$\leqslant CN^n \int_0^\infty \alpha^{p-1} |E_\alpha|\, d\alpha = CN^n \int_{\mathbf{R}^n} (u^*(x))^p dx,$$

which estimates u^*_N. To prove the more refined estimate on u^{**}, just note that $u^{**}(x) \leqslant \sup_{2^m = N\geqslant 1} (N^{-M} u^*_N(x))$, so that

$$\int_{\mathbf{R}^n}(u^{**}(x))^p\,dx \leqslant \sum_{m=1}^{\infty}\int_{\mathbf{R}^n}(2^{-Mm}u_N^*(x))^p\,dx$$

$$\leqslant C\sum_{m=1}^{\infty}2^{(n-Mp)m}\int_{\mathbf{R}^n}(u^*(x))^p dx \leqslant C\int_{\mathbf{R}^n}(u^*(x))^p dx < \infty. \quad \text{Q.E.D.}$$

Remarks

1. The theorem holds when the harmonic function u is real-valued or complex-valued, or more generally if u takes its values in a Hilbert space. It is to be understood of course that then the symbol $|\cdot|$ stands for the norm in that Hilbert space, whenever that is appropriate. In fact an examination of the argument above shows that all the estimates save one that are made hold in the even more general case of Banach-space valued functions, with $|\cdot|$ designating the norm. The exceptional estimate, valid only in the Hilbert space context, is the identity $\Delta(|u|^2)=2|\nabla u|^2$. This identity follows immediately from the scalar-valued case by passing to an orthonormal base of the Hilbert space. The Hilbert space variant of the theorem will be used in § 8 below.

2. The estimates (7.2) and (7.6), linking the distribution functions of u^* and $S(u)$, can be used to prove other inequalities relating u^* and $S(u)$. In particular the function $\Phi(t)=t^p$, $0<p<\infty$, which is used in theorem 8 can be replaced by a variety of others, such as $\Phi(t)=(\log(t+1))^{\delta}$, $\delta>0$. (1)

8. Characterizations of H^p

We begin by defining the H^p spaces, when $0<p<\infty$. For our purposes it will be convenient to adopt the following point of view. The elements of each H^p space will consist of (complex-valued) harmonic functions $u(x,t)$ defined on $\mathbf{R}_+^{n+1}$, satisfying certain additional conditions. We shall specify these additional conditions in stages.

1. *Case when* $1<p<\infty$.

$$u\in H^p \Leftrightarrow \sup_{t>0}\int_{\mathbf{R}^n}|u(x,t)|^p\,dx = \|u\|_{H^p}^p < \infty.$$

It is well known (see e.g. [25, Chapter II]) that with this definition $u\in H^p \Leftrightarrow u$ is the Poisson integral of an $f\in L^p$, and $\|u\|_{H^p}=\|f\|_p$.

2. *Case when* $(n-1)/n<p<\infty$.

We say $u\in H^p$ if there is a $(n+1)$tuple of harmonic functions $u=u_0, u_1, \ldots, u_n$, on $\mathbf{R}_+^{n+1}$ so that with $t=x_0$, this $(n+1)$tuple satisfies the equations of conjugacy

(1) A systematic study of such Φ inequalities has since been made by Burkholder and Gundy [1].

$$\begin{cases} \dfrac{\partial u_j}{\partial x_i} = \dfrac{\partial u_i}{\partial x}, & 0 \leqslant i, j \leqslant n \\ \sum_{j=0}^{n} \dfrac{\partial u_j}{\partial x_j} = 0 \end{cases} \tag{8.1}$$

and the condition $\sup_{t>0} \int_{\mathbf{R}^n} (\sum_{j=0}^n |u_j(x,t)|^2)^{p/2} dx = \|u\|_{H^p}^p < \infty$. We remark first, that in this case $u = u_0$ uniquely determines $u_1, \ldots, u_n$ as the reader may easily verify. Thus it is consistent to speak of u_0 as the element of H^p, instead of the $n+1$ tuple $u_0, u_1, \ldots, u_n$, as is usually done. Secondly, this definition is consistent with the previous one when $1 < p < \infty$, and gives an equivalent norm; this last statement is equivalent with the L^p boundedness of the Riesz transforms. Of course the "raison d'être" for this definition (originally given by [24]) is the fact that $(\sum_{j=0}^n |u_j(x)|^2)^{p/2}$ is sub-harmonic when $p \geqslant (n-1)/n$. Similar considerations motivate the following general definition. (1)

3. *General case,* $p_k = (n-1)/(n-1+k) < p$, *k a positive integer.*

For the general case we consider tensor functions of rank k, satisfying additional conditions as follows. The components of the tensors are written as $u_{j_1, j_2, \ldots j_k}$, where j's range between 0 and n. We require that the tensor is symmetric in the k indices, and that its traces vanish; the latter condition can be expressed as follows, in view of the symmetry of the indices,

$$\sum_{j=0}^{n} u_{jjj_3, \ldots, j_n} = 0 \quad \text{all } j_3, j_4, \ldots, j_n.$$

Now to the differential equations satisfied by these functions. We consider the tensor of rank $k+1$ obtained from our tensor by passing to its gradient, namely $u_{j_1 \ldots j_k j_{k+1}} = \partial/\partial x_{j_{k+1}} (u_{j_1 \ldots j_k})$. Then the equations analogous to (8.1) are precisely the statement that this tensor of rank $k+1$ is symmetric in all indices, and all its traces vanish. (See [6] and [26]). Finally we say that a harmonic function u is in H^p if there exists a tensor of rank k of the above type, with the property that $u(x,t) = u_{0,\ldots,0}(x,t)$ and $\sup_{t>0} \int_{\mathbf{R}^n} (\sum_{(j)} |u_{(j)}(x,t)|^2)^{p/2} dx = \|u\|_{H^p}^p < \infty$.

The fact that these definitions of H^p, with different k, are consistent is not obvious, but is contained in the theorem below, the main result of this section.

THEOREM 9. *Suppose $u(x,t)$ is harmonic in $\mathbf{R}_+^{n+1}$. Then $u \in H^p$ if and only if the non-tangential max.function $u^* \in L^p$. Also the definitions given above for H^p are all mutually consistent, and the resulting H^p "norms" are all equivalent.* (2) *Moreover* $\|u^*\|_p \approx \|u\|_{H^p}$.

(1) That the definition given for H^1 agrees with the notion used in parts II and III, see [21; p. 221].

(2) Note that $\|\cdot\|_{H^p}$ is actually a norm only when $p \geqslant 1$.

We shall prove this by showing that whenever $u \in H^p$ (defined according to definition 3, with some fixed k), then $u^* \in L^p$; and next, whenever $u^* \in L^p$ it follows that $u \in H^p$, when defined with respect to any k.

Suppose then that $u \in H^p$. The key point is that $(\sum_{(j)} |u_{(j)}(x,t)|^2)^{p_k/2}$ is sub-harmonic if $p_k = (n-1)/(n-1+k)$, see [6] and [26]. Then by the arguments in [25, chapter VI] there exists a harmonic majorant $h(x,t)$, so that $(\sum_{(j)} |u_{(j)}(x,t)|^2)^{p_k/2} \leq h(x,t)$ and

$$\sup_{t>0} \int_{\mathbf{R}^n} (h(x,t))^{p/p_k} dx = \sup_{t>0} \int_{\mathbf{R}^n} \left(\sum_{(t)} |u_{(j)}(x)|^2 \right)^{p/2} dx = \|u\|_{H^p}^p.$$

Since $p/p_k > 1$, $h(x,t)$ is the Poisson integral of an L^{p/p_k} function $h(x)$, with $\|h\|_{p/p_k} = \|u\|_{H^p}^{p_k}$. Now $|u(x,t)| \leq (\sum_{(j)} |u_{(j)}(x,t)|^2)^{\frac{1}{2}} \leq (h(x,t))^{1/p_k}$. Thus $u^*(x) \leq (h^*(x))^{1/p_k} \leq C(M(h)(x))^{1/p_k}$. Finally then

$$\int_{\mathbf{R}^n} (u^*(x))^p dx \leq C \int_{\mathbf{R}^n} (Mh(x))^{p/p_k} \leq C \int_{\mathbf{R}^n} (h(x))^{p/p_k} dx = C \|u\|_{H^p}^p.$$

So we have proved that $u^* \in L^p$, and $\|u^*\|_p \leq A \|u\|_{H^p}$.

Next suppose $u^* \in L^p$. Then it follows from Lemma 3 in § 9 that $\|u(x,t)\|_\infty = O(t^{-\delta})$, $\delta > 0$, as $t \to \infty$, and hence by a standard argument, that $\|(\partial/\partial x)^\alpha u(x,t)\|_\infty = O(t^{-\delta - |\alpha|})$, $t \to \infty$, where $(\partial/\partial x)^\alpha$ is any differential moniomial of order $|\alpha|$ (see e.g. [21; 143]). This will allow us to define the "conjugates of order k" of $u(x,t)$. In fact observe that if $k \geq 1$

$$u(x,t) = \frac{(-1)^k}{(k-1)!} \int_t^\infty (s-t)^{k-1} \frac{\partial^k}{\partial s^k} (u(x,s))\, ds, \quad t > 0.$$

The integral on the right converges, because of the observation we have just made, and also has as value $u(x,t)$, since $u(x,t)$, $(\partial/\partial t)\, u(x,t), \dots, (\partial^{k-1}/\partial t^{k-1})$, $u(x,t) \to 0$, as $t \to \infty$.

Now for any k-tuple of indecies $(j) = (j_1, j_2, \dots, j_k)$, each $0 \leq j_r \leq n$, we define $u_{(j)}(x,t)$ by

$$u_{(j)}(x,t) = \frac{(-1)^k}{(k-1)!} \int_t^\infty (s-t)^{k-1} \left(\frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_k}} \right) (u(x,s))\, ds, \tag{8.2}$$

where $\partial/\partial x_0 = \partial/\partial s$. These integrals again converge, and the tensor-function $u_{(j)}$ satisfies the identities of symmetry and vanishing traces required in definition 3. Moreover $u_{(0,\dots,0)} = u$. If we invoke Theorem 8, we see that the assumption $u^* \in L^p$ implies $S(u) \in L^p$. The main point now is that $S(u_{(j)})(x) \leq CS(u)(x)$, as long as the cone defining $S(u_j)$ is strictly interior to that of $S(u)$, which follows by repeated application of the lemma in [21, p. 213]. (The results stated there are for truncated cones, but there is no problem in passing to the corresponding inequality for non-truncated cones). However from (8.2) it is clear that $u_{(j)}(x,t) \to 0$, as $t \to \infty$; thus when we invoke Theorem 8 again we get

$$\|u^*_{(j)}\|_p \leqslant C\|S(u_{(j)})\|_p \leqslant C\|S(u)\|_p \leqslant C\|u^*\|_p$$

and hence
$$\sup_{t>0} \int_{\mathbf{R}^n} \left(\sum_{(j)} |u_{(j)}(x,t)|^2 \right)^{p/2} dx \leqslant C \sum_{(j)} \|u^*_{(j)}\|_p^p \leqslant C\|u^*\|_p^p.$$

This proves that $u \in H^p$ and

$$\|u\|_{H^p} \leqslant A\|u^*\|_p.$$

Since this part was carried out for any k, the proof of the theorem is therefore complete. An immediate consequence of Theorems 8 and 9 is the following (for which see Calderón [3] and Segovia [16]).

COROLLARY 1. *Let $u(x,t)$ be harmonic in $\mathbf{R}^{n+1}_+$. Then $u \in H^p$, $0<p<\infty$, if and only if $S(u) \in L^p$, and $u(x,t) \to 0$, as $t \to \infty$.*

We next show that the non-tangential max.function can be replaced by its "radial" analogue in characterizing H^p. In fact, define u^+ by $u^+(x) = \sup_{t>0} |u(x,t)|$.

