A PROOF OF A CONJECTURE OF LOEWNER AND OF THE CONJECTURE OF CARATHEODORY ON UMBILIC POINTS

(Dedicated to the Memory of Charles Loewner)

BY

CHARLES J. TITUS

University of Michigan, Ann Arbor, Mich. 48104, USA

1. Introduction

The Loewner Conjecture was motivated by the study of umbilic points on surfaces and by various other geometrical investigations concerning the qualitative theory of differential and integral operators (see especially Loewner [8]). Let u be a real analytic function on the disk D, $x^2+y^2<1$; with $2\partial_{\bar z}=\partial_x+i\partial_y$, think of the iterates $\partial_{\bar z}^n u$ of $\partial_{\bar z}$ on acting u as vector fields on D.

Loewner Conjecture (about 1950).

If the vector field $\partial_{\bar z}^n u$, $u\in C^\omega(D,\mathbf{R})$, $n\geqslant 1$, has an isolated zero at the origin, then the index of $\partial_{\bar z}^n u$ at the origin is not greater than n.

For $n=1$ this conjecture follows directly from standard techniques in differential equations (see Lefschetz [6]). For $n=2$ it is the key lemma required for a proof of the Caratheodory Conjecture (see Hamburger [3, 4], Bol [1], Klotz [5]).

Caratheodory Conjecture. *Every convex real analytic imbedding of S^2 in $\mathbf{E}^2$ has at least two umbilic points.*

With a proof of the Loewner Conjecture for $n=2$ the work of Hamburger together with standard more recent work in differential geometry will show that every real analytic immersion of S^2 in $\mathbf{E}^3$ has at least two umbilic points so that the convexity condition is in fact irrelevant.

The main difficulty in the proof of these conjectures occurs because the multiplicity of the zero of the vector field may be arbitrarily large; with standard conditions of genericity imposed the proofs of the Loewner Conjecture become relatively trivial. However, possibly with $n=1$ excepted, present day methods using approximation from the generic cases are nowhere near adequate. In fact a proof including the non-generic cases describes constraints on the limits of the generic cases. Thus, for example with $n=1$, a proof of the Loewner Conjecture implies that if the gradient ∇u has an isolated zero at the origin and u is the limit of Morse functions v then the sum of the indices at the zeros of the vector fields ∇v which approach the origin must be less than or equal to 1.

Geometrically, for $n \geqslant 3$, the Loewner Conjecture is related to generalizations of the Caratheodory Conjecture concerning the existence of higher order singularities on surfaces immersed in higher dimensional Euclidean spaces (Little [7]).

Because of the complexity of previous methods for $n=2$ (and even $n=1$) and because other substantial difficulties occur for $n \geqslant 3$ a very different approach seemed required. In this paper in the early part, the first six sections, we make a thorough qualitative study of ordinary differential operators with constant coefficients (which is closely related to the action of $\partial_{\bar{z}}^n$ on homogeneous polynomials); in the later part we develop a perturbation theory which extends the qualitative study to the partial differential operators involved in the Loewner Conjecture. It is however the detailed geometric study in the early part that makes the perturbation extension tractable.

We will prove the Loewner Conjecture in a sharpened form involving the linear factors of the first "consequential" homogeneous polynomial in the expansion of u; this sharpened form apparently gives new information even when $n=1$.

For $u \in C^\omega(D, \mathbf{R})$ let $u=u_p+\ldots+u_q+\ldots$ be the expansion into forms (=homogeneous polynomials) u_q of degree q. We may assume that none of the forms u_q is anihilated by the operator $\partial_{\bar{z}}^n$. Let $L_k u_p$ be the number of real linear factors of u_p of multiplicity at least k. (note that $L_k u_p \geqslant L_{k+1} u_p$).

The principal results of this paper are contained in the following pair of theorems.

THEOREM. *Given $u \in C^\omega(D, \mathbf{R})$ so that the vector field $\partial_{\bar{z}}^n u$ has an isolated zero at the origin choose the form u_p in the expansion of u with lowest degree such that $\partial_{\bar{z}}^n u_p \neq 0$ and then the index, $\Omega_0 \partial_{\bar{z}}^n u$, of the vector field at the origin satisfies the inequality:*

(a) *for $n \leqslant p \leqslant 2n-1$,*

$$\Omega_0 \partial_{\bar{z}}^n u \leqslant n-(L_1 u_p+\ldots+L_{p-n+1} u_p)+(L_{n+1} u_p+\ldots+L_p u_p),$$

(b) *for $2n \leqslant p$,*

$$\Omega_0 \partial_{\bar{z}}^n u \leqslant n-(L_1 u_p+\ldots+L_n u_p)+(L_{n+1} u_p+\ldots+L_{2n} u_p).$$

Since u is real, $\partial_z u$ is the complex conjugate of $\partial_{\bar{z}} u$ and one has the equivalent

Dual theorem. *Given $u \in C^\omega(D, \mathbf{R})$ so that the vector field $\partial_z^n u$ has an isolated zero at the origin, choose the form u_p in the expansion of u with lowest degree such that $\partial_z^n u_p \neq 0$. Then the index, $\Omega_0 \partial_z^n u$, of the vector field at the origin satisfies the inequality:*

(a) for $n \leqslant p \leqslant 2n-1$,

$$\Omega_0 \partial_z^n u \geqslant -n + (L_1 u_p + \ldots + L_{p-n+1} u_p) - (L_{n+1} u_p + \ldots + L_{2n} u_p),$$

(b) *for* $2n \leqslant p$,

$$\Omega_0 \partial_z^n u \geqslant -n + (L_1 u_p + \ldots + L_n u_p) - (L_{n+1} u_p + \ldots + L_{2n} u_p).$$

Under the same hypotheses one also has, by simple degree considerations, that

$$|\Omega_0 \partial_{\bar z}^n u| \leqslant p - n,$$

and

$$|\Omega_0 \partial_z^n u| \leqslant p - n.$$

When $p \leqslant 2n$ these imply the Loewner Conjecture and with $p \leqslant 2n-1$, the inequalities (a) are independent of these inequalities. When $p \geqslant 2n$ the inequalities (a) imply these in equalities.

Before proceeding a few examples may be helpful.

Example 1. Let $u = (z\bar z)^n$ so that u has no real linear factors (i.e., of the form $az + \bar a \bar z$) and then $\partial_{\bar z}^n u = n!\, z^n$ which has index $\Omega_0 = n$.

Example 2. Let $u = (z + \bar z)^n$ so that $L_1 u = \ldots = L_n u = n$ and $L_{n+1} u = 0$ and then $\partial_{\bar z}^n u = n!$ which has index $\Omega_0 = 0$.

Example 3. Let $u = (z + \bar z)^{2n} + (\text{higher order terms})$ and then $L_1 u_p = \ldots = L_{2n} u_p = 1$ so that by the theorem

$$\Omega_0 \partial_{\bar z}^n u \leqslant n - n + n = n.$$

Example 4. Let $u = (z + \bar z)^{2n} (iz - i\bar z)^n (z\bar z)^n + (\text{higher order terms})$ and then $L_1 u_p = \ldots = L_n u_p = 2$, $L_{n+1} u_p = \ldots = L_{2n} u_p = 1$ so that by the Theorem

$$\Omega_0 \partial_{\bar z}^n u \leqslant n - 2n = n = 0.$$

In the following we will actually prove the Dual Theorem since the inequalities involved in the proof tend to take a more intuitive form.

2. Preliminary notation and definitions

All mappings, manifolds and structures are real analytic. The theory is built on the oriented Euclidean plane $\mathbf{E}$ which will often be identified with the complex line $\mathbf{C}$. Let D be the closed unit disk in $\mathbf{E}$ about the origin and S^1 the naturally oriented circle that bounds

D. Let $\mathbf{P}$ be the real projective line thought of as the bundle of non-oriented lines through the origin of $\mathbf{E}$, where $\mathbf{E}$ induces the orientation and real analytic structure on $\mathbf{P}$; thus $\mathbf{P}$ is parameterized by the angle measured from a fixed line and taken modulo π.

One of the most useful technical devices involved is called the projective winding number of a mapping in $C^{\omega}(S^1, \mathbf{E})$. First we define the mapping $*$: $\mathbf{E}-\{0\}\to\mathbf{P}$, $v\mapsto v^*$, where v^* is the line through v and the origin. For $\zeta\in C^{\omega}(S^1, \mathbf{E})$ define the *multiplicity function* ζ: $S^1\to\mathbf{Z}$, $\theta\mapsto\mu_\zeta(\theta)$ (or simply $\mu(\theta)$ when the dependence on ζ is clear), where $\mu_\zeta(\theta)$ is the order of the first non-vanishing derivative of ζ at θ. So $\mu_\zeta(\theta)$ is simply the multiplicity of the zero of ζ at θ and we have $\zeta^{(\mu)}(\theta)\neq 0$. Next, define ζ^*: $S^1\to\mathbf{P}$, $\theta\mapsto[\zeta^{(\mu)}(\theta)]^*$; the mapping ζ^* is seen to be real analytic. Finally, the *projective winding number* of ζ about the origin is defined as one half the topological degree of the mapping ζ^*: $S^1\to\mathbf{P}$; it is denoted by $\omega_0^*\zeta$. Note that $\omega_0^*\zeta$ is defined whether or not ζ passes through the origin, that it takes half integer values and that, when ζ is never zero, it is the usual winding number.

The following explicit formulas for $\omega^*\zeta$ are not difficult to derive from standard sources and will be taken as known. The derivative of $\Theta=\arg\zeta^*$ with respect to θ is

$$(\arg\zeta^*)'=\frac{1}{\mu+1}\,\frac{\zeta^{(\mu)}\wedge\zeta^{(\mu+1)}}{\|\zeta^{(\mu)}\|^2};$$

here and throughout this paper $v\wedge w$ denotes the signed area det (v, w).

From differential degree theory (see e.g. Milnor [8]), for any v that is a regular value of ζ^* (i.e. $\zeta^*(\theta)=v^*$ only if $(\arg\zeta^*)'(\theta)\neq 0$ or ζ^* does not hit v^*) one has

$$\omega_0^*\zeta=\tfrac{1}{2}\Sigma\{\operatorname{sgn}[(\arg\zeta^*)'(\theta)]\,|\,\zeta^*(\theta)=v^*\}.$$

Similar to the Cauchy formula for the usual winding number, one has

$$\omega_0^*\zeta=\frac{1}{2\pi}\int_0^{2\pi}(\arg\zeta^*)'$$

3. The group $\mathcal{G}$ and the semigroup $\mathcal{S}$

We will define a group $\mathcal{G}$ and a semigroup $\mathcal{S}$ and two actions of the group which are central to the entire approach. The "algebraic" action is defined in section 4; it describes the geometry of the Euclidean algorithm and of the Sturm theory for separating pairs of polynomials. The "differential" action is defined in section 5 and includes the action of ∇^n on forms as a special case.

Let, as usual, $\mathbf{R}[x]$ be the ring of real polynomials and $\mathbf{E}[x]$ the (free) $\mathbf{R}[x]$-module spanned by a positive basis, e_1, e_2 in $\mathbf{E}$; thus $\mathbf{E}[x]=\{\alpha e_1+\beta e_2\,|\,\alpha, \beta\in\mathbf{R}[x]\}$. We will often think of $\mathbf{E}[x]$ and $\mathbf{R}[x]$ as subsets of the spaces of mappings $C^{\omega}(\mathbf{R}, \mathbf{E})$ and $C^{\omega}(\mathbf{R})$.

On $R[x]\times \mathbf{E}$ we define $(\alpha, v)=(\beta, w)$ if and only if there exist real a and b, note both zero, so that $av=bw$ and $b^2\alpha=a^2\beta$, (i.e., $b^2\alpha(x)=a^2\beta(x)$). It follows with a little computation that this is an equivalence relation and we note

$$(0, 0)=(\beta, w)\Leftrightarrow \beta=0 \quad \text{or} \quad w=0,$$

and, for $\alpha\neq 0$ and $v\neq 0$,

$$\{(\beta, w)\,|\,(\beta, w)=(\alpha, v)\}=\left\{\left(a^2\alpha, \frac{1}{a}v\right)\,|\,a\in \mathbf{R}, a\neq 0\right\}.$$

Let $\mathcal{G}(v)$ be the set $\{(\alpha, v)\,|\,\alpha\in\mathbf{R}[x]\}$ together with the binary relation

$$(\beta, v)(\alpha, v)=(\alpha+\beta, v),$$

both definitions making sense for equivalence classes. Each $\mathcal{G}(v)$ is then an abelian group with the identity represented by $(0, 0)=(0, v)$ and inverse by $(\alpha, v)^{-1}=(-\alpha, v)$. Note, with $v\neq 0$, $w\neq 0$, that $\mathcal{G}(v)=\mathcal{G}(w)$ if and only if v and w are dependent over $\mathbf{R}(\Leftrightarrow v\wedge w=0)$.

The group $\mathcal{G}$ itself is defined as the (finite) free product of the abelian groups $\mathcal{G}(v)$ and thus, for every $G\in\mathcal{G}-\{I\}$ there is a unique sequence of groups $\mathcal{G}(v_i)$ and elements in the $\mathcal{G}(v_i)$ represented by (α_i, v_i) so that G is the (reduced) product

$$G=(\alpha_n, v_n)\ldots(\alpha_1, v_1) \quad \text{where} \quad \alpha_j\neq 0 \quad \text{and} \quad v_{j+1}\wedge v_j\neq 0.$$

The number of factors in the (reduced) product is called the length of G.

There is a "constant coefficient" subgroup $\mathcal{G}_0\subset\mathcal{G}$ which is important. It is defined in the same way but we begin with the equivalence relation on $\mathbf{R}\times\mathbf{E}$ instead of $\mathbf{R}[x]\times\mathbf{E}$. Thus each element in $\mathcal{G}_0-\{I\}$ is represented by the (reduced product)

$$G=(a_n, v_n)\ldots(a_1, v_1) \quad \text{where} \quad a_j\in\mathbf{R},\ a_j\neq 0,\ v_{j+1}\wedge v_j\neq 0.$$

The semigroup $\mathcal{S}\subset\mathcal{G}$ is simply the set containing the identity $I=(0, 0)$ and the reduced products

$$S=(\alpha_n, v_n)\ldots(\alpha_1, v_1) \quad \text{where} \quad \alpha_j\geqslant 0,\ a_j\neq 0,\ v_{j+1}\wedge v_j\neq 0.$$

This too makes sense for equivalence classes since $\alpha\geqslant 0$, $\alpha\neq 0$ and $(\alpha, v)=(\beta, w)$ imply that $\beta\geqslant 0$ and $\beta\neq 0$.

