COVARIANT REPRESENTATIONS OF C^*-ALGEBRAS AND THEIR LOCALLY COMPACT AUTOMORPHISM GROUPS

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Introduction

If a locally compact group G acts on a C^*-algebra A as an automorphism group, then one can construct the crossed product $C^*(A, G)$, the covariance algebra in the sense of [5], of A by G in a way similar to the construction of a group C^*-algebra. In fact, the crossed product $C^*(A, G)$ of A by G is the enveloping C^*-algebra of the Banach *-algebra $L^1(A, G)$ of all Bochner integrable A-valued measurable functions on G with respect to the left Haar measure of G, where the *-algebraic structure of $L^1(A, G)$ is defined as follows

$$(xy)(s) = \int_G x(t)\, t(y(t^{-1}s))\, dt$$

for every pair $x, y \in L^1(A, G)$ and $s \in G$,

$$x^*(s) = \frac{1}{\Delta(s)}\, s(x(s^{-1}))^*$$

for every $x \in L^1(A, G)$ and $s \in G$, where dt and $\Delta(s)$ denote the left Haar measure and its modular function of G respectively. As it is shown in [5], if (π, U) is a covariant representation of (A, G) on a Hilbert space H (Cf. Def. 3.1), there exists a unique representation $\varrho_{(\pi, U)}$ of $C^*(A, G)$ on the same space H such that

$$\varrho_{(\pi, U)}(x) = \int_G \pi(x(s))\, U(s)\, ds$$

for each $x \in L^1(A, G)$. Further, the map $(\pi, U) \to \varrho_{(\pi, U)}$ sets up a one-to-one correspondence between the set of all covariant representations and the set of all *-representations of

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$C^*(A, G)$. The von Neumann algebra $\mathbf{M}(\pi, U)$ generated by $\{\pi(x), U(s); x\in A, s\in G\}$ is the weak closure of $\varrho_{(\pi, U)}(C^*(A, G))$. Therefore, the study of representations of $C^*(A, G)$ is, in a sense, equivalent to that of covariant representations of (A, G).

Recently, a number of works on the crossed product of a C^*-algebra by its automorphism group has been published by several authors—for example, by G. Zeller-Meier [23], A. Guichardet [10] and by S. Doplicher, D. Kastler and D. Robinson [5].

In this paper, we shall study covariant representations of (A, G), following G. W. Mackey's important theory of induced representations of groups. § 1 and § 2 are just preliminaries for the later study and the main part of this paper is §§ 4–8. In § 1, we shall study transformation groups of a topological space or of a Borel space. In § 2, we shall study the action of G on the space of representations of A. In § 3, we shall construct induced covariant representations of (A, G) following the construction of induced representations of locally compact groups given by G. W. Mackey [15]. In § 4, we shall study the converse process of what is discussed in § 3. That is, studying a system of imprimitivity for a covariant representation (Cf. Def. 4.1), we shall reduce the study of covariant representation of (A, G) to that of covariant representation of a subsystem (A, G_0) of (A, G) under a certain assumption. Applying the results of § 4, we shall show, in § 5, that the type of a covariant representation (π, U) is completely determined by the type of the associated projective representation of the associated subgroup of G under the hypothesis that the representation is of type I and that the central system of imprimitivity is transitive (Cf. Theorem 5.3). § 6 will be devoted to the study of covariant representations of a GCR-algebra and its smooth automorphism group as an application of the results of § 5. In § 7, we shall show the results concerning the induced covariant representation corresponding to the subgroup theorem in the theory of induced representation of a locally compact group. In § 8, we shall show the existence of non-type I covariant representation under a certain condition.

1. Transformation groups

Suppose that G is a topological (resp. Borel) group; that is, G is a topological (resp. Borel) space and a group such that the map $(s, t)\in G\times G\rightarrow s^{-1}t\in G$ is continuous (resp. Borelian). When G is a topological group, G is often considered as a Borel group equipped with the Borel structure determined by its topology. Let Γ be a topological (resp. Borel) space. Suppose that a homomorphism of G into the group of all homeomorphisms (resp. Borel-automorphisms) of Γ is given, denoting the homeomorphism (resp. Borel-automorphism) of Γ corresponding to $s\in G$ by $\gamma\in\Gamma\rightarrow\gamma s\in\Gamma$. If the map: $(\gamma, s)\in\Gamma\times G\rightarrow\gamma s\in\Gamma$ is continuous (resp. Borelian), then G is said to be a topological (resp. Borel) transformation group of Γ, or Γ is said to be a topological (resp. Borel) G-space.

In many cases, there are difficulties in proving the joint continuity of the map $(\gamma, s) \in \Gamma \times G \to \gamma s \in \Gamma$, even if its separate continuity is easily verified. However, we can get rid of such difficulties in our study by making use of the following theorem based on Baire's category theorem.

THEOREM 1.1. *If G is a Baire's space as a topological space and if Γ is metrizable, then the separate continuity of the map $(\gamma, s) \in \Gamma \times G \to \gamma s \in \Gamma$ automatically implies its joint continuity.*

This is easily proved from [2; Exercise 23, p. 118], but, for the convenience of the reader, we shall give the proof.

Proof. Let d be a distance function of Γ. For $\varepsilon > 0$, $\gamma_0 \in \Gamma$ and $s_0 \in G$, let $f(\gamma_0, s_0, \varepsilon)$ be the supremum of numbers $\delta > 0$ such that $d(\gamma_0, \gamma_1) < \delta$ implies $d(\gamma_0 s_0, \gamma_1 s_0) \leqslant \varepsilon$. If $f(\gamma_0, s_0, \varepsilon) < k$, then there exists a $\gamma_1 \in \Gamma$ such that $d(\gamma_0, \gamma_1) < k$ and $d(\gamma_0 s_0, \gamma_1 s_0) > \varepsilon$. There exists a neighborhood U of s_0 in G such that $d(\gamma_0 s, \gamma_1 s) > \varepsilon$ for every $s \in U$. Hence we have $f(\gamma_0, s, \varepsilon) < k$ for every $s \in U$, so that the function $s \in G \to f(\gamma_0, s, \varepsilon)$ is upper semi-continuous for every $\gamma_0 \in \Gamma$ and $\varepsilon > 0$. The continuity of the map: $\gamma \in \Gamma \to \gamma s \in \Gamma$ implies $f(\gamma, s, \varepsilon) \neq 0$ for every $\gamma \in \Gamma$ and $s \in G$. Let γ_0 be an arbitrary fixed element of Γ. Putting

$$G(\gamma_0; \varepsilon, n) = \{s \in G;\ f(\gamma_0, s, \varepsilon) \geqslant 1/n\},$$

$G(\gamma_0; \varepsilon, n)$ is closed and $\bigcup_{n=1}^{\infty} G(\gamma_0; \varepsilon, n) = G$. By the hypothesis for G, $\bigcup_{n=1}^{\infty} G(\gamma_0; \varepsilon, n)^i = G(\gamma_0, \varepsilon)$ is open and dense in G, where S^i means the interior of S for each subset S of G. Hence $G(\gamma_0) = \bigcap_{n=1}^{\infty} G(\gamma_0, 1/n)$ is a dense subset of G with the maigre complement. If an s_0 belongs to $G(\gamma_0)$, then for each n there exists a neighborhood U of s_0 and an m such that U is contained in $G(\gamma_0; 1/n, 1/m)$, that is, $f(\gamma_0, s, 1/n) \geqslant 1/m$ for every $s \in U$. Hence we have $d(\gamma_0 s, \gamma s) < 1/n$ whenever $d(\gamma_0, \gamma) < 1/m$. It follows that

$$d(\gamma_0 s_0, \gamma s) \leqslant d(\gamma_0 s_0, \gamma_0 s) + d(\gamma_0 s, \gamma s) < \frac{1}{n} + d(\gamma_0 s, \gamma s)$$

whenever $d(\gamma_0, \gamma) < 1/m$. Therefore, if U is chosen sufficiently small, then we have $d(\gamma_0 s_0, \gamma s) < 2/n$ whenever $d(\gamma_0, \gamma) < 1/m$ and $s \in U$, so that the map $(\gamma, s) \to \gamma s$ is jointly continuous at (γ_0, s_0) for every $s_0 \in G(\gamma_0)$. For an arbitrary element s_1 of G, considering the following sequence of maps

$$(\gamma, s) \in \Gamma \times G \to (\gamma, s s_1^{-1} s_0) \in \Gamma \times G \to \gamma s s_1^{-1} s_0 \in \Gamma,$$

we can conclude that the map $(\gamma, s)\in\Gamma\times G\to\gamma s\in\Gamma$ is jointly continuous at (γ_0, s_1). Since γ_0 is arbitrary, Γ becomes a topological G-space. This completes the proof.

THEOREM 1.2. *If G is a separable locally compact group acting on a countably separated Borel space Γ as a Borel transformation group, then for each γ of Γ the stability group $G_\gamma=\{s\in G; \gamma s=\gamma\}$ of G at γ is closed and the natural map of the right coset space $G_\gamma\backslash G$ onto the orbit γG is a Borel isomorphism.*

Proof. Let E_n be a countable separating family of Borel sets of Γ. Let φ_γ denote the natural map $G_\gamma s\in G_\gamma\backslash G\to\gamma s\in\Gamma$. Then φ_γ is a one-to-one Borel mapping with respect to the quotient Borel structure of $G_\gamma\backslash G$ and the Borel structure of Γ, so that the family $\varphi_\gamma^{-1}(E_n)$ becomes a countable separating family of Borel sets of $G_\gamma\backslash G$, that is $G_\gamma\backslash G$ is countably separated. Hence G_γ is closed by [16; Theorem 7.2]. Therefore, the Borel space $G_\gamma\backslash G$ is standard and φ_γ is a one-to-one Borel map of the standard Borel space $G_\gamma\backslash G$ into the countably separated Borel space Γ, so that φ_γ becomes a Borel isomorphism by [16; Theorem 3.2.]. This completes the proof.

By a measure μ on a Borel space Γ, we shall mean a complete measure determined by a σ-finite measure on the Borel sets of Γ. Let $C(\mu)$ denote the set of all measures equivalent to μ in the sense of absolute continuity.

Let G be a locally compact group and Γ a Borel G-space. For a measure μ on Γ and $s\in G$, let $s(\mu)$ denote the measure defined by $s(\mu)(E)=\mu(Es)$ for each Borel set E of Γ. If $s(\mu)$ and μ are equivalent, that is, if they have the same null sets for every $s\in G$, μ is said to be quasi-invariant (under G) and the measure space (Γ, μ) is said to be a G-measure space. For a quasi-invariant measure μ, let $\lambda(\gamma, s)$ denote the Radon–Nikodym derivative of $s(\mu)$ with respect to μ, that is,

$$\int f(\gamma s)\,\lambda(\gamma, s)\,d\mu(\gamma)=\int f(\gamma)\,d\mu(\gamma) \tag{1}$$

for every Borel function f on Γ. It is clear that the function $\lambda(\gamma, s)$ satisfies the condition

$$\lambda(\gamma, st)=\lambda(\gamma, s)\lambda(\gamma s, t) \tag{2}$$

for every pair s, $t\in G$ and almost every $\gamma\in\Gamma$.

If a quasi-invariant measure μ on a Borel G-space Γ satisfies the condition that $\mu(E)=0$ or $\mu(\Gamma-E)=0$ for every Borel set E of Γ with $\mu(E\Delta Es)=0$ for every $s\in G$, then μ is said to be ergodic, where Δ means the symmetric difference of sets. The notion of ergodicity does not depend on the measure μ itself, but only on the class $C(\mu)$. When μ is quasi-invariant and ergodic, the class $C(\mu)$ is said to be a quasi-orbit of G following

G. W. Mackey [17]. If G_0 is a closed subgroup of G, there exists a unique class C of the quasi-invariant measure on the right coset space $G_0\backslash G$ which is the image of the class of Haar measures under the canonical map $s \in G \to G_0 s \in G_0\backslash G$. In a Borel G-space Γ, each orbit γG carries a unique class of quasi-invariant measures since γG is isomorphic to the Borel G-space $G_\gamma\backslash G$ by Theorem 1.2. The transitivity of the action of G yields its ergodicity, so that the class of quasi-invariant measures concentrated on an orbit is a quasi-orbit, which is said to be transitive or merely orbit. The notion of quasi-orbit, defined above, is a generalization of that of a measure concentrated on an orbit.

