A GENERALIZATION TO OVERDETERMINED SYSTEMS OF THE NOTION OF DIAGONAL OPERATORS

I. *Elliptic operators*

BY

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Introduction

One of the problems in the theory of overdetermined systems of linear partial differential equations is to prove the existence of local solutions. If $\mathcal{D}$ is a differential operator, we would like to determine when we can solve the inhomogeneous equation $\mathcal{D}u=v$. In general, it is necessary that v satisfy a compatibility condition $\mathcal{D}'v=0$ for some operator $\mathcal{D}'$. We would like to prove that this compatibility condition is not only necessary but also sufficient for the existence of local solutions. That is, if $\underline{E}$, $\underline{F}$, and $\underline{G}$ are the sheaves of germs of differentiable sections of the vector bundles E, F, and G, where $\mathcal{D}$: $\underline{E}\to\underline{F}$ and $\mathcal{D}'$: $\underline{F}\to\underline{G}$, then the complex of sheaves,

$$0 \longrightarrow \theta \longrightarrow \underline{E} \xrightarrow{\mathcal{D}} \underline{F} \xrightarrow{\mathcal{D}'} \underline{G} \tag{1}$$

is exact, where θ is the sheaf of solutions of the homogeneous equation.

D. C. Spencer [7] has shown that, granted certain reasonable assumptions about $\mathcal{D}$, there exists a complex

$$0 \longrightarrow \theta \longrightarrow \underline{C}^0 \xrightarrow{D^0} \underline{C}^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} \underline{C}^n \longrightarrow 0 \tag{2}$$

of sheaves and of first order differential operators such that the cohomology of (2) at $\underline{C}^1$ is the same as the cohomology of (1) at $\underline{F}$. Thus, it is sufficient to consider the Spencer sequence of $\mathcal{D}$.

In general, the Spencer sequence is not exact, but we would like to show that it is when $\mathcal{D}$ satisfies some other conditions, such as ellipticity. Even in this case, however, it has not been shown that the cohomology of the Spencer sequence is finite dimensional.

In this paper, we consider several properties of a homological condition on the δ-complex of $\mathcal{D}$, which we call the δ-estimate. It seems that this condition is a reasonable generalization to overdetermined systems of the notion of "diagonal" or "uncoupled" operators. With ellipticity, it guarantees that the D-Neumann problem for $\mathcal{D}$ is solvable and that the Spencer sequence is exact.

If $\mathcal{D}$: $\underline{E} \to \underline{F}$ is a differential operator of order k, then the highest order part of $\mathcal{D}$ may be considered a map $\sigma(\mathcal{D})$: $S^k T^* \otimes E \to F$. The kernel of this map we call g_k. We may define the prolongation g_{k+l}, $l \geq 1$, of g_k and obtain the complex

$$0 \longrightarrow g_{k+2} \xrightarrow{\delta} T^* \otimes g_{k+1} \xrightarrow{\delta} \Lambda^2 T^* \otimes g_k.$$

We assign metrics to T^* and E, which then induce metrics on the g's. Then we may define the δ-estimate.

Definition. A differential operator $\mathcal{D}$ of order k satisfies the δ-estimate if and only if

$$\|\delta x\|^2 \geq \tfrac{1}{2}(k+1)^2 \|x\|^2 \quad \text{for all } x \in T^* \otimes g_{k+1} \cap \ker \delta^*.$$

We shall prove (Theorem II.3.1) that this is equivalent to the following definition.

Definition. $\mathcal{D}$ satisfies the δ-estimate if and only if in the sequence

$$0 \longrightarrow g_2^0 \xrightarrow{\delta} T^* \otimes g_1^0 \xrightarrow{\delta} \Lambda^2 T^* \otimes C^0,$$

$$\|\delta x\|^2 \geq \tfrac{1}{2} \|x\|^2 \quad \text{for all } x \in T^* \otimes g_1^0 \cap \ker \delta^*.$$

Here g_1^0 is the kernel of $\sigma(D^0)$, where D^0 is the first operator of the Spencer sequence.

Since it is more convenient to work with the Spencer sequence than with (1), we shall take the second definition as the definition of the δ-estimate, keeping in mind that we shall prove that it is equivalent to the first in Theorem II.3.1.

The δ-estimate, although it can be stated entirely in terms of the operator D^0, actually gives estimates for the other operators D^l, $l \geq 1$, in the Spencer sequence. In fact, we have that in the sequence

$$0 \longrightarrow g_2^l \xrightarrow{\delta} T^* \otimes g_1^l \xrightarrow{\delta} \Lambda^2 T^* \otimes C^l,$$

$$\|\delta x\|^2 \geq \tfrac{1}{2} \|x\|^2 \quad \text{for all } x \in T^* \otimes g_1^l \cap \ker \delta^*.$$

Here g_1^l is the kernel of $\sigma(D^l)$.

In the course of proving these estimates, we shall prove that if $\mathcal{D}$ satisfies the δ-estimate, then g_{k+1} is involutive, where k is the order of $\mathcal{D}$.

The importance of these estimates is that they enable us, in Chapter III, to prove that the Kohn–Nirenberg estimate holds for the Spencer sequence, and therefore that the D-Neumann problem is solvable and the Spencer sequence is exact.

The δ-estimate was discovered by I. M. Singer, who recognized its role in the proof of the Kohn–Nirenberg estimate,

$$^{\partial\Omega}\|u\|^2 \leqslant c\{^{\Omega}\|(D^0)^*u\|^2 + {}^{\Omega}\|D^1u\|^2 + {}^{\Omega}\|u\|^2\}$$

for $$u \in \Gamma(\Omega, C^1) \text{ and } u \in \text{domain } (D^0)^*.$$

W. J. Sweeney published the first proof of the Kohn-Nirenberg estimate for elliptic variable coefficient operators satisfying the δ-estimate, as well as the proof of his stronger estimate (Theorem III.2.1). V. W. Guillemin proved that the δ-estimate implies that g_{k+1} is involutive, but his unpublished proof is quite different from the one presented here. The new results in Chapters II and III are extensions of the δ-estimate to estimates on

$$\Lambda^{l-1}T^* \otimes g^0_{k+1} \to \Lambda^l T^* \otimes g^0_k \to \Lambda^{l+1}T^* \otimes g^0_{k-1},$$

which give a new proof of Guillemin's involutiveness theorem with a weaker hypothesis discussed in Section II.1, and which enable us to prove the Kohn–Nirenberg estimate and Sweeney's estimate for all $l \geqslant 1$, which gives exactness of the Spencer sequence.

We state without proof the justification for considering the δ-estimate to be a reasonable generalization to over-determined systems of the notion of "diagonal" or "uncoupled" operators. To see this, we must consider the Guillemin normal form of the operator D^0. Suppose that locally we have a foliation $\mathfrak{F}$ of the manifold which is given in local coordinates by $x^i = \text{const.}$, $i = 1, \ldots, m$, and suppose that the leaves of the foliation are non-characteristic; that is, $U \otimes C^0 \cap g^0_1 = 0$ where U is the sub-bundle of T^* which annihilates the tangent spaces of the leaves of the foliation (and is generated by $dx^1, \ldots, dx^m$). Then

$$D^0 = D_0 + \sum_{i=1}^{m} \sigma_{dx^i}(D^0)\, D_i,$$

where D_0: $C^0 \to C^1$ is an operator tangential to the foliation and D_i: $\underline{C}^0 \to \underline{C}^0$ is $\partial/\partial x_i + L_i$, where L_i is tangential to the foliation. There exist operators D'_i and D'_{ij}, $1 \leqslant i, j \leqslant m$, such that

$$D_0 D_i = D'_i D_0 \quad 1 \leqslant i \leqslant m$$

and $$[D_i, D_j] = D'_{ij} D_0 \quad 1 \leqslant i, j \leqslant m.$$

Therefore, for $\xi \in T^*$, the symbol maps $\sigma_\xi(D_i)$, $1 \leqslant i \leqslant m$, are a commuting set of linear maps on the kernel of $\sigma_\xi(D_0)$. The δ-estimate implies that each $\sigma_\xi(D_i)$ restricted to the kernel of $\sigma_\xi(D_0)$ is normal, and hence that the $\sigma_\xi(D_i)$'s may be diagonalized simultaneously on the kernel of $\sigma_\xi(D_0)$. However, this condition does not in general imply the δ-estimate.

If $\mathcal{D}$ is hyperbolic then the maps $\sigma_\xi(D_i)$, $1 \leqslant i \leqslant m$, are symmetric on the kernel of $\sigma_\xi(D_0)$, so there is justification in asserting that hyperbolic overdetermined systems satisfying the δ-estimate are a generalization of symmetric hyperbolic systems. Details will appear in a subsequent paper.