COROLLARY 2. *Let u be harmonic in $\mathbf{R}^{n+1}$. Then $u \in H^p$, $0<p<\infty$, if and only if $u^+ \in L^p$. Moreover $\|u\|_{H^p} \approx \|u^+\|_p$.*

In view of what we know already, and the fact that $u^+(x) \leqslant u^*(x)$, it suffices to prove that
$$\|u^*\|_p \leqslant A_p \|u^+\|_p \quad 0<p<\infty. \tag{8.3}$$

For simplicity of notation, let us assume that the aperture of the cone defining u^* is 1. Now if $(y,t) \in \Gamma(x)$, then $|x-y|<t$; hence if $B(y,t)$ denotes the ball of radius t centered at (y,t), its projection on $\mathbf{R}^n$ is contained in the ball of radius $2t$ centered at x. But by the mean-value property expressed in Lemma 2 in § 9, we have

$$|u(y,t)|^{p/2} \leqslant Ct^{-n-1} \int_{B(y,t)} |u(z,t')|^{p/2}\, dz\, dt'$$
$$\leqslant Ct^{-n-1} \int_{B(y,t)} |u^+(z)|^{p/2}\, dz\, dt' \leqslant Ct^{-n} \int_{|x-z|<2t} |u^+(z)|^{p/2}\, dz.$$

Thus $(u^*(x))^{p/2} \leqslant CM[(u^+)^{p/2}](x)$, and by the maximal theorem for L^2 we have

$$\int_{\mathbf{R}^n} (u^*(x))^p\, dx \leqslant C \int_{\mathbf{R}^n} [M(u^+)^{p/2}]^2\, dx \leqslant C \int_{\mathbf{R}^n} (u^+(x))^p\, dx,$$

proving (8.3) and the corollary.

Remark. For another argument leading to the proof of Corollary 2 see the reasoning in Theorem 11 below.

In the same spirit as Corollary 2, the S function (which is non-tangential), can be replaced by the g function (its radial analogue), $g(u)(x) = (\int_0^\infty |\nabla u(x, t)|^2 t\,dt)^{\frac{1}{2}}$.

COROLLARY 3. *Let u be harmonic in $\mathbf{R}^n$. Then $u \in H^p$, $0 < p < \infty$ if and only if $g(u) \in L^p$, and $u(x, t) \to 0$ as $t \to \infty$.*

It clearly suffices to show that

$$\|S(u)(x)\|_p \leqslant C\|g(u)(x)\|_p, \quad 0 < p < \infty. \tag{8.4}$$

The reverse inequality is of course a consequence of the pointwise estimate $g(u)(x) \leqslant CS(u)(x)$.

We may simplify matters, as in the proof of Theorem 8, by assuming that u is the Poisson integral of an L^2 function. Once the a priori inequality (8.4) is proved for such functions, the general case follows by the limiting argument already given above.

Let $\mathcal{H}$ be the Hilbert space defined as follows

$$\mathcal{H} = \left\{ \varphi = (\varphi_0, \dots, \varphi_n) : \|\varphi\|^2 = \sum_{i=0}^{n} \int_0^\infty |\varphi_i(s)|^2 s\,ds < \infty \right\}$$

whenever $u(x, t)$ is a harmonic function on $\mathbf{R}_+^{n+1}$, we shall define another harmonic function U, this time with values in $\mathcal{H}$. Write for each (x, t)

$$U(x, t) = \nabla u(x, t + s).$$

Thus
$$|U(x, t)| = \left(\int_0^\infty |\nabla u(x, t + s)|^2 s\,ds \right)^{\frac{1}{2}} \leqslant g(u)(x),$$

and hence $U^+(x) = \sup_{t>0} |U(x, t)| = g(u)(x) \in L^p$. The Hilbert space analogue of (8.3) and Theorem 8 (see Remark 1 at the end of § 7) thus give us

$$\|S(U)(x)\|_p \leqslant C\|g(u)(x)\|_p. \tag{8.5}$$

Writing out $S(U)$ we have

$$(S(U)(x))^2 = \iint_{|x-y|<t} \left\{ \int_0^\infty |\nabla^2 u(y, t + s)|^2 s\,ds \right\} t^{1-n}\,dt\,dy.$$

(We have taken $\Gamma(x) = \{(y, t) : |x - y| < t\}$, to be the cone defining S). Two of the integrations above may be assimilated into one, and if we use the simple estimate

$$\int_a^u (u - t)\, t^{1-n}\,dt \geqslant c u^{3-n}, \quad \text{for } u \geqslant 2a,$$

we obtain
$$(S(U)(x))^2 \geqslant C \iint_{2|x-y|<t} |\nabla^2 u(y,t)|^2 t^{3-n}\, dt\, dy. \tag{8.6}$$

We now invoke the simple estimate

$$\iint_{2|x-y|<t} |\nabla u(y,t)|^2 t^{1-n}\, dt\, dy \leqslant C \iint_{2|x-y|<t} |\nabla^2 u(y,t)|^2 t^{3-n}\, dt\, dy \tag{8.7}$$

whenever $\nabla u(y,t) \to 0$ as $t \to \infty$ (the latter is a consequence of the fact that u is the Poisson integral of an L^2 function).

This inequality is stated in [21, p. 216]. Unfortunately the proof outlined there is correct only when $n=1$. We take this opportunity to give a correct proof. It suffices to demonstrate (8.7) for $x=0$. Let ϱ be any unit vector in $\mathbf{R}^{n+1}_+$ which lies in the cone $\{(y,t)\colon 2|y|<t\}$. Then clearly

$$|(\nabla u)(\varrho s)| \leqslant \int_s^\infty |\nabla^2 u(\varrho t)|\, dt, \quad \text{if } s>0.$$

By Hardy's inequality

$$\int_0^\infty |\nabla u|^2 (\varrho s)\, s\, ds \leqslant \int_0^\infty |\nabla^2 u(\varrho t)|^2 t^3\, dt,$$

and a final integration over all unit vectors ϱ lying in the cone gives (8.7); in combination with (8.6) and (8.5) this concludes the proof of the corollary. Q.E.D.

Remark. There is a similar result for the functions $g_1(u)$ and $g_x(u)$ (defined in terms of the t or x derivatives of u only), but the proof is somewhat more elaborate.

9. Lemmas for harmonic functions

In this section we have gathered several results on harmonic functions some of which have already been used, and others that we will apply later.

LEMMA 2. *Suppose B is a ball in $\mathbf{R}^{n+1}$, with center (x^0, t^0). Let u be harmonic in B and continuous on the closure of B. For any $p>0$,*

$$|u(x^0,t^0)|^p \leqslant C_p \frac{1}{|B|} \int_B |u(x,t)|^p\, dx\, dt. \tag{9.1}$$

This lemma is of course standard when $p \geqslant 1$ (then $C_p = 1$). When $p<1$ the result is essentially due to Hardy and Littlewood [12], where other closely related questions are studied.

In proving (9.1) when $p<1$ we may assume that B is the unit ball centered at the origin, and that $\int_B |u(x,t)|^p dx\,dt=1$. Let us write

$$m_p(r)=\left(\int_{|x|^2+t^2=r}|u(x,t)|^p d\sigma\right)^{1/p},\quad 0<r<1,$$

and $m_\infty(r)$ for the sup $|u(x,t)|$ taken over the sphere of radius r. We may also assume that $m_\infty(r)\geqslant 1$, all $0<r<1$, for otherwise there is nothing to prove. By Hölder's inequality

$$m_1(r)\leqslant (m_p(r))^{1-\theta}(m_\infty(r))^\theta,\quad 0<r<1, \text{ with } 0<\theta<1,$$

since $p<1$, as we assumed. By standard estimates for the Poisson kernel of the sphere, we have $m_\infty(\varrho)\leqslant A(1-\varrho r^{-1})^{-n}m_1(r)$, wherever $0<\varrho<r$. Now take $\varrho=r^a$, with $a>1$; a will be chosen to be sufficiently close to 1 near the end of the proof. Insert this estimate in the above, take the logarithm of both sides and integrate. The result is,

$$\int_{\frac{1}{2}}^1 \log m_\infty(r^a)\frac{dr}{r}\leqslant C_a+\theta\int_{\frac{1}{2}}^0 \log m_\infty(r)\frac{dr}{r}+(1-\theta)\int_{\frac{1}{2}}^1 \log m_p(r)\frac{dr}{r}.$$

The last integral above is bounded by a constant, since $\int_0^1 (m_p(r))^p r\,dr=1$ by assumption. If we make the appropriate change of variables in the integral on the left side, then we have

$$\left(\frac{1}{a}\right)\int_{(\frac{1}{2})^a}^1 \log m_\infty(r)\frac{dr}{r}\leqslant C'_a+\theta\int_{\frac{1}{2}}^1 \log m_\infty(r)\frac{dr}{r}. \tag{9.2}$$

Since we assumed $m_\infty(r)\geqslant 1$, it follows that

$$\int_{(\frac{1}{2})^a}^1 \log m_\infty(r)\frac{dr}{r}\geqslant \int_{\frac{1}{2}}^1 \log m_\infty(r)\frac{dr}{r}.$$

Choose now a, close enough to 1, so that $1/a>\theta$. By (9.2) then $\int_{(\frac{1}{2})^a}^1 \log m_\infty(r)\,r^{-1}dr\leqslant C'_\theta$, and hence for at least one r_0, $m_\infty(r_0)\leqslant C=C_p$, which gives (9.1) by the maximum principle. Q.E.D.

LEMMA 3. *Suppose $u(x,t)$ is harmonic in $\mathbf{R}_+^{n+1}$, and for some p, $0<p<\infty$,*

$$\sup_{t>0}\int_{\mathbf{R}^n}|u(x,t)|^p dx<\infty,$$

then $\sup_{x\in\mathbf{R}^n}|u(x,t)|\leqslant At^{-n/p}$, $0<t<\infty$.

By Lemma 2, if B denote the ball of radius t centered at (x,t), then

$$|u(x,t)|^p \leqslant C_p t^{-n-1} \iint_B |u(z,t')|^p \, dz \, dt' \leqslant C_p t^{-n-1} \int_0^{2t} \left\{ \int_{\mathbf{R}^n} |u(z,t)|^p \, dz \right\} dt$$

$$\leqslant C_p t^{-n} \sup_{t>0} \int_{\mathbf{R}^n} |u(z,t)|^p \, dz.$$

Thus,
$$|u(x,t)| \leqslant C_p t^{-n/p} \sup_{t>0} \left(\int_{\mathbf{R}^n} |u(x,t)|^p \, dx \right)^{1/p}, \quad 0 < p < \infty \tag{9.3}$$

and lemma 3 is proved. Q.E.D.