2. Locally compact automophism groups of C^*-algebras

Let G be a locally compact group and A a C^*-algebra. If there exists a map

$$(s, x) \in G \times A \to s(x) \in A$$

satisfying the following conditions:

(1) *for each* $s \in G$, *the map* $x \in A \to s(x) \in A$ *is a *-automorphism of* A, *and* $s_1(s_2(x)) = s_1 s_2(x)$ *for each* $s_1, s_2 \in G$ *and* $x \in A$,

(2) *for each* $x \in A$ *and* $\varepsilon > 0$, *there exists a neighborhood* U *of the unit* e *of* G *such that* $\|s(x) - x\| < \varepsilon$ *for every* $s \in U$,

then G is said to be a *locally compact automorphism group* of A.

Let H be a Hilbert space and let $R(A{:}\, H)$ denote the set of all representations of A on H. Considering the *-strong operator topology in the full operator algebra $B(H)$ on H, we take for topology in $\mathrm{Rep}(A{:}\, H)$ the simple convergence topology. For each $s \in G$ and $\pi \in \mathrm{Rep}(A{:}\, H)$, we define an action of s to π by

$$(\pi s)(x) = \pi(s(x)) \tag{3}$$

for every $x \in A$. Then it follows at once that the map $(\pi, s) \in \mathrm{Rep}(A{:}\, H) \times G \to \pi s \in \mathrm{Rep}(A{:}H)$ is separately continuous. Since $\mathrm{Rep}(A{:}\, H)$ is metrizable whenever A and H are separable, we get the following result from Theorem 1.1.

THEOREM 2.1. *If A and H are separable, then* $\mathrm{Rep}(A{:}\, H)$ *becomes a topological G-space automatically for every locally compact automorphism group G of A.*

Let $\mathcal{U}(H)$ (or simply $\mathcal{U}$) denote the group of all unitary operators on H. For $u \in \mathcal{U}$ we shall define an action of u to $\pi \in \mathrm{Rep}(A{:}\, H)$ by

$$(u\pi)(x) = u\pi(x)u^{-1} \tag{4}$$

for every $x \in A$. Then it follows at once that

$$(u\pi)s = u(\pi s) \tag{5}$$

for every $(u, \pi, s) \in \mathcal{U} \times \mathrm{Rep}(A: H) \times G$. Suppose that the dimension of H is infinite. Then the action of G on $\mathrm{Rep}(A: H)$ satisfies the following:

$$(\pi_1 \oplus_j \pi_2)\, s = \pi_1\, s \oplus_j \pi_2\, s \tag{6}$$

for every $(\pi_1, \pi_2, s) \in \mathrm{Rep}(A: H) \times \mathrm{Rep}(A: H) \times G$,

$$p(\pi s) = p(\pi) \tag{7}$$

for every $(\pi, s) \in \mathrm{Rep}\,(A: H) \times G$, where the definition of the j-direct sum $\pi_1 \oplus_j \pi_2$ of $\pi_1, \pi_2 \in \mathrm{Rep}\,(A: H)$ and the definition of the projection $p(\pi)$ are given in [21]. Hence G becomes an automorphism group of the topological algebraic system Rep $(A: H)$.

As a converse of Theorem 2.1, we get the following

THEOREM 2.2. *When A and H are separable, if a locally compact group G acts on* Rep $(A: H)$ *as a topological transformation group satisfying conditions* (4)–(7), *G becomes an automorphism group of A and the action of G on* Rep $(A: H)$ *coincides with the action given by Theorem 2.1.*

Proof. By [21; Theorem 4], A is represented as the C^*-algebra of all continuous admissible operator fields on Rep $(A; H)$, so that we can define an action of $s \in G$ to $x \in A$ by

$$s(x)(\pi) = x(\pi s), \quad \pi \in \mathrm{Rep}\,(A: H). \tag{8}$$

Then $s(x)$ is a continuous admissible operator field on Rep $(A: H)$, so that $s(x)$ belongs to A. It is easily verified that s is an automorphism of A. Considering the Borel structure of $B(H)$ generated by the $*$-strong operator topology, we shall consider a Borel structure of A which is the smallest Borel structure in which the function $x \in A \to x(\pi) \in B(H)$ becomes a Borel function for every $\pi \in \mathrm{Rep}\,(A: H)$. Since the Borel structure of $B(H)$ is generated by the weak operator topology, the Borel structure of A, just defined, is generated by the family of functions $x \in A \to (x(\pi)\xi \,|\, \eta)$ for each $\pi \in \mathrm{Rep}\,(A: H)$, $\xi, \eta \in H$. By [21; Lemma 2], the Borel structure of A is generated by the $\sigma(A, A^*)$-topology, so that it is countably separated. The norm topology of A is polish and also finer than the $\sigma(A, A^*)$-topology, so that the Borel structure of A is generated by the norm topology. For each $\pi \in \mathrm{Rep}\,(A: H)$ and $x \in A$, the map $s \in G \to s(x)(\pi) = x(\pi s) \in B(H)$ is continuous, so that the map $s \in G \to s(x) \in A$

is a Borel function for each $x \in A$. Hence the action of G on A becomes a measurable representation of G on a separable Banach space A, so that we have $\|s(x)-x\| \to 0$ for each $x \in A$ as s converges to e in G. This completes the proof.

Let A^* denote the conjugate space of a C^*-algebra A and Σ^* its unit ball. For an $s \in G$ and an $\omega \in A^*$, we shall define an action of s to ω by

$$\langle \omega \cdot s, x \rangle = \langle \omega, s(x) \rangle \tag{9}$$

for each x of A, where $\langle \omega, x \rangle$ means the value of ω at x. If A is separable, then the weak *-topology of Σ^* is metrizable. Since the map $(\omega, s) \in \Sigma^* \times G \to \omega s \in \Sigma^*$ is separately continuous, Σ^* becomes a topological G-space by Theorem 1.1. In the following of this section, we shall assume the separability of A. Let U_A denote the group of all unitary elements of the C^*-algebra A_I obtained by adjunction of a unit to A. For $u \in U_A$ and $\omega \in A^*$, we shall define a functional ω^u by

$$\langle \omega^u, x \rangle = \langle \omega, uxu^{-1} \rangle \tag{10}$$

for each $x \in A$, recalling that A is an ideal of A_I. Then we have, for each $x \in A$,

$$\begin{aligned} \langle \omega^u s, x \rangle &= \langle \omega, us(x)u^{-1} \rangle = \langle \omega, s(s^{-1}(u)xs^{-1}(u^{-1})) \rangle \\ &= \langle ws, s^{-1}(u)xs^{-1}(u^{-1}) \rangle = \langle (\omega s)^{s^{-1}(u)}, x \rangle, \end{aligned}$$

so that we get

$$\omega^u s = (\omega \cdot s)^{s^{-1}(u)} \tag{11}$$

for each $\omega \in A^*$ and $s \in G$. Let $\mathfrak{P}$ denote the set of all pure states of A. Then $\mathfrak{P}$ becomes a G_δ-set in Σ^*, so that $\mathfrak{P}$ is a polish space in the weak *-topology. For each $\omega \in \mathfrak{P}$, let π_ω denote the cyclic representation of A induced by ω. Then, for each pair $\omega_1, \omega_2 \in \mathfrak{P}$, "$\pi_{\omega_1} \simeq \pi_{\omega_2}$" is equivalent to "there exists $u \in U_A$ such that $\omega_1^u = \omega_2$".

Let $\hat{A}$ be the dual space of A, that is, the set of unitary equivalence classes of all irreducible representations of A. In $\hat{A}$, we shall consider the Fell-topology as a topological space and the Mackey Borel structure as a Borel space. Then the map $\omega \in \mathfrak{P} \to \hat{\pi}_\omega \in \hat{A}$ is open by [8; Theorem 3], where $\hat{\pi}$ means the unitary equivalence class containing a representation π, so that we can identify $\hat{A}$ with the quotient topological space $\mathfrak{P}/U_A$ under the natural correspondence. For $(\omega_0, s_0) \in \mathfrak{P} \times G$, let V be an open neighborhood of $\omega_0 s_0$ which is invariant under the action of U_A. By the continuity of the map $(\omega, s) \in \mathfrak{P} \times G \to \omega s \in \mathfrak{P}$, there exist open neighborhoods W' of ω_0 and U of s_0 such that $W'U \subset V$. For each $\omega \in W'$, $s \in U$ and $u \in U_A$, $\omega^u s = (\omega s)^{s^{-1}(u)}$ belongs to V by U_A-invariance of V, so that $(\bigcup_{u \in U_A} W'^u) U \subset V$. Putting $W = \bigcup_{u \in U_A} W^u$, we can conclude that there exist open neighborhoods W of ω_0 and U of s_0 such that $WU \subset V$ and W is invariant under U_A. Therefore, the quotient topological space $\mathfrak{D}/U_A$ becomes a topological G-space. Recalling that $\pi_{\omega\cdot s}=\pi_\omega s$ for every $\omega\in\mathfrak{D}$ and $s\in G$, G becomes naturally a topological transformation group of the dual space $\hat{A}$ of A. Thus, we get the following

THEOREM 2.3. *For an irreducible representation π of A and $s\in G$, defining an action of s to the unitary equivalence class $\hat{\pi}$ of π by $\hat{\pi}\cdot s=\widehat{\pi\cdot s}$, G becomes a topological transformation group of the dual space $\hat{A}$ of A equipped with the Fell topology.*

THEOREM 2.4. *If A is GCR, then G becomes a Borel transformation group of the dual space $\hat{A}$ of A equipped with the Mackey Borel structure. Hence for each ζ of $\hat{A}$, the stability group G_ζ of G at ζ is a closed subgroup of G.*

Proof. If A is GCR, then the Borel structure of A is generated by its Fell topology and is standard by [6]. Hence our assertion follows from Theorem 1.2 and 2.3 at once. This completes the proof.

Now let A be a separable C^*-algebra and G a separable locally compact automorphism group of A and H a separable Hilbert space. Then the space Rep $(A: H)$ of all representations of A on H becomes a polish space. Let Irr $(A: H)$ denote the space of all non-degenerate irreducible representations of A on H. Then Irr $(A: H)$ is a G_δ-set in Rep $(A: H)$, so that it is also polish. Considering the natural action of the cartesian product group $\mathfrak{U}\times G$ on Irr $(A: H)$, $\mathfrak{U}\times G$ becomes a polish transformation group of the polish space Irr $(A: H)$.

LEMMA 2.1. *The action of $\mathfrak{U}\times G$ on* Irr $(A: H)$ *satisfies condition D in the sense of E. G. Effros* [24]. *That is, for each neighborhoods M of I in $\mathfrak{U}$ and U of e in G, there exist neighborhoods N of I in $\mathfrak{U}$ and V of e in G with the following property: If $\{Q_m\}$ is a decreasing basis of open neighborhoods of any element π of* Irr $(A: H)$, *then $\bigcap_{m=1}^\infty(\overline{NQ_mV})\subset M\pi U$, where $\bar{S}$ means the closure of S for any subset S of* Irr $(A: H)$.

Proof. By [24; Lemma 4.1], the action of $\mathfrak{U}$ on Irr $(A: H)$ satisfies condition D, so that there exists a neighborhood N of I in $\mathfrak{U}$ such that $\bigcap_{m=1}^\infty(\overline{NQ_m})\subset M\pi$ for every decreasing basis $\{Q_m\}$ of open neighborhoods of $\pi\in$ Irr $(A: H)$. Let V be a compact neighborhood of e contained in U. Take an arbitrary point $\pi_0\in\bigcap_{m=1}^\infty(\overline{NQ_mV})$. Then we can choose sequences $\{u_m\}$, $\{s_m\}$ and $\{\pi_m\}$ such that $u_m\in N$, $\pi_m\in Q_m$, $s_m\in V$ for each $m=1, 2, \ldots$ and $\lim u_m\pi_ms_m=\pi_0$. By the compactness of V, taking a subsequence of $\{s_m\}$ if it is necessary, we may assume that $\{s_m\}$ converges to an $s_0\in V$. Since $\lim u_m\pi_m=\lim (u_m\pi_ms_m)s_m^{-1}=\pi_0s_0^{-1}$, $\pi_0s_0^{-1}$ belongs to $\bigcap_{m=1}^\infty(\overline{NQ_m})$. From the assumption for N, we have

$$\pi_0 s_0^{-1} \in \bigcap_{m=1}^{\infty} (\overline{NQ_m}) \subset M\pi.$$

Therefore π_0 belongs to $M\pi U$, which implies that $\bigcap_{m=1}^{\infty}(\overline{NQ_m V}) \subset M\pi U$. This completes the proof.

Combining the results of E. G. Effros [24] and Lemma 2.1, we get the following

THEOREM 2.5. *If A is a separable C^*-algebra and G is a separable locally compact automorphism group of A, then the following are equivalent:*

(i) *The quotient Borel space $\hat{A}/G$ is countably separated.*
(ii) *$\hat{A}$ has no non-transitive quasi-orbit of G.*
(iii) *The quotient topological space $\hat{A}/G$ is almost Hausdorff.*

Proof. The implication (i) $\Rightarrow$ (ii) is easily proved by the argument of p. 126 in [25].