We also have the important "constant coefficient" semigroup $\mathcal{S}_0=\mathcal{S}\cap\mathcal{G}_0$ which contains the identity and the reduced products

$$S=(a_n, v_n)\ldots(a_1, v_1),\ a_j\in\mathbf{R},\ a_j>0,\ v_{j+1}\wedge v_j\neq 0.$$

PROPOSITION 1. *$\mathcal{S}\cap\mathcal{S}^{-1}=\{I\}$ and $\mathcal{S}_0\cap\mathcal{S}_0^{-1}=\{I\}$.*

Proof. If $(\alpha_m, v_m) \ldots (\alpha_1, v_1) = (\beta_n, w_n) \ldots (\beta_1, w_1)$ where $m > 1$ and all $\beta_k < 0$ and $\alpha_k > 0$, then $m = n$ and $0 < \alpha_1 = \beta_1 < 0$ a contradiction. This proof works for both semigroups $\mathcal{S}$ and $\mathcal{S}_0$.

PROPOSITION 2. *The group $\mathcal{G}$ is generated by $\mathcal{S}$ and $\mathcal{S}^{-1}$; the same is true for $\mathcal{G}_0$ and $\mathcal{S}_0$.*

Proof. Consider $G = (\alpha_n, v_n) \ldots (\alpha_1, v_1) \in \mathcal{G}$ of length n (so $a_j \neq 0$ and $v_{j+1} \wedge v_j \neq 0$). If an α_k changes sign write

$$(\alpha_k, v_k) = (\alpha_k^2 - \alpha_k + 2, v_k)(-\alpha_k^2 + 2\alpha_k - 2, v_k),$$

where then the left factor is in $\mathcal{S}$ and the other is in $\mathcal{S}^{-1}$. If α_k does not change sign, leave the factor (α_k, v_k) unchanged.

4. Algebraic action of $\mathcal{G}$ on $\mathbf{E}[x]$

Given an (α, v), in the abelian group $\mathcal{G}(v)$, define the operator (α, v): $\mathbf{E}[x] \to \mathbf{E}[x]$, $\zeta \mapsto (\alpha, v)\zeta$, by

$$[(\alpha, v)\zeta](x) \equiv \zeta(x) + x\alpha(x)(v \wedge \zeta(x))v;$$

or, in less explicit notation,

$$(\alpha, v)\zeta = \zeta + x\alpha(v \wedge \zeta)v.$$

This makes sense for the equivalence classes and is easily seen to be an action of the abelian group $\mathcal{G}(v)$ on $\mathbf{E}[x]$ wherein $(0, 0)\zeta = \zeta$ for all ζ and $(\beta, v)[(\alpha, v)\zeta] = (\alpha + \beta, \zeta)$ for all ζ. We have then defined, by composition, the action of the whole group $\mathcal{G}$ on $\mathbf{E}[x]$ and, analogously, the actions of $\mathcal{S}$, $\mathcal{G}_0$ and $\mathcal{S}_0$.

PROPOSITION 3. (*A geometric form of the Euclidean algorithm.*) *For any $\zeta \in \mathbf{E}[x]$;* (a) *there exists a $G \in \mathcal{G}$, a monic polynomial $\alpha_0 \in \mathbf{R}[x]$ and a vector $v_0 \in \mathbf{E}$ such that $\zeta = G\alpha_0 v_0$;* (b) *G, α_0 and v_0 are uniquely determined;* (c) *α_0 is the monic polynomial of highest degree that divides ζ.*

Proof. Given ζ choose a positive basis e_1, e_2 so that $\zeta = P_1 e_1 + P_2 e_2$ with $P_i \in \mathbf{R}[x]$ and $\deg P_1 < \deg P_2 = \deg \zeta$. By the Euclidean algorithm we have the existence of Q_j and P_j such that

$$P_j = Q_j P_{j+1} - P_{j+2}, \quad j = 1, \ldots, p,$$

with $\deg P_{j+1} < \deg P_j$ and $P_{p+2} = 0$ (the minus sign for the remainder is a useful convention). We write this in the form

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} Q_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} Q_p & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{p+1} \\ 0 \end{pmatrix}.$$

Choose $\alpha_k \in \mathbf{R}[x]$ and $a_k \in \mathbf{R}$ so that $Q_{p-k+1} = x\alpha_k + a_k$, $k = 1, \ldots, p$, and let $\alpha_0 = P_{p+1}$; we have then

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} x\alpha_p + a_p & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x\alpha_1 + a_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & x\alpha_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_p & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & x\alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}.$$

Let $T(a_k)$ be the linear transformation represented by

$$\begin{pmatrix} a_k & -1 \\ 1 & 0 \end{pmatrix}.$$

Then we have from the above

$$\zeta = (\alpha_p, e_1)\, T(a_p) \ldots (\alpha_1, e_1)\, T(a_1)\, \alpha_0 e_1.$$

Before proceeding further, we need the following identity for linear transformations T with $\det T = 1$; namely,

$$(a, Tv) = T(a, v)\, T^{-1}.$$

To derive the identity we compute

$$\begin{aligned}(a, Tv)\zeta &= \zeta + a(Tv \wedge \zeta)\, Tv = \zeta + a(Tv \wedge TT^{-1}\zeta)\, Tv \\ &= \zeta + a(v \wedge T^{-1}\zeta)\, Tv = T[T^{-1}\zeta + a(v \wedge T^{-1}\zeta)\, v] \\ &= T(a, v)\, T^{-1}\zeta.\end{aligned}$$

Let $T_k = T(a_p) \ldots T(a_k)$ and we rewrite ζ in the form

$$= (\alpha_p, e_1)\, T_p (\alpha_{p-1}, e_1)\, T_p^{-1} \ldots T_2 (\alpha_1, e_1)\, T_2^{-1} T_1 \alpha_0 e_1,$$

which, using the identity just derived, gives

$$\zeta = (\alpha_p, e_1)(\alpha_{p-1}, T_p e_1) \ldots (\alpha_1, T_2 e_1)\, \alpha_0 T_1 e_1.$$

This completes the proof of part (a); the proofs for parts (b) and (c) are omitted since they follow directly form this form and the Euclidean algorithm.

Next we take as known the notion of the Cauchy Index of a polynomial $w \in \mathbf{C}[z]$; it is defined as $\frac{1}{2}(\#^+ - \#^-)$, where $\#^+[\#^-]$ is the number of roots in the open upper [lower] complex half plane. Write $w=P_1+iP_2$, $P_j\in\mathbf{R}[z]$ and, using the Q_k as defined in the proof of Proposition 3, define SGN Q_k to be equal to ± 1 if deg Q_k is odd and $\operatorname{sgn} Q_k(+\infty)=\pm 1$ and to be equal to 0 if deg Q_k is even. The following result is known (see e.g. [2], pages 205–208).

PROPOSITION 4. *The Cauchy Index of $w\in\mathbf{C}[z]$ is given by*

$$\tfrac{1}{2}(\#^+-\#^-)=\tfrac{1}{2}(\operatorname{sgn} Q_1+\ldots+\operatorname{sgn} Q_p).$$

We note next that for $w\in\mathbf{C}[z]$ we can write, with e_1, e_2 a positive basis, $P_1+iP_2=w$, $P_j\in\mathbf{R}[z]$, and then $\zeta=P_1e_1+P_2e_2$. If we think of ζ as a mapping from $\mathbf{R}$ to $\mathbf{E}$ ($P_j\in\mathbf{R}[x]$) it follows directly that $\omega_0^*\zeta$ is the Cauchy Index of w. Also, with the polynomials α_k as in the proof of Proposition 3, we define $\operatorname{sgn}\alpha_k=\mathrm{SGN}\,Q_k$ (whence $\operatorname{sgn}\alpha_k$ is equal to ± 1 if deg α_k is even and $\operatorname{sgn}\alpha_k(+\infty)=\pm 1$ and $\operatorname{sgn}\alpha_k=0$ if deg α_k is odd). Let $\mathbf{R}_\infty$ be the 1 point compactification of $\mathbf{R}$ which is naturally isomorphic to $\mathbf{P}$ and, for $\zeta\in\mathbf{E}[x]$, define $\omega_0^*\zeta$ as the topological degree of the mappings ζ^*: $\mathbf{R}_\infty\to\mathbf{P}$. Also analogous to the formulae at the end of section 2,

$$\omega_0^*\zeta=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\arg\zeta]'.$$

Putting these facts and Proposition 4 together we have

PROPOSITION 5. *Given $\zeta\in\mathbf{E}[x]$, $\zeta=P_1e_1+P_2e_2$ and $\zeta=(\alpha_p, v_p)\ldots(\alpha_1, v_1)\alpha_0v_0$, one has*

(a) $$\omega^*\zeta=-\tfrac{1}{2}(\operatorname{sgn}\alpha_1+\ldots+\operatorname{sgn}\alpha_p),\quad \textit{with}$$

(b) $$w=P_1+iP_2.$$

The Cauchy Index of $w=\frac{1}{2}(\#^+-\#^-)=\omega_0^*\zeta$.

Classically, with $P_i\in\mathbf{R}[x]$, one says that P_2 separates P_1 positively [negatively] if deg $P_1=1+\deg P_2$, the roots of P_1 and P_2 are all real and simple, the roots of P_2 separate (interlace) the roots of P_1, and the product of the highest coefficients is positive [negative]. To have a more geometric definition we define, for $\zeta\in\mathbf{E}[x]$: ζ is *positively* [*negatively*] *separating* if there exists a positive basis e_1, e_2 of $\mathbf{E}$ so that, with $\zeta=P_1e_1+P_2w_2$, P_2 separates P_1 positively [negatively]. We can now state the principal properties of the algebraic action of S_0 on $\mathbf{E}[x]$.

PROPOSITION 6. (*Characterization of the algebraic action of S_0*). *The following conditions on $\zeta\in\mathbf{E}[x]$ are equivalent:*

(a) *ζ is positively [negatively] separating;*

(b) *there exists an $S\in S_0[S_0^-]$, $v_0\in\mathbf{E}$ such that $\zeta=Sv_0$ (i.e. $\exists\,\alpha_j\in\mathbf{R}$ $\alpha_j>0[\alpha_j<0]$ and $\exists\, v_{j+1}\wedge v_j\neq 0\ni\zeta=(\alpha_n, v_n)\ldots(\alpha_1, v_1)\alpha_0v_0$);*

(c) $\omega^*\zeta = -\frac{1}{2}\deg\zeta \quad [\frac{1}{2}\deg\zeta]$;

(d) *there is a positive basis* e_1, e_2 *of* $\mathbf{E}$ *so that with* $\zeta = P_1e_1 + P_2e_2, P_j \in \mathbf{R}[x]$, *the polynomial* $w = P_1 + iP_2$ *has all of its roots in the open lower [upper] half plane.*

Proof. We prove (a)$\Leftrightarrow$(b), (b)$\Leftrightarrow$(c) and (c)$\Leftrightarrow$(d); the proofs are given only for positive separation since the other case is completely analogous.

First, (a)$\Rightarrow$(b). There is a positive basis e_1, e_2 of $\mathbf{E}$ so that, with $\zeta = P_1e_1 + P_2e_2$, P_2 separates P_1 positively. We write

$$P_1 = Q_1P_2 - P_3$$

and will show next that P_3 separates P_2 positively. For, since $\deg P_1 = 1 + \deg P_2$, we have $Q_1 = \alpha x + a$ with $\alpha, a \in \mathbf{R}$ and $\alpha > 0$. Let c_i be the roots of P_2, $c_{n-1} > \dots > c_i$ and let b_1, b_2, b_3 be the leading coefficients of P_1, P_2, P_3. Then

$$\operatorname{sgn} P_1(c_i) = -\operatorname{sgn} P_3(c_i) = (-1)^i \operatorname{sgn} b_1 = (-1)^i \operatorname{sgn} b_2,$$

and so

$$\operatorname{sgn} P_3(c_1) = (-1)^{i+1} \operatorname{sgn} b_2.$$

This shows that P_3 has $n-2$ simple roots interlacing those of P_2 and that $b_2b_3 > 0$ so that P_3 separates P_2 positively. The proof can now be completed by induction. For using the notation as before, we have $\zeta = (\alpha_n, v_n) \dots (\alpha_1, v_1)\alpha_0v_0$ and have shown that $\alpha_n \in \mathbf{R}$, $\alpha_n > 0$, and that $(\alpha_{n-1}, v_{n-1}) \dots (\alpha_1, v_1)$ is separating. Continuing we have that all the $\alpha_j \in \mathbf{R}$, $\alpha_j > 0$ and thus $\zeta = Sv_0$ with $S \in \mathcal{S}_0$.

Second, (b)$\Rightarrow$(a). Write $\zeta = (\alpha_n, v_n) \dots (\alpha_1, v_1)v_0$ with $\alpha_j \in \mathbf{R}$, $\alpha_j > 0$. As in the previous paragraph we have $P_1 = Q_1P_2 - P_3$ and note that the identities there also establish that if P_3 separates P_2 positively, then P_2 separates P_1 positively. Again the proof is completed directly by induction on the degree of ζ.

Third, (b)$\Rightarrow$(c). From Proposition 5(a) we have that $\omega^*\zeta = -\frac{1}{2}(\operatorname{sgn}\alpha_1 + \dots + \operatorname{sgn}\alpha_n)$, but all of the $\alpha_n \in \mathbf{R}$ and thus have even degree and, since every $\alpha_j > 0$;, $\omega_0^*\zeta = -\frac{1}{2}\deg\zeta$.