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I. Preliminaries

0. Introduction

In Sections 1 through 4 we define the δ-cohomology and the Spencer sequence. For proofs of the theorems in these sections, the reader should consult Goldschmidt [1]. Another introduction to the formal theory of linear overdetermined systems of partial differential equations, as well as motivation for the Spencer sequence, appears in the survey article by Spencer [7], on which portions of this chapter are based.

In Section 5 we define elliptic operators and complexes, and state Quillen's theorem, which guarantees that the Spencer sequence of an elliptic operator is an elliptic complex. In Section 6 we define inner products on the fibers of various bundles. In Section 7 we calculate the eigenvalues of the formal Laplacian operator, which we must know for several of the proofs in Chapters II and III.

1. Jets

Let X be a differentiable manifold of dimension n. Since we shall confine ourselves to the C^∞ differentiable category, "differentiable" here means "differentiable of class C^∞". If E is a complex (differentiable) vector bundle over X, we denote, for each non-negative integer k, by $J_k(E)$ the vector bundle over X of k-jets of E. The fiber of $J_k(E)$ over a point x of X is the quotient of the space of germs of sections of E at x by the subspace of germs which vanish to order $k+1$. We identify $J_0(E)$ with E, and denote by π: $J_k(E) \to X$ and π_{k-1}: $J_k(E) \to J_{k-1}(E)$ the natural projections. The sheaf of germs of (differentiable) sections of E we call $\underline{E}$. We denote by j_k: $\underline{E} \to \underline{J_k(E)}$ the map which takes germs of sections of E into their k-jets.

We denote by T^* the complexified cotangent bundle of X, and by S^kT^*, Λ^lT^*, and $\otimes^mT^*$ the k-tuple symmetric product of T^*, the l-tuple exterior product of T^*, and the

m-tuple tensor product of T^*, respectively. There is a natural vector bundle morphism i: $S^kT^*\otimes E\to J_k(E)$, and the sequence

$$0 \longrightarrow S^kT^*\otimes E \xrightarrow{\;i\;} J_k(E) \xrightarrow{\pi_{k-1}} J_{k-1}(E) \longrightarrow 0 \tag{1.1}$$

is exact.

2. Differential operators and their prolongations

Let E and F be vector bundles over X, and let φ: $J_k(E)\to F$ be a morphism of vector bundles. Then φ induces a sheaf morphism φ: $\underline{J_k(E)}\to \underline{F}$.

Definition 2.1. The symbol $\sigma(\varphi)$ is the composition

$$\sigma(\varphi)=\varphi\circ i\colon S^kT^*\otimes E\to F.$$

Definition 2.2. A sheaf morphism $\mathcal{D}$: $\underline{E}\to\underline{F}$ is called a differential operator (from E to F) of order k if the triangle

$$\begin{tikzcd} \underline{E} \arrow[r, "j_k"] \arrow[dr, "\mathcal{D}"'] & \underline{J_k(E)} \arrow[d, "\varphi"] \\ & \underline{F} \end{tikzcd}$$

commutes; i.e., if $\mathcal{D}=\varphi\circ j_k$, where φ: $J_k(E)\to F$ is a bundle morphism. The symbol $\sigma(\mathcal{D})$ of $\mathcal{D}$ is the symbol of φ; i.e., $\sigma(\mathcal{D})=\sigma(\varphi)$.

Definition 2.3. The lth prolongation $p_l(\varphi)$: $J_{k+l}(E)\to J_l(F)$ of φ is the unique morphism of vector bundles such that the following diagram commutes:

$$\begin{tikzcd} \underline{J_{k+l}(E)} \arrow[r, "p_l(\varphi)"] & \underline{J_l(F)} \\ \underline{E} \arrow[u, "j_{k+l}"] \arrow[r, "\varphi\circ j_k"] & \underline{F} \arrow[u, "j_l"'] \end{tikzcd}$$

The differential operator $\mathcal{D}_l=j_l\circ\mathcal{D}=j_l\circ\varphi\circ j_k$: $\underline{E}\to\underline{J_l(F)}$ is the lth prolongation of the operator $\mathcal{D}=\varphi\circ j_k$: $\underline{E}\to\underline{F}$. We shall sometimes write $p_l(\mathcal{D})=p_l(\varphi)$ and $p(\mathcal{D})=p_0(\varphi)=\varphi$.

In particular, let 1_k: $J_k(E)\to J_k(E)$ be the identity map. (The corresponding differential operator is then j_k: $\underline{E}\to\underline{J_k(E)}$.) The lth prolongation of the map 1_k is a monomorphism of vector bundles

$$p_l(1_k)\colon J_{k+l}(E)\to J_l(J_k(E)),$$

and we identify $J_{k+l}(E)$ with its image in $J_l(J_k(E))$.

Definition 2.4. For $l\geqslant 1$, define

$$\sigma_l(\varphi): S^{k+l}T^* \otimes E \to S^l T^* \otimes F$$

to be the unique morphism of vector bundles such that the following diagram is exact and commutative:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ S^{k+l}T^* \otimes E & \xrightarrow{\sigma_l(\varphi)} & S^l T^* \otimes F \\ \downarrow & & \downarrow \\ J_{k+l}(E) & \xrightarrow{p_l(\varphi)} & J_l(F) \\ \downarrow & & \downarrow \\ J_{k+l-1}(E) & \xrightarrow{p_{l-1}(\varphi)} & J_{l-1}(F) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

We set $\sigma_0(\varphi) = \sigma(\varphi)$, and $\sigma_l(\mathcal{D}) = \sigma_l(p(\mathcal{D}))$. If $\xi \in T^*$, we define $\sigma_\xi(\mathcal{D})$ by letting $\sigma_\xi(\mathcal{D})(e) = \sigma(\mathcal{D})(\xi^k \otimes e)$.

Definition 2.5. A homogeneous linear partial differential equation R_k of order k on E is a subbundle of $J_k(E)$. A solution of R_k is a section e of E over an open set $U \subset X$ such that $j_k(e)(x) \in R_k$ for all $x \in U$. The lth prolongation of R_k is the subset

$$R_{k+l} = J_l(R_k) \cap J_{k+l}(E)$$

of $J_{k+l}(E)$ where both $J_l(R_k)$ and $J_{k+l}(E)$ are regarded as subsets of $J_l(J_k(E))$.

If φ: $J_k(E) \to F$ is a bundle morphism of locally constant rank with R_k as kernel, we say that R_k is the equation associated to the differential operator $\varphi \circ j_k$: $\underline{E} \to \underline{F}$. Conversely, given a sub-bundle R_k of $J_k(E)$, we can find a vector bundle F and a morphism φ: $J_k(E) \to F$ of locally constant ran such that $R_k = \ker \varphi$. We set $R_{k-l} = J_{k-l}(E)$ for $1 \leqslant l \leqslant k$.

It is easily seen that R_{k+l} is the kernel of $p_l(\varphi)$; i.e.,

$$0 \longrightarrow R_{k+l} \longrightarrow J_{k+l}(E) \xrightarrow{p_l(\varphi)} J_l(F)$$

is exact.

Let $g_{k+l} \subset S^{k+l}T^* \otimes E$ be the kernel of the map π_{k+l-1}: $R_{k+l} \to R_{k+l-1}$; i.e., for $l \geqslant 0$,

$$0 \longrightarrow g_{k+l} \longrightarrow R_{k+l} \xrightarrow{\pi_{k+l-1}} R_{k+l-1}$$

is exact. Then g_{k+l} is also the kernel of $\sigma_l(\varphi)$: $S^{k+l}T^* \otimes E \to S^l T^* \otimes F$; i.e.,

$$0 \longrightarrow g_{k+l} \longrightarrow S^{k+l}T^* \otimes E \xrightarrow{\sigma_l(\varphi)} S^l T^* \otimes F$$

is exact.

Definition 2.6. We call g_k the symbol of the equation R_k and set $g_{k-l} = S^{k-l}T^* \otimes E$ for $1 \leqslant l \leqslant k$.

It is important to note that R_{k+l} and g_{k+l} are families of vector spaces over X and are not necessarily vector bundles for $l \geqslant 0$ (with the exception of R_k, a vector bundle by definition).

Definition 2.7. Let $\mathcal{D}=\varphi\circ j_k$: $\underline{E}\to\underline{F}$ be a differential operator of order k, and let $R_k=\ker\varphi$. Then $\mathcal{D}$ is formally integrable if, for $l\geqslant 0$, R_{k+l} is a vector bundle and π_{k+l}: $R_{k+l+1}\to R_{k+l}$ is surjective.