Let H_n be an n-dimensional Hilbert space for $n=1, 2, \dots, \infty$ and $\mathcal{U}_n$ the group of all unitary operators on H_n. Putting $\hat{A}_n = \{\zeta \in A;$ dimension of $\zeta = n\}$, $\hat{A}_n$ can be regarded as the quotient topological (or Borel) space Irr $(A\colon H_n)/\mathcal{U}_n$. Since $\bigcup_{k=1}^{n} \hat{A}_k$ is a closed subspace of $\hat{A}$ and $\hat{A}_n$ is invariant under the action of G for each $n=1, 2, \dots, \infty$, it is sufficient to prove our assertions for $\hat{A}_n$ instead of the whole space $\hat{A}$. For each $\pi \in$ Irr $(A\colon H)$, let $\zeta(\pi)$ denote the class of all irreducible representations of $\hat{A}$ on H_n unitarily equivalent to π. For each $\zeta \in \hat{A}_n$, let $\varphi(\zeta)$ denote the orbit ζG of G at ζ. Each subset S of $\hat{A}_n/G$ is open (resp. Borelian) if and only if $\varphi^{-1}(S)$ is open (resp. Borelian) in $\hat{A}_n$. Also each subset E of $\hat{A}_n$ is open (resp. Borelian) if and only if $\zeta^{-1}(E)$ is open (resp. Borelian), so that each subset S of $\hat{A}/G$ is open (resp. Borelian) if and only if $\zeta^{-1}\varphi^{-1}(S)$ is open in Irr $(A\colon H)$ (resp. Borelian in Irr $(A\colon H_n)$). Hence the space $\hat{A}/G$ becomes the quotient space under the composed map $\psi = \varphi \circ \zeta$. On the other hand, ψ can be regarded as the canonical map of Irr $(A\colon H_n)$ onto the orbit space Irr $(A\colon H_n)/(\mathcal{U}_n \times G)$. Therefore, by Lemma 2.1 and [24; Theorem 2.9], the assertions (i) and (iii) are equivalent.

(ii) $\Rightarrow$ (i). Suppose the quotient Borel space $\hat{A}_n/G$ is not countably separated. By [24; Theorem 2.6], Irr $(A\colon H_n)$ has a non-transitive ergodic quasi-invariant measure μ with total mass one under the action of $\mathcal{U} \times G$. Define a measure ν in $\hat{A}_n$ by $\nu(E) = \mu(\zeta^{-1}(E))$ for each Borel set E of $\hat{A}_n$. Then ν becomes an ergodic measure in $\hat{A}_n$ under the action of G. Suppose that there exists a point $\zeta_0 \in \hat{A}_n$ with $\nu(\zeta_0 G) = 1$. Since $\zeta^{-1}(\zeta_0 G) = \mathcal{U}_n \pi_0 G$ for any π_0 of ζ_0 and $\mu(\zeta^{-1}(\zeta_0 G)) = \nu(\zeta_0 G) = 1$, μ is concentrated in the orbit $\mathcal{U}\pi_0 G$ of $\mathcal{U} \times G$ at π_0, which contradicts the non-transitivity of μ. Therefore, ν is a non-transitive ergodic measure in $\hat{A}_n$. This completes the proof.

Under the same conditions as in Theorem 2.5, we shall study an irreducible representation π of A on a separable Hilbert space H such that $\pi\cdot s\simeq\pi$ for each s of G. There exists a unitary operator $V(s)$ on H for each $s\in G$ such that $\pi s=V(s)\pi$. For each pair s, $t\in G$, we have

$$V(st)\pi=\pi(st)=(V(s)\pi)t=V(s)(\pi t)=V(s)V(t)\pi,$$

which implies that $V(t)^{-1}V(s)^{-1}V(st)$ commutes with $\pi(A)$. By the irreducibility of π, $V(t)^{-1}V(s)^{-1}V(st)$ is a scalar $\sigma(s,t)$ of modulus one. As in the arguments of [17; Theorem 8.2], we may choose $V(s)$ such that $V(\cdot)$ is a $\mathcal{U}(H)$-valued Borel function over G, and so σ becomes a multiplier in the sense of G. W. Mackey [17] and V is a σ-representation of G. Moreover, the multiplier σ and the σ-representation V are uniquely determined within equivalence. Thus we get the following

THEOREM 2.6. *Let A be a separable C^*-algebra and G a separable locally compact automorphism group of A. If π is an irreducible representation of A on a separable Hilbert space H such that πs is unitarily equivalent to π for each $s\in G$, then there exist a unique multiplier σ for G and a σ-representation V of G on H, unique within equivalence, such that $V(s)\pi=\pi\cdot s$ for each $s\in G$.*

3. Induced covariant representation of C^*-algebras and their locally compact automorphism groups

Let A be a C^*-algebra and G a locally compact automorphism group of A. On the connection between a representation of A and of G, we shall state the following.

Definition 3.1. If a representation π of A on a Hilbert space H and a unitary representation U of G on the same space H satisfy the condition

$$U(s)\pi(x)U(s)^{-1}=\pi s(x) \tag{1}$$

for every $x\in A$ and $s\in G$, then the couple (π, U) is said to be a covariant representation of (A, G). According to the type of the von Neumann algebra $\mathbf{M}(\pi, U)$ generated by $\{\pi(x), U(s); x\in A, s\in G\}$, the covariant representation (π, U) is said to be a factor representation, a type I representation and so on.

In the following, we shall assume the separability for A, G and H.

Let G_0 be a closed subgroup of G and let Γ denote the right coset space $G_0\backslash G$. Then Γ becomes a locally compact G-space. Let μ be a fixed finite quasi-invariant measure on Γ and λ the associated λ-function on Γ in the sense of G. W. Mackey [15]. Let (π_0, L) be a covariant representation of (A, G_0) on a separable Hilbert space H_0. Making use of Mackey's construction of the induced representation of G, we shall construct a new covariant representation (π, U) of (A, G), which is called the induced covariant representation. Take the unitary representation U^L of G induced by L as a unitary representation U of G. Let H denote the representation space of U^L. Then H is the space of all H_0-valued Borel function ξ on G satisfying the conditions:

$$\xi(st) = L(s)\xi(t) \tag{2}$$

for every $s \in G_0$ and $t \in G$,

$$\int_\Gamma \|\zeta(s)\|^2 d\mu(\dot{s}) < \infty. \tag{3}$$

The last condition (3) is easily justified by the fact that $\|\xi(st)\| = \|L(s)\xi(t)\| = \|\xi(t)\|$ for every $s \in G_0$ and $t \in G$, so that $\|\xi(s)\|$ can be interpreted as a function on Γ. For each $x \in A$, we shall define an operator $\pi(x)$ on H by;

$$(\pi(x)\xi)(t) = \pi_0 t(x)\xi(t) \tag{4}$$

for each $\xi \in H$ and $t \in G$. By the fact that

$$(\pi(x)\xi)(st) = \pi_0 st(x)\xi(st) = \{L(s)\pi_0 t(x)L(s)^{-1}\}L(s)\xi(t) = L(s)(\pi(x)\xi)(t)$$

for every $\xi \in H$, $s \in G_0$, $t \in G$ and $x \in A$, $\pi(x)\xi$ belongs to H. It is clear that $\pi(x)$ is bounded and that π preserves all algebraic operations of A, so that we get the representation π of A on H. The arguments in [15], which prove the independence of U^L on the choice of quasi-invariant measure μ, also show the independence of π on the measure. The covariance of (π, U) is proved at once by the following

$$\begin{aligned} U(s)\pi(x)U(s^{-1})\xi(t) &= \lambda(t, s)^{\frac{1}{2}}(\pi(x)U(s^{-1})\xi)(ts) \\ &= \lambda(t, s)^{\frac{1}{2}}\pi_0 ts(x)(U(s^{-1})\xi)(ts) \\ &= \pi_0 ts(x)\xi(t) = (\pi s(x)\xi)(t) \end{aligned}$$

for every $\xi \in H$, s, $t \in G$ and $x \in A$.

Definition 3.2. The couple (π, U), just defined, is said to be the covariant representation of (A, G) induced by the covariant representation (π_0, L) of (A, G_0).

4. Systems of imprimitivity for covariant representations

In this section we shall keep the basic hypothesis of countability condition for C^*-algebras, locally compact groups, Hilbert spaces and so on. We shall study the converse process of that described in § 3. That is, we shall study when a given covariant representation of (A, G) comes from a covariant representation of a subsystem (A, G_0) as an induced covariant representation. For the purpose, we shall make a definition following G. W. Mackey.

Definition 4.1. Let (π, U) be a covariant representation of a C^*-algebra A and a locally compact automorphism group G of A on a Hilbert space H. Let $\mathbf{A}$ be a commutative von Neumann algebra acting on H. If $\mathbf{A}$ satisfies the conditions:

$$\mathbf{A} \text{ is contained in the commutant } \pi(A)' \text{ of } \pi(A), \tag{1}$$

$$U(s)\mathbf{A}U(s^{-1}) = \mathbf{A} \quad \text{for every} \quad s \in G, \tag{2}$$

then $\mathbf{A}$ is said to be a system of imprimitivity for (π, U). Also, if each fixed element of $\mathbf{A}$, under the automorphism $x \in \mathbf{A} \to U(s)xU(s^{-1}) \in \mathbf{A}$ for all s of G, becomes a scalar, then $\mathbf{A}$ is said to be an ergodic system of imprimitivity for (π, U).

Suppose that a covariant representation (π, U) of (A, G) on H is induced by a covariant representation (π_0, L) of (A, G_0) on H_0 as in Definition 3.2. For each $f \in L^\infty(\Gamma, \mu)$, define an operator $i(f)$ on H by $(i(f)\xi)(s) = f(\dot{s})\xi(s)$ for every $\xi \in H$ and $s \in G$. Then the algebra $\mathbf{A}$ consisting of all $i(f)$'s becomes a system of imprimitivity for (π, U). We shall call $\mathbf{A}$ the canonical system of imprimitivity.

If a commutative von Neumann algebra $\mathbf{A}$ on H is a system of imprimitivity for (π, U) then by [18; Theorem 1 and 2] there exists an essentially unique standard Borel measure G-space (Γ, μ) and an isomorphism i of the algebra $L^\infty(\Gamma, \mu)$ of all essentially bounded measurable functions over (Γ, μ) onto $\mathbf{A}$ such that

$$U(s)\, i(f)\, U(s^{-1}) = i(s(f)) \tag{3}$$

for each $f \in L^\infty(\Gamma, \mu)$ where $s(f)$ means the function of $L^\infty(\Gamma, \mu)$ defined by $s(f)(\gamma) = f(\gamma s)$ for $\gamma \in \Gamma$. In other words, the couple (i, U) is, in a sense, a covariant representation of $(L^\infty(\Gamma, \mu), G)$ on H, considering G as an automorphism group of the algebra $L^\infty(\Gamma, \mu)$. The action of G on $L^\infty(\Gamma, \mu)$, however, does not satisfy condition (2.2). In the above situation, the system of imprimitivity $\mathbf{A}$ for (π, U) is said to be based on the G-measure space (Γ, μ) with respect to i. Then the ergodicity of the system of imprimitivity $\hat{A}$ is equivalent to that of the action of G on (Γ, μ).

Definition 4.2. If the quasi-invariant measure μ of Γ is transitive, then $\mathbf{A}$ is said to be a transitive system of imprimitivity for (π, U).

The above definition does not depend on the choice of G-space (Γ, μ) by [18], but it is determined by the triple $(\mathbf{A}, G, U)$.

For the study of the connection between a covariant representation and a system of imprimitivity, we shall briefly sketch the arguments employed by G. W. Mackey in [17].

Let $\mathbf{A}$ be a system of imprimitivity for a covariant representation (π, U) of (A, G). Since $\mathbf{A}$ is commutative, $\mathbf{A}$ is the center of the commutant $\mathbf{A}'$ of $\mathbf{A}$. Hence there exists a unique family $\{e_n\}$ of orthogonal projections of $\mathbf{A}$ such that $\Sigma_n e_n = I$ and, for each $n = 1, 2, ..., \infty$, $\mathbf{A}' e_n$ is spatially isomorphic, as a von Neumann algebra, to the tensor product $L^\infty(E_n, \mu) \otimes B_n$ of the von Neumann algebra $L^\infty(E_n, \mu)$ on the Hilbert space $L^2(E_n, \mu)$ and the full operator algebra B_n on an n-dimensional Hilbert space H_n, where E_n is a Borel set in Γ associated with $i^{-1}(e_n)$. Each projection e_n is invariant under every spatial automorphism of $\mathbf{A}$, so that we may assume that each Borel set E_n is G-invariant by [18; Theorem 3]. Hence we can, without loss of generality, limit ourselves to the case when $\mathbf{A}'$ is spatially isomorphic to $L^\infty(\Gamma, \mu) \otimes B_n$ for some n. In this case, $\mathbf{A}$ is said to have *uniform multiplicity n*. If $\mathbf{A}$ is an ergodic system of imprimitivity the situation certainly will fall into this case. If $\mathbf{A}$ has uniform multiplicity n, then the situation becomes as follows:

(4) The space H is the Hilbert space of all square integrable H_n-valued functions over (Γ, μ).