Fourth, (c)$\Rightarrow$(b). Given ζ we write by Proposition 3, $\zeta = (\alpha_p, v_p) \dots (\alpha_1, v_1)\alpha_0v_0$, $\alpha_i \in \mathbf{R}[x]$. But from Proposition 5(a) and our hypothesis we have

$$-\omega_0^*\zeta = \tfrac{1}{2}\deg\zeta = \tfrac{1}{2}(\operatorname{sgn}\alpha_1 + \dots + \operatorname{sgn}\alpha_p),$$

which, since $\deg\alpha_1 + \dots + \deg\alpha_p + p = \deg\zeta$, implies $\deg\alpha_j = 0$ for all j (thus $\alpha_j \in \mathbf{R}$) and $\operatorname{sgn}\alpha_j > 0$ and thus that $\zeta = Sv_0$ with $S \in \mathcal{S}_0$.

Fifth, (c)$\Leftrightarrow$(d). From Proposition 5(b) we have $\omega^*\zeta = \frac{1}{2}(\#^+ - \#^-)$. So, since (b)$\Leftrightarrow$(c),

$\frac{1}{2}(\#^+ - \#^-) = -\frac{1}{2}\deg \zeta$. But of course $\#^+ + \#^- \leqslant \deg \zeta$ and since $\#^+ + \#^- \leqslant -\#^+ + \#^-$ if and only if $\#^+ = 0$ and $\#^- = \deg \zeta$, the proof of Proposition 6 is complete.

5. Differential action of $\mathcal{G}$ on $C^\omega(S^1, \mathbf{E})$

There are interesting actions of the group $\mathcal{G}$ on many spaces but four our purposes here we need only the action of the "constant coefficient" subgroup $\mathcal{G}_0 \subset \mathcal{G}$ on the space $C^\omega(S^1, \mathbf{E})$ of real analytic mappings from the circle to the plane.

Given $(a, v) \in \mathcal{G}_0$ (so $a \in \mathbf{R}$, $v \in \mathbf{E}$) define the operator

$$(a, v): C^\omega(S^1, \mathbf{E}) \to C^\omega(S^1, \mathbf{E}),\ \zeta \mapsto (a, v)\zeta,$$

by

$$(a, v)\,\zeta = \zeta + a(v \wedge \zeta')\,v, \quad \zeta' = \frac{d\zeta}{d\theta};$$

note that this operator makes sense for our equivalence classes since $(a, v)\zeta = (b, w)\zeta$ for all ζ if and only if $(a, v) = (b, w)$. It follows directly that we have defined an action of the abelian group $\mathcal{G}_0(v)$ on $C^\omega(S^1, \mathbf{E})$ for clearly $(0, v)\zeta = (0, 0)\zeta = \zeta$ for all ζ and also

$$\begin{aligned}(b, v)(a, v)\zeta &= \zeta + a(v \wedge \zeta)v + b(v \wedge \zeta')v + ba(v \wedge \zeta'')(v \wedge v)v \\ &= \zeta + (a+b)(v \wedge \zeta')v = (a+b, v)\zeta\end{aligned}$$

for all ζ. And, since $\mathcal{G}_0$ is the free product of the abelian groups $\mathcal{G}_0(v)$, we have in fact defined an action of the whole group $\mathcal{G}_0$ simply by composition of the actions of the abelian groups $\mathcal{G}_0(v)$.

Recalling the definition of the multiplicity function μ in section 2, we state the first of the four properties of the action of $\mathcal{G}_0$.

PROPOSITION 7. (Effectiveness of the action.) *If $G \in \mathcal{G}_0$ and $G\zeta = \zeta$ for all $\zeta \in C^\omega(S^1, \mathbf{E})$, then $G = I$.*

Proof. Choose ζ such that $(\mu\zeta)(\theta_0) = n =$ length of G; now if $n \geqslant 1$, $\zeta(\theta_0) = 0$ and, with a short computation,

$$(G\zeta)(\theta_0) = a_n \ldots a_1(v_n \wedge v_{n-1}) \ldots (v_2 \wedge v_1)\zeta^{(n)}(\theta_0) \neq 0,$$

which contradicts the assumption that $G\zeta = \zeta$. So $n = 0$ and $G = I$.

PROPOSITION 8. (Concerning the multiplicities of zeros.) *Given $G \in \mathcal{G}_0$, $G = (a_n, v_n) \ldots (a_1, v_1)$, of length $n \geqslant 1$ and given $\zeta \in C^\omega(S^1, \mathbf{E})$ and $\theta \in S^1$ such that $\zeta^*(\theta) \neq v^*$, then*

(a) $(\mu\zeta)(\theta) = m \geqslant n \Rightarrow (\mu G\zeta)(\theta) = m - n,$

(b) $(\mu\zeta)(\theta) = m \leqslant n - 1 \Rightarrow (\mu G\zeta)(\theta) \leqslant n - m - 1.$

Proof. For part (a) suppose first that the length of G is 1 and so

$$G\zeta = \zeta = a_1(v_1 \wedge \zeta')v_1,$$

and by hypothesis $(\mu\zeta)(\theta) = m \geqslant 1$; thus we have at θ,

$$(G\zeta)^{(m-1)}(\theta) = \zeta^{(m-1)}(\theta) + a_1(v_1 \wedge \zeta^{(m)}(\theta))v_1 = a_1(v_1 \wedge \zeta^{(m)}(\theta))v_1.$$

Further, $v_1 \wedge \zeta^{(m)}(\theta) \neq 0$ since $\zeta^*(\theta) \neq v_1$ and so the above implies that $(\mu G\zeta)(\theta) \leqslant m-1$. But $(G\zeta)^{(k)}(\theta) = 0$ for $k \leqslant m-2$ and so $(\mu G\zeta)(\theta) = m-1$ as we needed to show. We have also shown that $(G\zeta)^*(\theta) = v_1^*$ and since $v_1^* \neq v_2^*$ for G of length $\geqslant 2$ the proof of part (a) follows by induction on the length of G.

For part (b) we proceed directly for any given $n \geqslant 1$. Let $G = (a_n, v_n) \ldots (a_1, v_1)$ be of length n and define $G_k = (a_k, v_k) \ldots (a_1, v_1)$. We are given $(\mu\zeta)(\theta) = m \leqslant n-1$ and $\zeta^*(\theta) \neq v^*$ so, by part (a),

$$(\mu G_m \zeta)(\theta) = m - m = 0, \text{ (i.e. } (G_m\zeta)(\theta) \neq 0),$$

and also

$$(G_m\zeta)^*\,(0) = v_m^* \quad (\neq v_{m+1}^*).$$

Using the fact that

$$(G_{m+1}\zeta)(\theta) = (G_m\zeta)(\theta) + a_{m+1}(v_{m+1} \wedge G_m\zeta'(\theta))v_{m+1},$$

we see that $(G_{m+1}\zeta)(\theta) \neq 0$ and thus that $(\mu G_{m+1}\zeta)(\theta) = 0$.

We need at this point the following lemma: If $(\mu\zeta)(\theta) = p$ and G has length 1 (no restriction on $\zeta^*(\theta)$), then $(\mu G\zeta)(\theta) \leqslant p+1$. Assuming this for the moment we have by using it over and over,

$$(\mu G_{m+2}\zeta)(\theta) \leqslant 1$$

$$\cdots\cdots\cdots\cdots$$

$$(\mu G_n\zeta)(\theta) \leqslant n - m + 1,$$

as we needed to show.

To prove the lemma, we consider the following cases:

(i) $\zeta^*(\theta) \neq v^*$;

(ii) $\zeta^*(\theta) = v^*$, and $(G\zeta)^{(p)}(\theta) \neq 0$;

(iii) $\zeta^*(\theta) = v^*$, and $(G\zeta)^{(p)}(\theta) = 0$.

For (i), if $p \geqslant 1$, part (a) implies $(\mu G\zeta)(\theta) = p-1$; if $p = 0$, then $\zeta(\theta) \neq 0$, $\zeta(\theta) \wedge v \neq 0$ and thus $(G\zeta)(\theta) = \zeta(\theta) + a(v \wedge \zeta'(\theta))v \neq 0$; so in either case $(\mu G\zeta)(\theta) \leqslant p$. For (ii), we have of course that $(\mu G\zeta)(\theta) \leqslant p$ immediately. For (iii) we have $\zeta^{(p)}(\theta) \neq 0$, $v \wedge \zeta^{(p)}(\theta) = 0$ and

$$(G\zeta)^{(p)}(\theta) = \zeta^{(p)}(\theta) + a(v \wedge \zeta^{(p+1)}(\theta))v = 0,$$

when $v \wedge \zeta^{(p+1)}(\theta) \neq 0$. Thus, from

$$(G\zeta)^{(p+1)}(\theta) = \zeta^{(p+1)}(\theta) + a(v \wedge \zeta^{(p+2)}(\theta))v$$

we see that $(G\zeta)^{(p+1)}(\theta) \neq 0$ which means that $(\mu G\zeta)(\theta) \leqslant p+1$. The proof of the lemma and therefore of Proposition 8 is complete.

PROPOSITION 9. (*Alignment at zeros.*) *If* $G = (a_n, v_n) \dots (a_1, v_1)$ *has length* $n \geqslant 1$ *and if* $\theta \in S^1$ *and* $\zeta \in C^\omega(S^1, \mathrm{E})$ *are such that* $\zeta^*(\theta) \neq v^*$, $(\mu\zeta)(\theta) \geqslant n$, *then*

$$(G\zeta)^*(\theta) = v_n^*.$$

Proof. (By induction on the length n.) For $n = 1$, we have $G\zeta = \zeta + a_1(v_1 \wedge \zeta')v_1$, $(\mu\zeta)(\theta) = m \geqslant 1$ and $v_1 \wedge \zeta^{(m)}(\theta) \neq 0$, so

$$(G\zeta)^{(k)}(\theta) = 0, \quad k \leqslant m-2,$$

and

$$(G\zeta)^{(m-1)}(\theta) = a_1(v_1 \wedge \zeta^{(m)}(\theta))v_1 \neq 0.$$

So $(G\zeta)^*(\theta) = v_1^*$ as we needed to show and further, since $v_1^* \neq v_2^* \neq \dots \neq v_n^*$ and since $(\mu G\zeta)(\theta) \geqslant (\mu\zeta)(\theta) - 1$ (by Proposition 8(a)) the proof can be completed by induction.

In the next proposition a special property of the differential action of the semigroup S_0 is studied. Let $\#A$ denote the number of elements in the set A.

PROPOSITION 10. (Monotonicity of ω^* under the action of S_0.)
Given $S \in S_0$ *of length* 1, *i.e.* $S = (a, v)$ *with* $a > 0$ *and* $v \neq 0$; *given* $\zeta \in C^\omega(S^1, \mathrm{E})$, *then*

$$\omega_0^* S\zeta \geqslant w_0^*\zeta + \tfrac{1}{2}\#\{\theta \mid (\mu S\zeta)(\theta) = (\mu\zeta)(\theta) - 1\} + \tfrac{1}{2}\#\{\theta \mid (\mu S\zeta)(\theta) = (\mu\zeta)(\theta) + 1\}.$$

Proof. To simplify notation let $S\zeta = \tilde{\zeta}$, $\mu\zeta = \mu$ and $\mu\tilde{\zeta} = \tilde{\mu}$. We first show that the circle is the disjoint union of three sets A, B, and C, where

$A = \{\theta \mid \tilde{\mu}(\theta) = \mu(\theta) - 1\}$, i.e., multiplicity of zero decreased by 1,

$B = \{\theta \mid \tilde{\mu}(\theta) = \mu(\theta) + 1\}$, i.e., multiplicity of zero increased by 1,

$C = \{\{\theta \mid \tilde{\mu}(\theta) = \mu(\theta)\}$, i.e., multiplicity invariant.

That A, B, and C are pairwise disjoint is obvious; that their union S^1 is seen as follows. Let $\mu(\theta) = m$, and since

$$\tilde{\zeta}^{(k)} = \zeta^{(k)} + a(v \wedge \zeta^{(k+1)})v,$$

it follows that

$$\tilde{\zeta}^{(k)}(\theta) = 0 \quad \text{for } k \leqslant m-2$$

and thus that $\tilde{\mu}(\theta) \geqslant m-1$. To complete the argument all we need to show is that $\tilde{\mu}(\theta) \leqslant m+1$; we show, if $\tilde{\mu}(\theta) \geqslant m+1$ then $\tilde{\mu}(\theta) = m+1$. With $\tilde{\mu}(\theta) \geqslant m+1$ we have

$$0 = \tilde{\zeta}^{(m)}(\theta) = \zeta^{(m)}(\theta) + a(v \wedge \zeta^{(m+1)}(\theta))v.$$

Thus, since $\zeta^{(m)}(\theta) \neq 0$, it follows that $\zeta^{(m+1)}(\theta) \neq 0$ and then that $\mu(\theta) \leqslant m+1$.

We define two more sets, the inverse images of v^* under ζ^* and $\tilde{\zeta}^*$, respectively:

$$W = \{\theta \,|\, \zeta^*(\theta) = v^*\} \quad (= \{\theta \,|\, \zeta^{(\mu)}(\theta) \wedge v = 0\}),$$

$$\tilde{W} = \{\theta \,|\, \tilde{\zeta}^*(\theta) = v^*\} \quad (= \{\theta \,|\, \tilde{\zeta}^{(\tilde{\mu})}(\theta) \wedge v = 0\}).$$

The idea of the proof is to compare the two sums

$$\omega^* \zeta = \tfrac{1}{2} \sum_{W} \operatorname{sgn} [\arg \zeta^*]'(\theta)$$

$$\omega^* \zeta = \tfrac{1}{2} \sum_{\tilde{W}} \operatorname{sgn} [\arg \tilde{\zeta}^*]'(\theta)$$

on the intersection of W and $\tilde{W}$ with the sets A, B and C.

First we establish three facts; that $A \cap W$ and $B \cap \tilde{W}$ are empty and that $C \cap W = C \cap \tilde{W}$.

$A \cap W$ is empty. We have $\mu(\theta) = \mu(\theta) - 1$ so that, with $m = \mu(\theta)$, $\zeta^{(m)}(\theta) \neq 0$ and $\tilde{\zeta}^{(m-1)}(\theta) \neq 0$; we also have, since $\theta \in W$, that $\zeta^{(m)}(\theta) \wedge v = 0$. Thus,

$$\tilde{\zeta}^{(m-1)}(\theta) = \zeta^{(m-1)}(\theta) + a(v \wedge \zeta^{(m)}(\theta))v = 0$$

which shows that $\tilde{\mu}(\theta) \geqslant m$ contradicting the assumption that $\theta \in A$.