Formal integrability means that the ranks of the prolongations of $\mathcal{D}$ are locally constant, and that formal solutions of the equation exist. A formal solution to the equation at a point $x\in X$ is a sequence $\{r_k, r_{k+1}, \ldots\}$ where $r_{k+l}\in R_{k+l}$ such that $\pi_{k+l}(r_{k+l+1})=r_{k+l}$ for all $l\geqslant 0$. It corresponds to a formal power series solution of the homogeneous equation.

3. The δ-cohomology

Let δ: $S^mT^*\to T^*\otimes S^{m-1}T^*$ be the unique linear map such that

$$\delta(\xi^1\cdot\ldots\cdot\xi^m)=\sum_{i=1}^{m}\xi^i\otimes(\xi^1\cdot\ldots\cdot\hat{\xi}^i\cdot\ldots\cdot\xi^m)$$

for all $\xi^1, \ldots, \xi^m\in T^*$. We extend δ to a linear map

$$\delta:\ \Lambda^lT^*\otimes S^mT^*\otimes E\to\Lambda^{l+1}T^*\otimes S^{m-1}T^*\otimes E$$

by setting $\delta(u\otimes v\otimes e)=u\wedge\delta v\otimes e$ if $u\in\Lambda^lT^*$, $v\in S^mT^*$, and $e\in E$. Clearly, $\delta^2=0$.

One can show that the following square commutes:

$$\begin{array}{ccc}
\Lambda^lT^*\otimes S^{k+m+1}T^*\otimes E & \xrightarrow{\sigma_{m+1}(\varphi)} & \Lambda^lT^*\otimes S^{m+1}T^*\otimes F\\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta}\\
\Lambda^{l+1}T^*\otimes S^{k+m}T^*\otimes E & \xrightarrow{\sigma_m(\varphi)} & \Lambda^{l+1}T^*\otimes S^mT^*\otimes F
\end{array}$$

From this we conclude that

$$\delta(\Lambda^lT^*\otimes g_{k+m+1})\subset\Lambda^{l+1}T^*\otimes g_{k+m}.$$

Thus, we have a complex for each $m\geqslant k$,

$$0\longrightarrow g_m\xrightarrow{\delta}T^*\otimes g_{m-1}\xrightarrow{\delta}\cdots\xrightarrow{\delta}\Lambda^{m-k}T^*\otimes g_k\xrightarrow{\delta}\Lambda^{m-k+1}T^*\otimes S^{k-1}T^*\otimes E. \tag{3.1$_m$}$$

Definition 3.1. The δ-cohomology of g_k is the cohomology of the sequences (3.1$_m$) where $m\geqslant k$. We denote by $H^{m-l,l}=H^{m-l,l}(g_k)$ the cohomology of the sequence (3.1$_m$) at $\Lambda^lT^*\otimes g_{m-l}$. We say that g_k is involutive if the sequences (3.1$_m$) are exact. We say that g_k is q-acyclic if $H^{m,l}=0$ for $m\geqslant k$, $0\leqslant l\leqslant q$.

We remark that, for each $m \geqslant k$, the sequence

$$0 \longrightarrow g_{m+1} \xrightarrow{\delta} T^* \otimes g_m \xrightarrow{\delta} \Lambda^2 T^* \otimes g_{m-1}$$

is easily seen to be exact. Because of this, it is possible to define g_m inductively by setting $g_{m+1} = T^* \otimes g_m \cap S^{m+1}T^* \otimes E$ for $m \geqslant k$, where $T^* \otimes g_m$ and $S^{m+1}T^* \otimes E$ are both considered to be subspaces of $T^* \otimes S^m T^* \otimes E$. We shall occasionally use this fact.

The following theorem states that if we prolong a differential operator sufficiently often, its symbol becomes involutive.

THEOREM 3.2. (δ-Poincaré lemma). *If the fiber dimension of E is $\leqslant e$, there exists an integer $\mu \geqslant k$, depending only on n (dimension of X), k (order of the differential operator), and e such that $H^{m,l} = 0$ for all $m \geqslant \mu$ and $l \geqslant 0$.*

Proof. See Sweeney [8].

4. The Spencer sequence

We wish to construct the Spencer sequence of R_k which is a complex

$$0 \longrightarrow \theta \xrightarrow{j_k} \underline{C}^0 \xrightarrow{D^0} \underline{C}^1 \xrightarrow{D^1} \dots \xrightarrow{D^{n-1}} \underline{C}^n \longrightarrow 0, \tag{4.1}$$

where D^l: $\underline{C}^l \to \underline{C}^{l+1}$ is a first order differential operator.

Let $C^0 = R_k$ and set $C^l = (\Lambda^l T^* \otimes C^0)/\delta(\Lambda^{l-1} T^* \otimes g_{k+1})$, for $l \geqslant 1$. The exterior multiplication map $T^* \otimes \Lambda^l T^* \to \Lambda^{l+1} T^*$ induces an epimorphism τ_l: $T^* \otimes C^l \to C^{l+1}$.

We have:

THEOREM 4.1. *Assume that the equation R_k is formally integrable and that C^l is a vector bundle, for $1 \leqslant l \leqslant r$. The following statements are equivalent:*

(i) *g_{k+1} is r-acyclic.*

(ii) *There exists a unique complex*

$$0 \longrightarrow \theta \xrightarrow{j_k} \underline{C}^0 \xrightarrow{D^0} \underline{C}^1 \xrightarrow{D^1} \dots \longrightarrow \underline{C}^{r-1} \xrightarrow{D^{r-1}} \underline{C}^r$$

satisfying the following properties:

(a) *The map D^l: $\underline{C}^l \to \underline{C}^{l+1}$ is a first order operator induced by a morphism of vector bundles ϱ_l: $J_1(C^l) \to C^{l+1}$, whose symbol is the morphism τ_l: $T^* \otimes C^l \to C^{l+1}$, $0 \leqslant l \leqslant r-1$.*

(b) *The complex is formally exact in the sense that the sequences*

$$0 \longrightarrow R_{k+m} \longrightarrow J_m(C^0) \xrightarrow{p_{m-1}(\varrho_0)} J_{m-1}(C^1) \xrightarrow{p_{m-2}(\varrho_1)} \dots \longrightarrow J_{m-r}(C^r)$$

are exact at R_{k+m} for $m \geqslant 1$, and at $J_{m-l}(C^l)$ for $m \geqslant l+1$, $0 \leqslant l \leqslant r-1$.

Proof. See Goldschmidt [1].

Remark. In Chapter II and all of Chapter III, except Theorem III.3.1, we shall not have to assume that R_k is formally integrable or that $C^l, l \geq 1$, is a vector bundle. We shall not be considering the operators D^l, but rather their symbols, $\sigma(D^l)$, which exist as maps of vector spaces at each point of P without these assumptions.

The operators D^l are essentially the difference of exterior differentiation and formal exterior differentiation (see Spencer [7]).

It is easy to show that the sequence

$$0 \longrightarrow \theta \xrightarrow{j_k} \underline{C}^0 \xrightarrow{D^0} \underline{C}^1$$

is exact, so we see that the solutions of the homogeneous equation $\mathcal{D}u = 0$ are the same as solutions of $D^0 u = 0$.

5. Elliptic complexes

Since T^* is the complexification of the real cotangent bundle, we can identify the sub-bundle consisting of real cotangent vectors. An operator $\mathcal{D}$: $\underline{E} \to \underline{F}$ is called elliptic if for every real cotangent vector ξ, the bundle morphism $\sigma_\xi(\mathcal{D})$: $E \to F$ is injective. A complex of operators

$$\underline{E}^0 \xrightarrow{\mathcal{D}_0} \underline{E}^1 \xrightarrow{\mathcal{D}_1} \underline{E}^2 \longrightarrow \dots \xrightarrow{\mathcal{D}_{n-1}} \underline{E}^n \longrightarrow 0$$

is called elliptic if for every real cotangent vector ξ, the complex of bundle morphisms

$$0 \longrightarrow E^0 \xrightarrow{\sigma_\xi(\mathcal{D}_0)} E^1 \xrightarrow{\sigma_\xi(\mathcal{D}_1)} E^2 \longrightarrow \dots \xrightarrow{\sigma_\xi(\mathcal{D}_{n-1})} E^n \longrightarrow 0$$

is exact.