The action of $\mathbf{A}$ on H is given by

$$(i(f)\xi)(\gamma) = f(\gamma)\xi(\gamma) \tag{5}$$

for each $f \in L^\infty(\Gamma, \mu)$ and $\xi \in H = L^2(H_n, \Gamma, \mu)$.

Let $\lambda(\gamma, s)$ be the function on $\Gamma \times G$ defined by condition (1.1). For each $s \in G$, we shall define an operator $V(s)$ on H by

$$(V(s)\xi)(\gamma) = \lambda(\gamma, s)^{\frac{1}{2}}\, \xi(\gamma s) \tag{6}$$

for every $\xi \in H$. Then, by [17; Theorem 5.3], $V(s)$ becomes a unitary representation of G on H. We also have

$$V(s)\, i(f)\, V(s^{-1}) = i(s(f)) \tag{7}$$

for every $s \in G$ and $f \in L^\infty(\Gamma, \mu)$. Combining equations (3) and (7), we find that the operator $W(s) = U(s)\, V(s)^{-1}$ belongs to $\mathbf{A}'$ for every $s \in G$, so that $W(s)$ is decomposable. Hence, for each $s \in G$, there exists a $\mathcal{U}(H_n)$-valued Borel function $W(\gamma, s)$ on (Γ, μ) such that $(W(s)\xi)(\gamma) = W(\gamma, s)\xi(\gamma)$ for every $\xi \in H$ and almost every $\gamma \in \Gamma$. From the equation

$$\begin{aligned}
(W(st)\xi)(\gamma) &= (U(st)\, V(st)^{-1}\xi)(\gamma) = (U(s)\, U(t)\, V(t)^{-1}\, V(s)^{-1}\xi)(\gamma) \\
&= (U(s)\, V(s)^{-1}\, V(s)\, W(t)\, V(s)^{-1}\xi)(\gamma) \\
&= W(\gamma, s)(V(s)\, W(t)\, V(s)^{-1}\xi)(\gamma) \\
&= \lambda(\gamma, s)^{\frac{1}{2}}\, W(\gamma, s)\, W(\gamma s, t)(V(s)^{-1}\xi)(\gamma s) \\
&= \lambda(\gamma, s)^{\frac{1}{2}}\, \lambda(\gamma s, s^{-1})^{\frac{1}{2}}\, W(\gamma, s)\, W(\gamma s, t)\xi(\gamma) \\
&= W(\gamma, s)\, W(\gamma s, t)\xi(\gamma)
\end{aligned}$$

for every $\xi \in H$ and almost every γ, it follows that

$$\left.\begin{aligned} W(\gamma, st) &= W(\gamma, s)\, W(\gamma s, t) \\ W(\gamma, e) &= I \end{aligned}\right\} \tag{8}$$

for every pair s, t in G and almost every γ. By [17; Theorem 5.6], the $\mathcal{U}(H_n)$-valued function $W(\gamma, s)$ on $\Gamma \times G$ may be chosen as a measurable function. Hence we have

$$(U(s)\xi)(\gamma) = \lambda(\gamma, s)^{\frac{1}{2}}\, W(\gamma, s)\xi(\gamma s) \tag{9}$$

for every $\xi \in H$, $s \in G$ and almost every $\gamma \in \Gamma$.

Let us come back to the study of π. Condition (1) yields that there exists a $\mathrm{Rep}(A; H_n)$-valued measurable function $\gamma \in \Gamma \to \pi_\gamma \in \mathrm{Rep}\,(A\colon H_n)$ such that

$$(\pi(x)\xi)(\gamma) = \pi_\gamma(x)\xi(\gamma) \tag{10}$$

for each $x \in A$, $\xi \in H$ and almost every γ. Then we have

$$\begin{aligned} (\pi s(x)\xi)(\gamma) &= (U(s)\pi(x)\, U(s^{-1})\xi)(\gamma) \\ &= \lambda(\gamma, s)^{\frac{1}{2}}\, W(\gamma, s)(\pi(x)\, U(s^{-1})\xi)(\gamma s) \\ &= \lambda(\gamma, s)^{\frac{1}{2}}\, W(\gamma, s)\pi_{\gamma s}(x)(U(s^{-1})\xi)(\gamma s) \\ &= \lambda(\gamma, s)^{\frac{1}{2}}\lambda(\gamma s, s^{-1})^{\frac{1}{2}}\, W(\gamma, s)\pi_{\gamma s}(x)\, W(\gamma s, s^{-1})\xi(\gamma) \\ &= W(\gamma, s)\pi_{\gamma s}(x)\, W(\gamma s, s^{-1})\xi(\gamma) \end{aligned}$$

for every $x \in A$, $s \in G$, $\xi \in H$ and almost every γ. Since it follows from (8) at once that $W(\gamma s, s^{-1}) = W(\gamma, s)^{-1}$ for every s of G and almost every γ, we have

$$\pi_\gamma s = W(\gamma, s)\cdot\pi_{\gamma s} \tag{11}$$

for each s of G and almost every γ. Now we are in the position to describe how the family $\{\pi_\gamma, W(\gamma, s); \gamma \in \Gamma, s \in G\}$ determines the covariant representation (π, U).

THEOREM 4.1. *Let (Γ, μ) be a standard G-measure space and H_0 a separable Hilbert space. Let H denote the Hilbert space of all H_0-valued square summable functions over (Γ, μ). Let $W(\gamma, s)$ be a $\mathcal{U}(H_0)$-valued measurable function on $\Gamma \times G$ and π_γ a* Rep $(A\colon H_0)$*-valued measurable function on Γ. If $W(\gamma, s)$ and π_γ satisfy conditions* (8) *and* (11), *then the couple (π, U) which is defined on H by equations* (9) *and* (10), *becomes a covariant representation of (A, G) and the algebra* **A** *given by equation* (5) *becomes a system of imprimitivity for (π, U).*

Conversely, if a commutative von Neumann algebra **A** *with uniform multiplicity is a system of imprimitivity for a covariant representation (π, U) of (A, G) on a Hilbert space H, then there exists a standard G-measure space (Γ, μ), a Hilbert space H_0, a $\mathcal{U}(H_0)$-valued measurable function $W(\gamma, s)$ of $\Gamma \times G$ and a* Rep $(A\colon H_0)$*-valued measurable function of π_γ as in the above such that (π, U) is unitarily equivalent to the covariant representation on the Hilbert space*

$L^2(H_0, \Gamma, \mu)$ of all H_0-valued square summable functions over Γ determined by $\{\pi_\gamma, W(\gamma, s);$ $\gamma \in \Gamma, s \in G\}$.

When a system of imprimitivity for a covariant representation is transitive, we can sharpen the above result as follows. By [17; Theorem 6.1], the associated G-measure space (Γ, μ) can be identified with the right coset space $G_0\backslash G$ of a closed subgroup G_0 as a G-space. Suppose that a system of imprimitivity **A** for a covariant representation (π, U) of (A, G) is based on the right coset space $G_0\backslash G$ of a closed subgroup of G. By [17; Theorem 6.6], the representation U of G is certainly induced by a unitary representation L of G_0 on a Hilbert space H_0. We shall, however, briefly sketch Mackey's arguments again for the study of the covariance of π and U.

Let $\gamma(s)$ denote the coset $G_0 s$ for each $s \in G$. We shall keep the notations of Theorem 4.1. Putting

$$\tilde{W}(s, t) = W(\gamma(s), t) \quad \text{and} \quad \tilde{\pi}_s = \pi_{\gamma(s)}$$

for each pair s, t in G, $W(s, t)$ and $\tilde{\pi}_s$ satisfy the following conditions:

For every pair s, t in G,

$$\tilde{W}(r, st) = \tilde{W}(r, s)\,\tilde{W}(rs, t) \quad \text{and} \quad \tilde{W}(r, e) = I \tag{12}$$

for almost every $r \in G$, and

$$\tilde{W}(rs, t) = \tilde{W}(s, t) \quad \text{and} \quad \tilde{\pi}_{rs} = \tilde{\pi}_s \tag{13}$$

for every $(r, s, t) \in G_0 \times G \times G$, for every $t \in G$

$$\tilde{W}(s, t)\,\tilde{\pi}_{st} = \tilde{\pi}_s \cdot t \tag{14}$$

for almost every $s \in G$. Then, by [17; Lemma 6.1 and 6.2], there exists a $\mathcal{U}(H_0)$-valued Borel function B on G and a unitary representation L of G_0 on H_0 such that

$$\tilde{W}(s, t) = B(s)^{-1} B(st) \tag{15}$$

for almost every pair s, t in G,

$$\tilde{B}(rs) = L(r)\,B(s) \tag{16}$$

for every $r \in G_0$ and every $s \in$ G. Further, these functions B and L are uniquely determined within suitable equivalence, that is, if B' and L' are another couple of $\mathcal{U}(H_0)$-valued Borel functions with the properties (15) and (16), there exists a unitary operator C such that $B'(s) = CB(s)$ for almost every $s \in G$ and such that $L'(s) = C^{-1}L(s)\,C$ for every $s \in G_0$. It follows from condition (14) that

$$(B(s)\,\tilde{\pi}_s)\,s^{-1} = B(st)\,\tilde{\pi}_{st}(st)^{-1} \tag{17}$$

for almost every pair s, t in G. Hence, the representation $B(s)\tilde{\pi}_s s^{-1}$ of A on H_0 is almost everywhere equal to the representation π_0 which is independent of s. Let α be a member of the unique invariant measure class in G with total mass one. Then we have, for every pair $\xi, \eta \in H_0$ and $x \in A$,

$$(\pi_0(x)\xi \mid \eta) = \int_G (B(s)\tilde{\pi}_s s^{-1}(x) B(s)^{-1}\xi \mid \eta)\, d\alpha(s).$$

For every $r \in G_0$, we have

$$(\pi_0 r(x)\ \xi \mid \eta) = \int_G (B(s)\tilde{\pi}_s s^{-1} r(x) B(s)^{-1}\xi \mid \eta)\, d\alpha(s) = \int_G (B(rs)\tilde{\pi}_{rs} s^{-1}(x) B(rs)^{-1}\xi \mid \eta)\, d\alpha'(s),$$

where α' is another measure in G with the same properties as α. The right side of the above integration becomes

$$\int_G (L(r) B(s)\tilde{\pi}_s s^{-1}(x) B(s)^{-1} L(r)^{-1}\xi \mid \eta)\, d\alpha'(s)$$
$$= \int_G (B(s)\tilde{\pi}_s s^{-1}(x) B(s)^{-1} L(r)^{-1}\xi \mid L(r)^{-1}\eta)\, d\alpha'(s).$$

Since the function of s under the integral sign is constant almost everywhere, the above integral is equal to $(\pi_0(x) L(r)^{-1}\xi \mid L(r)^{-1}\eta)$. Hence we have

$$\pi_0 \cdot r = L(r) \cdot \pi_0 \tag{18}$$

for every $r \in G_0$. Therefore (π_0, L) is a covariant representation of (A, G_0). Since $B(s)\tilde{\pi}_s s^{-1}$ is equal to π_0 almost everywhere, we have

$$B(s)\tilde{\pi}_s = \pi_0 \cdot s \tag{19}$$

for almost every $s \in G$. Now we are in the position to show the relation between the covariant representation (π, U) of (A, G) and the covariant representation (π_0, L) of (A, G_0).

THEOREM 4.2. *In the above situation, the covariant representation* (π, U) *of* (A, G) *is unitarily equivalent to* $(\tilde{\pi}, \tilde{U})$, *the one induced by the covariant representation* (π_0, L) *of* (A, G_0).