$B \cap \tilde{W}$ is empty. We have $\tilde{\mu}(\theta) = \mu(\theta) + 1$ so that, with $\mu(\theta) = m$, $\tilde{\zeta}^{(m)}(\theta) = 0$, $\zeta^{(m)}(\theta) \neq 0$; we also have that $\tilde{\zeta}^{(m+1)}(\theta) \wedge v = 0$. Thus

$$0 = \tilde{\zeta}^{(m)}(\theta) = \zeta^{(m)}(\theta) + a(v \wedge \zeta^{(m+1)}(\theta))v,$$

which implies that $v \wedge \zeta^{(m+1)}(\theta) \neq 0$. But from the equation

$$\tilde{\zeta}^{(m+1)}(\theta) = \zeta^{(m+1)}(\theta) + a(v \wedge \zeta^{(m+2)})v$$

we obtain

$$v \wedge \tilde{\zeta}^{(m+1)}(\theta) = v \wedge \zeta^{(m+1)}(\theta),$$

and thus that $v \wedge \tilde{\zeta}^{(m+1)}(\theta) \neq 0$ which contradicts the assumption that $\theta \in \tilde{W}$.

$C \cap W = C \cap \tilde{W}$. We have $\mu(\theta) = \mu(\theta) = m$. From

$$\tilde{\zeta}^{(m)}(\theta) = \zeta^{(m)}(\theta) + a(v \wedge \zeta^{(m+1)}(\theta))v$$

we obtain

$$v \wedge \tilde{\zeta}^{(m)}(\theta) = v \wedge \zeta^{(m)}(\theta),$$

which shows, for $\theta \in C$, that $\theta \in W$ if and only if $\theta \in \tilde{W}$.

The above three facts show that

$$\omega\tilde{\zeta} - \omega\zeta = \tfrac{1}{2} \sum_{A \cap \tilde{W}} \operatorname{sgn} [\arg \tilde{\zeta}^*]'(\theta) - \tfrac{1}{2} \sum_{B \cap W} \operatorname{sgn} [\arg \zeta^*]'(\theta)$$
$$+ \tfrac{1}{2} \sum_{C \cap W = C \cap \tilde{W}} \{\operatorname{sgn} [\arg \tilde{\zeta}^*]'(\theta) - \operatorname{sgn} [\arg \zeta^*]'(\theta)\}.$$

We proceed by estimating these three sums.

Sum on $A \cap \tilde{W}$. We have $\mu(\theta) = m \geqslant 1$, $\tilde{\mu}(\theta) = m-1$ and $\tilde{\zeta}^{(m-1)}(\theta) \wedge v = 0$. So

$$0 \neq \tilde{\zeta}^{(m-1)}(\theta) = \zeta^{(m-1)}(\theta) + a(v \wedge \zeta^{(m)}(\theta))v = a(v \wedge \zeta^{(m)}(\theta))v$$

implies $v \wedge \zeta^{(m)}(\theta) \neq 0$. Thus

$$\tilde{\zeta}^{(m-1)}(\theta) \wedge \tilde{\zeta}^{(m)}(\theta) = [a(v \wedge \zeta^{(m)}(\theta))v] \wedge [\zeta^{(m)}(\theta) + a(v\zeta^{(m+1)}(\theta))v]$$
$$= a[v \wedge \zeta^{(m)}(\theta)]^2 > 0,$$

and then
$$[\arg \tilde{\zeta}^*]'(\theta) > 0.$$

The sum on $A \cap \tilde{W}$ is therefore computed and we have

$$\sum_{A \cap \tilde{W}} \operatorname{sgn} [\arg \tilde{\zeta}^*]'(\theta) = \#(A \cap \tilde{W}).$$

Sum on $B \cap W$. We have $\mu(\theta) = m \geqslant 0$, $\tilde{\mu}(\theta) = m+1$ and $\zeta^{(m)}(\theta) \wedge v = 0$. Since $\tilde{\zeta}^{(m)}(\theta) = 0$ we have

$$0 = \tilde{\zeta}^{(m)}(\theta) \wedge \tilde{\zeta}^{(m+1)}(\theta) = \zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) + a[v \wedge \zeta^{(m+1)}(\theta)]^2$$

and also
$$0 = \tilde{\zeta}^{(m)}(\theta) = \zeta^{(m)}(\theta) + a[v \wedge \zeta^{(m+1)}(\theta)]v.$$

This last equation, since $\zeta^{(m)}(\theta) \neq 0$, implies that $v \wedge \zeta^{(m+1)}(\theta) \neq 0$ and then, with the first equation, that

$$\zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) = -a[v \wedge \zeta^{(m+1)}(\theta)]^2 < 0,$$

which means that
$$[\arg \zeta^*]'(\theta) < 0.$$

So the sum on $B \cap W$ is also computed and we have

$$\sum_{B \cap W} \operatorname{sgn} [\arg \zeta^*]'(\theta) = -\#(B \cap W).$$

Sum on $C \cap W = C \cap \tilde{W}$. We have $\mu(\theta) = \tilde{\mu}(\theta) = m \geqslant 0$, $\zeta^{(m)}(\theta) \wedge v = 0$ and $\tilde{\zeta}^{(m)}(\theta) \wedge v = 0$. First we compute

$$\tilde{\zeta}^{(m)}(\theta) \wedge \tilde{\zeta}^{(m+1)}(\theta) = [\zeta^{(m)}(\theta) + a(v \wedge \zeta^{(m+1)}(\theta))] \wedge [\zeta^{(m+1)}(\theta) + a(v \wedge \zeta^{(m+2)}(\theta))v]$$
$$= \zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) + a[v \wedge \zeta^{(m+1)}(\theta)]^2$$
$$= [\zeta^{(m)}(\theta) + a(v \wedge \zeta^{(m+1)}(\theta))v] \wedge \zeta^{(m+1)}(\theta).$$

Since $\zeta^{(m)}(\theta) \wedge v = 0$ this shows that $\tilde{\zeta}^{(m)}(\theta) \wedge \tilde{\zeta}^{(m+1)}(\theta) = 0$ if and only if $\zeta^{(m+1)}(\theta) \wedge v = 0$, and this if and only if $\zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) = 0$. We write C as the disjoint union of C_1 and C_2 and have then the two cases to consider:

$$C_1 = \{\theta \,|\, [\arg \zeta^*]'(\theta) \neq 0,\ [\arg \tilde{\zeta}^*]'(\theta) \neq 0\}$$

and
$$C_2 = \{\theta \,|\, [\arg \zeta^*]'(\theta) = [\arg \tilde{\zeta}^*]'(\theta) = 0\}.$$

Sum on $C_1 \cap W = C_1 \cap \tilde{W}$. We have $\zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) \neq 0$ and $\tilde{\zeta}^{(m)}(\theta) \wedge \tilde{\zeta}^{(m+1)}(\theta) \neq 0$. A few lines above we obtained

$$\tilde{\zeta}^{(m)}(\theta) \wedge \tilde{\zeta}^{(m+1)}(\theta) = \zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) + a[v \wedge \zeta^{(m+1)}(\theta)]^2$$

which implies that
$$\operatorname{sgn} [\arg \tilde{\zeta}^*]'(\theta) \geqslant \operatorname{sgn} [\arg \zeta^*]'(\theta).$$

Thus
$$\sum_{C_1 \cap W} \operatorname{sgn} [\arg \tilde{\zeta}^*]'(\theta) - \operatorname{sgn} [\arg \zeta^*]'(\theta) \geqslant 0.$$

Sum on $C_2 \cap W = C_2 \cap \tilde{W}$. We have as on C_1 that $\mu(\theta) = \tilde{\mu}(\theta) = m \geqslant 0$, $\zeta^{(m)}(\theta) \wedge v = 0$ and $\tilde{\zeta}^{(m)}(\theta) \wedge v = 0$; but here we also have $\zeta^{(m)}(\theta) \wedge \zeta^{(m+1)}(\theta) = \tilde{\zeta}^{(m)}(\theta) \wedge \tilde{\zeta}^{(m+1)}(\theta) = 0$. In this case a more delicate local study is required. To facilitate this we will make use of the following which is a simple extension of the sum formula which allows for non-regular values.

LEMMA: *Define*

$$\varepsilon_\zeta(\theta) = \begin{cases} 0 & \textit{if} \quad [\mu(\arg \zeta^*)](\theta) \quad \textit{is even and positive} \\ \pm 1 & \textit{if} \quad [\mu(\arg \zeta^*)](\theta) \quad \textit{is odd and} \quad \operatorname{sgn} [\arg \zeta^*]^{(\mu)}(\theta) = \pm 1. \end{cases}$$

Then
$$\omega^* \zeta = \tfrac{1}{2} \sum_{\zeta^*(\theta) = v^*} \varepsilon_\zeta(\theta).$$

Next, supposing $\theta \in C_2 \cap W$ to be $\theta = 0$ we write ζ locally in the form

$$\zeta(\theta) \equiv (a_0 \theta^m + \dots) v + (b_0 \theta^p + \dots) w, \quad v \wedge w = 1,$$

where, since $\zeta^{(m)}(\theta) \wedge v = \zeta^{(m+1)}(\theta) \wedge v = 0$, it follows taking $a_0 b_0 \neq 0$ that $p \geqslant m+2$. We compute

$$(\zeta \wedge \zeta')(\theta) \equiv a_0 b_0 (p-m) \theta^{m+p-1} + \dots$$

$$(v \wedge \zeta)(\theta) \equiv b_0 \theta^p + \dots$$

$$(v \wedge \zeta')(\theta) \equiv p b_0 \theta^{p-1} + \dots$$

$$(v \wedge \zeta'')(\theta) \equiv p(p-1) \theta^{p-2} + \dots$$

and
$$(\tilde{\zeta} \wedge \tilde{\zeta}')(\theta) \equiv (\zeta \wedge \zeta')(\theta) + a[v \wedge \zeta'(\theta)]^2 - a[v \wedge \zeta(\theta)][v \wedge \zeta'(\theta)]$$

$$\equiv a_0 b_0 (p-m)\, \theta^{m+p-1} + \ldots + b_0^2 p \theta^{2p-2} + \ldots. \tag{**}$$

Thus, since $m+p-1<2p-2$ (we have $p \geqslant m+2$), the local behavior of ζ and $\tilde{\zeta}$ are the same in the sense of the lemma; i.e.,

$$\varepsilon_\zeta(\theta) = \varepsilon_{\tilde{\zeta}}(\theta) \quad \text{for} \quad \theta \in C_2 \cap W.$$

So if we interpret this part of the sum formulas in the extended sense of the lemma we have

$$\sum_{C_2 \cap W} [\varepsilon_{\tilde{\zeta}}(\theta) - \varepsilon_\zeta(\theta)] = 0.$$

We have shown, adding up the inequalities obtained, that

$$\omega^*\tilde{\zeta} - \omega^*\zeta \geqslant \tfrac{1}{2}\#(A \cap \tilde{W}) + \tfrac{1}{2}\#(B \cap W),$$

which completes the proof of Proposition 10.

PROPOSITION 11. *Given $S \in \mathcal{S}_0$ of length n, $S=(a_n, v_n) \ldots (a_1, v_1)$, and given $\zeta \in C^\omega(S^1, \mathbf{E})$ such that $\zeta(\theta)=0$ only if $\zeta^*(\theta) \neq v_1^*$, then*

$$\omega^* S\zeta \geqslant \omega^*\zeta + \tfrac{1}{2}\Sigma\{\min[n, (\mu\zeta)(\theta)] \,|\, \zeta(\theta)=0\} + \tfrac{1}{2}\Sigma\{(\mu S\zeta)(\theta) \,|\, (S\zeta)=0, (\mu\zeta)(\theta) \leqslant n-1\}.$$

Proof. If we take note of two facts, the proof can then be given by induction on the length of S. First we note (using assumption that $\zeta(\theta)=0 \Rightarrow \zeta^*(\theta) \neq v_1^*$) that

$$A \cap \tilde{W} = \{\theta \,|\, \zeta(\theta)=0 \quad \text{and} \quad \zeta^*(\theta) \neq v_1^*\},$$

as is seen in the proof of Proposition 10 concerned with the sum on $A \cap W$. Second, because of Proposition 9, we have that $(a_1, v_1)\zeta$ satisfies the hypothesis of this Proposition; so we can induct until the number of factors is equal to either n or the multiplicity of ζ at θ. Using Proposition 8, we see that if $(\mu S\zeta)(\theta)=p$ and $(\mu\zeta)(\theta) \leqslant n-1$, then there had to be at least p steps of the type

$$(a_k, v_k) \ldots (a_1, v_1)\zeta \mapsto (a_{k+1}, v_{k+1}) \ldots (a_1, v_1)\zeta,$$

where θ is in $B \cap W$.

COROLLARY. *Given $\zeta \in \mathcal{S}_0$ of length n, $S=(a_n, v_n) \ldots (a_1, v_1)$, and given $\zeta \in C^\omega(S^1, \mathbf{E})$ of the form $\zeta = f v_0$, $f \in C^\omega(S^1, \mathbf{R})$, $v_0 \in \mathbf{E}$, $v_0 \wedge v_1 \neq 0$, then*

$$\omega^* S\zeta \geqslant \tfrac{1}{2}\Sigma\{\min[n, (\mu f)(\theta) \,|\, f(\theta)=0\} + \tfrac{1}{2}\Sigma\{(\mu S f v_0)(\theta) \,|\, (S f v_0)(\theta)=0, (\mu f)(\theta) \leqslant n-1\}.$$

Proof. Since f is real and $v_0 \wedge v_1 \neq 0$ the hypotheses of Proposition 11 are satisfied.

6. Relation between the actions of S_0 and ∂_z^n

Next we study the relation between the action of ∂_z^n on real forms (= homogeneous polynomials) $u_q \in C^\omega(D, \mathbf{R})$ and the differential action of the semigroup S_0 on $\zeta \in C^\omega(S_1, \mathbf{E})$ where $\zeta = f_q v_0$ with $f_q(\theta) \equiv u_q(\cos\theta, \sin\theta)$ and $v_0 \in \mathbf{E}$.