If there are metrics in the fibers of the bundles E^i, we may define the formal adjoints $\mathcal{D}_i^*$ of the operators $\mathcal{D}_i$ and the generalized Laplacian

$$\mathcal{D}_{i-1}\mathcal{D}_{i-1}^* + \mathcal{D}_i^*\mathcal{D}_i \colon \underline{E}^i \to \underline{E}^i.$$

If ξ is a real cotangent vector, and if $\mathcal{D}_i$ is first order, $\sigma_\xi(\mathcal{D}_i^*) = -\sigma_\xi(\mathcal{D}_i)^*$. Therefore,

$$-\sigma_\xi(\mathcal{D}_{i-1}\mathcal{D}_{i-1}^* + \mathcal{D}_i^*\mathcal{D}_i) = \sigma_\xi(\mathcal{D}_{i-1})\sigma_\xi(\mathcal{D}_{i-1})^* + \sigma_\xi(\mathcal{D}_i)^*\sigma_\xi(\mathcal{D}_i),$$

which is injective if the symbol sequence is exact. Therefore the generalized Laplacian of an elliptic complex is a determined elliptic operator.

The following theorem due to Quillen guarantees that the Spencer sequence of an elliptic operator is an elliptic complex.

THEOREM 5.1. (Quillen). *If g_{k+1} is involutive, and if $\xi \in T^*$ is a non-zero covector, then the following conditions are equivalent:*

(i) *The sequence*

$$0 \longrightarrow E \xrightarrow{\sigma_\xi(D)} F$$

is exact.

(ii) *The sequence*

$$0 \longrightarrow C^0 \xrightarrow{\sigma_\xi(D^0)} C^1$$

is exact.

(iii) *The sequence*

$$0 \longrightarrow C^0 \xrightarrow{\sigma_\xi(D^0)} C^1 \xrightarrow{\sigma_\xi(D^1)} \dots \xrightarrow{\sigma_\xi(D^{n-1})} C^n \longrightarrow 0$$

is exact.

Proof. See Quillen [5], Goldschmidt [1], or Sweeney [8].

6. Extension of metrics

We assume that we are given inner products on the fibers of T^* and E. We shall extend these to inner products on $\Lambda^l T^* \otimes S^m T^* \otimes E$; C^l, $0 \leqslant l \leqslant n$; and all the other spaces we shall consider.

If $V_1, \dots, V_m$ are finite dimensional complex hermitian spaces, we obtain an inner product on $V_1 \otimes \dots \otimes V_m$ by setting

$$\langle v_1 \otimes \dots \otimes v_m, w_1 \otimes \dots \otimes w_m \rangle = \langle v_1, w_1 \rangle \dots \langle v_m, w_m \rangle$$

and extending linearly. Therefore we have an inner product on $\otimes^m T^*$. We define an inner product on $S^m T^*$ as follows. Let α be the monomorphism of $S^m T^*$ into $\otimes^m T^*$ generated by

$$\alpha(\xi^1 \dots \xi^m) = \frac{1}{m!} \sum \xi^{\pi(1)} \otimes \dots \otimes \xi^{\pi(m)}, \quad \pi \in S(m),$$

where $S(m)$ is the permutation group on $\{1, \dots, m\}$. Then α induces an inner product on $S^m T^*$. Similarly, let β: $\Lambda^l T^* \to \otimes T^*$ be given by

$$\beta(\xi^1 \Lambda \dots \Lambda \xi^l) = \frac{1}{l!} \sum (-1)^\pi \xi^{\pi(1)} \otimes \dots \otimes \xi^{\pi(l)}, \quad \pi \in S(l).$$

This induces an inner product on $\Lambda^l T^*$.

We now extend the metrics to $J_k(E)$. This cannot be done canonically; we must choose a splitting p of the following exact sequence for each $k \geqslant 1$:

$$0 \to S^k T^* \otimes E \underset{p}{\overset{i}{\rightleftarrows}} J_k(E) \xrightarrow{\pi_{k-1}} J_{k-1}(E) \to 0.$$

For $k=1$, the choice of such a splitting is equivalent to the choice of a connection on E. Furthermore, given connections on E and on T^* (say the Riemannian connection) we can define canonically a splitting of the above sequence for every k, and therefore a canonical isomorphism

$$J_k(E) \cong \oplus S^l T^* \otimes E, \quad (l=0, \dots, k).$$

This clearly induces an inner product on $J_k(E)$ for which the maps i and π_{k-1} are isometric injection and projection, respectively. See Palais [4, Chap. IV, § 9] for details. This inner product induces one on $R_k = C^0$, which gives us one on $\Lambda^l T^* \otimes C^0$. Since C^l may be identified with the orthocomplement of $\delta(\Lambda^{l-1} T^* \otimes g_{k+1})$ in $\Lambda^l T^* \otimes C^0$, we assign it the inner product it has as a subspace of $\Lambda^l T^* \otimes C^0$.

These inner products and the volume element on X allow us to define L_2 inner products on sections of bundles. If F is a bundle over X and if Ω is a compact manifold-with-boundary contained in X, we define $\Gamma(\Omega, F)$ to be the space of sections of F over Ω which can be extended to smooth sections over some neighborhood of Ω. Then if $e, f \in \Gamma(\Omega, F)$ we define

$${}^{\Omega}\langle e, f\rangle = \int_{\Omega} \langle e, f\rangle \, dv,$$

and we define

$${}^{\partial\Omega}\langle e, f\rangle = \int_{\partial\Omega} \langle e, f\rangle \, da,$$

where da is the induced volume element on $\partial\Omega$.

7. The eigenvalues of the formal Laplacian

Consider the exact sequence

$$0 \longrightarrow S^m T^* \xrightarrow{\delta} T^* \otimes S^{m-1} T^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \Lambda^{m-1} T^* \otimes T^* \xrightarrow{\delta} \Lambda^m T^* \longrightarrow 0.$$

Since we have inner products on all spaces, we may define the adjoint δ^* of δ, and the map

$$\delta^* \delta + \delta\delta^* \colon S^{m-l} T^* \otimes \Lambda^l T^* \to S^{m-l} T^* \otimes \Lambda^l T^*.$$

Since δ is the formal analogue of exterior differentiation, we call $\delta^*\delta + \delta\delta^*$ the formal Laplacian.

The proofs in Chapters II and III require that we know the eigenvalues of this map, so we calculate them here. The reader, if he wishes to avoid the calculations, may omit this section after reading the statement of Theorem 7.1 and Corollary 7.2.

THEOREM 7.1. *$S^mT^*\otimes\Lambda^lT^*$ is the sum of the eigenspaces* $\ker\delta$ *and* $\ker\delta^*$*; on* $\ker\delta$, *the eigenvalue of $\delta^*\delta+\delta\delta^*$ is $(m+1)(m+l)/l$, and on* $\ker\delta^*$ *the eigenvalue is $m(m+l)/(l+1)$.*

COROLLARY 7.2. *On $S^mT^*\otimes\Lambda^lT^*$, the identity map is equal to*

$$\frac{l+1}{m(m+l)}\,\delta^*\delta+\frac{l}{(m+1)(m+l)}\,\delta\delta^*.$$

The corollary is an obvious consequence of the theorem. To prove the theorem, we need a series of computational lemmas which we give without proof.

LEMMA 7.3. *If $(\cdot\xi)^m$ is the symmetric product of ξ with itself m times, and if $(\otimes\xi)^m$ is the tensor product, then $\alpha((\cdot\xi)^m)=(\otimes\xi)^m$, where α is as defined in Section 6. Therefore we shall write $(\xi)^k$ for both $(\cdot\xi)^k$ and $(\otimes\xi)^k$.*

LEMMA 7.4. *If β is the map defined in Section 6, then*

$$\langle\xi^1\wedge\ldots\wedge\xi^l,\zeta^1\wedge\ldots\wedge\zeta^l\rangle=\langle\xi^1\otimes\ldots\otimes\xi^l,\beta(\zeta^1\wedge\ldots\wedge\zeta^l)\rangle.$$

LEMMA 7.5.

$$\beta(\xi^1\wedge\ldots\wedge\xi^l)=\frac{1}{l}\sum_{i=1}^{l}(-1)^{i-1}\xi^i\otimes\beta(\xi^1\wedge\ldots\wedge\hat{\xi}^i\wedge\ldots\wedge\xi^l).$$

LEMMA 7.6. $$\langle(\xi)^m,(\zeta)^{m-1}\zeta'\rangle=\langle\xi,\zeta\rangle^{m-1}\langle\xi,\zeta'\rangle.$$

LEMMA 7.7.

$$\delta^*((\xi)^{m-1}\otimes\xi^1\wedge\ldots\wedge\xi^{l+1})=\frac{m}{l+1}\sum_{i=1}^{l+1}(-1)^{i-1}(\xi)^{m-1}\xi^i\otimes\xi^1\wedge\ldots\wedge\hat{\xi}^i\wedge\ldots\wedge\xi^{l+1}.$$