Proof. Let $\tilde{H}$ denote the representation space of the induced covariant representation $(\tilde{\pi}, \tilde{U})$. For each $\xi \in H$, let $\tilde{\xi}$ denote the H_0-valued function on G defined by $\tilde{\xi}(s) = B(s)\xi(\gamma(s))$ for every $s \in G$. It is not so difficult to show that the operator V, defined by $V\xi = \tilde{\xi}$ for ξ $\xi \in H$, becomes an isometry of H onto $\tilde{H}$. Then we have, for each ξ of H and every $s \in G$,

$$\begin{aligned}(VU(s)\xi)(r) &= B(r)(U(s)\xi)(\gamma(r))\\ &= \lambda(\gamma(r), s)^{\frac{1}{2}} B(r)\, W(\gamma(r), s)\xi(\gamma(r)s)\\ &= \lambda(\gamma(r), s)^{\frac{1}{2}} B(r)\, \tilde{W}(r, s)\xi(\gamma(rs))\\ &= \lambda(\gamma(r), s)^{\frac{1}{2}} B(rs)\xi(\gamma(rs))\\ &= (U^L(s) V\xi)(r)\end{aligned}$$

for almost every $r \in G$. Also we have, for each $x \in A$ and each $\xi \in H$,

$$\begin{aligned}(V\pi(x)\xi)(r) &= B(r)(\pi(x)\xi)(\gamma(r))\\ &= B(r)\tilde{\pi}_r(x)\xi(\gamma(r))\\ &= \pi_0\cdot r(x)\, B(r)\xi(\gamma(r))\\ &= \pi_0\cdot r(x)(V\xi)(r) = (\tilde{\pi}(x)\, V\xi)(r)\end{aligned}$$

for almost every $r \in G$. It follows that the isometry V carries the unitary equivalence between (π, U) and $(\tilde{\pi}, \tilde{U})$. This completes the proof.

In the last part of this section, we shall study the commutant algebra $\mathbf{M}(\pi, U)'$ of the covariant representation (π, U) in the situation of Theorem 4.2.

Suppose that T is an abitary operator in $\mathbf{M}(\pi, U)' \cap \mathbf{A}'$. Since T is decomposable, there is a measurable $B(H_0)$-valued function $T(\gamma)$ on Γ such that $(T\xi)(\gamma) = T(\gamma)\xi(\gamma)$ for every $\xi \in H$ and almost every $\gamma \in \Gamma$. For each $s \in G$, we have

$$(U(s)\, T\xi)(\gamma) = \lambda(\gamma, s)^{\frac{1}{2}}\, W(\gamma, s)\, T(\gamma s)\xi(\gamma s)$$

and also

$$(TU(s)\xi)(\gamma) = \lambda(\gamma, s)^{\frac{1}{2}}\, T(\gamma)\, W(\gamma, s)\xi(\gamma s)$$

for each $\xi \in H$ and almost every $\gamma \in \Gamma$, so that we have

$$T(\gamma)\, W(\gamma, s) = W(\gamma, s)\, T(\gamma s) \tag{20}$$

for almost every $\gamma \in \Gamma$. Putting $\tilde{T}(s) = T(\gamma(s))$ for every $s \in G$, we get $\tilde{T}(rs) = \tilde{T}(s)$ for every $r \in G_0$ and every $s \in G$. Also for each $s \in G$, we have

$$\tilde{T}(r)\, B(r)^{-1} B(rs) = B(r)^{-1} B(rs)\, \tilde{T}(rs)$$

for almost every $r \in G$. Therefore, we have

$$B(r)\, \tilde{T}(r)\, B(r)^{-1} = B(rs)\, \tilde{T}(rs)\, B(rs)^{-1} \tag{21}$$

for almost every pair r, s in G, that is, the $B(H_0)$-valued Borel function $B(r)\,\tilde{T}(r)\,B(r)^{-1}$ on G becomes a constant operator T_0 of $B(H_0)$ almost everywhere. Since T belongs to $\mathbf{M}(\pi, U)' \cap \mathbf{A}'$, almost every $T(\gamma)$ belongs to $\pi_\gamma(A)'$, so that almost every $\tilde{T}(s)$ belongs to

$\tilde{\pi}_s(A)'$. It is clear that $\tilde{\pi}_s s^{-1}(A)''=\tilde{\pi}_s(A)''$ for every $s\in G$ and that $B(s)\tilde{T}(s)B(s)^{-1}$ belongs to the commutant $\mathbf{M}(B(s)\tilde{\pi}_s)'$ of $(B(s)\tilde{\pi}_s)(A)$ for almost every $s\in G$, so that T_0 belongs to the commutant $\mathbf{M}(\pi_0)'$ of $\pi_0(A)$. For each $r\in G_0$ and each pair $\xi, \eta\in H_0$, we have

$$\begin{aligned}(L(r)T_0\xi|\eta) &= (T_0\xi|L(r)^{-1}\eta) = \int_G (B(s)\tilde{T}(s)B(s)^{-1}\xi|L(r)^{-1}\eta)\,d\alpha(s)\\ &= \int_G (L(r)B(s)\tilde{T}(s)B(s)^{-1}\xi|\eta)\,d\alpha(s) = \int_G (B(rs)\tilde{T}(s)B(s)^{-1}\xi|\eta)\,d\alpha(s)\\ &= \int_G (B(s)\tilde{T}(r^{-1}s)B(r^{-1}s)^{-1}\xi|\eta)\,d\alpha'(s) = \int_G (B(s)\tilde{T}(s)B(s)^{-1}L(r)\xi|\eta)\,d\alpha'(s),\end{aligned}$$

where α and α' mean the measures on G taken up on page 288. By the same reason as in the arguments there, the above integral is equal to $(T_0L(r)\xi|\eta)$. Therefore, T_0 belongs to the commutant $\mathbf{M}(L)'$ of $\{L(r); r\in G_0\}$. Let Φ denote the map $T\in\mathbf{M}(\pi, U)'\cap\mathbf{A}'\to T_0\in\mathbf{M}(\pi_0, L)'$.

THEOREM 4.3. *In the same situation as in Theorem 4.2, the von Neumann algebra* $\mathbf{M}(\pi, U)'\cap\mathbf{A}'$ *is isomorphic to* $\mathbf{M}(\pi_0, L)'$ *under the isomorphism* Φ.

Proof. We shall use the preceeding notations. We have, for each $\tilde{\xi}\in\tilde{H}$,

$$(VTV^{-1}\tilde{\xi})(s) = B(s)(TV^{-1}\tilde{\xi})(\gamma(s)) = B(s)T(\gamma(s))(V^{-1}\tilde{\xi})(\gamma(s)) = B(s)\tilde{T}(s)B(s)^{-1}\tilde{\xi}(s) = T_0\tilde{\xi}(s)$$

for almost every $s\in G$. Therefore, VTV^{-1} becomes $T_0\otimes 1$ on the space $\tilde{H}$, under a suitable interpretation. Hence Φ is an isomorphism of $\mathbf{M}(\pi, U)'\cap\mathbf{A}'$ into $\mathbf{M}(\pi_0, L)'$. For each T_0 of $\mathbf{M}(\pi_0, L)'$, we shall define the operator $\tilde{T}$ on $\tilde{H}$ by $\tilde{T}\tilde{\xi}(s)=T_0\tilde{\xi}(s)$ for every $\tilde{\xi}\in\tilde{H}$ and $s\in G$. Then it is clear that $\tilde{T}$ belongs to $\mathbf{M}(\tilde{\pi}, U^L)'$ and that $V^{-1}\tilde{T}V$ belongs to $\mathbf{M}(\pi, U)'\cap\mathbf{A}'$. Moreover, we can easily show that $\Phi(V^{-1}TV)=T_0$. This completes the proof.

5. Central systems of imprimitivity for covariant representations

We shall in the following keep the basic countability assumptions. Let (π, U) be a covariant representation on a Hilbert space H of a C^*-algebra A and its locally compact automorphism group G. Let $\mathbf{Z}(\pi)$ denote the center of the von Neumann algebra $\pi(A)''$ generated by $\pi(A)$. Then, $\mathbf{Z}(\pi)$ automatically becomes a system of imprimitivity for (π, U), which is said to be the central system of imprimitivity for (π, U). Applying the results of § 4 to the central system of imprimitivity $\mathbf{Z}(\pi)$ for (π, U), we shall make a detailed study of the covariant representation (π, U). Since each fixed element of $\mathbf{Z}(\pi)$, under the automorphism; $T\in\mathbf{Z}(\pi)\to U(s)TU(s)^{-1}\in\mathbf{Z}(\pi)$ for every $s\in G$, belongs to the center of the von Neumann algebra $\mathbf{M}(\pi, U)$ generated by $\{\pi(x), U(s); x\in A, s\in G\}$, we get the following

THEOREM 5.1. *Under the above conditions and if the covariant representation* (π, U) *is a covariant factor representation of* (A, G), *then the central system of imprimitivity* $\mathbf{Z}(\pi)$ *for* (π, U) *is ergodic.*

If the central system of imprimitivity $\mathbf{Z}(\pi)$ is transitive, it is rather easy to describe the covariant representation (π, U) in terms of the associated covariant representation (π_0, L) of some associated subsystem (A, G_0). Applying Theorem 4.3, we get the following

THEOREM 5.2. *In the same situation as in Theorem 5.1, if the central system of imprimitivity* $\mathbf{Z}(\pi)$ *for a covariant representation* (π, U) *of* (A, G) *is transitive, then there exists a unique closed subgroup* G_0 *of* G *and a unique covariant representation* (π_0, L) *of the subsystem* (A, G_0) *such that* (π, U) *is induced by* (π_0, L), *where the uniqueness is up to equivalence. Moreover, in this case, the commutant algebra* $\mathbf{M}(\pi, U)'$ *of* $\{\pi(x), U(s); x \in A, s \in G\}$ *is isomorphic to the commutant algebra* $\mathbf{M}(\pi_0, L)'$ *of* $\{\pi_0(x), L(s); x \in A, s \in G_0\}$ *under the canonical isomorphism* Φ *and* π_0 *is a factor representation.*

In the following, we shall study a covariant representation (π, U) of (A, G) under the hypothesis that $\mathbf{Z}(\pi)$ is transitive and that the associated representation π of A, in the sense of Theorem 5.2, is of type I. For the purpose, we need the following lemma.

LEMMA 5.1. *Let* G_0 *be a separable locally compact automorphism group of a separable* C^**-algebra* A. *Let* (π_0, L) *be a covariant representation of* (A, G_0) *on a Hilbert space* H_0. *If* π_0 *is a type I factor representation, then there exist a multiplier* σ *for* G_0 *in the sense of G. W. Mackey* [17], *a* σ*-representation* L^1 *(resp.* σ^{-1}*-representation* L^2*) of* G_0 *on a Hilbert space* H_1 *(resp.* H_2*) and an irreducible representation* π_1 *of* A *on* H_1 *such that*

$$H_0 = H_1 \otimes H_2, \quad L = L^1 \otimes L^2, \quad \pi_0(x) = \pi_1(x) \otimes 1, \quad x \in A,$$

and

$$L^1(s)\pi_1 = \pi_1 s \quad \textit{for each } s \in G.$$

Proof. By the hypothesis for π_0, there exist two Hilbert spaces H_1 and H_2 and an irreducible representation π_1 of A on H_1 such that $H = H_1 \otimes H_2$ and $\pi_0(x) = \pi_1(x) \otimes 1$ for every $x \in A$. Since $\pi_0 s = L(s)\pi_0$, $\pi_0 s$ is unitarily equivalent to π_0 for each $s \in G_0$. π_1 is quasi-equivalent to π_0 and so is $\pi_1 s$ to $\pi_0 s$, so that $\pi_1 s$ and π_1 are quasi-equivalent for each $s \in G_0$. Since $\pi_1 s$ is irreducible for every $s \in G_0$, π_1 and $\pi_1 s$ are unitarily equivalent. Therefore, by Theorem 2.6, there exist a unique multiplier σ for G_0 and a σ-representation L^1 of G_0 on H_1 such that $L^1(s)\pi_1 = \pi_1 s$ for every $s \in G$. Since $(L^1(s) \otimes 1)\pi_0 = \pi_0 s = L(s)\pi_0$ for each $s \in G$, $(L^1(s)^{-1} \otimes 1)L(s)$ commutes with $\pi_0(A)$. Hence there exists, for each $s \in G$, a unitary operator $L^2(s)$ on H_2 such that $(L^1(s)^{-1} \otimes 1)L(s) = 1 \otimes L^2(s)$. The maps $s \in G \to L^2(s)$ and $s \in G \to L(s)$ are Borel functions so that the map $s \in G \to L^2(s)$ is a Borel function. Since $L(s) = L^1(s) \otimes L^2(s)$

is an ordinary representation, L^2 becomes a σ^{-1}-representation of G on H_2. This completes the proof.