In this and later sections the dependence of the various symbols on the order n of the differential operator ∂_z^n will often be suppressed since n may be considered as fixed throughout. Also we will often make the formal identification between $\mathbf{E}$ and $\mathbf{C}$ by choosing a positive basis e_1, e_2 and identifying (e_1, e_2) with $(1, i)$; e.g., we will write, for $c \in \mathbf{C}$, $v \in \mathbf{E}$, simply $c = v$ ($\mathbf{C} \equiv \mathbf{E}$) meaning, with $v = ae_1 + be_2$, that $\operatorname{Re} c = a$ and $\operatorname{Im} c = b$.

Let ϱ, θ be polar coordinates on $\mathbf{C}$, $z = x + iy = \varrho e^{i\theta} = e^w$ we have the standard operator identities

$$\partial_w = z\partial_z = \tfrac{1}{2}(x+iy)(\partial_x - i\partial_y) = \tfrac{1}{2}(\varrho\partial_\varrho - i\partial_\theta).$$

One establishes by induction the operator identity

$$z^n \partial_z^n = [\partial_w - (n-1)] \dots [\partial_w - 1]\, \partial_w, \quad n \geqslant 1. \tag{1}$$

The action of ∂_w on a form u_q is given by

$$\partial_w u_q = \tfrac{1}{2}(q u_q - i\partial_\theta u_q)$$

and thus the action is that of an ordinary differential operator in $d/d\theta$ with constant coefficients.

With $\varrho^q f_q(\theta) \equiv u_q(\varrho\cos\theta, \varrho\sin\theta)$ and $\partial = d/d\theta$ we compute directly from (1)

$$z^n \partial_z^n u_q = 2^{-n} \varrho^q \prod_{\sigma=0}^{n-1} [(q - 2\sigma) - i\partial]\, f_q. \tag{2}$$

Define the operator L_q by

$$L_q(\partial) \equiv 2^{-n} \prod_{\sigma=0}^{n-1} [(q - 2\sigma) - i\partial]. \tag{3}$$

Then (2) may be written

$$z^n \partial_z^n u_q = \varrho^q L_q f_q.$$

Note that L_q, as a polynomial in $\mathbf{C}[\partial]$, has all of its roots in the open lower half plane when $q \geqslant 2n-1$. When $n \leqslant q \leqslant 2n-2$ it turns out, as we shall see, that some roots appear as conjugate pairs and all the rest are in the open lower half plane. This indicates that the actions of L_q on f_q is, with $\mathbf{C} \equiv \mathbf{E}$, closely connected to the action of an $S \in S_0$ on $f_q v_0$, $v \in \mathbf{E}$. To make this connection precise we first show

PROPOSITION 12. *The operator L_q factors, $L_q = N_q \lambda_q$, where λ_q is an operator with real coefficients, in the following way (we always suppose $q \geqslant n$):*

(a) *for $q \leqslant 2n-2$ and q even,*

$$N_q(\partial) \equiv (-i)^{2n-q-1} \prod_{\sigma=0}^{q-n} [(q-2\sigma) - i\partial], \quad \textit{of order} \quad q-n+1,$$

$$\lambda_q(\partial) \equiv \partial \prod_{\sigma=\frac{1}{2}(q+2)}^{n-1} [(q-2\sigma)^2 + \partial^2], \quad \textit{of order} \quad 2n-q-1;$$

(b) *for $q \leqslant 2n-2$ and q odd,*

$$N_q(\partial) \equiv 2^{-n} (-i)^{2n-q-1} \prod_{\sigma=0}^{q-n} [(q-2\sigma) - i\partial], \quad \textit{of order} \quad q-n+1,$$

$$\lambda_q(\partial) \equiv \partial \prod_{\sigma=\frac{1}{2}(q+1)}^{n-1} [(q-2\sigma)^2 + \partial^2], \quad \textit{of order} \quad 2n-q+1;$$

(c) *for $2n-1 \leqslant q$*

$$N_q(\partial) = 2^{-2} \prod_{\sigma=0}^{n-1} [(q-2\sigma) - i\partial] = L_q(\partial), \quad \textit{of order } n,$$

$$\lambda_q(\partial) = 1, \quad \textit{of order } 0.$$

Proof. The proof is a straightforward computation and is omitted.

At this point a few more definitions will be useful. Given $S = (a_m, v_m) \dots (a_1, v_1) \in \mathcal{S}_0$ with length m ($\equiv v_j \wedge v_{j+1} \neq 0$) and a $w \in \mathbf{E}$ call S *initially independent* of w if $v_1^* \neq w^*$ ($\equiv v_1 \wedge w \neq 0$) and call S *terminally independent* of w if $v_m^* \neq w^*$. Given a polynomial ζ in $\mathbf{C}[x]$ (or $\mathbf{E}[x]$) define $\zeta^*(\infty)$ to be the line

$$\zeta^*(\infty) = \lim_{\substack{x \in \mathbf{R} \\ |x| \to \infty}} \left[\frac{\zeta(x)}{|\zeta(x)|}\right]^*.$$

Proposition 13. *Given L_q, as in (3), there exists a unique pair, $S_q \in \mathcal{S}_0$ and $w_q \in \mathbf{E}$, such that, with $L_q = N_q \lambda_q$ as in Proposition 12,*

(a) *for algebraic action (L_q, $N_q \in \mathbf{C}[\partial]$, $\lambda_q \in \mathbf{R}[\partial]$, $S_q w_q \in \mathbf{E}[\partial]$), and $\mathbf{C} \equiv \mathbf{E}$,*

(a_1) $L_q = N_q \lambda_q = S_q \lambda_q w_q$, *length of $S_q = \min\,[q-n+1, n]$,*

(a_2) *S_q is initially independent of w_q,*

(a_3) *S_q is terminally dependent on $L_q(\infty)$;*

(b) *for differential action on $f \in C^\omega(S^1, \mathbf{R})$ and $\mathbf{C} \equiv \mathbf{E}$,*

$$L_q f \equiv N_q \lambda_q f = S_q(\lambda_q f) w_q, \textit{ for all } f.$$

Proof. Part (a_1) follows immediately from Proposition 6 and this together with the definition of the differential action of $\mathcal{S}_0$ gives part (b). Write S_q in (reduced) form, $S_q = (a_m, v_m) \dots (a_1, v_1)$ where $v_j \wedge v_{j+1} \neq 0$, and then, again by Proposition 6, we have

$$\deg N_q = \deg S_q \lambda_q w_q = \deg S_q w_q + \deg \lambda_q.$$

But if S_q were initially dependent on w_q we would have, since then (a_1, v_1) $\lambda_q = \lambda_q$, that

$$\deg S_q \lambda_q w_q < \deg N,$$

a contradiction. Part (a_2) is proved.

Part (a_3) follows immediately from part (a_1) and the proof of Proposition 13 is complete.

At this point, note that the action of ∂_z^n on $u \in C^\omega(D, \mathbf{R})$ can be written, $\mathbf{C} \equiv \mathbf{E}$,

$$z^n \partial_z^n u = \sum_{q=p}^{\infty} z^n \partial_z^n u_q = \sum_{q=p}^{\infty} \varrho^q L_q f_q = \sum_{q=p}^{\infty} \varrho^q N_q \lambda_q f_q = \sum_{q=p}^{\infty} \varrho^q S_q \lambda_q f_q w_q.$$

This is to say that alg action of the N_q on $\mathbf{C}[\partial]$ is equivalent to algebraic action of S on $\mathbf{E}[\partial]$. They are not! One can find polynomials $w \in \mathbf{C}[\partial]$, $\zeta \in \mathbf{E}[\partial]$ with $w = \zeta (C \equiv \mathbf{E})$ and an N_q such that

$$N_q w \notin S_0 \zeta.$$

The point is that N_q operates on real polynomials in the same way as S_0 operating on real polynomials times a fixed vector. It is now seen, that in this sense, the proof of the Theorem is reduced to the study of perturbations of this form since

$$\Omega_0 \partial_z^n u = \lim_{\varrho \to 0} \omega^* \sum_{q=p}^{\infty} \varrho^q S_q \lambda_q f_q w_q.$$

7. Preparation for perturbation

PROPOSITION 14′. (A simple reduction of multiplicities). *Given L_q and f_q and $\theta \in S^1$, it follows that*

$$(\mu L_q f_q)(\theta) \leqslant q - n.$$

Proof. The proof is given using the fact that $\partial_z^n u_q = p^q L_q f_q$ and that each real linear factor (of form $az + \bar{a}\bar{z}$) of $\partial_z^n u_q$ corresponds to zeros of $L_q f_q$ at some θ and $\theta + \pi$. Let

$$u_q = \sum_{\sigma=0}^{q} c_\sigma z^\sigma \bar{z}^{q-\sigma}, \; c_\sigma \in \mathbf{C},$$

where $c_{q-\sigma} = \bar{c}_\sigma$ (so that u_q is real). Then

$$z^n \partial_z^n u_q = \frac{z^n}{n!} \sum_{\sigma=n}^{q} \sigma! \, c_\sigma z^{\sigma-n} \bar{z}^{q-\sigma}$$

and so $z^n \partial^n u_q$ can have no more than $q-n$ real linear factors which shows in particular that, for every θ, $(\mu L_q f_q)(\theta) \leqslant q - n$.

PROPOSITION 14″.

Given $(\mu f_q)(\theta) \geqslant n$ *then* $(\mu L_q f_q)(\theta) = (\mu f_q)(\theta) - n$.

Proof. As above suppose a linear factor of u_q, corresponding to θ, $\theta + \pi$ for f_q, is $az + \bar{a}\bar{z}$. Then, with $(\mu f_q)(\theta) = m$

$$u_q = (az + \bar{a}\bar{z})^m v, \ \deg v = q - m.$$

A direct computation of $z^n \partial_z^n u_q (= L_q f_q)$ gives the proof.

PROPOSITION 15. (Alignment at zeros.)

If $(\mu f_q)(\theta) \geqslant \min [q - n + 1, n]$ *then, with* $\mathbf{C} \equiv \mathbf{E}$,

$$[(L_q f_q)(\theta)]^* = [(S_q \lambda_q f_q w_q)(\theta)]^* = [(-i)^n]^*$$

for all $q (\geqslant n)$.

Proof. We know from Proposition 13 (a_3) and Proposition 9 that alignment for all q is with $L_q^*(\infty)$; and $L_q^*(\infty) = [-i^n]^*$.

The next result describes a very special and important property of the form of the perturbations, as displayed just before Proposition 14; the ones we must study.

PROPOSITION 16. (Monotonicity properties of the projective derivative.)

For $\theta \in S^1$ *and* $(\mu f_q)(\theta) = m \geqslant n$,

$$[\arg (L_q f_q)^*]'(\theta) = \frac{n(q-n+1)}{(m-n+1)}.$$

Thus the value of the projective derivative $(q \geqslant n,\ m \geqslant n)$ is always positive, depends only on the order n in ∂_z^n, the multiplicity of the zero at θ, m, and on the (homogeneous) degree of the form u_q. As a sequence it is strictly increasing in q and strictly decreasing in m.

Proof. First we note, directly from the definition, that the projective derivative of a curve $\zeta \in C^\omega(S^1, \mathbf{E})$ is invariant if the curve is multiplied by any real functions $g \in C^\omega(S^1, \mathbf{R})$. Next we normalize by choosing $\theta = 0$ and write f near $\theta = 0$ as $f(\theta) \equiv \theta^m g(\theta)$ where $g(\theta) \neq 0$; then, near $\theta = 0$,

$$\begin{aligned}
2^n L_q f &= \prod_{\sigma=0}^{n-1} [(q - 2\sigma) - i\partial]\,\theta^m g \\
&= (-i)^n [\theta^m g]^{(n)} + (-i)^{n-1} n(q-n+1)[\theta^m g]^{(n-1)} + \dots \\
&= (-i)^n \left[\frac{m!}{(m-n)!}\theta^{m-n} g(0) + \frac{m!}{(m-n+1)!}\theta^{m-n+1} g'(0) + o(\theta^{m-n+1})\right] \\
&\qquad + (-i)^{n-1} n(q-n+1)\,[\theta^{m-n+1} g(0) + o(\theta^{m-n+1})] \\
&= (-i)^{n-1}\frac{m!}{(m-n+1)!}\theta^{m-n}\{[n(q-n+1)g(0)]\,\theta + [(m-n+1)g(0) + g'(0)\,\theta]\,i + \theta^{m-n+2}h(\theta)\}
\end{aligned}$$

we have after factoring out by

$$\frac{(-1)^{n-1}m!}{(m-n+1)!}\theta^{m-n}$$

the form,

$$\zeta(\theta)=[n(q-n+1)g(0)]\theta+[(m-n+1)g(0)+g'(0)\theta]i+\theta^2R(\theta).$$

To simplify notation we write $\zeta(\theta)=A\theta+[B+C\theta]i+\theta^2R(\theta)$ and then compute the projective derivative of ζ at $\theta=0$. In complex form the projective derivative is

$$[\arg\zeta^*]'=\operatorname{Im}\frac{\bar{\zeta}\,\zeta}{|\zeta|^2}.$$

So

$$[\arg\zeta^*]'=\operatorname{Im}\frac{[A\theta-(B+C\theta)\,i+\theta^2R]\,[A-C\theta i+\theta R_1]}{(A\theta)^2+(B+C\theta)^2}$$

which at $\theta=0$ establishes Proposition 16:

$$[\arg\zeta^*]'(0)=\frac{A}{B}=n\,\frac{q-n+1}{m-n+1}.$$

8. Localization and perturbation

In this section we will often make use of the hypotheses of the Dual Theorem: that $\partial_z^n u$ has an isolated singular point at the origin and that u_p is the lowest order form such that $\partial^n u_p \neq 0$.