LEMMA 4. *Suppose u satisfies the conditions of the above lemma. Then $\lim_{t\to 0} u(x,t) = f(x)$ exists in the sense of tempered distributions.*

We must show that there exists a tempered distribution $f(x)$, so that whenever φ belongs to the space of testing functions

$$\lim_{t\to 0} \int_{\mathbf{R}^n} u(x,t)\, \varphi(x)\, dx = \int_{\mathbf{R}^n} f(x)\, \varphi(x)\, dx.$$

This is of course well known when $p \geqslant 1$. Thus if $\infty > p > 1$ the convergence is also in L^p norm, while if $p=1$ the convergence is in the weak* topology of finite measures. We may assume therefore that $p \leqslant 1$. Let u_δ be defined by $u_\delta(x,t) = u(x, t+\delta)$, for $(x,t) \in \mathbf{R}^{n+1}_+$, and $\delta > 0$. Then because of Lemma 3 $\sup_{t>0} \int_{\mathbf{R}^n} |u_\delta(x,t| \, dx < \infty$, and so u_δ is the Poisson integral of a finite measure (see e.g. [25, Chapter II]); this measure is the weak limit of $u_\delta(x,t) = u(x, t+\delta)$, as $t \to 0$, and hence is the integrable function $u_k(x,0) = u(x,\delta)$. By the Fourier transform, $\hat{u}_\delta(\xi,t) = \hat{u}_\delta(\xi,0)\, e^{-2\pi|\xi|t}$, i.e. $\hat{u}(\xi, t+\delta) = \hat{u}_\delta(\xi,0)\, e^{-2\pi|\xi|t} = \hat{u}_0(\xi)\, e^{-2\pi|\xi|(t+\delta)}$, for an appropriate continuous function $\hat{u}_0(\xi)$. Also $|\hat{u}_0(\xi) e^{-2\pi|\xi|t}| \leqslant \int_{\mathbf{R}^n} |u(x,t)| \, dx \leqslant At^{-n[p^{-1}-1]}$, $t>0$ by (9.3). Thus $|\hat{u}_0(\xi)| \leqslant A|\xi|^N$ (with $N = n[p^{-1}-1]$). However

$$\int_{\mathbf{R}^n} u(x,t)\, \overline{\varphi(x)}\, dx = \int_{\mathbf{R}^n} \hat{u}_0(\xi)\, e^{-2\pi|\xi|t}\, \hat{\varphi}(\xi)\, d\xi$$

and so it is clear that $\lim_{t\to 0} u(x,t) = f$ exists where f is the inverse Fourier transform of the tempered function $\hat{u}_0$. Q.E.D.

Remark. Observe that the boundary value f uniquely determines u. For the proof shows that if $f=0$ then $u=0$.

The lemma implies in particular that whenever $\varphi \in S$, the functional

$$u \to \lim_{t\to 0} \int_{\mathbf{R}^n} u(x,t)\, \varphi(x)\, dx$$

is continuous on H^p and also

$$\left| \lim_{t\to 0} \int_{\mathbf{R}^n} u(x,t)\, \varphi(x)\, dx \right| \leqslant A \, \|u\|_{H^p}.$$

We shall make a more accurate statement momentarily, but for this we shall need another lemma (which will also be useful later).

LEMMA 5. *Suppose* $1<p_0<r<\infty$, $r/p_0=1+\lambda$. *If* u *is the Poisson integral of an* $f\in L^{p_0}(\mathbf{R}^n)$, *then*

$$\left(\int_{\mathbf{R}^{n+1}_+} t^{\lambda n}|u(x,t)|^r \frac{dx\,dt}{t}\right)^{1/r} \leqslant A\,\|f\|_{p_0}. \tag{9.4}$$

The lemma is known in even more general form, see Flett [11[a]]. Here we shall give an alternate proof for the special case we will need.

The assumptions require that $0<\lambda<\infty$. It will suffice to prove the inequality for λ positive and sufficiently small (say $0<\lambda<1$); for if it is true for λ_1 it also follows for λ_2, whenever $\lambda_2>\lambda_1$, which we can see by writing

$$\int_{\mathbf{R}^{n+1}_+} t^{\lambda_2 n}|u(x,t)|^{r_2}\frac{dx\,dt}{t} \leqslant \sup_{(x,t)} t^{(\lambda_2-\lambda_1)n}\,|u(x,t)|^{r_2-r_1}\int_{\mathbf{R}^{n+1}_+} t^{\lambda_1 n}|u(x,t)|^{r_1}\frac{dx\,dt}{t}$$

and invoking the inequality (9.3).

Assume, then, that $0<\lambda<1$, and that in addition $f\geqslant 0$. We have

$$\int_{\mathbf{R}^{n+1}_+} t^{\lambda n}u^r(x,t)\frac{dx\,dt}{t} \leqslant \int_{\mathbf{R}^n}\sup_{t>0}\,(u(x,t))^{r-1}\left\{\int_0^\infty t^{\lambda n}u(x,t)\frac{dt}{t}\right\}dx$$

$$\leqslant C\int_{\mathbf{R}^n}(M(f)(x))^{r-1}I_{\lambda n}(f)(x)\,dx$$

Here
$$(I_{\lambda n}f)(x)=\frac{1}{\gamma(\lambda n)}\int_{\mathbf{R}^n} f(x-y)\,|y|^{-n+\lambda n}\,dy,$$

and the last inequality results from the observation that

$$\int_0^\infty t^{\lambda n}P_t(x)\frac{dt}{t}=c_n\int_0^\infty \frac{t^{\lambda n}}{(|x|^2+t^2)^{(n+1)/2}}\,dt=c\,|x|^{-n+n\lambda}.$$

Hölder's inequality and Sobolev's theorem [21; 119] on fractional integration then gives

$$\int_{\mathbf{R}^n}(Mf)^{r-1}I_{\lambda n}(f)\,dx \leqslant \left(\int_{\mathbf{R}^n}(Mf)^{p_0}dx\right)^{(r-1)/p_0}\left(\int_{\mathbf{R}^n}(I_{\lambda n}(f))^q\,dx\right)^{1/q}$$
$$\leqslant A\,\|f\|_{p_0}^{r-1}A\,\|f\|_{p_0}=A\,\|f\|_{p_0}^r.$$

Here, q is the exponent conjugate to $p_0/(r-1)$, so $q^{-1}=1-(r-1)\,p_0^{-1}=p_0^{-1}-\lambda$, which allows the application of the fractional integration theorem. Q.E.D.

For the next lemma recall the spaces $\Lambda_\alpha(\mathbf{R}^n)$, $\alpha>0$; (see [21; 141]). Briefly stated, when $0<\alpha<1$, Λ_α consists of the continuous and bounded functions φ on $\mathbf{R}^n$ which satisfy the Lipschitz condition $|\varphi(x)-\varphi(y)|\leqslant M|x-y|^{\alpha}$; for general β, Λ_β is the image of Λ_α under the mapping $\varphi\to\mathfrak{J}_{\beta-\alpha}(\varphi)$. Here $(\mathfrak{J}_{\beta-\alpha}(\varphi))^{\wedge}(\xi)=(1+4\pi^2|\xi|^2)^{-(\beta-\alpha)/2}\hat{\varphi}(\xi)$.(1) It can be proved that the dual of H^p, $0<p<1$, can be identified with Λ_α where $\alpha=n[p^{-1}-1]$. (In the case $n=1$ the theorem is proved in Duren, Romberg and Shields [7]). In the general case the essential part of this duality is given by the following inequality.

LEMMA 6. *Let $u\in H^p$, and suppose $f=\lim_{t\to 0}u(x,t)$ in the sense of distributions. If $0<p<1$, and $\alpha=n[p^{-1}-1]$. Then*

$$\left|\int_{\mathbf{R}^n} f\varphi\,dx\right|\leqslant A\,\|u\|_{H^p}\|\varphi\|_{\Lambda_\alpha},\quad \text{if } \varphi\in\mathcal{S}. \tag{9.5}$$

We have already seen that

$$\sup_{t>t_0>0}\int_{\mathbf{R}^n}|u(x,t)|\,dx<\infty$$

and thus replacing $u(x,t)$ by $u(x,t+t_0)$, we may assume that u is the Poisson integral of an L^1 function and reduce the problem to proving the a priori inequality

$$\left|\int_{\mathbf{R}} f\varphi\,dx\right|\leqslant A\,\|u\|_{H^p}\|\varphi\|_{\Lambda_\alpha}. \tag{9.5'}$$

where $u=\text{P.I.}(f)$. Let $\varphi(x,t)$ be the Poisson integral of φ, then one can verify that for each integer $k\geqslant 1$

$$\int_{\mathbf{R}^n} f\varphi\,dx=c_k\int_{\mathbf{R}^{n+1}_+}t^{2k-1}\frac{\partial^k u(x,t)}{\partial t^k}\,\frac{\partial^k\varphi(x,t)}{\partial t^k}\,dx\,dt.$$

(In fact if $\hat{f}$ and $\hat{\varphi}$ are respectively the Fourier transforms of f and φ, then the above identity is the same as

$$\int_{\mathbf{R}^n}\hat{f}(\xi)\,\hat{\varphi}(-\xi)\,d\xi=c_k\int_{\mathbf{R}^n}\hat{f}(\xi)\,\hat{\varphi}(-\xi)\,(-2\pi|\xi|)^{2k}\left\{\int_0^\infty t^{2k-1}e^{-4\pi|\xi|t}\,dt\right\}d\xi\quad\text{with } c_k=\frac{2^{2k}}{\Gamma(2k)}\Bigg).$$

By a basic porperty of the space Λ_α (see [21, p. 145]), we know that $\|\partial^k\varphi(x,t)/\partial t^k\|_\infty\leqslant At^{-k+\alpha}\|\varphi\|_{\Lambda_\alpha}$, whenever $k>\alpha$. Thus

$$\left|\int f\varphi\,dx\right|\leqslant A\left(\int_0^\infty t^{k+\alpha-1}\left\|\frac{\partial^k u(x,t)}{\partial t^k}\right\|_1 dt\right)\|\varphi\|_{\Lambda_\alpha}$$

and it suffices to prove that

(1) The reader should be warned that the spaces Λ_1 so defined are called Λ_1^* by Zygmund [28].

$$\int_0^\infty t^{k+\alpha-1}\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_1 dt \leqslant A\|u\|_{H^p}. \tag{9.6}$$

We shall prove (9.6) first in the case $k=0$. Using the argument of the proof of Theorem 9, we choose k so that $p_k=(n-1)/(n-1+k)<p$, and let $h(x,t)$ be the harmonic majorant described there. Recall that $u(x,t)\leqslant(h(x,t))^{1/p_k}$, and h is the Poisson integral of a function in L^{p/p_k}. Now use Lemma 5 with h in place of u, $r=p_k^{-1}$, $p_0=p/p_k$; then $1+\lambda=rp_0^{-1}=p^{-1}$, and so $\lambda=p^{-1}-1$. Thus with $\alpha=n\lambda=n[p^{-1}-1]$, we have

$$\int_{\mathbf{R}^n}\int_0^\infty t^{\alpha-1}|u(x,t)|\,dx\,dt\leqslant\int_{\mathbf{R}^n}\int_0^\infty t^{\lambda n}(h(x,t))^r\frac{dx\,dt}{t}\leqslant A\|h\|^r_{p/p_k}=A\|u\|_{H^p}$$

which proves (9.6) when $k=0$. The case for $k>0$ follows from this case by observing that

$$\left\|\frac{\partial^t}{\partial t^k}u(x,t)\right\|_1\leqslant\left\|\left(\frac{\partial^k}{\partial s^k}P(x,s)\right)_{s=t/2}\right\|_1\left\|u\left(x,\frac{t}{2}\right)\right\|_1\leqslant At^{-k}\left\|u\left(x,\frac{t}{2}\right)\right\|_1. \qquad \text{Q.E.D.}$$

Remark. It is true, conversely, that whenever, say, $\varphi\in S$, then $|\int_{\mathbf{R}^n}f\varphi\,dx|\leqslant A\|u\|_{H^p}$, all $u\in H^p$, implies $\varphi\in\Lambda_\alpha$. To see this, take $u_{t_0}(x,t)=(\partial^k/\partial t^k)P_{t+t_0}(x)$ where $P_t(x)$ is the Poisson kernel, and k is sufficiently large ($k>n[p^{-1}-1]$). Then $u_{t_0}\in H^p$ and $\|u_{t_0}\|_{H^p}\approx At_0^{-k+\alpha}$. So the condition implies that $\|\partial^k\varphi(x,t_0)/\partial t^k\|_\infty\leqslant At_0^{-k+\alpha}$ which means $\varphi\in\Lambda_\alpha$.