LEMMA 5.2. *Let (A, G_0) be a couple as in Lemma 5.1. and let (π_1, L^1) be a couple of an irreducible representation of A and a σ-representation of G_0 on the same Hilbert space H_1 for a multiplier σ for G_0 such that $L^1(s)\pi_1 = \pi_1 s$ for every $s \in G_0$. Then, for each σ^{-1}-representation L^2 on a Hilbert space H_2, putting $H_0 = H_1 \otimes H_2$, $\pi_0(x) = \pi_1(x) \otimes 1$ for each $x \in A$ and $L = L^1 \otimes L^2$, the map $L^2 \to (\pi_0, L)$ sets up a one-to-one correspondence between the set of σ^{-1}-representations of G_0 and the set of all covariant representations of the form $(\pi_1 \otimes 1, L^1 \otimes L^2)$ of (A, G_0).*

Moreover, the commutant $\mathbf{M}(\pi_0, L)'$ is isomorphic to the commutant $\mathbf{M}(L^2)'$ of $L^2(G)$ under the canonical correspondence.

Proof. Except for the last assertion our theorem has been proved already. For each T of $B(H_2)$, we shall put $\Phi(T) = 1 \otimes T$ in $H_0 = H_1 \otimes H_2$. Then it is clear that $\Phi(T)$ belongs to $\mathbf{M}(\pi_0, L)'$ for each T of $\mathbf{M}(L^2)'$. Suppose T_0 is an operator of $\mathbf{M}(\pi_0, L)'$. Since $\mathbf{M}(\pi_0, L)' = \mathbf{M}(\pi_0)' \cap \mathbf{M}(L)'$, there exists a unique T of $B(H_2)$ such that $\Phi(T) = T_0$. From the fact that

$$\begin{aligned}\Phi(L^2(s)\,TL^2(s)^{-1}) &= 1 \otimes (L^2(s)\,TL^2(s)^{-1}) \\ &= (L^1(s) \otimes L^2(s))(1 \otimes T)(L^1(s)^{-1} \otimes L^2(s)^{-1}) \\ &= L(s)\,T_0 L(s)^{-1} = T_0 = \Phi(T)\end{aligned}$$

for each $s \in G_0$, we have $L^2(s)\,TL^2(s)^{-1} = T$, which implies that T belongs to $\mathbf{M}(L^2)'$. This completes the proof.

Combining Theorem 5.2, Lemma 5.1 and 5.2, we get the following at once.

THEOREM 5.3. *In the same situation as in Theorem 5.1, suppose that the central system of imprimitivity $\mathbf{Z}(\pi)$ for a covariant representation (π, U) of (A, G) on a Hilbert space H is transitive and that the representation π is of type* I. *Then there exist an irreducible representation π_1 of A on a Hilbert space H_1, a closed subgroup G_0 of G, a multiplier σ for G_0, a σ-representation L^1 of G_0 on H_1 and a σ^{-1}-representation L^2 of G_0 on a Hilbert space H_2 with the following properties:*

(1) *The couple (π_0, L), defined by*

$$\pi_0(x) = \pi_1(x) \otimes 1 \text{ on } H_0 = H_1 \otimes H_2 \text{ for } x \in A,$$
$$L(s) = L^1(s) \otimes L^2(s) \text{ on } H_0 \text{ for } s \in G,$$

is a covariant representation of (A, G_0).

(2) *(π, U) is unitarily equivalent to the covariant representation induced by (π_0, L).*

(3) *The commutant* $\mathbf{M}(\pi, U)'$ *of* $\{\pi(x), U(s); x \in A, s \in G\}$ *is isomorphic to the commutant* $\mathbf{M}(L^2)'$ *of* $\{L^2(s); s \in G_0\}$.

Therefore, the type of (π, U) is completely determined by that of the associated projective representation L^2 of the associated closed subgroup G_0.

6. Covariant representations of GCR-algebras and their locally compact smooth automorphism groups

In this section, let A denote a separable GCR-algebra and G a separable locally compact automorphism group of A. Then G becomes a Borel transormation group of the dual space $\hat{A}$ of A by Theorem 2.4. When the action of G on $\hat{A}$ is smooth, that is, when the quotient Borel space $\hat{A}/G$ is countably separated, G is said to be *smooth* or *smoothly* acting on A. In this section, we shall assume the smoothness of G. For each $\zeta \in \hat{A}$ the stability group G_ζ of G at ζ is a closed subgroup of G and the orbit ζG of G at ζ is Borel isomorphic to the right coset space $G_\zeta \backslash G$ under the canonical map. Let π_ζ be an irreducible representation of A contained in ζ on a Hilbert space H_ζ. Since $\hat{\pi}_\zeta s = \widehat{\pi_\zeta s} = \zeta s = \zeta = \hat{\pi}_\zeta$ for each $s \in G_\zeta$, $\pi_\zeta s$ is unitarily equivalent to π_ζ for each $s \in G_\zeta$. Then, by Theorem 2.6, there exist a unique multiplier for G_ζ and a unique σ_ζ-representation L_ζ of G_ζ on H_ζ such that $L_\zeta(s)\pi_\zeta = \pi_\zeta s$ for each $s \in G_\zeta$. It is easy to show that the multiplier σ_ζ and the projective representation L_ζ do not depend on the choice of π_ζ but only on the class ζ within equivalence.

Let (π, U) be a covariant representation of (A, G) on a separable Hilbert space H. Suppose that the representation π has uniform multiplicity and also that the central system of imprimitivity $\mathbf{Z}(\pi)$ for (π, U) has a uniform multiplicity. (In general, every covariant representation is a direct sum of a number of disjoint covariant representations of the above form.) Let (Γ, μ), H_0, $\{\pi_\gamma; \gamma \in \Gamma\}$ and $\{W(\gamma, s); \gamma \in \Gamma, s \in G\}$ be as in Theorem 4.1. Then almost every π_γ is a type I factor representation of A on H_0 and $\{\pi_\gamma; \gamma \in \Gamma\}$ are mutually disjoint except for a null set of Γ. Besides, for each $s \in G$, we have $\pi_\gamma s = W(\gamma, s)\pi_{\gamma s}$ for almost every $\gamma \in \Gamma$. Since A is GCR, we can identify the quasi-dual space $\tilde{A}$ of A with the dual space $\hat{A}$ of A under the canonical map. For each $\gamma \in \Gamma$ such that π_γ is a factor representation, let $\zeta(\gamma)$ denote the element of $\hat{A}$ which corresponds to the quasi-equivalence class of π_γ. Then the map $\gamma \in \Gamma \to \zeta(\gamma) \in \hat{A}$ is a one-to-one Borel map of Γ into $\hat{A}$ defined almost everywhere. For $s \in G$, if $\zeta(\gamma)$ and $\zeta(\gamma s)$ are defined and $\pi_\gamma s = W(\gamma, s)\pi_{\gamma s}$ then $\zeta(\gamma s) = \zeta(\gamma)s$. Define a Borel measure ν on $\hat{A}$ by $\nu(S) = \mu(\zeta^{-1}(S))$ for each Borel set S of $\hat{A}$. For a function f on $\hat{A}$, define a function $\zeta^*(f)$ on Γ by

$$\zeta^*(f)(\gamma) = \begin{cases} f(\zeta(\gamma)) & \text{if } \zeta(\gamma) \text{ is defined,} \\ 0 & \text{if } \zeta(\gamma) \text{ is not defined.} \end{cases}$$

Since ζ is one-to-one almost everywhere, ζ^* is an isomorphism of $L^\infty(\hat{A}, \nu)$ onto $L^\infty(\Gamma, \mu)$. Besides, for each $s \in G$ and each $f \in L^\infty(\hat{A}, \nu)$, we have

$$s(\zeta^*(f))(\gamma) = \zeta^*(f)(\gamma s) = f(\zeta(\gamma)s) = s(f)(\zeta(\gamma)) = \zeta^*(s(f))(\gamma)$$

for almost every $\gamma \in \Gamma$, so that $s(\zeta^*(f)) = \zeta^*(s(f))$ for every $s \in G$ and $f \in L^\infty(\hat{A}, \nu)$. Putting $j(f) = i(\zeta^*(f))$ for each $f \in L^\infty(\hat{A}, \nu)$, where i is the isomorphism of $L^\infty(\Gamma, \mu)$ onto $\mathbf{Z}(\pi)$ defined in § 4, we have $U(s)j(f)U(s)^{-1} = j(s(f))$ for each $f \in L^\infty(\hat{A}, \nu)$ and $s \in G$. Therefore, the base space (Γ, μ) of the system of imprimitivity $\mathbf{Z}(\pi)$ can be replaced by $(\hat{A}, \nu)$. By the hypothesis for the action of G on $\hat{A}$, every quasi-orbit of G in $\hat{A}$ is transitive. Therefore, if (π, U) is a covariant factor representation, then the central system of imprimitivity $\mathbf{Z}(\pi)$ for (π, U) becomes transitive. Thus, we can apply Theorem 5.3 to (π, U) in this case.

THEOREM 6.1. *Suppose that A is a separable GCR-algebra and that G is a separable locally compact smooth automorphism group of A. Then every covariant factor representation (π, U) of (A, G) is induced by a covariant representation (π_ζ, L_ζ) of (A, G_ζ) for a certain point ζ of $\hat{A}$ such that π_ζ is a factor representation of A which is quasi-equivalent to a member of ζ and such that G_ζ is the stability group of G at ζ. Moreover, every covariant representation of (A, G) is of type* I *if and only if, for each $\zeta \in A$, every σ_ζ-representation of G_ζ is of type* I, *where σ_ζ means the associated multiplier for the stability group G_ζ of G at ζ which is canonically determined. Therefore, the crossed product $C^*(A, G)$ of A by G is GCR if and only if every σ_ζ-representation of G_ζ is of type* I *for every $\zeta \in \hat{A}$.*

Thus the study of covariant representations is reduced to that of projective representations of stability groups. Unfortunately, the study of projective representations of locally compact groups is not easy, even if the considered group is abelian or of type I. So we shall limit ourselves to a simple case in the last part of this section.

Let A be the algebra $C(H)$ of all compact operators on a separable Hilbert space H. Let G be a separable locally compact group. If G acts on A as an automorphism group of A, then the stability group of G at the point of A becomes the whole group G, since $\hat{A}$ is reduced to one point. To give a projective representation of G on H is the same as to define an action of G on A as an automorphism group.

THEOREM 6.2. *In the above situation, if G is a connected nilpotent Lie group, then every covariant representation of (A, G) is of type* I.

Proof. Let σ be the associated multiplier for G. It is sufficient, by Theorem 6.1, to show that every σ-representation of G is of type I. Let G^σ denote the cartesian product of G and the one-dimensional torus group T^1. For each pair (s, λ), (t, μ) in G^σ, define the product $(s, \lambda)(t, \mu)$ by $(s, \lambda)(t, \mu) = (st, \lambda\mu(\sigma(s, t))^{-1})$. Then G becomes a locally compact group with the normal subgroup $K = \{(e, \lambda);\ |\lambda| = 1\}$ such that $G^\sigma/K \cong G$ by [17]. Since K and G are connected Lie groups, G itself is a connected Lie group by [11; Theorem 7]. K is also the center of G^σ. Therefore G^σ is a connected nilpotent Lie group. By [12], G^σ is of type I, that is, every σ-representation of G is of type I. This completes the proof.

Applying Theorem 4.1 to another simple case, we get the following

THEOREM 6.3. *Let A be a separable GCR-algebra and G a separable locally compact smooth automorphism group of A. If G acts freely on the dual space $\hat{A}$ of A, that is, if every stability group of G at a point of $\hat{A}$ is reduced to the trivial subgroup $\{e\}$, then every covariant representation of (A, G) is of type* I. *Therefore, the crossed product $C^*(A, G)$ of A by G is GCR in this case.*

7. The subgroup theorem for induced covariant representations

We have already seen that the theory of covariant representations is quite similar to that of induced representations of locally compact groups. In fact, we can show that a number of results in the theory of induced representations of locally compact groups correspond to the results in the theory of covariant representations. In this section we shall only show the result corresponding to the subgroup theorem which is one of the most important results in the theory of induced representations of locally compact groups.

Let G_1 and G_2 be two regularly related subgroups, in the sense of G. W. Mackey [15], of a separable locally compact group G. Let $\mathcal{D}$ denote the double $G_1 : G_2$ coset space $G_1\backslash G/G_2$. G_1 and G_2 are regularly related, so $\mathcal{D}$ becomes a standard Borel space as quotient space. For each $s \in G$, let $d(s)$ denote the double coset $G_1 s G_2$ and $\gamma(s)$ the right coset $G_1 s$. For any finite measure ν on G with same null sets as the Haar measure, we define a measure ν_0 on $\mathcal{D}$ by $\nu_0(E) = \nu(d^{-1}(E))$ for every Borel subset E of $\mathcal{D}$. The measure class $C(\nu_0)$ in $\mathcal{D}$ does not depend on the choice of ν, that is, the measure class $C(\nu_0)$ is uniquely determined by the measure class of the Haar measure.