Define $\mathcal{E}_n$ and E_q by

$$\mathcal{E}_n=z^n\partial_z^n u,\quad E_q=L_q f_q,$$

so that $(\mathbf{C}\equiv\mathbf{E})$,

$$\mathcal{E}_n=z^n\partial_z^n u+\sum_{q=p}^{\infty}\varrho^q L_q f_q=\sum_{q=p}^{\infty}\varrho^q S_q\lambda_q f_q w_q=\sum_{q=p}^{\infty}\varrho^q E_q. \tag{1}$$

The integral formula for the projective winding number of the curves $\mathcal{E}(\varrho,-)$ and the definition of the index of the vector field $\mathcal{E}_n$ at the origin gives

$$\Omega_0 z^n\partial_z^n u=\Omega_0\mathcal{E}_n=\lim_{\varrho\to 0}\omega_0^*\mathcal{E}_n(\varrho,-)=\lim_{\varrho\to 0}\frac{1}{2\pi}\int_0^{2\pi}[\arg\mathcal{E}_n^*(\varrho,-)]'. \tag{2}$$

At each θ with $E_p(\theta)=0$ define the functional V_θ by

$$V_\theta\mathcal{E}_n=\lim_{\substack{h\to 0\\(h>0)}}\lim_{\varrho\to 0}\frac{1}{2\pi}\int_{\theta-h}^{\theta+h}[\arg\mathcal{E}_n^*(\varrho,-)]', \tag{3}$$

and we have by a straightforward limit computation that

$$\Omega_0\mathcal{E}_n=\sum\{V_\theta\mathcal{E}_n\,|\,E_p(\theta)=0\}+\omega_0^*E_p. \tag{4}$$

The results of sections 6 and 7 give inequalities on $\omega_0^* E_p$; in this section we will derive inequalities on the perturbation term $\Sigma\{V_\theta \mathcal{E}_n \mid E_p(\theta)=0\}$.

For $E_p(\theta_j)=0$ let $x=\theta-\theta_j$ and define

$$F_q^j(x) \equiv E_q(\theta_j+x),$$
$$\mathcal{J}_n^j(\varrho, x) \equiv \sum_{q=p}^{\infty} \varrho^q F_q^j(x) \equiv \mathcal{E}_n(\varrho, \theta_j+x). \tag{5}$$

Formula (4) now takes the form

$$\Omega_0 \mathcal{E}_n = \sum \{V_0 \mathcal{J}_n^j \mid E_p(\theta_j)=0\} + \omega_0^* E_p. \tag{6}$$

In most of the following θ_j may be thought of as a fixed zero of E_p and, for notational simplicity, the dependence of the various terms on θ_j will often be suppressed.

From here on we suppose some familiarity with the elementary theory of algebraic curves, especially properties of the Newton Polygon ($\equiv NP$); see, e.g., [10]. Consider the expansion of a real analytic function $g(\varrho, x)$ about $\varrho=x=0$ with terms $ax^\alpha\varrho^\beta$ and let $\{(\alpha,\beta)\}_0$ be the set of all the integer points (α,β) that occur as exponents except for $(0,0)$. The Newton Polygon of the function g, NP of g, consists of the finite sequence of lines $\mathfrak{L}$ and the subset $\{(\alpha_\sigma,\beta_\sigma)\}\subset\{(\alpha,\beta)\}_0$ such that

each line in $\mathfrak{L}$ contains at least two points of $\{(\alpha,\beta)\}_0$, (7a)

each line in $\mathfrak{L}$ separates the origin $(0,0)$ from the points in $\{(\alpha,\beta)\}_0$, not on that line, (7b)

with the equation of the k^{th} line, $\varepsilon_k\alpha+\beta=r_k$, $\varepsilon_k>0$, $r_k>0$, the lines in $\mathfrak{L}$ are ordered so that the sequence $\{\varepsilon_k\}$ is increasing (and thus that $\{r_k\}$ is increasing). (7c)

Thus the lines in $\mathfrak{L}$ and the points $\{(\alpha_\sigma,\beta_\sigma)\}$ in the NP satisfy:

$$\varepsilon_1\alpha_\sigma+\beta_\sigma=r_1 \quad \text{for} \quad \sigma=1,\dots,t_1. \tag{8}$$
$$\cdots\cdots\cdots\cdots$$
$$\varepsilon_s\alpha_\sigma+\beta_\sigma=r_s \quad \text{for} \quad \sigma=t_{s-1},\dots,t_s;$$

where

$$0<\varepsilon_1<\dots<\varepsilon_s, \quad r_1<\dots<r_s<0, \tag{8}$$
$$\alpha_1>\dots>\alpha_{t_s}\geqslant 0, \quad p\leqslant\beta_1<\dots<\beta_{t_s}.$$

The reason for ordering the lines in $\mathfrak{L}$ as in (7c) is that the NP will be used to study the functional V_0 which involves an iterated limit with $\varrho\to 0$ before $h\to 0$.

There exist rotations on C (which leave projective winding numbers as well as V_0 invariant) so that for sufficiently small $\varrho>0$ we have using (5) that the imaginary axis $(=i^*)$ is a regular value of $(\mathfrak{J}_n^j)^*(\varrho, -)$; (9a)

with $\{(\alpha_\sigma, \beta_\sigma)\}$, the points in the NP of the function $g = \operatorname{Re} \mathfrak{J}_n^j$, (9b)

$$\operatorname{Re} F_{\beta_\sigma}^j(x) \equiv [a_\sigma + o(x)]\, x^{\alpha_\sigma},\ a_\sigma \neq 0,$$

$$\operatorname{Im} F_{\beta_\sigma}^j(x) \equiv [b_\sigma x^{\gamma_\sigma} + o(x^{\gamma_\sigma})]\, x^{\alpha_\sigma},\ b_\sigma \neq 0,\ \gamma_\sigma \geqslant 0;$$

this means a choice of line, (i^*) after rotation, so that

$$(F_{\beta_\sigma}^j)^*(0) \neq i^*.$$

The following is a simple extension of the sum formula for ω_0^* as in section 2 where $\mathbf{R}_\infty = \mathbf{R} \cup \{\infty\}$ is the natural one point compactification of $\mathbf{R}$ so that $\mathbf{R}_\infty$ is real analytically diffeomorphic to S^1. The proof is omitted.

PROPOSITION 17′. *Let $H(y)$ be a polynomial curve, H: $\mathbf{R} \to \mathbf{C}$, and v^* a regular value of H^*, then*

(a) *if* $v^* = H^*(\infty)$,

$$\omega_0^* H = \tfrac{1}{2} \sum \{\operatorname{sgn}\,[\arg H^*]'(y) \mid H^*(y) = v^*,\ y \in \mathbf{R}_\infty\};$$

(b) *if* $v^* \neq H^*(\infty)$,

$$\omega_0^* H = \tfrac{1}{2} \{\operatorname{sgn}\,[\arg H^*]'(y) \mid H^*(y) = v^*,\ y \in \mathbf{R}\}.$$

PROPOSITION 17″. *With $\mathfrak{J} = \mathfrak{J}_n^j$, let v^* be a regular value of $\mathfrak{J}^*(\varrho, -)$ for sufficiently small $\varrho > 0$, then,*

$$V_0 \mathfrak{J}_n^j = \lim_{\substack{h \to 0 \\ (h>0)}} \lim_{\varrho \to 0} \tfrac{1}{2} \sum \{\operatorname{sgn}\,[\arg \mathfrak{J}^*(\varrho, -)]'(x) \mid \mathfrak{J}^*(\varrho, x) = v^*,\ |x| \leqslant h\}.$$

Proof. Using the Theorem of Rouché there exists a truncation of $\mathfrak{J}$ giving a polynomial $\hat{\mathfrak{J}}$ so that $V_0 \hat{\mathfrak{J}} = V_0 \mathfrak{J}$. From Proposition 17′,

$$\omega_0^* \hat{\mathfrak{J}}(\varrho, -) = \tfrac{1}{2} \sum \{\operatorname{sgn}\,[\arg \hat{\mathfrak{J}}^*(\varrho, -)]'(x) \mid \hat{\mathfrak{J}}^*(\varrho, x) = v^*,\ x \in \mathbf{R}_\infty\}. \tag{10}$$

From the definition of V_0,

$$V_0 \mathfrak{J}_n^j = V_0 \hat{\mathfrak{J}} = \lim_{\varrho \to 0} \hat{\mathfrak{J}}(\varrho, -) - \omega_0^* \hat{\mathfrak{J}}(0, -),$$

which together with (10) completes the proof.

Define G_σ and $\mathcal{G}_k$ with (α_0, β_0) chosen as in (9), by

$$G_\sigma(\varrho, x) = (a_\sigma + i b_\sigma x^{\gamma_\sigma})\, x^{\alpha_\sigma} \varrho^{\beta_\sigma}, \quad \sigma = 1, \dots, t_s.$$

$$\mathcal{G}_k = \sum_{\sigma=t_{k-1}}^{t_k} G_\sigma(\varrho, x), \quad \sum \mathcal{G}_k = \mathcal{G}.$$

PROPOSITION 18. *With* $\{(\alpha_\sigma, \beta_\sigma)$ *the set of points in the* NP *of* Re $\mathcal{J}_n^j = 0$ *and with a rotation chosen as in* (9)

$$V_0 \mathcal{J}_n^j = V_0 \sum_{\sigma=1}^{t_s} G_\sigma = V_0 \sum_{k=1}^{s} \mathcal{G}_k.$$

Proof. By (9a) the imaginary axis is a regular value of $(\mathcal{J}_n^j)^*$. Using Proposition 17″ we need only to show that

$$[V_0 \mathcal{J}_n^j] = \lim_{\substack{h\to 0\\(h>0)}} \lim_{\varrho\to 0} \tfrac{1}{2} \sum \{\operatorname{sgn} [\arg \mathcal{G}^*(\varrho, -)]'(x) \mid \operatorname{Re} \mathcal{G}(\varrho, x) = 0, |x| \leqslant h\}, \tag{11 a}$$

where

$$\operatorname{Re} \mathcal{G}(\varrho, x) \equiv \sum_{\sigma=0}^{t_s} a_\sigma x^{\alpha_\sigma} \varrho^{\beta_\sigma}, \tag{11 b}$$

$$\operatorname{sgn} [\arg \mathcal{G}^*(\varrho, -)]'(x) \equiv -\operatorname{sgn} \left(\sum_{\sigma=1}^{t_s} \alpha_\sigma a_\sigma x^{\alpha_\sigma - 1} \varrho^{\beta_\sigma} \right) \left(\sum_{\sigma=1}^{t_s} b_\sigma x^{\gamma_\sigma + \alpha_\sigma} \varrho^{\beta_\sigma} \right). \tag{11 c}$$

But the standard majorizing properties of the NP show that on the real zero branches of $\Sigma a_\sigma x^{\alpha_\sigma} \varrho^{\beta_\sigma} = 0$ the signs involved in the sum above are exactly the signs of

$$[\arg (\mathcal{J}_n^j)^*]' = -\left(\frac{\partial}{\partial x} \operatorname{Re} \mathcal{J}_n^j\right) (\operatorname{Im} \mathcal{J}_n^j)$$

on the zero branches of Re $\mathcal{J}_n^j = 0$.

Now, let $x = y\varrho^{\varepsilon_k}$ and define the polynomial curves H_k by

$$\varrho^{r_k} H_k(\varrho, y) \equiv \sum_{\sigma=t_{k-1}}^{t_k} G_\sigma(\varrho, x) = \sum_{\sigma=t_{k-1}}^{t_k} (a_\sigma + i b_\sigma x^{\gamma_\sigma}) x^{\alpha_\sigma} \varrho^{\beta_\sigma}$$

where we have used the equations $t_k \alpha_\sigma + \beta_\sigma = r_k$ from the NP for Re $\mathcal{J}_n^j = 0$. We have established using this and Proposition 18 that

$$V_0 \mathcal{J}_n^j = V_0 \sum_{k=1}^{s} \varrho^{r_k} H_k = V_0 \sum_{k=1}^{s} \sum_{\sigma=1_{k-1}}^{t_k} G_\sigma. \tag{12}$$

PROPOSITION 19. *With the* H_k *as above,*

$$V_0 \mathcal{J}_n^j = \sum_{k=1}^{s} \{\lim_{\varrho\to 0} \omega_0^* H_k(\varrho, -) - \tfrac{1}{2} \operatorname{sgn} [\arg H_k^*(\varrho, -)]'(\infty)\}.$$

Proof. Apply Proposition 17″ to the polynomial curves

$$\sum_{k=1}^{s} \varrho^{r_k} H_k - \sum_{\sigma=1}^{t_s} G_\sigma = \mathcal{G},$$

where we are concerned with the real zero branches of

$$\operatorname{Re} \mathcal{G} = \sum_{\sigma=1}^{t_s} a_\sigma x^{\alpha_\sigma} \varrho^{\beta_\sigma} = 0;$$

the branches of lowest initial degree in ϱ are of the form

$$x = y\varrho^{\varepsilon_1} + o(\varrho^{\varepsilon_1}) \quad \text{where} \quad \operatorname{Re} H_1(y) = \sum_{\sigma=1}^{t_1} a_\sigma y^{\alpha_\sigma} = 0, \quad y \in \mathbf{R}.$$

But the real zero branches of the next lowest initial degree are of the form

$$x = y\varrho^{\varepsilon_2} + o(\varrho^{\varepsilon_2}) \quad \text{where} \quad \operatorname{Re} H_2(y) = 0, \quad y \in \mathbf{R};$$

and continuing this process we obtain the lowest terms of all the real zero branches; i.e., they are given by

$$x = y\varrho^{\varepsilon_k} + o(\varrho^{\varepsilon_k}), \quad k = 1, \ldots, s, \tag{13a}$$

where

$$\operatorname{Re} H_k(y) = \sum_{\sigma=t_{k-1}}^{t_k} a_\sigma y^{\alpha_\sigma} = 0, \quad y \in \mathbf{R}. \tag{13b}$$

From Proposition 17′ we have, for sufficiently small $\varrho > 0$,

$$\begin{aligned}\omega_0^* H_k(\varrho, -) &= \tfrac{1}{2} \sum \{\operatorname{sgn} [\arg H_k^*(\varrho, -)]'(y) \mid \operatorname{Re} H(y) = 0, \quad y \in \mathbf{R}_\infty\} \\ &= \tfrac{1}{2} \sum \{\operatorname{sgn} [\arg H_k^*(\varrho, -)]'(y) \mid \operatorname{Re} H(y) = 0, \quad y \in \mathbf{R}\} \\ &\qquad + \tfrac{1}{2} \operatorname{sgn} [\arg H_k^*(\varrho, -)]'(\infty);\end{aligned}$$

from the process in (13) we have for sufficiently small $\varrho > 0$

$$V_0 \mathcal{J}_n^j = \sum_{k=1}^{s} \sum \{\operatorname{sgn} [\arg H_k^*(\varrho, -)]'(y) \mid \operatorname{Re} H_k(y) = 0, \quad y \in \mathbf{R}\}.$$

These two facts together with Proposition 17″ complete the proof.