10. Passage to "arbitrary" approximate identies

In section 8 we saw that functions in L^p can be characterized in terms of the max.-functions of (what amounts to) the Poission integral of their boundary values. It will be shown here that the Poisson kernel can be replaced by arbitrary "smooth" functions, which are sufficiently small at infinity. In this way we are led to one of our main results namely that the H^p classes can be characterized without any recourse to analytic functions, conjugacy of harmonic functions, Poisson integrals, etc., and have an intrinsic "real-variable" meaning of their own. Our analysis in this section will be rather "fine"; our results will be in the nature of best possible, or nearly so. In the last part of this paper we take up these results again; we obtain there an alternative (less precise but more elementary) derivation which nevertheless allows us to obtain the full converse of Theorem 10 below.

We shall consider the elements $u\in H^p$ in terms of their boundary values f, according to Lemmas 4 and 6 above. Our result states that whenever φ is sufficiently smooth and small at infinity, and $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$, then $u\in H^p$ implies that $\sup_{\varepsilon>0}|(f*\varphi_\varepsilon)(x)|\in L^p$. We state the required conditions on φ. For a fixed $\alpha\geqslant 0$, we require

$$\sup_{x\in\mathbf{R}^n}(1+|x|)^{n+|\gamma|+\eta}\left|\frac{\partial^\gamma}{\partial x^\gamma}\varphi(x)\right|<\infty,\ \text{all}\ |\gamma|\leqslant\alpha,\ \text{some}\ \eta>0. \tag{10.1}$$

(10.1) is the full requirement in case α is a positive integer. Alternatively, if $k<\alpha<k+1$, with k a positive integer we assume (10.1), and in addition the following condition on all $\tilde{\varphi}(x)=(\partial^\gamma/\partial x^\gamma)\,\varphi(x)$, where $|\gamma|=k$.

$$\sup_{|x|\geqslant 2|y|}(1+|x|)^{n+\alpha+\eta}\frac{|\tilde{\varphi}(x-y)-\tilde{\varphi}(x)|}{|y|^{\alpha-|\gamma|}}<\infty, \quad \text{some } \eta>0. \tag{10.2}$$

These conditions, although complicated in appearance, say (very roughly) that $\varphi\in C^{(\alpha)}$, and that its αth derivatives are $O(1+|x|)^{-n-\alpha-\eta}$, as $|x|\to\infty$.

THEOREM 10. *Suppose $f\in H^p$, $f=\lim_{t\to 0}u(x,t)$, and φ satisfies the conditions* (10.1) *and* (10.2) *above with some $\alpha>n[p^{-1}-1]$. (We require $\alpha=0$, if $p>1$). Then*

$$\sup_{\varepsilon>0}|(f*\varphi_\varepsilon)(x)|\in L^p, \quad \textit{and} \quad \|\sup_{\varepsilon>0}|f*\varphi_\varepsilon|\|_p\leqslant A\|u\|_{H^p}.$$

This theorem, in a more precise form, will be a consequence of two lemmas. To state these lemmas we require some notation. For fixed $\alpha\geqslant 0$, let $\mathcal{B}$ be the class of φ which satisfy (10.1) and (10.2), and such that the quantities appearing as the left sides of these inequalities are bounded by 1. For any $u=u(x,t)$ harmonic in $\mathbf{R}^{n+1}_+$, define

$$M_\lambda(u)(x^0)=\sup_{0<h<\infty}\left\{\frac{\int_{T(x^0,h)}|u(x,t)|\,t^{\lambda n}\frac{dx\,dt}{t}}{\int_{T(x^0,h)}t^{\lambda n}\frac{dx\,dt}{t}}\right\}$$

where $T(x^0,h)=\{(x,t):|x-x^0|<h,\ 0<t<h\}$.

LEMMA 7. $\qquad \sup_{\varphi\in\mathcal{B}}\sup_{\varepsilon>0}|(\varphi_\varepsilon*f)(x)|\leqslant AM_\lambda(u)(x)$, *if* $n\lambda<\alpha,\alpha>0$.

LEMMA 8. *Suppose $0<p_0<1$, $p_0^{-1}=1+\lambda$, thus $0<\lambda<\infty$. Then the mapping $u\to M_\lambda(u)$ is of "weak-type (p_0,p_0)", and "strong type (p,p)", whenever $p_0<p<\infty$. This means,*

$$|\{x:M_\lambda(u)>\eta\}|\leqslant A\eta^{-p_0}\|u\|^{p_0}_{H^{p_0}}, \quad \textit{all } \eta>0$$

and

$$\|M_\lambda(u)\|_p\leqslant A_p\|u\|_{H^p}, \quad p_0<p<\infty.$$

Remarks

1. Clearly only the second conclusion of Lemma 8 is needed for Theorem 10. We state the weak-type result for the sake of completeness.

2. The Poisson kernel satisfies the conditions (10.1) and (10.2), (for each α), which is as it should be. Observe also that $u^*(x)\leqslant cM_\lambda(u)(x)$, for each $\lambda>0$, as a simple argument

involving the mean-value property shows. Thus Lemma 8 gives a refinement of the result involving the non-tangential max.function.

3. These results are essentially sharp. For example, if $p<p_0$, there exists $u \in H^p$, so that $M_\lambda(u)=\infty$ everywhere. Alo the condition $\alpha>n[p^{-1}-1]$ cannot be much relaxed. While Theorem 4 (in § 3) shows that when $p=1$ we may replace the Lipschitz conditions of positive order by a Dini condition, the result would be false if $\alpha<n[p^{-1}-1]$ in Theorem 10.

Proof of Lemma 7. This is easy. The simplest case arises if α is integral and more precisely, an even positive integer $\alpha=k=2l$. Let $\Phi=(-\Delta)^l\varphi$. Then as is easily verified

$$(f*\varphi_\varepsilon)(x)=\frac{\varepsilon^{-n-k}}{\Gamma(k)}\iint_{\mathbf{R}^{n+1}_+}u(x-y,t)\,\Phi\left(\frac{y}{\varepsilon}\right)t^{k-1}\,dt\,dy. \tag{10.3}$$

(Just take the Fourier transform of both sides. For the left side we obtain $\hat{f}(\xi)\,\hat{\varphi}(\varepsilon\xi)$, and for the right side $(\varepsilon^{-k}/\Gamma(k))\;\hat{f}(\xi)\,\hat{\Phi}(\varepsilon\xi)\{\int_0^\infty e^{-2\pi|\xi|t}t^{k-1}dt\}=\hat{f}(\xi)\hat{\varphi}(\varepsilon\xi)$, if we keep in mind the fact that $\hat{\Phi}(\xi)=(2\pi|\xi|)^k\hat{\varphi}(\xi)$.)

Next write

$$\iint_{\mathbf{R}^{n+1}_+}=\iint_{T(0,\varepsilon)}+\sum_{j=1}^{\infty}\int_{\{T(0,2^j\varepsilon)-T(0,2^{j-1}\varepsilon)\}}.$$

In view of our assumptions on φ, $|\Phi(y/\varepsilon)|\leqslant A$ on $T(0,\varepsilon)$ and $|\Phi(y/\varepsilon)|\leqslant A\,2^{-j[n+k+\eta]}$, for $(y,t)\in T(0,2^j\varepsilon)-T(0,2^{j-1}\varepsilon)$. Moreover $\int_{T(0,2^j\varepsilon)}t^{k-1}\,dx\,dt=c\,\varepsilon^{n+k}\,2^{j[n+k]}$. Inserting this in the above and using the definition of M_λ, with $n\lambda=k$, gives

$$|(f*\varphi_\varepsilon)(x)|\leqslant A\left(\sum_{j=0}^{\infty}2^{-j\eta}\right)M_\lambda(u)(x).$$

Passing to the sup. over ε we obtain Lemma 7, in the case k is an even positive integer (in fact here in the sharper form that $n\lambda=k$, instead of $n\lambda<k$).

Next assume $0<\alpha<2$, and $n\lambda<\alpha$. Fix a β so that $n\lambda<\beta<\alpha$. We write down the analogous identity as (10.3)

$$(f*\varphi_\varepsilon)(x)=\frac{\varepsilon^{-n-\beta}}{\Gamma(\beta)}\iint_{\mathbf{R}^{n+1}_+}u(x-y,t)\,\Phi\left(\frac{y}{\varepsilon}\right)t^{\beta-1}\,dt\,dy \tag{10.3}$$

except now Φ is the βth derivative of φ, more precisely $\hat{\Phi}(\xi)=(2\pi|\xi|)^\beta\hat{\varphi}(\xi)$. Then all we need to show is that as a consequence of our assumptions (10.1) and (10.2) on we have

$$|\Phi(x)|\leqslant A(1+|x|)^{-n-\beta-\eta}. \tag{10.4}$$

But we know that

$$\Phi(x)=c_\beta\int_{\mathbf{R}^n}[\varphi(x+y)+\varphi(x-y)-2\,\varphi(x)]\frac{dy}{|y|^{n+\beta}},\quad \text{(see [21, 162])}.$$

Break up the range of integration into the sets $|x|<2|y|$ and $|x|\geqslant 2|y|$. For the first set use the estimate (10.1) for φ; for the second set the estimates (10.1) (and (10.2) if necessary), show that $|\varphi(x+y)+\varphi(x-y)-2\varphi(x)| \leqslant A|y|^a(1+|x|)^{-n-a-\eta}$, when $|x| \geqslant 2|y|$. It follows immediately that the inequality (10.4) for Φ holds, and so the proof of the lemma can be concluded as in the previous case by showing that $\sup_{\varepsilon\to 0}|(f*\varphi_\varepsilon)(x)| \leqslant AM_{\lambda'}(u)(x)$ with $n\lambda'=\beta$; this implies the stated result for M_λ. The case of general α is a simple combination of the ideas described for α even integral and $0<\alpha<2$; details are left to the interested reader. Observe that the case $\alpha=0$ corresponds to the usual maximal function with $1<p$. Q.E.D.

We come now to the proof of Lemma 8. It will be an immediate consequence of harmonic majorization and the following lemma about Poisson integrals of L^{p_0} functions.

Suppose $1<p_0<r$ where $r/p_0=1+\lambda$. Let $f\in L^{p_0}(R^n)$, and u its Poisson integral

LEMMA 9. *The mapping*

$$f \to \sup_{h>0} \frac{\left(\iint_{T(x,h)} t^{\lambda n}|u(y,t)|^r \frac{dy\,dt}{t}\right)^{1/r}}{\left(\iint_{T(x,h)} t^{\lambda n}\frac{dx\,dt}{t}\right)^{1/r}} = \mathcal{N}_\lambda^r(f)$$

is of weak type (p_0, p_0), *and of strong type* (p,p) *if* $p_0<p\leqslant\infty$.

Proof. The result is obvious when $p=\infty$, and so by the Marcinkiewicz interpolation theorem it suffices to prove the weak type (p_0,p_0) result. This has some connection with the tangential maximal function considered in [18] and [21, p. 236 § 4.5], but we shall follow the spirit of the argument given in [9] for the sharp estimates of the g_λ^* function.

We shall prove that if $f\in L^{p_0}(\mathbf{R}^n)$, then for an appropriate large constant C

$$|\{x: \mathcal{N}_\lambda^r(f)(x)>\alpha\}| \leqslant C\alpha^{-p_0}\|f\|_{p_0}^{p_0}, \quad \text{all } \alpha>0.$$

We may assume $f\geqslant 0$. We set up a modified Calderón–Zygmund decomposition for $(f)^{p_0}$ (as in [21, pp. 19, 169]) as follows. Let $\Omega=\{x\colon (M(f)^{p_0})(x)>\alpha^{p_0}\}$. Then $|\Omega| \leqslant C\alpha^{-p_0}\|f\|_{p_0}^{p_0}$. Let $\{Q_k\}$ the disjoint family of cubes guaranteed by Whitney's lemma, whose union covers Ω and whose diameters are comparable to their distances from $\complement\Omega$. We then have

$$\frac{1}{|Q_k|}\int_{Q_k} f^{p_0}\,dx \leqslant C\alpha^{p_0}.$$

Of course $f\leqslant\alpha$ in $\complement\Omega$. Let Q_k^* be the cube with the same center as Q_k, but expanded by a fixed factor of 6/5. Then no point is contained in more than N of the Q_k^* (e.g. $N=(12)^n$). Also let $\tilde{Q}_k$ denote the cube with the same center as Q_k but expanded by a factor of 2. We write $\tilde{\Omega}=\bigcup_k\tilde{Q}_k$. Since $|\tilde{\Omega}|\leqslant C\alpha^{-p_0}\|f\|_{p_0}^{p_0}$, it will suffice to estimate $\mathcal{N}_\lambda^r(f)$ on $\complement\tilde{\Omega}$.