THEOREM 7.1. *In the above situation, let G be an automorphism group of a separable C^*-algebra A. Let (π_1, L_1) be a covariant representation of (A, G_1) on a separable Hilbert space H_1 and (π, U) the covariant representation of (A, G) on the Hilbert space H induced by (π_1, L_1). For each $s_0 \in G$, let $G_1(s_0)$ denote the subgroup $s_0^{-1} G_1 s_0$ and (π_{s_0}, L_{s_0}) the covariant representation of (A, G_2) on the Hilbert space H_{s_0} induced by the covariant representation of $(A, G_1(s_0) \cap G_2)$;*

$$(x, t) \in A \times (G_1(s_0) \cap G_2) \to (\pi_1 s_0(x),\ L_1(s_0 t s_0^{-1})).$$

Then (π_{s_0}, L_{s_0}) is uniquely determined, up to unitary equivalence, by the double coset $d(s_0)$.

Hence we may write $(\pi_{s_0}, L_{s_0})=(\pi_{d_0}, L_{d_0})$ *where* $d_0=d(s_0)$. *Also, the restriction* $(\pi, U)|_{(A, G_2)}$ *of the induced covariant representation* (π, U) *to the couple* (A, G_2) *is decomposed into a direct integration over* $(\mathcal{D}, \nu_0)$

$$(\pi, U)|_{(A, G_2)}=\int_{\mathcal{D}}^{\oplus}(\pi_d, L_d)\,d\nu(d). \tag{1}$$

Proof. Let μ denote the measure in $G_1\backslash G=\Gamma$ defined by $\mu(E)=\nu(\gamma^{-1}(E))$ for each Borel set E of Γ. For each γ of Γ, let $k(\gamma)$ denote the double coset of $\mathcal{D}$ containing γ. Then k is a Borel map of Γ onto $\mathcal{D}$ and $k(\mu)=\nu_0$. Since Γ and $\mathcal{D}$ are standard, μ is represented by direct integration with respect to the map k

$$\mu=\int_{\mathcal{D}}\mu_d\,d\nu_0(d), \tag{2}$$

where each μ_d is concentrated on the orbit $k^{-1}(d)$ and quasi-invariant under the action of G_2 by [15; Lemma 11, 5].

Since Γ is a standard G-space, Γ is obviously a standard G_2-space. Putting $\gamma_0=\gamma(s_0)$, the orbit of G_2 at γ_0 becomes the double coset $G_1s_0G_2=d_0$ and the stability group of G_2 at γ_0 becomes $G_1(s_0)\cap G_2$. Therefore, the orbit d_0 is isomorphic to the right coset space $(G_1(s_0)\cap G_2)\backslash G_2$ as a Borel G_2-space under the map k

$$\gamma_0 t\in d_0\to(G_1(s_0)\cap G_2)t\in(G_1(s_0)\cap G_2)\backslash G_2, \quad t\in G_2.$$

Let μ_{d_0} denote a unique quasi-invariant measure on the orbit d_0. Let H_{d_0} denote the space of all H_1-valued Borel functions ξ on the double coset $G_1s_0G_2$ satisfying the conditions

$$\xi(st)=L_1(s)\xi(t) \quad \text{for every } s\in G_1 \text{ and } t\in G_1s_0G_2, \tag{3}$$

$$\int_{d_0}\|\xi(s)\|^2\,d\mu_{d_0}(\dot{s})<\infty, \tag{4}$$

where $\dot{s}$ means the right coset $\gamma(s)$. Then H_{d_0} becomes a Hilbert space with the norm defined by

$$\|\xi\|=\left\{\int_{d_0}\|\xi(s)\|^2\,d\mu_{d_0}(\dot{s})\right\}^{\frac{1}{2}}.$$

On the space H_{d_0}, we define representations π_{d_0} and L_{d_0} of A and G_2 respectively by

$$\left.\begin{aligned}(\pi_{d_0}(s)\,\xi)\,(s)&=\pi_1\,s(x)\,\xi(s)\\ (L_{d_0}(t)\,\xi)\,(s)&=\lambda(s,t)^{\frac{1}{2}}\,\xi(st)\end{aligned}\right\} \tag{5}$$

for each $x\in A$, $t\in G_2$, $s\in G_1s_0G_2$ and $\xi\in H_{d_0}$, where λ means the λ-function on $G_1s_0G_2\times G_2$ in the sense of G. W. Mackey [15]. By the equalities

$$\begin{aligned}(L_{d_0}(t)\pi_{d_0}(x)\,L_{d_0}(t)^{-1}\xi)\,(s) &= \lambda(s,t)^{\frac{1}{2}}\,(\pi_{d_0}(x)L_{d_0}(t)^{-1}\xi)\,(st)\\ &= \lambda(s,t)^{\frac{1}{2}}\pi_1\,st(x)\,(L_{d_0}(t)^{-1}\xi)\,(st)\\ &= \pi_1 st(x)\xi(s)\end{aligned}$$

for every $t\in G_2$, $x\in A$ and $\xi\in H_{d_0}$, the couple (π_{d_0}, L_{d_0}) is covariant. For each $\xi\in H_{d_0}$, putting $\tilde{\xi}(t)=\xi(s_0t)$ for $t\in G_2$, $\tilde{\xi}$ belongs to the space H_{s_0} and the map V: $\xi\rightarrow\tilde{\xi}$ sets up the unitary equivalence between L_{d_0} and L_{s_0} by [15; Lemma 6.1]. Moreover, we have

$$(V\pi_{d_0}(x)\,V^{-1}\tilde{\xi})\,(t) = (\pi_{d_0}(x)\,V^{-1}\tilde{\xi})\,(s_0t) = \pi_1 s_0 t(x)\,(V^{-1}\tilde{\xi})\,(s_0t) = \pi_1 s_0 t(x)\tilde{\xi}(t)$$

for every $x\in A$, $\tilde{\xi}\in H_{s_0}$ and almost every $t\in G_2$, so that the map V also sets up the unitary equivalence between π_{d_0} and π_{s_0}. The construction of (π_{d_0}, L_{d_0}) does not depend on the element s_0 itself, but on the double coset d_0, so that the first part of our assertion has been established.

For each function $f\in L^\infty(\Gamma,\mu)$ (resp. $f\in L^\infty(\mathcal{D},\nu_0)$), we define the operator $i(f)$ (resp. $j(f)$) on H by

$$i(f)\xi(s) = f(\gamma(s))\xi(s) \qquad (\text{resp. } j(f)\xi(s) = f(d(s))\xi(s))$$

for every $\xi\in H$ and $s\in G$. Let $\mathbf{A}_\Gamma$ (resp. $\mathbf{A}_\mathcal{D}$) denote the set consisting of all $i(f)$'s (resp. $j(f)$'s). Then $\mathbf{A}_\Gamma$ (resp. $\mathbf{A}_\mathcal{D}$) becomes a system of imprimitivity for the restriction $(\pi, U)|_{(A,G_2)}$ of (π, U) based on the G-measure space (Γ,μ) (resp. $(\mathcal{D},\nu_0)$). Besides, $\mathbf{A}_\mathcal{D}$ is a von Neumann subalgebra of $\mathbf{A}_\Gamma$. Since the function $s\rightarrow f(d(s))$, $f\in L^\infty(\mathcal{D},\nu_0)$, is constant on each double $G_1:G_2$ coset, $\mathbf{A}_\mathcal{D}$ commutes with $\{\pi(x), U(s); x\in A, s\in G_2\}$. Hence $(\pi, U)|_{(A,G_2)}$ is decomposed into the direct integration

$$\pi = \int_\mathcal{D}^\oplus \pi^d d\nu_0(d) \quad \text{and} \quad U|_{G_2} = \int_\mathcal{D}^\oplus U^d d\nu_0(d)$$

with respect to the diagonal algebra $\mathbf{A}_\mathcal{D}$. For each $f\in L^\infty(\mathcal{D},\nu_0)$ and $g\in L^\infty(\Gamma,\mu)$ we have

$$\int_\Gamma f(k(\gamma))\,g(\gamma)\,d\mu(\gamma) = \int_\mathcal{D} f(d)\left(\int_d g(\gamma)\,d\mu_d(\gamma)\right)d\nu_0(d)$$

by equation (2). Therefore, the direct integral decomposition of H with respect to the diagonal algebra $\mathbf{A}_\Gamma$ becomes

$$H = \int_\Gamma^\oplus H(\gamma)\,d\mu(\gamma) = \int_\mathcal{D}^\oplus\left(\int_d^\oplus H(\gamma)\,d\mu_d(\gamma)\right)d\nu_0(d).$$

Besides, $\mathbf{A}_\mathcal{D}$ is the diagonal algebra of the first direct integral in the last integration. Let $\mathbf{A}_d$ denote the diagonal algebra of the direct integral $\int_d^\oplus H(\gamma)\,d\mu_d(\gamma)$. Then we have

$$\mathbf{A}_\Gamma = \int_\mathcal{D}^\oplus \mathbf{A}_d\,d\nu_0(d),$$

Since $\mathbf{A}_\Gamma$ is a system of imprimitivity for $(\pi, U)|_{(A, G_2)}$ and since $\mathbf{A}_\mathcal{D}$ commutes with $(\pi, U)|_{(A, G_2)}$ $\mathbf{A}_d$ becomes a system of imprimitivity for $(\pi^d, U^d)|_{(A, G_2)}$ based on the G_2-measure space (d, μ_d) for ν_0-almost every $d \in \mathcal{D}$. Since G_2 acts on every $d \in \mathcal{D}$ transitively and since the stability group of G_2 at each $\gamma(s_0)$ of d is $G_2 \cap s_0^{-1} G_1 s_0 = G_1(s_0) \cap G_2$, the covariant representation $(\pi^d, U^d)|_{(A, G_2)}$ is unitarily equivalent to the induced covariant representation (π_d, L_d) for almost every $d \in \mathcal{D}$ by Theorem 4.2. This completes the proof.

Now we shall show an application of Theorem 7.1. Suppose that A is a separable GCR-algebra and that G is a separable locally compact smooth automorphism group of A as in Theorem 6.1. We shall use the notation of Theorem 6.1. For each subgroup G_0 of G, multiplier σ_0 for G_0 and $s_0 \in G_0$, putting $G_0(s_0) = s_0^{-1} G_0 s_0$ and $\sigma_0^{s_0}(s, t) = \sigma_0(s_0 s s_0^{-1}, s_0 t s_0^{-1})$ for each pair s, t in $G_0(s_0)$, $\sigma_0^{s_0}$ becomes a mutiplier for $G_0(s_0)$. Take a point $\zeta \in \hat{A}$ and an element $s_0 \in G$. Then it is clear that $G_{\zeta s_0} = G_\zeta(s_0)$. Let π_ζ^1 be an irreducible representation of A belonging to ζ and L_ζ^1 a σ_ζ-representation of G_ζ such that $L_\zeta^1(s)\pi_\zeta^1 = \pi_\zeta^1 s$ for every $s \in G_\zeta$. Then we have

$$L_\zeta^1(s_0 s s_0^{-1})\, \pi_\zeta^1 s_0 = \pi_\zeta^1 s_0 s$$

for every $s \in G_\zeta(s_0)$ and the map $s \in G_\zeta(s_0) \to L_\zeta^1(s_0 s s_0^{-1})$ is a $\sigma_\zeta^{s_0}$-representation of $G_\zeta(s_0)$, so that $\sigma_\zeta^{s_0}$ is equivalent to $\sigma_{\zeta s_0}$. Let L_ζ^2 be an arbitrary irreducible σ_ζ^{-1}-representation of G_ζ. Putting $L_\zeta = L_\zeta^1 \otimes L_\zeta^2$ and $\pi_\zeta(x) = \pi_\zeta^1(x) \otimes 1$, we get a covariant representation (π_ζ, L_ζ) of (A, G_ζ). By Theorem 5.3, the covariant representation (π, U) of (A, G) induced by (π_ζ, L_ζ) is irreducible. By Theorem 6.1, every irreducible covariant representation of (A, G) is obtained in this way. Applying Theorem 7.1 to (A, G_ζ) and (A, G), (π_ζ, L_ζ) and $(\pi_\zeta s_0, L_\zeta^{s_0})$ induce unitarily equivalent covariant representations of (A, G) where $L_\zeta^{s_0}$ is the unitary representation of $G_\zeta(s_0)$ given by $L_\zeta(s) = L_\zeta(s_0 s s_0^{-1})$ for $s \in G_\zeta(s_0)$. Therefore, if we define an action of G on the space $\bigcup_{\zeta \in \hat{A}} (G_\zeta, \sigma_s^{-1})^\wedge$ by $\widehat{L_\zeta^2} s_0 = \widehat{L_\zeta^{2s_0}}$ for every L_ζ^2 of $(G_\zeta, \sigma_s^{-1})^\wedge$, where $(G_\zeta, \sigma_s^{-1})^\wedge$ denotes the space of all unitarily equivalence class of irreducible σ_s^{-1}-representation of G_ζ and $L_\zeta^{2s_0}$ means a $(\sigma_\zeta^{s_0})^{-1}$-representation of $G_\zeta(s_0)$ given by $L_\zeta^{2s_0}(s) = L_\zeta^2(s_0 s s_0^{-1})$ for $s \in G_\zeta(s_0)$, then we get the following

THEOREM 7.2. *In the above situation, there exists a one-to-one correspondence between the dual space of the crossed product $C^*(A, G)$ of A by G and the space of all orbits of G in $\bigcup_{\zeta \in \hat{A}} (G_\zeta, \sigma_\zeta^{-1})^\wedge$. In particular, if G acts freely on $\hat{A}$, then there exists a one-to-one correspondence between the dual space of $C^*(A, G)$ and the orbit space $\hat{A}/G$.*

8. Non-type I covariant representations

Let A be a separable GCR-algebra and G a separable locally compact automorphism group of A. If G is a smooth automorphism group, the existence problem of non-type I covariant representations is reduced to that of non-type I projective representations of the stability subgroups of G at any point of $\hat{A}$ by Theorem 6.1. Therefore our attention is concentrated on the case when G is a non-smooth automorphism group. But then the existence problem of non-type I covariant representations is in general not easy, so we shall show the existence of non-type I covariant representation only in a special case.