Proof. Since $S^mT^*\otimes\Lambda^lT^*$ is generated by elements of the form $(\zeta)^m\otimes\zeta^1\wedge\ldots\wedge\zeta^l$, it is sufficient to verify that

$$\langle\delta((\zeta)^m\otimes\zeta^1\wedge\ldots\wedge\zeta^l),(\xi)^{m-1}\otimes\xi^1\wedge\ldots\wedge\xi^{l+1}\rangle$$
$$=\langle(\zeta)^m\otimes\zeta^1\wedge\ldots\wedge\zeta^l,\frac{m}{l+1}\sum_{i=1}^{l+1}(-1)^{i-1}(\xi)^{m-1}\xi^i\otimes\xi^1\wedge\ldots\wedge\hat{\xi}^i\wedge\ldots\wedge\xi^{l+1}\rangle. \quad (7.1)$$

The left-hand side is

$$m\langle\zeta,\xi\rangle^{m-1}\langle\zeta\otimes\zeta^1\otimes\ldots\otimes\zeta^l,\beta(\xi^1\wedge\ldots\wedge\xi^{l+1})\rangle$$

by Lemma 7.6 and Lemma 7.4. By Lemma 7.5, this becomes

$$m\langle\zeta,\xi\rangle^{m-1}\langle\zeta\otimes\ldots\otimes\zeta^{l},\ \frac{1}{l+1}\sum_{i=1}^{l+1}(-1)^{i-1}\xi^{i}\otimes\beta(\xi^{1}\wedge\ldots\wedge\hat{\xi}^{i}\wedge\ldots\wedge\xi^{l+1})\rangle$$

which is equal to the right-hand side of (7.1). Q.e.d.

Now we proceed to the proof of Theorem 7.1. Any $x\in S^{m}T^{*}\otimes\Lambda^{l}T^{*}$ may be written $x=\sum_j x_j$, where $x_j=(\xi_j^0)^m\otimes\xi_j^1\wedge\ldots\wedge\xi_j^l$. Then

$$\delta x=\sum_j\delta x_j=m\sum_j(\xi_j^0)^{m-1}\otimes\xi_j^0\wedge\xi_j^1\wedge\ldots\wedge\xi_j^l$$

$$\delta^*\delta x=m\sum_j\frac{m}{l+1}\sum_{i=0}^{l}(-1)^i(\xi_j^0)^{m-1}\xi_j^i\otimes\xi_j^0\wedge\ldots\wedge\hat{\xi}_j^i\wedge\ldots\wedge\xi_j^l$$

$$\delta^*x=\sum_j\frac{m+1}{l}\sum_{i=1}^{l}(-1)^{i-1}(\xi_j^0)^m\xi_j^i\otimes\xi_j^1\wedge\ldots\wedge\hat{\xi}_j^i\wedge\ldots\wedge\xi_j^l$$

$$\delta\delta^*x=\sum_j\frac{m+1}{l}\sum_{i=1}^{l}(-1)^{i-1}[m(\xi_j^0)^{m-1}\xi_j^i\otimes\xi_j^0\wedge\ldots\wedge\hat{\xi}_j^i\wedge\ldots\wedge\xi_j^l]+(m+1)\,x.$$

If $x\in\ker\delta^*$, then $0=(l/(m+1))\,\delta\delta^*x$, so

$$lx=m\sum_j\sum_{i=1}^{l}(-1)^i\,(\xi_j^0)^{m-1}\xi_j^i\otimes\xi_j^0\wedge\ldots\wedge\hat{\xi}_j^i\wedge\ldots\wedge\xi_j^l.$$

Then $((l+1)/m)\,\delta^*\delta x=lx+mx$, so $\delta^*\delta x=(m(m+l)/(l+1))\,x$.

If $x\in\ker\delta$, then $0=((l+1)/m)\,\delta^*\delta x$, so

$$0=m\sum_j\sum_{i=0}^{l}(-1)^i\,(\xi_j^0)^{m-1}\xi_j^i\otimes\xi_j^0\wedge\ldots\wedge\hat{\xi}_j^i\wedge\ldots\wedge\xi_j^l.$$

Then $(l/(m+1))\,\delta\delta^*x=(m+l)\,x$, so $\delta\delta^*x=((m+1)\,(m+l)/l)\,x$. Q.e.d.

II. The δ-estimate

0. Introduction

The purpose of Chapters II and III is to prove the exactness of the Spencer sequence.

THEOREM III.3.1. *If $\mathcal{D}$: $\underline{E}\to\underline{F}$ is a formally integrable elliptic differential operator which satisfies the δ-estimate, then:*

(i) *The Spencer sequence*

$$0\longrightarrow\theta\longrightarrow\underline{C}^0\xrightarrow{D^0}\underline{C}^1\xrightarrow{D^1}\cdots\xrightarrow{D^{n-1}}\underline{C}^n\longrightarrow 0$$

is exact, and is a fine resolution of the sheaf θ of germs of solutions of the homogeneous equation $\mathcal{D}u=0$. Consequently the cohomology of

$$0 \longrightarrow \Gamma(X, C^0) \xrightarrow{D^0} \Gamma(X, C^1) \xrightarrow{D^1} \dots \xrightarrow{D^{n-1}} \Gamma(X, C^n) \longrightarrow 0$$

is isomorphic to the cohomology of the manifold X with coefficients in θ.

(ii) *There exists an operator $\mathcal{D}'$: $\underline{F} \to \underline{G}$ such that the sequence*

$$\underline{E} \xrightarrow{\mathcal{D}} \underline{F} \xrightarrow{\mathcal{D}'} \underline{G}$$

is exact.

The only difficult part of the proof is to show that on small, suitably convex domains the D-Neumann problem is solvable. In order to solve the D-Neumann problem on a domain Ω, it is sufficient to prove the Kohn–Nirenberg estimate:

There exists a constant c such that for all $u \in \Gamma(\Omega, C^l)$ in the domain of $(D^{l-1})^*$

$$^{\partial\Omega}\|u\|^2 \leqslant c\{^{\Omega}\|(D^{l-1})^* u\|^2 + {}^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|u\|^2\}.$$

This is sufficient to prove that the cohomology

$$\Gamma(\Omega, C^0) \xrightarrow{D^0} \Gamma(\Omega, C^1) \xrightarrow{D^1} \dots \xrightarrow{D^{n-1}} \Gamma(\Omega, C^n) \longrightarrow 0$$

is isomorphic to the harmonic space $\mathbf{H} = \Sigma \mathbf{H}^l$ on Ω, and that the harmonic space is finite dimensional. To prove that the harmonic space is zero, we need the following estimate, due to Sweeney [10]: There exists a constant c such that

$$^{\Omega}\|u\|^2_{\frac{3}{2}} \leqslant c\{^{\Omega}\|(D^{l-1})^* u\|^2_1 + {}^{\Omega}\|D^l u\|^2_1\}$$

for all $u \in \Gamma(\Omega, C^l)$ in the domain of $(D^{l-1})^*$.

When we attempt to prove these estimates, we find that the only obstacle is the possibility that the integral of a certain bilinear form may be negative. The role of the δ-estimate is that it guarantees that this bilinear form and, *a fortiori*, its integral are non-negative. Thus the estimates hold.

Here is a brief outline of the argument and results of Chapter II. We start with the following definition.

Definition. $\mathcal{D}$ satisfies the δ-estimate if and only if, in the sequence

$$0 \longrightarrow g_2^0 \xrightarrow{\delta} T^* \otimes g_1^0 \xrightarrow{\delta} \Lambda^2 T^* \otimes C^0,$$

$$\|\delta x\|^2 \geqslant \tfrac{1}{2}\|x\|^2 \quad \text{for all } x \in T^* \otimes g_1^0 \cap \ker \delta^*.$$

Here $g_1^0 = \ker \sigma(D^0)$. This definition is equivalent to the following statement, which shows that to verify the δ-estimate, we need not construct the Spencer sequence.

THEOREM 3.1. *A differential operator $\mathcal{D}$ of order k satisfies the δ-estimate if and only if*

$$\|\delta x\|^2 \geqslant \tfrac{1}{2}(k+1)^2 \|x\|^2 \quad \textit{for all } x \in T^* \otimes g_{k+1} \cap \ker \delta^*.$$

Although we have begun with an estimate in the sequence

$$0 \to g_2^0 \to T^* \otimes g_1^0 \to \Lambda^2 T^* \otimes C^0$$

only, we shall see that it implies estimates in the sequences

$$\Lambda^{l-1} T^* \otimes g_{k+1}^0 \to \Lambda^l T^* \otimes g_k^0 \to \Lambda^{l+1} T^* \otimes g_{k-1}^0$$

for l and $k \geqslant 1$. In particular we shall prove the following in Section II.1.