Write first $f = f_0 + \sum_{k=1}^{\infty} f_k$. Here $f_0 = f\chi_{\complement\Omega}$, and $f_k = f\chi_{Q_k}$, $k \geq 1$, where χ_E denotes the characteristic function of the set E. Write also $u_j(x, t)$ for the Poisson integral for f_j. Since $0 \leq f_0 \leq \alpha$, then $\mathcal{N}_\lambda^r(f_0)(x) \leq \alpha$, all x, and hence $\{x: \mathcal{N}_\lambda^r(f)(x) > \alpha\} = \emptyset$. We need to estimate therefore $\mathcal{N}_\lambda^r(\sum_{k=1}^{\infty} f_k)$.

Write $U(x, t) = \sum_{k=1}^{\infty} \chi_{Q_k^*}(x) u_k(x, t)$, and $V(x, t) = \sum_{k=1}^{\infty}(1 - \chi_{Q_k^*}(x)) u_k(x, t)$. Then $U + V$ is the Poisson integral of $\sum_{k=1}^{\infty} f_k$, and it suffices to estimate

$$\frac{\iint_{T(x,h)} t^{\lambda n} (U(y,t))^r \frac{dy\,dt}{t}}{\iint_{T(x,h)} t^{\lambda n} \frac{dx\,dt}{t}}, \quad x \in \complement\tilde{\Omega}, \tag{10.5}$$

and the analogous expression with U replaced by V.

Since the cubes Q_k^* have the bounded intersection property,

$$\iint_{T(x,h)} t^{\lambda n} (U(y,t))^r \frac{dy\,dt}{t} \leq N^{r-1} \sum_{k=1}^{\infty} \iint_{T(x,h)} \chi_{Q_k^*}(y) (u_k(y,t))^r t^{\lambda n} \frac{dy\,dt}{t}.$$

For each integral appearing in the sum in the right-hand side there are two possibilities

(i) Either $T(x, h) \cap \{Q_k^* \times (0, \infty)\} = \emptyset$,

and then the summand is zero.

(ii) Or $T(x, h) \cap \{Q_k^* \times (0, \infty\} \neq \emptyset$;

by the geometry of the situation, since $x \notin \tilde{Q}_k$, it follows that then $h \geq C|x - x^k|$, where x^k is the center of Q_k. In this case we replace the integral over $T(x, h)$ by the integral over all of $\mathbf{R}_+^{n+1}$ and invoke the estimate (9.4), giving

$$\iint_{\mathbf{R}_+^{n+1}} (u_k(y,t))^r t^{\lambda n} \frac{dy\,dt}{t} \leq A \|f_k\|_{p_0}^r \leq C |Q_k|^{r/p_0} \alpha^r.$$

However for the kth term

$$\iint_{T(x,h)} t^{\lambda n} \frac{dy\,dt}{t} = ch^{\lambda n + n} \geq c|x - x^k|^{\lambda n + n}.$$

Thus, altogether as an estimate for (10.5) we have

$$C\alpha^r \sum_{k=1}^{\infty} \frac{|Q_k|^{r/p_0}}{|x - x^k|^{n\lambda + n}} \leq C\alpha^r \int \frac{(\delta(y))^{\lambda n} dy}{|x - y|^{\lambda n + n}} = C\alpha^r I(x).$$

$\delta(y)$ denotes the distance of y from $\complement\Omega$. The first inequality above follows from the fact that the diameters of the Q_k are comparable to their distances from $\complement\Omega$, and that

$|x-x^k|\approx|x-y|$, if $y\in Q_k$, $x\notin\tilde{Q}_k$. The expression $I(x)$ is the familiar Marcinkiewicz integral involving the distance function, and we have therefore $|\{x\in\complement\tilde{\Omega}: C\alpha^r I(x)>\alpha^r\}|\leqslant C\int_{\complement\Omega} I(x)\,dx\leqslant C|\Omega|\leqslant C\alpha^{-p_0}\|f\|_{p_0}^{p_0}$, by [21, 16]. We have thus obtained the correct estimate for (10.5). To prove the analogue of (10.5) with V in place of U, observe that since $1-\chi_{Q_k^*}$ is non-vanishing only in $\complement Q_k^*$, and u_k is the Poisson integral of a function supported on Q_k it follows that $(1-\chi_{Q_k^*})u_k(x,t)\leqslant CU_k(x,t)$ where U_k is the Poisson integral of the function which is constant on Q_k and has the same mean-value on Q_k as $f\chi_{Q_k}=f_k$. Thus $(1-\chi_{Q_k^*})u_k(x,t)\leqslant C$ Poisson integral of $\alpha\chi_{Q_k}$. Altogether then

$$V(x,t)=\sum(1-\chi_{Q_k^*})\,u_k(x,t)\leqslant C\alpha\,\text{P.I.}(\chi_\Omega)\leqslant C\alpha,$$

and so we have reduced matters here to the trivial estimate for L^∞. Gathering all these estimates together we have

$$|\{x:\mathcal{N}_\lambda^r(f)(x)>2\alpha\}|\leqslant|\tilde{\Omega}|+|\{x\in\complement\tilde{\Omega}:\mathcal{N}_\lambda^r(f_0)(x)>\alpha\}|$$
$$+|\{x\in\complement\tilde{\Omega}:\mathcal{N}_\lambda^r(\sum_{k=1}^{\infty}f_k)(x)>\alpha\}|\leqslant C\alpha^{-p_0}\|f\|_{p_0}^{p_0}.$$

This concludes the proof of Lemma 9.

Proof of Lemma 8.

We repeat the argument of harmonic majorization used several times before. We have $u(x,t)\leqslant h^{1/p_k}(x,t)$ where $h(x,t)$ is the Poisson integral of an L^{p/p_k} function f and $\|u\|_{H^p}=\|f\|_{p/p_k}^{1/p_k}$. We set then $r=1/p_k$, and apply Lemma 9, with of course $h^{1/p}(x,t)$ in place of $|u(x,t)|^r$. The critical relation then becomes $p_0^{-1}=1+\lambda$, and so Lemma 8 follows from Lemma 9.

With this we have also concluded the proof of Theorem 10. By the theory of H^p spaces we know that whenever $u\in H^p$, $\lim_{t\to 0}u(x,t)$ exists almost everywhere and also dominatedly in L^p (see [25, Chapter VI], and also Theorem 9 in Section 8). Let us call this pointwise limit $f(x)$, (mindful of the possible ambiguities this may cause since we also called f the distributional limit of $u(x,t)$ as $t\to 0$).

COROLLARY. *Let $u\in H^p$, and suppose φ satisfies the conditions of Theorem 10. Assume in addition that $\int_{\mathbf{R}^n}\varphi\,dx=1$. Then $\lim_{\varepsilon\to 0}(u*\varphi_\varepsilon)(x)$ exists and equals $f(x)$ for almost every x.*

In fact write $u_\delta(x,t)=u(x,t+\delta)$. Then $u=u_\delta+(u-u_\delta)$, and $f=u_\delta(\cdot,0)+(f-u_\delta(\cdot,0))$. So $f*\varphi_\varepsilon=u_\delta(\cdot,0)*\varphi_\varepsilon+(f-u_\delta(\cdot,0))*\varphi_\varepsilon$, as $\varepsilon\to 0$, $u_\delta(\cdot,0)*\varphi_\varepsilon\to u_\delta(\cdot,0)$ everywhere; but by Theorem 10 $\|\sup_{\varepsilon>0}(f-u_\delta(\cdot,0))*\varphi_\varepsilon\|_p\leqslant A\|u-u_\delta\|_{H^p}\to 0$ as $\delta\to 0$, and so the corollary follows by standard arguments.(1)

(1) The fact that $\|u-u_\delta\|_{H^p}\to 0$, as $\delta\to 0$, when $u\in H^p$ is most easily proved by using the fact that $\int_{\mathbf{R}^n}(\sum_{(j)}|u_{(j)}(x,t)|^2)^{p/2}dx$ is a decreasing function in $t>0$. Compare [24].

V. Real-variable theory of H^p

11. Equivalence of several definitions

The extension of the theorem of Burkholder–Gundy–Silverstein (Theorem 9 in Section 8) shows that H^p arises naturally as a space of harmonic functions, free from notions of conjugacy. Theorem 10 goes further, and suggests that the H^p spaces are utterly intrinsic—they arise as soon as we ask simple questions about regularizing distributions with approximate identities. From this point of view the special rôle of the Poisson kernel fades into the background.

In this section we shall show how to carry out much of H^p theory by purely real variable methods. By our methods H^p can be treated in many ways like L^p, with certain natural changes. As a result, we can sharpen some known H^p theorems, and also prove results unattainable by earlier techniques.

We begin with an indication of our goals. Fix $\varphi \in \mathcal{S}$ with $\int_{\mathbf{R}^n}\varphi(x)\,dx = 1$. We are tempted to say that a tempered distribution f is of the class H^p $(0<p<\infty)$, if the maximal function $\sup_{t>0}|\varphi_t * f(\cdot)|$ belongs to $L^p(\varphi_t(x)=t^{-n}\varphi(x/t))$. For this definition to be significant, the resulting class H^p would have to be independent of the given $\varphi\in\mathcal{S}$ we started from, and if so H^p would be intrinsically defined. In fact we would also want that H^p to be the same as the H^p space studied in parts II–IV. Fortunately all this turns out to be so.

The assertions just made have interesting consequences. For instance consider the H^p space defined in terms of the heat equation in $\mathbf{R}^{n+1}_+$ (there $\varphi(x)=(2\pi)^{-n/2}e^{-\frac{1}{2}|x|^2}$). A solution of $\partial u/\partial t = \Delta_x u$ belongs to $H^p_{\text{heat}}(\mathbf{R}^{n+1}_+)$ if the maximal function $\sup_{t>0}|u(x,t)|$ belongs to L^p. Then $H^p_{\text{heat}}(\mathbf{R}^{n+1}_+)$ is really the same as the ordinary H^p, which arises from Laplace's equation. More precisely the functions in both H^p spaces have the same "boundary values" on $\mathbf{R}^n$.

All these claims are immediate from the following theorem.

THEOREM 11. *Fix $0<p<\infty$. For any tempered distribution f, the following are equivalent.*

(A) $u^+(x)=\sup\limits_{t>0}|\varphi_t * f(x)|\in L^p$ *for some $\varphi\in\mathcal{S}$ satisfying $\int_{\mathbf{R}^n}\varphi(x)\,dx=1$.*

(B) $u^*(x)=\sup\limits_{|x-y|<t}|\varphi_t * f(y)|\in L^p$ *for some φ as above.*

(C) $f^*(x)=\sup\limits_{\Phi\in\mathcal{A}}\sup\limits_{|x-y|<t}|\Phi_t * f(y)|\in L^p$, *where*

$$\mathcal{A}=\left\{\Phi\in\mathcal{S}\,\Big|\,\int_{\mathbf{R}^n}(1+|x|)^{N_0}\left(\sum_{|\alpha|\leqslant N_0}\left|\frac{\partial^\alpha}{\partial x^\alpha}\Phi(x)\right|^2\right)dx\leqslant 1\right\}$$

and N_0 is a large number depending only on p and n.