If G acts non-smoothly on A, then there exists a non-transitive ergodic quasi-invariant finite measure μ on $\hat{A}$ by Theorem 2.6.

THEOREM 8.1. *Suppose that G is a separable locally compact non-smooth automorphism group of a separable GCR-algebra A. Let μ be a non-transitive ergodic measure on $\hat{A}$. If there exists a G-invariant non-null Borel set E in $\hat{A}$ such that the stability group G_ζ of G at each $\zeta \in E$ is reduced to the trivial group $\{e\}$, then there exists a non-type I covariant factor representation of (A, G).*

Proof. By the ergodicity of μ, μ is concentrated on E. For a separable Hilbert space H, Irr $(A:H)/\mathcal{U}(H)$ can be imbedded in $\hat{A}$ and it is invariant under the action of G, so that we may assume that E is contained in Irr $(A:H_0)/\mathcal{U}(H_0)$ for some separable Hilbert space H_0. Let $\zeta \in E \to \pi_\zeta \in \mathrm{Irr}\,(A: H_0)$ be a Borel cross-section. Let H be the Hilbert space consisting of all H_0-valued square summable functions on $E \times G$ with respect to the product measure of μ and the right Haar measure ds of G. Take a representation U of G and π of A on H defined by

$$\left.\begin{aligned} (U(t)\,\xi)\,(\xi, s) &= \xi(\zeta, st) \\ (\pi(x)\,\xi)\,(\zeta, s) &= \pi_\zeta s(x)\,\xi(\zeta, s) \end{aligned}\right\} \tag{1}$$

for each $x \in A$, $s, t \in G$, $\xi \in H$ and $\zeta \in E$. Then (π, U) becomes a covariant representation of (A, G) on H. We shall show that (π, U) is a non-type I covariant factor representation.

Let H_1 denote the Hilbert space consisting of all H_0-valued square summable Borel functions on G with respect to the right Haar measure. For each $\zeta \in E$, let (π_ζ^1, U_ζ^1) be a covariant representation of (A, G) on H_1 defined by

$$(U_\zeta^1(t)\,\xi)\,(s) = \xi(st) \quad \text{and} \quad (\pi_\zeta^1(x)\,\xi)\,(s) = \pi_\zeta s(x)\,\xi(s) \tag{2}$$

for every $x \in A$, $s, t \in G$ and $\xi \in H_1$. Then (π_ζ^1, U_ζ^1) is the covariant representation induced by the covariant representation (π_ζ, ι) of $(A, \{e\})$ on H_0 where ι means the trivial representation of the trivial subgroup $\{e\}$ of G. For each $\zeta \in E$, $\pi_\zeta s$ and $\pi_\zeta t$ are disjoint irreducible representations for each distinct pair s, t in G by the hypothesis for the action of G on E, so that π_ζ^1 becomes a representation which is multiplicity free in the sense of G. W. Mackey [16]. Therefore, the canonical system of imprimitivity for (π_ζ^1, U_ζ^1) is central, so that the commutant $\mathbf{M}(\pi_\zeta^1, U_\zeta^1)'$ is isomorphic to the commutant $\mathbf{M}(\pi_\zeta, \iota)'$ which is reduced to the algebra of scalar multiples. That is, (π_ζ^1, U_ζ^1) is irreducible.

By Fubini's theorem, the Hilbert space H can be regarded as the Hilbert space consisting of all H_1-valued square summable Borel functions on E with respect to the measure μ. Comparing equation (1) with equation (2), we find that (π, U) is represented by the direct integral

$$(\pi, U) = \int_E^{\oplus} (\pi_\zeta^1, U_\zeta^1)\, d\pi(\zeta). \tag{3}$$

Take $(\zeta_0, s_0) \in E \times G$. From the definition of the action of G on $\hat{A}$, there exists a unitary operator V_0 on H_0 with $V_0\pi_{\zeta_0 s_0} = \pi_{\zeta_0} s_0$. It follows at once that $(\pi_{\zeta_0} s_0)s = V_0(\pi_{\zeta_0 s_0} s)$ for every $s \in G$. If we define a unitary operator V on H_1 by $(V\xi)(s) = V_0\xi(s)$ for each $\xi \in H_1$ and $s \in G$, V sets up a unitary equivalence between $(\pi_{\zeta_0 s_0}^1, U_{\zeta_0 s_0}^1)$ and $(\pi_{\zeta_0}^1, U_{\zeta_0}^1)$. If ζ_1 and ζ_2 lie on different orbits of G in E, then the direct integral decompositions of $\pi_{\zeta_1}^1$ and $\pi_{\zeta_2}^1$,

$$\pi_{\zeta_1}^1 = \int_G^{\oplus} \pi_{\zeta_1} s\, ds \quad \text{and} \quad \pi_{\zeta_2}^1 = \int_G^{\oplus} \pi_{\zeta_2} s\, ds, \tag{4}$$

have no common component, which yields that $\pi_{\zeta_1}^1$, and $\pi_{\zeta_2}^1$ are disjoint representations of A. Thus, $(\pi_{\zeta_1}^1, U_{\zeta_1}^1)$ and $(\pi_{\zeta_2}^1, U_{\zeta_2}^1)$ are unitarily equivalent for a pair ζ_1, ζ_2 in E if and only if ζ_1 and ζ_2 lie on the same orbit of G. Therefore, if we define an equivalence relation "$\zeta_1 \sim \zeta_2$" by "$(\pi_{\zeta_1}^1, U_{\zeta_1}^1) \simeq (\pi_{\zeta_2}^1, U_{\zeta_2}^1)$", then the quotient Borel space $E/$"$\sim$" is not countably separated because of the ergodicity of the measure μ.

For each function $f \in L^\infty(E, \mu)$, define an operator $i(f)$ on H by $i(f)\xi(\zeta, s) = f(\zeta)\xi(\zeta, s)$ for every $\xi \in H$ and $(\zeta, s) \in E \times G$. Then the algebra $\mathbf{A}$ consisting of all $i(f)$'s becomes a diagonal algebra of the direct integral (3) which is a completely rough subalgebra of the commutant $\mathbf{M}(\pi, U)'$ in the sense of [26; Definition 2.1]. Since each component of the direct integral (3) is irreducible, $\mathbf{A}$ is a completely rough maximal abelian subalgebra of the commutant $\mathbf{M}(\pi, U)'$. Every maximal abelian subalgebra of a type I von Neumann algebra is smooth, so that $\mathbf{M}(\pi, U)'$ is not of type I, equivalently $\mathbf{M}(\pi, U)$ is not of type I. If z is a non-trivial central projection of $\mathbf{M}(\pi, U)$, $\mathbf{A}z$ and $\mathbf{A}(I-z)$ are unrelated in the sense of [26; Definition 3.2] by [27]. But this is impossible by the ergodicity of the action of G on (E, μ). Therefore, $\mathbf{M}(\pi, U)'$ is a factor. This completes the proof.

COROLLARY. *Suppose that G is a one-parameter automorphism group of a separable GCR-algebra A. Then the following are equivalent:*

(i) *There exists a non-type* I *factor covariant representation of (A, G).*

(ii) *The action of G on A is not smooth, that is, the quotient Borel space $\hat{A}/G$ is not countably separated.*

Proof. Any proper closed subgroup of G is cyclic and discrete. Hence every projective representation of a closed subgroup of G is of type I. Therefore, (i) implies (ii) by Theorem 6.1.

(ii) $\Rightarrow$ (i). Let μ be a non-transitive ergodic measure of $\hat{A}$ with total mass one. By [1; Proposition 2.4, p. 70], there exists a G-invariant Borel set E, which carries the measure μ, such that the stability group of G in E does not depend on the point of E. Let G_0 denote the constant stability group. If G_0 is not reduced to the trivial group $\{e\}$, then the quotient group G/G_0 becomes a compact group. The action of G induces naturally an action of the compact group G/G_0 on E, so that E becomes a standard G/G_0-space. Hence every ergodic measure on E is transitive by the compactness of G/G_0, which contradicts the non-transitivity of μ. Therefore G_0 is reduced to the trivial group $\{e\}$, which is the case of Theorem 8.1. This completes the proof.

9. Appendix

Throughout this paper, we have been assuming that the basic C^*-algebra is of type I. Of course, the study in the case of non-type I is important, although, without the type I assumption for the basic C^*-algebra, we are far from the detail at present. For example, it is plausible that a pair (A, G) of a non-type I C^*-algebra A and its locally compact automorphism group G could have a non-type I covariant representation. But this guess is not true. At the Fifth Symposium on Functional Analysis (in Japan) held in Sendai on August 1–3, 1967, Professor O. Takenouchi pointed out to the author the following curious example.

Let H be the famous example of 5-dimensional non-type I solvable Lie group due to F. I. Mautner. That is, H is the cartesian product set $C^2 \times R$ of the complex number field C and the real number field R, and the multiplication in H is defined by the equation

$$(z_1, w_1, s_1)(z_2, w_2, s_2) = (z_1 + e^{2\pi i s_1} z_2,\ w_1 + e^{2\pi \alpha i s_1} w_2,\ s_1 + s_2)$$

for $(z_i, w_i, s_i) \in H$ $i = 1, 2$, where α is an irrational real number. Let A denote the group C^*-algebra $C^*(H)$ of H. Then A is not of type I. Let G denote the additive group of real numbers. For each $t \in G$, define an action of t on H as an automorphism by

$$(z, w, s)t = (z, e^{2\pi i t} w, s) \quad \text{for} \quad (z, w, s) \in H.$$

Let K denote the semi-direct product HG of H and G. Then K becomes the cartesian product set $C^2 \times R^2$ with the multiplication

$$(z_1, w_1, s_1, t_1)(z_2, w_2, s_2, t_2) = (z_1 + e^{2\pi i s_1} z_2,\ w_1 + e^{2\pi i(\alpha s_1 + t_1)} w_2,\ s_1 + s_2,\ t_1 + t_2)$$

for $(z_i, w_i, s_i, t_i) \in C^2 \times R^2$. Define a map σ of $C^2 \times R^2$ onto $C^2 \times R^2$ by

$$\sigma(z, w, s, t) = (z, w, s, \alpha s+t) \quad \text{for} \quad (z, w, s, t) \in C^2 \times R^2.$$

Then we easily see that σ is an isomorphism of K onto the group $K' = C^2 \times R^2$ with the multiplication

$$(z_1, w_1, s_1, t_1)(z_2, w_2, s_2, t_2) = (z_1 + e^{2\pi i s_1} z_2, w_1 + e^{2\pi i t_1} w_2, s_1 + s_2, t_1 + t_2)$$

for $(z_i, w_i, s_i, t_i) \in K'$ $i=1, 2$. Since K' is the direct product of the group $H' = C \times R$ and itself with the multiplication

$$(z_1, s_1)(z_2, s_2) = (z_1 + e^{2\pi i s_1} z_2, s_1 + s_2)$$

for $(z_i, s_i) \in C \times R$ $i=1, 2$ and H' is of type I, K' is also of type I and then so is K. The action of G on H induces naturally that of G on the group C^*-algebra $A = C^*(H)$ of H. The crossed product of A by G is nothing else but the group C^*-algebra $C^*(K)$ of the group K, so every covariant representation of (A, G) is of type I despite the non-type I property of A.

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