THEOREM 1.6. *If $\mathcal{D}$ satisfies the δ-estimate, then $\|\delta x\|^2 \geqslant (k^2/l)\|x\|^2$ for all $x \in \Lambda^{l-1} T^* \otimes g_k^0 \cap \ker \delta^*$.*

As an immediate consequence we conclude that the sequences

$$0 \to g_m^0 \to T^* \otimes g_{m-1}^0 \to \dots \Lambda^{m-1} T^* \otimes g_1^0 \to \Lambda^m T^* \otimes C^0$$

are exact. Hence g_1^0 is involutive and, equivalently, g_{k+1} is involutive, which is the conclusion of Theorem 1.7.

In Section II.2, using the estimates of Theorem 1.6, we extend the δ-estimate to all operators in the Spencer sequence.

THEOREM 2.1. *If $\mathcal{D}$ satisfies the δ-estimate, then $\|\delta x\|^2 \geqslant \frac{1}{2}\|x\|^2$ for all $x \in T^* \otimes g_1^l \cap \ker \delta^*$.*

Thus, by assuming the δ-estimate on the symbol of D^0 we obtain the same estimate on the symbol of D^l. This is exactly what we shall need in Chapter III to prove the Kohn–Nirenberg estimate for each $l \geqslant 1$.

1. The δ-estimate and involutiveness

The proofs in this and following sections are diagram chases with norms added, and are simplified if we observe that if $f\colon V \to W$ is a linear map of complex hermitian spaces and λ is the least eigenvalue of f^*f, then $\lambda \geqslant 0$ and $\|fv\|^2 \geqslant \lambda \|v\|^2$ for all $v \in V$.

The δ-complexes

$$0 \to g_m^0 \to T^* \otimes g_{m-1}^0 \to \dots \to \Lambda^{m-1} T^* \otimes g_1^0 \to \Lambda^m T^* \otimes C^0 \tag{1.1$_m$}$$

are related to each other by the following diagram:

$$
\begin{tikzcd}
 & & & & & 0 \arrow[d] \\
 & & & & 0 \arrow[d] & S^mT^*\otimes C^0 \arrow[d, "\varepsilon"] \\
 & & & 0 \arrow[r] & S^{m-1}T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & S^{m-1}T^*\otimes T^*\otimes C^0 \arrow[d, "\varepsilon"] \\
 & & & \cdots \arrow[r, "\delta"] & S^{m-2}T^*\otimes T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & S^{m-2}T^*\otimes \Lambda^2T^*\otimes C^0 \arrow[d, "\varepsilon"] \\
 & & 0 \arrow[d] & & \vdots \arrow[d, "\varepsilon"] & \vdots \arrow[d, "\varepsilon"] \\
 & 0 \arrow[r] & T^*\otimes g_{m-1}^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \cdots \arrow[r, "\delta"] & T^*\otimes \Lambda^{m-2}T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & T^*\otimes \Lambda^{m-1}T^*\otimes C^0 \arrow[d, "\varepsilon"] \\
0 \arrow[r] & g_m^0 \arrow[r, "\delta"] & T^*\otimes g_{m-1}^0 \arrow[r, "\delta"] \arrow[d] & \cdots \arrow[r, "\delta"] & \Lambda^{m-1}T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d] & \Lambda^mT^*\otimes C^0 \arrow[d] \\
 & & 0 & & 0 & 0
\end{tikzcd}
\tag{1.2$_m$}
$$

Each row is the usual δ-complex with a symmetric tensor space added, on which δ acts as the identity. The vertical maps, denoted by ε, act on $ST^*\otimes\Lambda T^*$ and as the identity on the g's and C^0. Clearly the diagram commutes and the columns are exact.

Diagram chases will relate the eigenvalues of $\delta^*\delta$ on the various spaces. These calculations, however, require several lemmas.

LEMMA 1.1. *The composition* $\delta^*\delta$,

$$g_{k+1}^0 \xrightarrow{\delta} T^*\otimes g_k^0 \xrightarrow{\delta^*} g_{k+1}^0,$$

is $(k+1)^2I$.

Proof. The above composition $\delta^*\delta$ is the composition $\delta^*\delta$ in the sequence

$$S^{k+1}T^*\otimes C^0 \xrightarrow{\delta} T^*\otimes S^kT^*\otimes C^0 \xrightarrow{\delta^*} S^{k+1}T^*\otimes C^0$$

restricted to g_{k+1}^0 followed by projection onto g_{k+1}^0. But since the eigenvalue of the formal Laplacian is $(k+1)^2$ on $S^{k+1}T^*\otimes C^0$, the composition $\delta^*\delta$ is $(k+1)^2I$ on $S^{k+1}T^*\otimes C^0$. Thus if we restrict it to g_{k+1}^0, the projection onto g_{k+1}^0 is the identity. The lemma results. Q.e.d.

Definition 1.2. Let

$$\eta\colon \Lambda^{l-1}T^*\otimes S^{k+1}T^*\otimes C^0 \to T^*\otimes\Lambda^{l-1}T^*\otimes S^kT^*\otimes C^0$$

be the unique linear map satisfying

$$\eta(x) = (k+1)\xi^l\otimes\xi^1\wedge\ldots\wedge\xi^{l-1}\otimes(\xi^l)^k$$

when $x=\xi^1\wedge\ldots\wedge\xi^{l-1}\otimes(\xi^l)^{k+1}$.

Since η is the identity on $\Lambda^{l-1}T^*$ and δ on $S^{k+1}T^*\otimes C^0$, it restricts to a map

$$\eta: \Lambda^{l-1}T^*\otimes g^0_{k+1} \to T^*\otimes\Lambda^{l-1}T^*\otimes g^0_k.$$

By Lemma 1.1, we know that if $x\in g^0_{k+1}$, then $\|\delta x\|^2 = \langle\delta^*\delta x, x\rangle = (k+1)^2\|x\|^2$, so for $x\in\Lambda^{l-1}T^*\otimes g^0_{k+1}$, we have $\|\eta x\|^2 = (k+1)^2\|x\|^2$.

In the diagram (1.2_m) we know that δ and ε commute, and therefore that δ^* and ε^* commute. The following lemma, which is important in almost all the proofs in this chapter, indicates by how much δ and ε^* fail to commute in a special case.

LEMMA 1.3. *In the following rectangle*

$$\begin{tikzcd}
T^*\otimes\Lambda^{l-2}T^*\otimes S^{k+1}T^*\otimes C^0 \arrow[r, "\delta"] & T^*\otimes\Lambda^{l-1}T^*\otimes S^kT^*\otimes C^0 \\
\Lambda^{l-1}T^*\otimes S^{k+1}T^*\otimes C^0 \arrow[u, "\varepsilon^*"] \arrow[ur, "\eta"] \arrow[r, "\delta"] & \Lambda^lT^*\otimes S^kT^*\otimes C^0 \arrow[u, "\varepsilon^*"']
\end{tikzcd}$$

we have

(i) $\varepsilon\eta = (-1)^{l-1}\delta$

and

(ii) $l\varepsilon^*\delta = (l-1)\delta\varepsilon^* + (-1)^{l-1}\eta$.

The diagram restricts to

$$\begin{tikzcd}
T^*\otimes\Lambda^{l-2}T^*\otimes g^0_{k+1} \arrow[r, "\delta"] & T^*\otimes\Lambda^{l-1}T^*\otimes g^0_k \\
\Lambda^{l-1}T^*\otimes g^0_{k+1} \arrow[u, "\varepsilon^*"] \arrow[ur, "\eta"] \arrow[r, "\delta"] & \Lambda^lT^*\otimes g^0_k \arrow[u, "\varepsilon^*"']
\end{tikzcd}$$

and (i) *and* (ii) *hold for this rectangle also.*

Proof of (i). Let $x = \xi^1\wedge\ldots\wedge\xi^{l-1}\otimes(\xi^l)^{k+1}\otimes c$. Then

$$\varepsilon\eta = (-1)^{l-1}(k+1)(\xi^1\wedge\ldots\wedge\xi^{l-1}\wedge\xi^l\otimes(\xi^l)^k\otimes c) = (-1)^{l-1}\delta x.$$

By linearity, we obtain (i) for all $x\in\Lambda^{l-1}T^*\otimes S^{k+1}T^*\otimes C^0$.