9. The semigroupoid action

Let $\mathcal{H}$ be the set of polynomial curves H, $H\colon \mathbf{R} \to \mathbf{C}$, such that

$$H \in \mathcal{H} \Leftrightarrow \deg \operatorname{Re} H \leqslant 1 + \deg \operatorname{Im} H.$$

We also write $H = P + iQ$ and

$$P = p_0 y^\gamma + \ldots p_\gamma, \quad Q = q_0 y^\delta + \ldots + p_\delta, \quad p_0 q_0 \neq 0, \quad \gamma \leqslant 1 + \delta.$$

Define the following operators on $\mathcal{H}$; note that some of the operators have domains that are proper subsets of $\mathcal{H}$ but that each operator maps its domain into $\mathcal{H}$:

(i) for $a_1, a_2 \in \mathbf{R}$, $a_1 a_2 > 0$, define $A(a_1, a_2)$ by

$$A(a_1, a_2)[P+iQ] = a_1 P + i a_2 Q;$$

(ii) for $b \in \mathbf{R}$, $q_0 + bp_0 > 0$ when $\gamma + 1 + \delta$, otherwise for arbitrary b, define $B(b)$ by

$$B(b)[P+iQ] = P + i(bxQ + P);$$

(iii) for $c \in \mathbf{R}$, $c \geqslant 0$, define $C(c)$ by

$$C(c)[P+iQ] = (P + icQ') + iQ;$$

(iv) for $d \in \mathbf{Z}$ and $d + (\mu H)(0) \geqslant 1$, define $D(d)$ by

$$D(d)[P+iQ] = (P+iQ)y^d.$$

Given a polynomial curve $H \in \mathcal{H}$ consider the orbit of H consisting of all polynomial curves TH where T is a finite composition of the operators A, B, C, D; when an operator B or D is involved the polycurve on which it operates must of course be in the appropriate domain of definition.

PROPOSITION 20. *For any $H \in \mathcal{H}$ and TH in the orbit of H*

$$\omega_0^* TH \geqslant \omega_0^* H.$$

Proof. The proof is given for the operators of each type. For (i), we simply note that $A(a_1, a_2)$ is a proper affine transformation on $\mathbf{C}(\mathbf{C} \equiv \mathbf{E})$ and that ω^* is invariant. For (ii), we take first a rotation so that the real axis is a regular value of H^* (although such a rotation need not leave $\mathcal{H}$ invariant it is used only to compute ω_0^*), and apply Proposition 17′. With $\operatorname{Im} H(y) = P(y) = 0$, $y \in \mathbf{R}$,

$$\operatorname{sgn}\,[\arg\,(BH)^*]'(y) = \operatorname{sgn} P(y)Q'(y) = \operatorname{sgn}\,[\arg H^*]'(y),$$

so that none of the signs, with $y \in \mathbf{R}$, is changed. But the condition that $q_0 + bp_0 > 0$ and the fact that $[\arg H^*]'(\infty)$ is invariant under rotations on $\mathbf{C}$ show that $\operatorname{sgn}\,[\arg\,(BH)^*](\infty) = +1$ and so ω_0^* can only increase under the action of $B(b)$. For (iii), apply Proposition 10 identifying the operator (a, v) there with (c, e_1), and then, with $e_1 \equiv 1$, $e_2 \equiv i$, we have

$$(c, e_1)[Pe_1 + Qe_2] = Pe_1 + Qe_2 = c(e_1 \wedge P'e_1 + Q'e_2)e_2 = (P + cQ')e_1 + Qe_2 = C(c)[P+iQ].$$

Finally, for (iv), the proof follows immediately from the fact that the projective derivative and therefore the projective winding number are invariant under multiplication by real polynomials.

PROPOSITION 21. *For certain $H \in \mathcal{H}$ any TH in the orbit of H satisfies special inequalities:*

(a) *for* $H = iQ$, $\qquad \omega_0^* TH \geq \frac{1}{2}\#\{y \mid Q(y)=0\} \geq 0;$

(b) *for* $H = P + iQ$, $\qquad \omega_0^* TH \geq -\frac{1}{2}(1+\deg P).$

Proof. For (a), note that $\omega^* iQ = 0$ and apply Proposition 20. For (b), we first show that

$$\omega_0^*(P+iQ) \geq -\tfrac{1}{2}\#\{y \mid Q(y)=0, y \in \mathbf{R}\} - \tfrac{1}{2}.$$

This follows immediately from Proposition 17′ if the imaginary axis is a regular value of $(P+iQ)^*$; if not, $P+iQ$ can be arbitrarily closely approximated by $\hat{P}+i\hat{Q}$ so that $\deg P = \deg \hat{P}$, $\deg Q = \deg \hat{Q}$, and so that the imaginary axis is a regular value of $(\hat{P}+i\hat{Q})^*$.

10. Some special perturbations

Recall Proposition 19,

$$V_0 \mathfrak{J}_n^j = V_0 \sum_{k=1}^{s} \sum_{\sigma=t_{k-1}}^{t_k} F_{\beta_\sigma}^j = \sum_{k=1}^{s} \{\lim_{\varrho\to 0} \omega_0^* H_k(\varrho, -) - \tfrac{1}{2}\operatorname{sgn}[\arg H_k^*(\varrho, -)]'(\infty)\}$$

and also, from the proof, that

$$V_0 \sum_{\sigma=t_{k-1}}^{t_k} F_{\beta_\sigma}^j = \lim_{\varrho\to 0} \omega_0^* H_k(\varrho, -) - \tfrac{1}{2}\operatorname{sgn}[\arg H_k^*(\varrho, -)]'(\infty)$$

so that we have the alternate statement of Proposition 19

PROPOSITION 19′.
$$V_0 \sum_{k=1}^{s} \sum_{\sigma=t_{k-1}}^{t_k} F_{\beta_\sigma}^j = \sum_{k=1}^{s} V_0 \sum_{\sigma=t_{k-1}}^{t_k} F_{\beta_\sigma}^j.$$

Now the semigroupoid Propositions of section 9 are especially useful under the special conditions (14) that follow; using these conditions we will show that the approximations in (9b) and thus the G_σ as in Proposition 18 take a very special form.

PROPOSITION 22. *Let $\{(\alpha_\sigma, \beta_\sigma)\}$ be the set of points in the NP for* $\operatorname{Re} \mathfrak{J}_n^j = 0$. *If*

$$(\mu f_{\beta_\sigma})(\theta_j) \geq n \quad \textit{for} \quad \sigma = t_{k-1}, \ldots, t_k \tag{14}$$

then $$V_0 \sum_{\sigma=t_{k-1}}^{t_k} F^j_{\beta_\sigma} \geqslant 0.$$

Proof. Under the conditions (14) we know by Proposition 15 after a rotation by $(-i)^n$ that
$$(F^j_{\beta_\sigma})^*(0) = 1^* \quad \text{for} \quad \sigma = t_{k-1}, \dots, t_k, \tag{15}$$
and by Proposition 16 that
$$[\arg (F^j_{\beta_\sigma})^*]'(0) = \frac{n(\beta_\sigma - n+1)}{m-n+1}, \quad m = (f_{\beta_\sigma})(\theta_j).$$
But by Proposition 14″, $\alpha_\sigma = m - n$, so that we have
$$[\arg (F^j_{\beta_\sigma})^*]'(0) = \frac{n(\beta_\sigma - n+1)}{\alpha_\sigma + 1}, \quad \sigma = t_{k-1}, \dots, t_k. \tag{16}$$

Referring to the approximation (9) we see that (15) implies that $\gamma_\sigma \geqslant 1$. Furthermore (16) then implies
$$\frac{n(\beta_\sigma - n+1)}{\alpha_{\sigma+1}} = [\arg (F^j_{\beta_\sigma})^*]'(0) = \left. \frac{\begin{vmatrix} a_\sigma & 0 \\ c_\sigma x^{\gamma_\sigma} & \gamma_\sigma b_\sigma x^{\gamma_\sigma - 1} \end{vmatrix}}{a_\sigma^2 + b_\sigma^2 x^{2\gamma_\sigma}} \right|_{x=0} = \left. \frac{b_\sigma \gamma_\sigma}{a_\sigma} x^{\gamma_\sigma - 1} \right|_{x=0};$$
but since the left hand side is never zero we have $\gamma_\sigma = 1$ and thus, recalling the definition of G_σ for Proposition 18,
$$G_\sigma(\varrho, x) = (a_\sigma + i b_\sigma x)\, x^{\alpha_\sigma} \varrho^{\beta_\sigma}, \quad a_\sigma \neq 0, b_\sigma \neq 0.$$
Note also then, by a simple computation, that
$$[\arg (F^j_{\beta_\sigma})^*]'(0) = [\arg G^*_\sigma]'(0) = \frac{b_\sigma}{a_\sigma}; \tag{17}$$
and also, since $\varepsilon_k \alpha_\sigma + \beta_\sigma = r_k$, that
$$\frac{n(n - \beta_\sigma + 1)}{\alpha_\sigma + 1} = \frac{-n\varepsilon_k \alpha_\sigma + n(r_k - n+1)}{\alpha_\sigma + 1}, \tag{18}$$
for $\sigma = t_{k-1}, \dots, t_k$. Gathering together (15, 16, 17, 18) we have
$$V_0 \sum_{\lambda=t_{k-1}}^{t_k} F^j_{\beta_\sigma} = \lim_{\varrho \to 0} \omega_0^* H_k(\varrho, -) - \tfrac{1}{2} \operatorname{sgn} [\arg H_k^*(\varrho, -)]'(\infty), \tag{19a}$$
where $$\varrho^{r_k} H_k(\varrho, y) = \sum_{\sigma=t_{k-1}}^{t_k} (a_\sigma + i b_\sigma y \varrho^{\varepsilon_k})\, y^{\alpha_\sigma} = G_k(\varrho, y\varrho^{\varepsilon_k}) \tag{19b}$$

and $$\frac{b_\sigma}{a_\sigma}=\frac{-\varepsilon_\kappa n\alpha_\sigma+n(r_k-n+1)}{\alpha_\sigma+1} \quad \text{for} \quad \sigma=t_{k-1},\ldots,t_k. \tag{19c}$$

Note, using a proper affine transformation, it follows that

$$\lim_{\varrho\to 0}\omega_0^*\hat{H}_k(\varrho,\,-)=\omega_0^*H_k \tag{20}$$

where $$\hat{H}_k(y)=\sum_{\sigma=1_{k-1}}^{t_k}(a_\sigma+ib_\sigma y)\,y^{\alpha_k};$$

also from the assumption (14) we have

$$[\arg H_k^*(\varrho,\,-)]'(\infty)=+\tfrac{1}{2}. \tag{21}$$

The next step in the proof of Proposition 22 is to apply the semigroupoid Propositions 20 and 21 (a). Let

$$Q(y)=y\sum_{\sigma=t_{k-1}}^{t_k}b_\sigma y^{\alpha_\sigma} \tag{22}$$

and define the transformation T as in section 9 by $T=A(a)\,B(b)\,C(c)$ where, writing $r=r_k,\ \varepsilon=\varepsilon_k$,

$$a=n(r-n+1),$$

$$b=\frac{-\varepsilon n}{n(r-n+1)+\varepsilon n},$$

$$c=1.$$

A straight forward computation shows that

$$T[iQ]=A(a)\,B(b)\,C(c)\,[iQ]=\hat{H}_k.$$

Assuming for the moment that $a>0$ and that $B(b)$ is defined on $C(c)[iQ]$ we proceed as follows. From (22), Q has least one real root and, since $\omega^*[iQ]=0$, Proposition 21(a) gives

$$\omega_0^*H_k=\omega_0^*T[iQ]\geqslant\tfrac{1}{2}.$$

But then (20) and (19a) give

$$V_0\sum_{\sigma=t_{k-1}}^{t_k}\varrho^{\beta_\sigma}F_{\beta_\sigma}^j=\omega_0^*H_k-\tfrac{1}{2}\,[\arg H_k^*(\varrho,\,-)]'(\infty)\geqslant 0$$

so that Proposition 22 is proved provided $A(a)$ and $B(b)$ are allowable operators. To see this; we first note that $(\mu f_{\beta_\sigma})(\theta_j)\leqslant\beta_\sigma$ since u_{β_σ} is a form of degree β_σ. From Proposition 14′, $\alpha_\sigma\leqslant\beta_\sigma-n$; so

$$r_k = \varepsilon_k \alpha_\sigma + \beta_\sigma \geqslant \varepsilon_k(\alpha_\sigma + 1) + n \tag{23}$$

and so
$$r_k - n + 1 \geqslant (\varepsilon_k + 1)\alpha_\sigma + 1 > 0$$

which shows that $a > 0$ and thus $A(a)$ is of correct form. To see that $B(b)$ is defined on $C(c)[iQ]$ we must show that $bc(\alpha_\sigma + 1) + 1 > 0$ but, with $r = r_k$, $\varepsilon = \varepsilon_k$,

$$bc(\alpha_0 + 1) + 1 = \frac{n(r-n+1) - \varepsilon n \alpha_\sigma}{n(r-n+1) + \varepsilon n}$$

which, by (23),
$$\geqslant \frac{(\alpha_0 + 1)}{(r - +1) + \varepsilon} > 0.$$

This completes the proof of Proposition 22.

We need to study one more special perturbation.