(D) *The distribution f arises as $lim_{t\to 0} u(x,t)$, where $u \in H^p$.*

In view of the above theorem we shall say, with a slight abuse of language, that a tempered distribution f satisfying the conclusions (A), (B), (C) or (D) is of class H^p.

Remarks 1. Condition (D) could alternately by phrased as follows: $P_t * f(x) = \lim_{\delta\to 0} \int_{\mathbf{R}^n} e^{-\delta|y|^2} P_t(y) f(x-y)\, dy$ exists and $\sup_{|x-y|<t} |(P_t * f)(y)| \in L^p$. Many other variants are possible.

2. We also have $\|u^+\|_p \approx \|u\|_{H^p}$, with similar equivalences for (B) and (C), as the proof of Theorem 11 shows.

3. The purpose of the class $\mathcal{A}$ in (C) is merely to fix some reasonable normalization for approximate identities. In all that follows, the "large number" N_0 defining $\mathcal{A}$ in (C) may change from one occurrence of $\mathcal{A}$ to the next. This justifies statements like "$\Phi \in \mathcal{A}$ implies $\partial\Phi/\partial x_1 \in \mathcal{A}$".

4. Condition (C) plays an important rôle in the real-variable theory for H^p. For, consider the simple problem of estimating $\int_{\mathbf{R}^n} f(x)\varphi(x)\,dx$ for $\varphi \in S$, $f \in H^p$ $(0 < p < 1)$. If $\varphi \in \mathcal{A}$, then of course $|\int_{\mathbf{R}^n} f(x)\varphi(x)\,dx| = |f * \varphi(0)| \leqslant f^*(0)$. More generally, if φ can be written in the form $\varphi(x) = A\, d^{-n}\Phi((x-x_0)/d)$ with $\Phi \in \mathcal{A}$, $x_0 \in \mathbf{R}^n$, A and $d > 0$, then $|\int_{\mathbf{R}^n} f(x)\varphi(x)dx| \leqslant A\, f^*(y)$ whenever $|x_0 - y| < d$. In other words,

$$\left|\int_{\mathbf{R}^n} f(x)\,\varphi(x)\,dx\right| \leqslant N(\varphi; x_0, d) \min_{|x_0-y|<d} f^*(y) \tag{11.1}$$

where

$$N(\varphi; x_0, d) = \min\left\{A > 0 \,|\, \varphi \text{ can be written as } \varphi(x) = A\, d^{-n}\Phi\left(\frac{x_0 - x}{d}\right) \text{ with } \Phi \in \mathcal{A}\right\}$$
$$\approx \left(\int_{\mathbf{R}^n}\left(1 + \frac{|x-x_0|}{d}\right)^{N_0}\left(\sum_{|\alpha|\leqslant N_0} d^{2n|\alpha|}\left|\frac{\partial^\alpha}{\partial x^\alpha}\varphi(x)\right|^2\right)dx\right)^{\frac{1}{2}}.$$

(11.1) holds for all $f \in H^p$, $\varphi \in S$, $x_0 \in \mathbf{R}^n$, $d > 0$. In practice, $N(\varphi; x_0, d)$ is easily computed. Roughly speaking, if φ is a "bump" function of "thickness" d, centered at x_0, then $N(\varphi; x_0, d)$ is essentially $\int_{\mathbf{R}^n} |\varphi(x)|\,dx$.

From (11.1) we obtain at once

$$\left|\int_{\mathbf{R}^n} f(x)\,\varphi(x)\,dx\right| \leqslant N(\varphi; x_0, d)\left(\frac{1}{d^n}\int_{|x-x_0|<d} (f^*(y))^p\,dy\right)^{1/p}. \tag{11.2}$$

Proof of Theorem 11. The idea is straight-forward. We simply have to formalize the plausible fact that all approximate identities are more or less alike, and that any one can be built up from any other. First we prove (B) ⇒ (C) ⇒ (A) ⇒ (B) and then we take care of (D).

(B) ⇒ (C). Set $u(x,t)=\varphi_t * f(x)$, assume $u^* \in L^p$, and fix a $\Phi \in \mathcal{A}$. According to Lemma 1 in § 7, the "tangential maximal function"

$$u^{**}(x) = \sup_{\substack{t>0\\ y\in \mathbf{R}^n}} |u(y,t)| \left(\frac{t}{|x-y|+t}\right)^N$$

also belongs to L^p. We shall prove that $\sup_{|x-y|<t} |\Phi_t * f(y)| \leqslant Cu^{**}(x)$ for all $x \in \mathbf{R}^n$.

Step 1. Assume first that Φ has the form $\Phi = \psi * \varphi_s$ with $\psi \in \mathcal{A}$ and $0 \leqslant s \leqslant 1$. Then for $|x-y|<t$, we have

$$|\Phi_t * f(y)| = |\psi_t * (\varphi_{st} * f)(y)| = \left|\int_{\mathbf{R}^n} \psi_t(y-z)\, u(z,st)\, dz\right|$$

$$\leqslant \int_{\mathbf{R}^n} |\psi_t(y-z)|\,|u(z,st)|\,dz \leqslant \int_{\mathbf{R}^n} |\psi_t(y-z)| \left[\left(\frac{st}{|x-z|+st}\right)^{-N} u^{**}(x)\right] dz \leqslant Cs^{-N}u^{**}(x),$$

since $\psi \in \mathcal{A}$. Thus, $\sup_{|x-y|<t} |\Phi_t * f(y)| \leqslant Cs^{-N}u^{**}(x)$ if $\Phi = \psi * \psi_s, \psi \in \mathcal{A}, 0 < s \leqslant 1$.

Step 2. Suppose $\Phi \in \mathcal{A}$ and $\hat{\Phi}(\xi)$ is supported in $|\xi| < 2^r$. Then Φ can be written as $\Phi = \psi * \varphi_s$ with $\psi \in \mathcal{A}$ and $s = c\cdot 2^{-r}$ (c small but independent of r). To see this, we simply set $\hat{\psi}(\xi) = \hat{\Phi}(\xi)/\hat{\varphi}(s\xi)$, and check that $\hat{\varphi}(s\xi) \neq 0$ in support $(\hat{\Phi}) \subseteq \{|\xi| < 2^r\}$. Actually, $|\hat{\varphi}(s\xi)| \geqslant \frac{1}{2}$ for $|\xi| < 2^r = c/s$, since $\hat{\varphi}$ is continuous, $\hat{\varphi}(0) = \int_{\mathbf{R}^n} \varphi(x)\,dx = 1$, and $|s\xi - 0| \leqslant c$ for $|\xi| < 2^r$. So $\Phi = \psi * \varphi_s$ as claimed. By the conclusion of step 1, we have $\sup_{|x-y|<t} |\Phi_t * f(y)| \leqslant C\cdot 2^{Nr}\, u^{**}(x)$ if $\Phi \in \mathcal{A}$ and $\hat{\Phi}(\xi)$ is supported in $\{|\xi| < 2^r\}$.

Step 3. Any $\Phi \in \mathcal{A}$ may be written in the form $\Phi = \sum_{r=0}^{\infty} C_r \Phi_{(r)}$, with $\Phi_{(r)} \in \mathcal{A}$, $\hat{\Phi}_{(r)}$ supported in $\{|\xi| < 2^r\}$, and $C_r = O(2^{-10Nr})$. (Simply cut up $\hat{\Phi}$.) By the conclusion of step 2,

$$\sup_{|x-y|<t} |\Phi_t * f(y)| \leqslant \sum_{r=0}^{\infty} |C_r| \sup_{|x-y|<t} |\Phi_{(r)t} * f(y)| \leqslant Cu^{**}(x).$$

Therefore, $f^*(x) \leqslant Cu^{**}(x)$ for all $x \in R^n$, so that $\|f^*\|_p \leqslant C\|u^{**}\|_p \leqslant C\|u^*\|_p < \infty$. This proves (B) ⇒ (C).

Remark. Taking $\Phi = \partial\varphi/\partial x_j$ above, we obtain $\|U^*\|_p \leqslant C\|u^*\|_p$ where $u(x,t) = \varphi_t * f(x)$ and $U(x,t) = t|\nabla_x u(x,t)|$. For technical reasons, we set

$$u^*_{\varepsilon N}(x) = \sup_{|x-y|<t<\varepsilon^{-1}} |u(y,t)| \left(\frac{t}{\varepsilon+t}\right)^N (1+\varepsilon|y|)^{-N}$$

and
$$U^*_{\varepsilon N}(x) = \sup_{|x-y|<t<\varepsilon^{-1}} t|\nabla_y u(y,t)| \left(\frac{t}{\varepsilon+t}\right) (1+\varepsilon|y|)^{-N},$$

and assert that $\|U^*_{\varepsilon N}\|_p \leqslant C\|u^*_{\varepsilon N}\|_p$ for all $f \in \mathcal{S}'$, with C independent of $0<\varepsilon<1$. (The proof simply copies that of (B) ⇒ (C) above, with obvious changes). As $\varepsilon \to 0$, $u^*_{\varepsilon N}(x) \nearrow u^*(x)$ and $U^*_{\varepsilon N}(x) \nearrow U^*(x)$ for all $x \in R^n$. However, given any $f \in \mathcal{S}'$, there is a large $N>0$ which makes $U^*_{\varepsilon N}(\cdot)$, $u^*_{\varepsilon N}(\cdot) \in L^\infty \cap L^p$ for all $\varepsilon>0$.

(C) ⇒ (A) is trivial.

(A) ⇒ (B). We adapt an alternate proof of Corollary 2 of Theorem 9, devised by D. Burkholder and R. Gundy. Set $u(x, t) = \varphi_t * f(x)$, and assume $u^+(x) = \sup_{t>0} |u(x, t)| \in L^p$. To prove that $u^*(x) = \sup_{|x-y|<t} |u(y, t)| \in L^p$, we shall dominate u^* by the function

$$M(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q (u^+(y))^r \, dy \right)^{1/r} \quad (0<r<p).$$

$\|M(\cdot)\|_p \leqslant C\|u^+\|_p$ by the maximal theorem.

Fix a large $N>0$ so that $u^*_{\varepsilon N}$, defined above, belongs to L^p for all $\varepsilon>0$. We want to prove $\|u^*_{\varepsilon N}\|_p \leqslant C\|M(\cdot)\|_p$ with C independent of ε. Rather than compare $u^*_{\varepsilon N}(x)$ with $M(x)$ for all $x \in R^n$, we shall restrict attention to x's in the "good" set $G_{\varepsilon N} = \{U^*_{\varepsilon N} \leqslant B u^*_{\varepsilon N}\}$. The set $G_{\varepsilon N}$ already captures most of the bad behavior of $u^*_{\varepsilon N}$, since

$$\int_{\mathbf{R}^n - G_{\varepsilon N}} (u^*_{\varepsilon N}(x))^p \, dx \leqslant \int_{\mathbf{R}^n - G_{\varepsilon N}} \left(\frac{U^*_{\varepsilon N}(x)}{B} \right)^p dx \leqslant \frac{1}{B^p} \int_{\mathbf{R}^n} (U^*_{\varepsilon N}(x))^p \, dx \leqslant \frac{C}{B^p} \int_{\mathbf{R}^n} (u^*_{\varepsilon N}(x))^p \, dx$$
$$\leqslant \frac{1}{2} \int_{\mathbf{R}^n} (u^*_{\varepsilon N}(x))^p \, dx$$

(by the remark) if B is large enough. Since $\varepsilon>0$, we know a priori that $\int_{\mathbf{R}^n} (u^*_{\varepsilon N}(x))^p \, dx < \infty$; hence, the above chain of inequalities shows that

$$\int_{\mathbf{R}^n} (u^*_{\varepsilon N}(x))^p \, dx \leqslant 2 \int_{G_{\varepsilon N}} (u^*_{\varepsilon N}(x))^p \, dx.$$

Therefore, we need only estimate $u^*_{\varepsilon N}(x)$ for $x \in G_{\varepsilon N}$.