Proof of (ii). Let x be as above. Then

$$l\varepsilon^*\delta x = (k+1)\left(\sum_{i=1}^{l}(-1)^{i-1}\xi^i\otimes\xi^1\wedge\ldots\wedge\hat{\xi}^i\wedge\ldots\wedge\xi^l\otimes(\xi^l)^k\otimes c\right).$$

Similarly,

$$(l-1)\,\delta\varepsilon^* x = (k+1)\left(\sum_{i=1}^{l-1}(-1)^{i-1}\xi^i\otimes\xi^1\wedge\ldots\wedge\hat{\xi}^i\wedge\ldots\wedge\xi^l\otimes(\xi^l)^k\otimes c\right).$$

Therefore,
$$(l\varepsilon^*\delta-(l-1)\delta\varepsilon^*)x=(-1)^{l-1}\eta x.$$

Then (ii) follows by linearity. That the diagram restricts as asserted is obvious. Q.e.d.

We may now state and prove the main theorems of this section.

THEOREM 1.4. *If*
$$\|\delta x\|^2\geqslant c(1,1)\,\|x\|^2 \quad \textit{for all } x\in T^*\otimes g_1^0\cap\ker\delta^*,$$
then
$$\|\delta x\|^2\geqslant c(1,k)\,\|x\|^2 \quad \textit{for all } x\in T^*\otimes g_k^0\cap\ker\delta^*,$$
where
$$c(1,k)=\left(1-\frac{(k-1)^2}{4\,c(1,k-1)}\right)k^2.$$

In particular, if $c(1,1)=\frac{1}{2}$, *then* $c(1,k)=\frac{1}{2}k^2$.

Remark. The recursive relationship of the c's indicates the naturality of the $\frac{1}{2}$ in the δ-estimate. It is the least constant which guarantees that $c(1,k)>0$ for all k.

Proof of Theorem 1.4. Consider the following diagram, which is part of diagram (1.2_{k+1}), for $k\geqslant 2$:

$$\begin{tikzcd}
 & 0 \arrow[r] & T^*\otimes g_k^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & T^*\otimes T^*\otimes g_{k-1}^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & T^*\otimes\Lambda^2T^*\otimes g_{k-2}^0 \arrow[d, "\varepsilon"] \\
0 \arrow[r] & g_{k+1}^0 \arrow[r, "\delta"] & T^*\otimes g_k^0 \arrow[r, "\delta"] & \Lambda^2T^*\otimes g_{k-1}^0 \arrow[r, "\delta"] & \Lambda^3T^*\otimes g_{k-2}^0
\end{tikzcd} \tag{1.3_{k+1}}$$

The proof is by induction on k. The theorem is trivially true if $k=1$. Assume that $k\geqslant 2$ and that the theorem is true for $k-1$. Let $x\in T^*\otimes g_k^0\cap\ker\delta^*$ be a non-zero eigenvector of $\delta^*\delta$ with eigenvalue λ. Since
$$g_{k+1}^0\to T^*\otimes g_k^0\to\Lambda^2T^*\otimes g_{k-1}^0$$
is exact, we know that $\lambda>0$.

By Lemma 1.1, $\delta^*\delta\varepsilon^*x/k^2=\varepsilon^*x$. Because δ^* and ε^* commute and $\delta^*\delta x=\lambda x$, $\delta^*\varepsilon^*\delta x/\lambda$ also equals ε^*x. Therefore $\delta^*(\delta\varepsilon^*x/k^2-\varepsilon^*\delta x/\lambda)=0$. This has two consequences. First,
$$\langle(\delta\varepsilon^*x/k^2-\varepsilon^*\delta x/\lambda),\,\delta\varepsilon^*x/k^2\rangle=0$$
so
$$\|\delta\varepsilon^*x/k^2\|^2=(1/\lambda k^2)\langle\varepsilon^*\delta x,\,\delta\varepsilon^*x\rangle. \tag{1.4}$$

Second, we can apply the inductive hypothesis to obtain
$$\|\delta(\delta\varepsilon^*x/k^2-\varepsilon^*\delta x/\lambda)\|^2\geqslant c(1,k-1)\,\|\delta\varepsilon^*x/k^2-\varepsilon^*\delta x/\lambda\|^2.$$

Since $\delta^2=0$,

$$\|\delta\varepsilon^*\delta x/\lambda\|^2 \geqslant c(1, k-1)\{\|\delta\varepsilon^* x/k^2\|^2-(2/\lambda k^2)\langle\delta\varepsilon^* x, \varepsilon^*\delta x\rangle+\|\varepsilon^*\delta x/\lambda\|^2\}.$$

By applying (1.4) to this we obtain

$$\|\delta\varepsilon^*\delta x/\lambda\|^2 \geqslant c(1, k-1)\{\|\varepsilon^*\delta x/\lambda\|^2-\|\delta\varepsilon^* x/k^2\|^2\}. \tag{1.5}$$

Since $\|\delta\varepsilon^* x/k^2\|^2=k^{-4}\langle\delta^*\delta\varepsilon^* x, \varepsilon^* x\rangle$, Lemma 1.1 implies that

$$\|\delta\varepsilon^* x/k^2\|^2 = k^{-2}\|\varepsilon^* x\|^2.$$

Since $\varepsilon\varepsilon^*$ on $T^*\otimes g_k^0$ is the identity map,

$$\|\delta\varepsilon^* x/k^2\|^2 = k^{-2}\|x\|^2,$$

and

$$\|\varepsilon^*\delta x/\lambda\| = \lambda^{-2}\langle\delta^*\delta x, x\rangle = \lambda^{-1}\|x\|^2.$$

With these substitutions, (1.5) becomes

$$\|\delta\varepsilon^*\delta x/\lambda\|^2 \geqslant c(1,k-1)(\lambda^{-1}-k^{-2})\|x\|^2. \tag{1.6}$$

Now apply Lemma 1.3 to δx with $l=3$. Then,

$$3\varepsilon^*\delta^2 x = 0 = 2\delta\varepsilon^*\delta x+(-1)^2\eta\delta x,$$

so

$$4\|\delta\varepsilon^*\delta x\|^2 = \|\eta\delta x\|^2 = \lambda(k-1)^2\|x\|^2.$$

Now (1.6) becomes

$$((k-1)^2/4\lambda)\|x\|^2 \geqslant c(1, k-1)(\lambda^{-1}-k^{-2})\|x\|^2. \tag{1.7}$$

Therefore,

$$\lambda \geqslant \left(\frac{(k-1)^2}{4\,c(1, k-1)}\right) k^2 = c(1, k).$$

Since all eigenvalues are bounded below by $c(1, k)$, we have that $\|\delta x\|^2 \geqslant c(1, k)\|x\|^2$.

This completes the inductive step, so the theorem follows. Q.e.d.

THEOREM 1.5. *If*

$$\|\delta x\|^2 \geqslant c(1, k)\|x\|^2 \quad \textit{for all } x\in T^*\otimes g_k^0\cap\ker\delta^*,$$

then

$$\|\delta x\|^2 \geqslant c(l, k)\|x\|^2 \quad \textit{for all } x\in\Lambda^l T^*\otimes g_k^0\cap\ker\delta^*,$$

where

$$c(l, k) = \frac{l^2 c(l-1, k)-k^2}{(l+1)(l-1)},$$

In particular, if $c(1, k)=\frac{1}{2}k^2$, *then* $c(l, k)=k^2/(l+1)$.

Remark.. Once again we see that $\frac{1}{2}$ is the lowest possible value for $c(1, 1)$ such that $c(l, k)>0$ for all l and k.

Proof of Theorem 1.5. Consider the following portion of diagram (1.2_{l+k}).

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & T^*\otimes\Lambda^{l-1}T^*\otimes g_k^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & T^*\otimes\Lambda^l T^*\otimes g_{k-1}^0 \arrow[d, "\varepsilon"] \\
\cdots \arrow[r, "\delta"] & \Lambda^l T^*\otimes g_k^0 \arrow[r, "\delta"] \arrow[d] & \Lambda^{l+1}T^*\otimes g_{k-1}^0 \arrow[d] \\
& 0 & 0
\end{tikzcd}$$

The proof is by induction on l. If $l=1$, the theorem is trivially true. Assume that $l\geqslant 2$ and that the theorem is true for $l-1$. Let $x\in\Lambda^l T^*\otimes g_k^0\cap\ker\delta^*$ be a non-zero eigenvector of $\delta^*\delta$ with eigenvalue λ. Then $\|\delta x\|^2=\lambda\|x\|^2$.