PROPOSITION 23. *Given t, $t_{k-1} < t \leqslant t_k$, where*

$$(\mu f_{\beta_\sigma})(\theta_j) \geqslant n, \quad t_{k-1} \leqslant \sigma \leqslant t-1, \quad (\mu f_{\beta_t})(\theta_j) \leqslant n-1;$$

then
$$V_0 \mathfrak{J}^j_n = V_0 \sum_{\sigma = t_{k-1}}^{t_k} G_\sigma \geqslant -\tfrac{1}{2}(1 + \alpha_t - \alpha_{t_k}).$$

Proof. Using the approximation (9), in the special form as in Proposition 22 when $t_{k-1} < \sigma \leqslant t-1$, we have

$$\sum_{\sigma=t_{k-1}}^{t_k} G_\sigma = \sum_{\sigma=t_{k-1}}^{t-1} (a_\sigma + ib_\sigma x)\, x^{\alpha_\sigma} \varrho^{\beta_\sigma} + \sum_{\sigma=t}^{t_k} (a_\sigma + ib_\sigma x^{\gamma_\sigma})\, x^{\alpha_\sigma} \varrho^{\beta_\sigma};$$

which, in turn, with $x = y\varrho^\varepsilon$, $\varepsilon = \varepsilon_k$, $r = r_k$, gives

$$\sum_{\sigma=t_{k-1}}^{t_k} G_\sigma = \varrho^r \sum_{\sigma=t_{k-1}}^{t-1} (a_\sigma + ib_\sigma \varrho^\varepsilon y)\, y^{\alpha_\sigma} + \varrho^r \sum_{\sigma=t}^{t_k} (a_\sigma + ib_\sigma \varrho^{\gamma_\sigma \varepsilon} y^{\gamma_\sigma})\, y^{\alpha_\sigma}. \tag{24}$$

We note that if $\gamma_\sigma \varepsilon > \varepsilon$ the term, as in the proof of Proposition 19, can be neglected; we may therefore assume that

$$0 \leqslant \gamma_\sigma \leqslant 1 \quad \text{for} \quad t \leqslant \sigma \leqslant t_k. \tag{25}$$

If we multiply (24) by $y^{-\alpha_{t_k}} \varrho^{-r}$ and then apply the proper affine transformation that sends $1 \mapsto 1$ and $i \to \varrho^{-\varepsilon} i$ (both operations leave V_0 invariant) we obtain

$$V_0 \sum_{\sigma=t_{k-1}}^{t_k} G_\sigma = V_0 \sum_{\sigma=t_{k-1}}^{t-1} (a_\sigma + ib_\sigma y)\, y^{\alpha_\sigma - \alpha_{t_k}} + \sum_{\sigma=t}^{t_k} (a_\sigma + ib_\sigma \varrho^{(\gamma_\sigma - 1)\varepsilon} y^{\gamma_\sigma})\, y^{\alpha_\sigma - \alpha_{t_k}}.$$

To simplify notation, let

$$A_1 + iB_1 = \sum_{\sigma=t_{k-1}}^{t-1} (a_\sigma + ib_\sigma y)\, y^{\alpha_\sigma - \alpha_{t_k}}$$

$$A_2 + iB_2 = \sum_{\sigma=t}^{t_k} (a_\sigma + ib_\sigma \varrho^{(\gamma_\sigma - 1)\varepsilon} y^{\gamma_\sigma})\, y^{\alpha_\sigma - \alpha_{t_k}}.$$

By our choice of t we have, as in Proposition 22, the existence of a semigroupoid operator T of the form $A(a)\,B(v)\,C(c)$ such that

$$T[iB_1] = A_1 + iB_1.$$

Now the operators T thought of as acting on *all* polynomial curves always have inverses; thus $T^{-1} = C(-c)\,B(-b)\,A(1/a)$, although T^{-1} need not, of course, be in the semigroupoid. Define

$$R + iS = T^{-1}(A_2 + iB_2)$$

whence

$$T[R + i(B_1 + S)] = (A_1 + A_2) + i(B_1 + B_2).$$

Because of conditions (25) on the γ_σ and the form of T, $T = A(a)\,B(b)\,C(c)$, it follows directly that $\deg R \leqslant \deg A_2 = \alpha_t - \alpha_{t_k}$. We now apply Proposition 21 (b) to $R + i(B_1 + S)$ with T as above and the proof is complete.

11. Proof of the theorems

Before putting it all together we need one simple inequality which is stated formally so that Propositions 22, 23, 24 together give the main idea of the method for estimating the perturbation term.

PROPOSITION 24. *Given s', $1 < s' \leqslant s$, then*

(a) $$V_0 \mathcal{J}_n^j \geqslant V_0 \sum_{k=1}^{s'} \sum_{\sigma=t_{k-1}}^{t_k} G_\sigma - \tfrac{1}{2}\alpha_{t_{s'}},$$

(b) $$V_0 \mathcal{J}_n^j \geqslant -\tfrac{1}{2}\alpha_1.$$

Proof. The proof of these facts follows immediately from the fact that the sequence $\{\alpha_0\}$ is strictly decreasing and that $\alpha_1 = (\mu E_p)(\theta_j)$.

PROPOSITION 25.

(a) *For* $(\mu f_{\beta_1})(\theta_j) \leqslant n-1, \quad (\beta_1 = p),$

$$V_{\theta_j} \mathcal{E}_n = V_0 \mathcal{J}_n^j \geqslant -\tfrac{1}{2}\alpha_1 = (\mu E_p)(\theta\,);$$

(b) *for* $(\mu f_{\beta_\sigma})(\theta_j) \geqslant n, \quad 1 \leqslant \sigma \leqslant t_s,$

$$V_{\theta_j}\mathcal{E}_n = V_0 \mathfrak{J}^j_n \geqslant 0;$$

(c) *for* $(\mu f_{\beta 1})(\theta_j) \geqslant n$ *and* t *such that* $(\mu f_{\beta_t})(\theta_j) \leqslant n-1,$

$$V_{\theta_j}\mathcal{E}_n = V_0 \mathfrak{J}^j_n \geqslant -\tfrac{1}{2}\min[n, p-n\,(\mu f_p)(\theta_j) - n].$$

Proof. Part (a) is Proposition 24(b). For part (b) we have from Proposition 22 that

$$V_0 \sum_{\sigma=t_{k-1}}^{t_k} G_\sigma \geqslant 0,$$

and thus, using Proposition 19′,

$$V_0 \mathfrak{J}^j_n = V_0 \sum_{\sigma=1}^{t_s} G_\sigma = \sum_{k=1}^{s} V_0 \sum_{t_{k-1}}^{k} G_\sigma \geqslant 0.$$

For part (c) we show first that $V_0 \mathfrak{J}^j_n \geqslant -\frac{1}{2}n$; from Proposition 8, $\alpha_t \leqslant n-1$ and then Proposition 23 completes the proof. Second, since $\{\alpha_0\}$ is strictly decreasing and since $\alpha_1 = (\mu E_p)(\theta_j) \leqslant p-n$ by Proposition 14′, it follows that $V_0 \mathfrak{J}^j_n \geqslant -\frac{1}{2}(p-n)$. Third, again since $V_0 \mathfrak{J}^j_n \geqslant -\frac{1}{2}\alpha_1$ and since, by Proposition 14″, $\alpha_1 = (\mu E_p)(\theta_j) \leqslant (\mu f_p)(\theta_j) - n$ it follows that $V_0 \mathfrak{J}^j_n \leqslant (\mu f_p)(\theta_j) - n$.

Next we simply sum the inequalities in Proposition 25 and obtain directly the complete estimate of the perturbation term.

PROPOSITION 26.

$$\sum V_{\theta_j}\mathcal{E}_n \geqslant -\tfrac{1}{2}\sum\{(\mu E_p)(\theta_j) \mid E_p(\theta_j) = 0\ (\mu f_p)(\theta_j) \leqslant n-1\}$$
$$-\tfrac{1}{2}\sum\{\min[p-n, n\,(\mu f_p)(\theta_j) - n] \mid E_p(\theta_j) = 0, (\mu f_p)(\theta_j) \geqslant n+1\}.$$

Using the definition $\mathcal{E}_n = z^n \partial_z^n u$ and also (4) in section 8 we have

$$\Omega_0 \partial_z^n u = -n + \omega_0^* E_p + \sum\{V_{\theta_j}\mathcal{E}_n \mid E_p(\theta_j) = 0\}. \tag{26}$$

We also have as an estimate on $\omega_0^* E_p$.

PROPOSITION 27.

$$\omega_0^* E_p \geqslant \tfrac{1}{2}\sum\{\min[n, p-n+1\ (\mu f_p)(\theta_j)] \mid f_p(\theta_j) = 0\}$$
$$+\tfrac{1}{2}\sum\{\mu E_p)(\theta_j) \mid E_p(\theta_j) = 0, \quad (\mu f_p)(\theta_j) \leqslant n-1\}.$$

Proof. From the fact that $E_p = L_p f_p = N_p \lambda_p f_p$ and that there exists by Proposition 13 an $S_p \in \mathrm{S}_0$ and $w \in \mathrm{E}$ such that $L_p f_p = S_p \lambda_p f_p w$ where the length of S_p is equal to $\min[n, p-n+1]$ the proof follows directly from the corollary to Proposition 11.

Now the identity (26) together with the inequalities in Propositions 26 and 27 give directly

PROPOSITION 28.

$$\Omega_0 \partial_z^n u \geq -n + \tfrac{1}{2}\Sigma\{\min[n, p-n+1, (\mu f_p)(\theta_j)] \mid f_p(\theta_j) = 0\}$$
$$-\tfrac{1}{2}\Sigma\{\min[n, p-n, (\mu f_p)(\theta_j) - n] \mid E_p(\theta_j) = 0, (\mu f_p)(\theta_j) \geq n+1\}.$$

Note again that a linear factor of u_p(or $\partial_z^n u_p$) of multiplicity μ means exactly that f_p (or E_p) has zero at θ and $\theta + \pi$ of multiplicity μ; thus Proposition 28 is equivalent to the Dual Theorem and so the proof of both the Theorem and the Dual Theorem is complete.

12. Remarks

1. The isolated singularity condition is not really important since in the non-isolated case a real analytic function can be factored out of $\partial_z^n u$, $u \in C^\omega(D, \mathbf{R})$, and the index Ω_0 remains invariant. The Loewner conjecture in the case that $u \in C^\infty(D, \mathbf{R})$ remains open; here some condition similar to that of an isolated singularity will no doubt be crucial.

2. Let $\mathcal{L}_n$ be a homogeneous polynomial

$$\mathcal{L}_n(z, \bar{z}) = \sum_{\sigma=0}^{n} c_\sigma z^{n-\sigma} \bar{z}^\sigma, \quad c_\sigma \in \mathbf{C}$$

and interpret $\mathcal{L}_n$ as a mapping from the real one dimensional projective space $\mathbf{P}(\mathbf{R})$ to itself. As such $\mathcal{L}_n$ has a topological degree $\delta\mathcal{L}_n$. Let $A(x, y)$ and $B(x, y)$ be real polynomials with $z = x + iy$ and choose $c \in \mathbf{C}$ so that $\mathcal{L}_n(z, \bar{z}) \equiv c[A(x, y) + iB(x, y)]$. With a little algebra one can establish that $\delta\mathcal{L}_n = -n$ if and only if $\mathcal{L}_n(z, \bar{z}) \equiv \Pi(a_\sigma z + b_\sigma \bar{z})$ with a_σ, $b_\sigma \in \mathbf{C}$ and $a_\sigma \bar{a}_\sigma - b_\sigma \bar{b}_\sigma > 0$, and this if and only if the polynomial $B(x, 1)$ separates $A(x, 1)$ positively. There is another Loewner Conjecture which states with $u \in C^\infty(D, \mathbf{R})$, $\delta\mathcal{L}_n = -n$ and an isolated singular point that

$$\Omega_0 \mathcal{L}_n(\partial_z, \partial_{\bar{z}}) u \geq -n.$$

This conjecture remains unsettled; an affirmative answer even when $u \in C^\omega(D, \mathbf{R})$ would be of considerable use in various differential geometric conjectures (see Little [7], Wall [15]).

The results obtained here for the qualitative properties of the differential action of S_0 should, with slight modifications, be sufficient but the perturbation theory and the relation between the differential actions of $\mathcal{L}_n$ and S_0 apparently present much more serious difficulties.

3. The differential action of the semigroups S_0 allows for considerable generalization. First, with real polynomials A_n and B_{n-1}, if B_{n-1} separates A_n positively then, with x, y, $f \in C^\infty(S^1, \mathbf{R})$, the differential equation $x = A_n[f]$ has for given x a unique solution f. Thus the "parametric" function f may be eliminated as in the classical Heavyside Calculus to obtain operators of the form $y = B_{n-1}[A_n^{-1}[x]] = (B_{n-1}A^{-1})[x]$. As is well known these "degenerate" operators effectively approximate, for example, the Hilbert Kernel Operator $y = Hx$ where

$$y(t) = -\frac{1}{2\mu} \mathcal{P} \int_{-\pi}^{\pi} \cot\left(\frac{s}{2}\right) x(t-s)\, dt.$$

It can then be shown, see [12], that for any C^∞ immersion of S^1 in E represented by (x, y) where $y = Hx$, can also be obtained by $y = Sx$ where S comes from a semigroup $\mathcal{S}$ involving real non-negative functions rather than the constant coefficients as for S_0. Since again S is a finite product of generators the theory of certain integral operators is in this sense combinatorialized.

Second, also see [12], the definition of the semigroup $\mathcal{S}$ and its differential action can be further extended so that it operates on the C^∞ mappings from an oriented manifold M of dimension n to an oriented manifold of dimension $n+1$ (replacing S^1 and E respectively). Suppose $\hat{M}$ is an oriented manifold of dimension $n+1$ so that $\partial\hat{M} = M$ and call $f\colon M \to N$ positively extendable to M if there exists a sensepreserving (roughly, non-negative Jacobian) extension $F\colon \hat{M} \to N$. One has, with the appropriate definitions, that positive extendability is an invariant under the differential action of such a semigroup.

This generalization has perhaps most of its interest in the fact that the target need not be a linear space since the operators are in "parametric" form; i.e., a principle reason for a linear target is to make the procedure for eliminating the parametric functions (mappings) easier to handle.

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