We claim that $u^*_{\varepsilon N}(x) \leqslant CM(x)$ for $x \in G_{\varepsilon N}$. To see this, pick $(y, t) \in \mathbf{R}^{n+1}_+$ satisfying $|x-y|<t<\varepsilon^{-1}$, $|u(y, t)(t/(\varepsilon+t))^N (1+\varepsilon|y|)^{-N}| \geqslant \frac{1}{2} u^*_{\varepsilon N}(x)$. Since $x \in G_{\varepsilon N}$, we have

$$t|\nabla_z u(z, t)| \left\{ \left(\frac{t}{\varepsilon+t} \right)^N (1+\varepsilon|z|)^{-N} \right\} \leqslant 2B|u(y, t)| \left\{ \left(\frac{t}{\varepsilon+t} \right)^N (1+\varepsilon|y|)^{-N} \right\}$$

for all z satisfying $|x-z|<t$. By the geometry of the situation, the two factors in braces are roughly the same, so that $t|\nabla_z u(z, t)| \leqslant C|u(y, t)|$ for all z with $|z-x|<t$. Consequently, $|u(\omega, t)| \geqslant \frac{1}{2}|u(y; t)|$ for all ω in the fat set $P = \{\omega \in R^n | \ |\omega-x|<t, |\omega-y|<t/2C\}$. In particular, $|u(\omega, t)| \geqslant \frac{1}{2}|u(y, t)|(t/(\varepsilon+t))^N(1+\varepsilon|y|)^{-N} \geqslant \frac{1}{4} u^*_{\varepsilon N}(x)$ for $\omega \in P$. From that key fact, $u^*_{\varepsilon N}(x) \leqslant M(x)$ is easy. We simply write

$$M^r(x) \geqslant \frac{c}{|B(x,2t)|}\int_{B(x,2t)} (u^+(\omega))^r\,d\omega$$
$$\geqslant \frac{c}{|B(x,2t)|}\int_{B(x,2t)} |u(\omega,t)|^r\,d\omega \geqslant c\left(\tfrac{1}{4}u^*_{\varepsilon N}(x)\right)^r \frac{|P|}{|B(x,2t)|} \geqslant c(u^*_{\varepsilon N}(x))^r.$$

Thus, $u^*_{\varepsilon N}(x) \leqslant CM(x)$ for $x\in G_{\varepsilon N}$ which implies that

$$\int_{\mathbf{R}^n} (u^*_{\varepsilon N}(x))^p\,dx \leqslant 2\int_{G_{\varepsilon N}} (u^*_{\varepsilon N}(x))^p\,dx$$
$$\leqslant C\int_{G_{\varepsilon N}} (M(x))^p dx \leqslant C\int_{\mathbf{R}^n}(M(x))^p\,dx \leqslant C\int_{\mathbf{R}^n}(u^+(x))^p\,dx,$$

with C independent of ε. Letting $\varepsilon \to 0$, we obtain $\|u^*\|_p \leqslant C\|u^+\|_p$, which completes the proof of (A) $\Rightarrow$ (B).

To conclude the proof of Theorem 1, we prove (C) $\Rightarrow$ (D) $\Rightarrow$ (A). Both steps are easy, since the Poisson kernel is "almost" in $\mathcal{A}$—it just doesn't decrease fast enough at infinity.

(C) $\Rightarrow$ (D). First of all, note that $P_t * f(x) = \lim_{\delta\to 0}\int_{\mathbf{R}^n} e^{-\delta|y|^2}P_t(y)\,f(x-y)\,dy \equiv \lim_{\delta\to 0}P_{t,\delta}*f(x)$ always exists for $(x,t)\in \mathbf{R}^{n+1}_+$, since $\lim_{\delta_1,\delta_2\to 0}|(P_{t,\delta_1}-P_{t,\delta_2})*f(x)|=0$ by (11.2). Details are left to the reader. To estimate $P_t * f(\cdot)$, just write $P_t = P_{t,\delta_0} + \sum_{k=1}^{\infty}(P_{t,\delta_k}-P_{t,\delta_{k-1}}) \equiv \sum_{k=0}^{\infty}P_t^k$, with $\delta_k = 2^{-2k}t^{-2}$. For $|x-y|<t$, (11.1) yields

$$|(P_t * f)(y)| \leqslant \sum_{k=0}^{\infty}|P_t^k * f(y)| \leqslant \sum_{k=0}^{\infty} N(P_t^k;0,2^k t)\,f^*(x) \leqslant Cf^*(x).$$

Thus $\sup_{|x-y|<t}|P_t * f(y)| \in L^p$, and $P_t * f$ satisfies the H^p characterization of Theorem 9 Hence f satisfies (D).

(D) $\Rightarrow$ (A). We manufacture a rapidly decreasing approximate identity from the Poisson kernel. Let $\varphi_t(x) \equiv \int_1^\infty \psi(t)\,P_{t/s}(x)\,ds$, where ψ is rapidly decreasing at infinity, and satisfies

$$\int_1^\infty s^k\psi(s)\,ds = \begin{cases}1 \text{ if } k=0\\ 0 \text{ if } k=1,2,3,\ldots.\end{cases}$$

(See [21, p. 183] for such a ψ). Then $\varphi_t \in \mathcal{S}$. To see this, note that $\varphi_t(x) = t^{-n}\varphi_1(x/t)$ and that $\hat\varphi_1(\xi) = \int_1^\infty \psi(s)\,e^{-s|\xi|}\,ds$ is automatically rapidly decreasing at infinity and smooth outside the origin. Near the origin, we expand $e^{-s|\xi|}$ in powers of $s|\xi|$ to obtain $\hat\varphi_1(\xi) = \sum_{k=0}^{N-1}(-1)^k/k!\,|\xi|^k\int_1^\infty s^k\psi(s)\,ds + O(|\xi|^N) = 1 + O(|\xi|^N)$ for each $N>0$. Therefore $\hat\varphi_1$ is also smooth at the origin, so that $\hat\varphi_1 \in \mathcal{S}$ and hence $\varphi_1\in\mathcal{S}$. Also, recall that $\int_{\mathbf{R}^n}\varphi_1(x)\,dx = \hat\varphi_1(0) = 1$. So φ_1 is as required in (A). However, $\sup_{t>0}|\varphi_t * f(x)| = \sup_{t>0}|\int_1^\infty \varphi(s)\,(P_{t/s} * f(x))\,ds| \leqslant \sup_{t>0}|P_t * f(x)\int_1^\infty|\psi(s)|\,ds \leqslant C\sup_{t>0}|P_t * f(x)|$ for all $x\in\mathbf{R}^n$. This proves (A).

Theorem 11 is proved. Q.E.D.

12. Applications

Theorem 11 provides us with great flexibility in studying the Fourier analysis of H^p spaces. Instead of relying exclusively on complex methods when $n=1$, or conjugate harmonic functions for general n, we can now apply the circle of ideas leading to the Calderón–Zygmund inequality. Often, this produces simpler proofs of more precise results than those previously known. To illustrate, we prove the analogue of the Calderón–Zygmund inequality for H^p by real-variable methods.

Let K be a tempered distribution whose Fourier transform is a bounded function $|\hat{K}(\xi)| \leqslant B$. Assume also that K is of class $C^{(N_0)}$ away from the origin, and

$$\left|\frac{\partial^\alpha K}{\partial x^\alpha}\right| \leqslant B|x|^{-n-|\alpha|}, \quad |\alpha| \leqslant N_0.$$

(N_0 is the index appearing in statement (C) of Theorem 11). Next let ψ be a fixed C^∞ function of compact support which is 1 in a neighborhood of the origin. Write $K_M = K\psi(x/M)$. Finally let φ be another fixed C^∞ function with compact support such that $\int \varphi\, dx = 1$. Write $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, and $K_{\varepsilon M} = K_M * \varphi_\varepsilon$. Then each $K_{\varepsilon M}$ is a C^∞ function of compact support; and the $K_{\varepsilon M}$ satisfy the conditions imposed on K, uniformly in ε and M.(1)

LEMMA 10. *Suppose $f \in L^1(\mathbf{R}^n)$, then $K*f = \lim_{\varepsilon\to 0,\, M\to\infty} K_{\varepsilon M} * f$ converges in the sense of tempered distributions, and the limit is independent of the choice of φ and ψ.*

This follows immediately from the fact, which the reader may easily verify, that $\hat{K}_{\varepsilon M}(\xi)$ converges boundedly to $\hat{K}(\xi)$, as $\varepsilon \to 0$ and $M \to \infty$.

We pass to the basic a priori inequality.

LEMMA 11. *Let f be a bounded C^∞ function on $\mathbf{R}^n$. Then $\|\sup_\varepsilon |K_{\varepsilon M} * f(\cdot)|\|_p \leqslant C_p \|f^*\|_p$ $(0<p<\infty)$, with C_p independent of M.*

Proof. We imitate the proof of the Calderón–Zygmund [5] inequality. Given $\alpha>0$, set $\Omega = \{f^*(x) > \alpha\}$. The proof of the Whitney extension theorem [21] exhibits a collection $\{Q_j\}$ of cubes, and a family of smooth functions $\{\varphi_j\}$ on R^n, with the properties:

(1) Ω is the disjoint union of the $\{Q_j\}$.

(1′) $\chi_\Omega = \sum_j \varphi_j$ and each $\varphi_j \geqslant 0$.

(2) distance $(R^n - \Omega, Q_j) \sim$ diameter $(Q_j) \equiv d_j$.

(2′) φ_j is supported in the cube Q_j expanded by the factor $\frac{6}{5}$, say.

(1) To see this, notice first that the last assertion is evident for K_M. It then follows for $\varphi_\varepsilon * K_M = K_{\varepsilon M}$ by the argument in [9, § 3].

(2″) φ_j has "thickness" d_j. That is, $\|\partial^\alpha \varphi_j/\partial x^\alpha\|_\infty \leqslant C_\alpha d_j^{-|\alpha|}$ for each multi-index α. Also, $a_j = \int_{\mathbf{R}^n} \varphi_j(x)\,dx \sim d_j^n$.

Denote the center of Q_j by x_j.

Now $f = f\chi_{\mathbf{R}^n-\Omega} + \sum_j f\varphi_j$. We shall replace f by the "good" function $\tilde{f} = f\chi_{\mathbf{R}^n-\Omega} + \sum_j b_j\varphi_j$, where b_j is a constant so chosen that $\int_{\mathbf{R}^n} b_i\varphi_j(x)\,dx = \int_{\mathbf{R}^n} f(x)\varphi_j(x)\,dx$. Here, φ_j is analogous to χ_{Q_j}, and b_j corresponds to the average of f over Q_j. We have $b_j = (1/a_j)\int_{\mathbf{R}^n} f(x)\varphi_j(x)\,dx$.

To estimate $\sup_{\varepsilon,M}|K_{\varepsilon M} * f(\cdot)|$, we shall study $|K_{\varepsilon M} * \tilde{f}|$ and $|K_{\varepsilon M} * (f-\tilde{f})|$. Let us begin with the first term. We claim that $\|\tilde{f}\|_\infty \leqslant C\alpha$. For, we know from (2) that $\inf_{|x_j-y|<d_j} f^*(y) \leqslant \alpha$ for each j. Elementary computations with (2′) and (2″) show that $N(\varphi_j/a_j; x_j, d_j) = O(1)$, so that $|b_j| = |\int_{\mathbf{R}^n}(1/a_j)\varphi_j(y) f(y)\,dy| \leqslant C \inf_{|x_j-y|<d_j} f^*(y) \leqslant C\alpha$. Therefore,

$$|\tilde{f}(x)| \leqslant |f(x)\chi_{\mathbf{R}^n-\Omega}(x)| + C\alpha\sum_j \varphi_j(x) = |f(x)|\chi_{\mathbf{R}^n-\Omega}(x) + C\alpha\chi_\Omega(x)$$

for all $x \in \mathbf{R}^n$, which proves that $\|\tilde{f}\|_\infty \leqslant C\alpha$. Now we can write

$$|\{\sup_\varepsilon |K_{\varepsilon M} * \tilde{f}(\cdot)| > \alpha\}| \leqslant \frac{\|\sup_\varepsilon |K_{\varepsilon M} * \tilde{f}(\cdot)|\|_2^2}{\alpha^2} \leqslant \frac{C\|\tilde{f}\|_2^2}{\alpha^2} \leqslant \frac{C}{\alpha^2}\int_{\mathbf{R}^n-\Omega} |f(y)|^2\,dy + C|\Omega|, \quad (12.1)$$

by virtue of our estimates for $\tilde{f}$. (12.1) is our basic estimate for $K_{\varepsilon M} * \tilde{f}$.

We turn to the "error" term

$$K_{\varepsilon M} * (f-\tilde{f})(x) = \sum_j K_{\varepsilon M} * (f\varphi_j - b_j\varphi_j)(x). \quad (12.2)$$

For fixed j,

$$\begin{aligned} K_{\varepsilon M} * (f\varphi_j - b_j\varphi_j)(x) &= \int_{\mathbf{R}^n} K_{\varepsilon M}(x-\omega)\varphi_j(\omega) f(\omega)\,d\omega - \int_{\mathbf{R}^n} K_{\varepsilon M}(x-y) b_j\varphi_j(y)\,dy \\ &= \int_{\mathbf{R}^n} K_{\varepsilon M}(x-\omega)\varphi_j(\omega) f(\omega)\,d\omega - \int_{\mathbf{R}^n} K_{\varepsilon M}(x-y)\left[\frac{1}{a_j}\int_{\mathbf{R}^n}\varphi_j(\omega) f(\omega)\,d\omega\right]\varphi_j(y)\,dy \\ &= \int_{\mathbf{R}^n} K_{\varepsilon M}(x-\omega)\varphi_j(\omega) f(\omega)\,d\omega - \int_{\mathbf{R}^n}\left[\frac{1}{a_j}\int_{\mathbf{R}^n} K_{\varepsilon M}(x-y)\varphi_j(y)\,dy\right]\varphi_j(\omega) f(\omega)\,d\omega \\ &= \int_{\mathbf{R}^n}\left\{K_{\varepsilon M}(x-\omega) - \left[\frac{1}{a_j}\int_{\mathbf{R}^n} K_{\varepsilon M}(x-y)\varphi_j(y)\,dy\right]\right\}\varphi_j(\omega) f(\omega)\,d\omega \equiv I_j. \end{aligned}$$

Here, $\{-\}$ plays the rôle of $\{K(x-y) - K(x-x_j)\}$ from standard Calderón–Zygmund proofs, and I_j is like $\int_{Q_j}\{K(x-\omega) - K(x-x_j)\} f(\omega)\,d\omega$. Elementary computations show that $N(\{-\}\varphi_j; x_j; d_j) \leqslant (Cd_j^{n+1})/|x-x_j|^{n+1}$ for each fixed $x \in \mathbf{R}^n - \Omega$. Therefore, by (11.1),

$$|I_j| = \left|\int_{\mathbf{R}^n}\{-\}\varphi_j(\omega) f(\omega)\,d\omega\right| \leqslant \frac{Cd_j^{n+1}}{|x-x_j|^{n+1}} \min_{|x_j-y|<d_j} f^*(y) \leqslant C\alpha\frac{d_j^{n+1}}{|x-x_j|^{n+1}}.$$

Putting this into (12.2) yields

$$\sup_\varepsilon |K_{\varepsilon M} * (f - \tilde{f})(x)| \leqslant C\alpha \sum_j \frac{d_j^{n+1}}{|x - x_j|^{n+1}} \equiv C\alpha\, \mu(x) \text{ for } x \notin \Omega. \tag{12.3}$$

$(\cdot)$ is a standard Marcinkiewicz "distance function integral", and we apply the well-known trick:

$$\int_{\mathbf{R}^n - \Omega} \mu(x)\, dx \leqslant \sum_j \int_{\mathbf{R}^n - \Omega_j} \frac{d_j^{n+1}}{|x - x_j|^{n+1}}\, dx \leqslant c \sum_j d_j^n = c\,|\Omega|,$$

so that $\mu(x) \leqslant 1$ except on a set of measure $\leqslant C|\Omega|$. From (12.3) we obtain $|\{\sup_\varepsilon |K_{\varepsilon M} * (f - \tilde{f})(\cdot)| > \alpha\}| \leqslant C|\Omega|$, and then from (12.1) we find $\lambda(\alpha) \equiv |\{\sup_\varepsilon |K_{\varepsilon M} * f(\cdot)| > \alpha\}| \leqslant (C/\alpha^2) \int_{\{f^* \leqslant \alpha\}} (f^*(y))^2\, dy + C|\{f^* > \alpha\}|$. Consequently,

$$\begin{aligned}
\int_{\mathbf{R}^n} (\sup_\varepsilon |K_{\varepsilon M} * f(y)|)^p\, dy &= C \int_0^\infty \alpha^{p-1} \lambda(\alpha)\, d\alpha \\
&\leqslant C \int_0^\infty \alpha^{p-1} |\{f^* > \alpha\}|\, d\alpha + C \int_0^\infty \alpha^{p-1} \frac{1}{\alpha^2} \int_{\{f^* \leqslant \alpha\}} (f^*(y))^2\, dy\, d\alpha \\
&= C \int_{\mathbf{R}^n} (f^*(y))^p\, dy + C \int_{\mathbf{R}^n} (f^*(y))^2 \int_{f^*(y)}^\infty \alpha^{p-3}\, d\alpha\, dy \\
&= C \int_{\mathbf{R}^n} (f^*(y))^p\, dy + C \int_{\mathbf{R}^n} (f^*(y))^2 (f^*(y))^{p-2}\, dy = C \int_{\mathbf{R}^n} (f^*(y))^p\, dy
\end{aligned}$$

as in the proof of Theorem 8. Thus, $\|\sup_\varepsilon |K_{\varepsilon M} * f(\cdot)|\|_p \leqslant C_p \|f^*\|_p$. Q.E.D.

Remark. Under the same conditions as Lemma 11 the proof also gives

$$\|\sup_t \sup_\varepsilon |\varphi_t * K_{\varepsilon M} * f(\cdot)|\|_p = \|\sup_t \sup_\varepsilon |\varphi_t * \varphi_\varepsilon * K_M * f|\|_p \leqslant C_p \|f^*\|_p.$$

Notice that all the constants do not depend on M.

THEOREM 12. *Let f be a distribution in H^p. Then $K * f = \lim_{\varepsilon \to 0, M \to \infty} K_{\varepsilon M} * f$ exists in the sense of distributions, and is in H^p. The limit is independent of the choice of ψ and φ. Also $f \to K * f$ is bounded on H^p.*

In proving the theorem we may assume $p \leqslant 1$, because only in this case are there any technical difficulties in deducing the theorem from Lemma 11.

Notice that whenever $f \in H^p$, $K_{\varepsilon M} * f \in H^p$. In fact $\varphi_t * K_{\varepsilon M} * f = K_{\varepsilon M} * (\varphi_t * f)$, and $\varphi_t * f$ is bounded and C^∞. So we can apply the remark at the end of Lemma 12, and also Theorem 11 (see Remark 2) to get $\|K_{\varepsilon M} * f\|_{H^p} \approx \|\sup_t |\varphi_t * K_{\varepsilon M} * f|\|_p \leqslant A\|f^*\|_p$.

This also shows that the mappings $f \to K_{\varepsilon M} * f$ are uniformly bounded, as mappings on H^p.

Next, as in the argument at the end of section 10, write $f=f_\delta+(f-f_\delta)$. Here $u(x,t)$ is the harmonic function whose boundary values are f, and $f_\delta(x)=u(x,\delta)$. Then we know that f_δ is $H^p\cap C^\infty$ and also in L^1. Write $K_{\varepsilon M}*f=K_{\varepsilon M}*f_\delta+K_{\varepsilon M}*(f-f_\delta)$. By Lemma 1, $K_{\varepsilon M}*f_\delta\to K*f_\delta$ in the sense of distributions. Also by what we have just seen $\|K_{\varepsilon M}(f-f_\delta)\|_{H^p}\leqslant A\|f-f_\delta\|_{H^p}\to 0$, as $\delta\to 0$. Altogether, then $K_{\varepsilon M}f$ converges in the sense of distributions as $\varepsilon\to 0$, and $M\to\infty$. To finish the proof, observe that

$$\int \sup_{\delta<t<\delta^{-1}} |\varphi_t*K_{\varepsilon M}*f|^p\,dx\leqslant A\,\|f\|^p_{H^p}$$

with A independent of δ, ε and M. Letting $\varepsilon\to 0$, and $M\to\infty$, and then $\delta\to 0$, gives us via Fatou's lemma that

$$\|\sup_{t>0}|\varphi_t*K*f|\|_p\leqslant A\,\|f\|_{H^p}$$

which proves that $K*f\in H^p$, and that the mapping $f\to K*f$ is bounded on H^p. Q.E.D.

Remarks. 1. The proof just given could easily be adapted to give boundedness on H^p under essentially sharp conditions on the kernel K. We in effect have to find the best possible N_0 in (11.1) and (11.2) and the definition of $\mathcal{A}$. We get this by using Theorem 10 instead of Theorem 11. Then the result can be formulated as follows: K is a tempered distribution whose Fourier transform is a bounded function. For any $\alpha>0$, we assume that K is of the class $C^{(k)}$ away from the origin, where k is the greatest integer $\leqslant\alpha$; and also

$$\left|\frac{\partial^\gamma}{\partial x^\gamma}K(x)\right|\leqslant A\,|x|^{-n-|\gamma|}\quad |\gamma|\leqslant k.$$

In addition, whenever $\tilde{K}$ is one of the derivatives of total order k of K, we assume that

$$|\tilde{K}(x-y)-\tilde{K}(x)|\leqslant A\,\frac{|y|^{\alpha-k}}{|x|^{n+\alpha}},\quad \text{if } 2|y|<|x|.$$

Then $f\to K*f=\lim_{\varepsilon\to 0,\,M\to\infty}K_{\varepsilon M}*f$ is bounded on H^p with $\alpha>n[p^{-1}-1]$. This refines and extends the results given by the "Littlewood–Paley" proof in [21].

2. The theorem applies in particular to the Riesz transforms and their products, and so in effect, it gives us a new proof of Theorem 9.

3. The methods can also be used to obtain results for H^p boundedness, $p<1$, for operators of the type arising in examples 1 and 2 of § 6. Let us, for instance, consider the operator T_λ. Suppose $0<p_0\leqslant 1$, $1/p_0-\frac{1}{2}=(b/n)\,[(\frac{1}{2}n+\lambda)/(b+\lambda)]$, with a, a', b and λ satisfying the relations described in § 6.

Then T_λ is bounded on H^p, for $p > p_0$. This statement is false for $p < p_0$; but the case $p = p_0$ is left open. The interest of this result is that it gives, as far as we know, the first example of an H^p inequality, $p < 1$ for operators of the type T_λ. Since we believe that this theorem is probably not the final result (when $p_0 = 1$, we know that the conclusion is valid for $p = p_0$, using part III), we shall not give the proof; we point out only that it is in the same spirit as Lemma 11, but adapts the techniques of [9], which gave the sharp "weak-type" results for $p_0 = 1$.

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