Since $\delta^*x=0$, by the commutation of δ^* and ε^* we have $\delta^*\varepsilon^*x=0$. Therefore we may apply the inductive hypothesis to obtain

$$\|\delta\varepsilon^*x\|^2\geqslant c(l-1,k)\,\|\varepsilon^*x\|^2=c(l-1,k)\,\|x\|^2. \tag{1.8}$$

By Lemma 1.3 (ii), we have

$$(l+1)\varepsilon^*\delta x=l\delta\varepsilon^*x+(-1)^l\eta x,$$

so
$$l^2\|\delta\varepsilon^*x\|^2=(l+1)^2\|\varepsilon^*\delta x\|^2-2(l+1)(-1)^l\langle\varepsilon^*\delta x,\eta x\rangle+\|\eta x\|^2.$$

By Lemma 1.3 (i),
$$\langle\varepsilon^*\delta x,\eta x\rangle=(-1)^l\|\delta x\|^2,$$

so
$$l^2\|\delta\varepsilon^*x\|^2=(l+1)^2\lambda\|x\|^2-2(l+1)\lambda\|x\|^2+k^2\|x\|^2.$$

Thus, (1.8) becomes
$$((l+1)(l-1)\lambda+k^2)\,\|x\|^2\geqslant l^2c(l-1,k)\,\|x\|^2$$

so
$$\lambda\geqslant\frac{l^2c(l-1,k)-k^2}{(l+1)(l-1)}=c(l,k).$$

Since the eigenvalues of $\delta^*\delta$ are bounded below by $c(l,k)$, we have that $\|\delta x\|^2\geqslant c(l,k)\,\|x\|^2$.

This completes the inductive step, and the theorem follows. Q.e.d.

Theorems 1.4 and 1.5 together imply

THEOREM 1.6. *If $\mathcal{D}$ satisfies the δ-estimate, then $\|\delta x\|^2\geqslant(k^2/l)\,\|x\|^2$ for all $x\in\Lambda^{l-1}T^*\otimes g_k^0\cap\ker\delta^*$.*

Remark. These estimates are the best possible, since there exist operators such that the above inequality actually becomes an equality for some x. The $\bar\partial$ operator in several complex variables is such an operator.

THEOREM 1.7. *If $\mathcal{D}$ satisfies the δ-estimate, then g_{k+1} is involutive.*

Proof. If the sequence

$$\Lambda^{l-2}T^*\otimes g^0_{k+1} \xrightarrow{\delta} \Lambda^{l-1}T^*\otimes g^0_k \xrightarrow{\delta} \Lambda^l T^*\otimes g^0_{k-1}$$

were not exact for some $l\geq 2$ and $k\geq 1$, there would exist an $x\in\Lambda^{l-1}T^*\otimes g^0_k$ such that $x\neq 0$, $\delta x=0$, and $\delta^*x=0$. But this contradicts Theorem 1.6. Thus g^0_1 is involutive. This is equivalent to g_{k+1} being involutive. The proof of this consists in showing that the obvious map from $\Lambda^l T^*\otimes S^{k+m}T^*\otimes E$ to $\Lambda^l T^*\otimes S^m T^*\otimes S^k T^*\otimes E$ gives an isomorphism between the δ-complexes for g_{k+1} and g^0_1. We omit the details. Q.e.d.

Remark. The hypothesis of Theorem 1.7 is stronger than necessary. Recall that the metrics on T^* and E have been given, and that all other metrics have been induced by these. Suppose that for every positive ε it is possible to find metrics on T^* and E such that with these metrics

$$\|\delta x\|^2\geq(\tfrac{1}{2}-\varepsilon)\|x\|^2 \quad \text{for all } x\in T^*\otimes g^0_1\cap\ker\delta^*.$$

Then g_{k+1} is involutive.

To see that

$$\Lambda^{l-k}T\otimes g^0_{k+1}\to\Lambda^l T^*\otimes g^0_k\to\Lambda^{l+1}T^*\otimes g^0_{k-1}$$

is exact, note that $c(l, k)$ is a continuous function of $c(1, 1)$. By choosing ε small enough so that $c(l, k)>0$ when $c(1, 1)=\frac{1}{2}-\varepsilon$, and by choosing metrics on T^* and E such that

$$\|\delta x\|^2\geq(\tfrac{1}{2}-\varepsilon)\|x\|^2 \quad \text{for all } x\in T^*\otimes g^0_1\cap\ker\delta^*$$

we can make $\ker\delta^*\cap\ker\delta=0$, so that the sequence is exact. Since this can be done for all l and k, g^0_1 is involutive, so g_{k+1} is involutive.

Thus we are led to conjecture that the converse is true; i.e., if g_{k+1} is involutive, then for any $\varepsilon>0$ there exist metrics on T^* and E such that

$$\|\delta x\|^2\geq(\tfrac{1}{2}-\varepsilon)\|x\|^2 \quad \text{for all } x\in T^*\otimes g^0_1\cap\ker\delta^*.$$

This conjecture is true for W. J. Sweeney's example of an involutive differential operator with a noncompact Dirichlet norm [9]. We know that this operator cannot satisfy the δ-estimate, since the results of Chapter III would then imply that the Dirichlet norm is compact. But the Sweeney operator does satisfy the conclusion of the conjecture.

2. The δ-estimate on g^l_1, $l>0$

THEOREM 2.1. *If $\mathcal{D}$ satisfies the δ-estimate, then*

$$\|\delta x\|^2\geq\tfrac{1}{2}\|x\|^2 \quad \textit{for all } x\in T^*\otimes g^l_1\cap\ker\delta^*,$$

where $g^l_1=\ker\sigma(D^l)$.

Proof. The theorem is a consequence of two lemmas.

LEMMA 2.2. *If $\mathcal{D}$ satisfies the δ-estimate and if $x \in g^{l+1}$ is an eigenvector of $\sigma_1(D^l)\sigma_1(D^l)^*$ with eigenvalue λ, then $\lambda \geqslant 2$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\varepsilon"] & \vdots \arrow[d, "\varepsilon"] \arrow[d, "x_8"' near end, "x_3" near end] & \vdots \arrow[d, "\sigma_2(D^{l-1})"] \arrow[d, "x_9"' near end, "x_4" near end] & \\
\cdots \arrow[r, "\delta"] & S^2T^*\otimes\Lambda^{l-1}T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] \arrow[d, "x_6" near end] & S^2T^*\otimes\Lambda^{l}T^*\otimes C^0 \arrow[r, "\pi"] \arrow[d, "\varepsilon"] \arrow[d, "x_7"' near end, "x_1" near end] & S^2T^*\otimes C^l \arrow[r] \arrow[d, "\sigma_1(D^l)"] \arrow[d, "x"' near end] & 0 \\
\cdots \arrow[r, "\delta"] & T^*\otimes\Lambda^{l}T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] \arrow[d, "x_5" near end] & T^*\otimes\Lambda^{l+1}T^*\otimes C^0 \arrow[r, "\pi"] \arrow[d, "\varepsilon"] \arrow[d, "x_2" near end] & T^*\otimes C^{l+1} \arrow[r] \arrow[d, "\sigma(D^{l+1})"] & 0 \\
\cdots \arrow[r, "\delta"] & \Lambda^{l+1}T^*\otimes g_1^0 \arrow[r, "\delta"] \arrow[d] & \Lambda^{l+2}T^*\otimes C^0 \arrow[r, "\pi"] \arrow[d] & C^{l+2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.1$_l$}$$

This diagram is a part of (1.2_{l+2}) which has been extended by adding the cokernels of the final δ maps. The first two columns are exact as usual, and the rows are exact since g_1^0 is involutive. A diagram chase shows that the last column is exact. The diagram commutes because (1.2_{l+2}) commutes and because the σ maps are induced by the ε maps.

Recall that the metrics on the spaces C^l were defined so that the π maps are projections (i.e., $\pi\pi^*$ is the identity). Recall also that ε^* from the bottom row to the middle is an isometry because $\varepsilon\varepsilon^* = I$ on the bottom row. We shall denote all of the σ maps by σ. The context will always make clear which one is meant. Now we proceed with the proof.

Suppose that $x \in g_1^{l+1}$ (i.e., $x \in T^*\otimes C^{l+1}$ and $\sigma x = 0$) and that $\sigma\sigma^* x = \lambda x$. Then by exactness of the last column, $\lambda > 0$. Let $x_4 = \sigma^* x/\lambda$. Then $\sigma x_4 = x$ and $\|x_4\|^2 = (1/\lambda)\|x\|^2$. Furthermore, x_4 is the element of least norm in $S^2T^*\otimes C^l$ which maps onto x. For, assume $\sigma x_4' = x$. Then by the exactness of the column, $x_4' - x_4 = \sigma y$. Thus

$$\langle x_4' - x_4, x_4\rangle = \langle \sigma y, \sigma^* x/\lambda\rangle = \langle \sigma\sigma y, x/\lambda\rangle = 0$$

so

$$\|x_4'\|^2 = \|x_4' - x_4\|^2 + \|x_4\|^2 \geqslant \|x_4\|^2.$$

Thus, x_4 has the least norm.

Let $x_1 = \pi^* x$. Then $\|x_1\|^2 = \|x\|^2$. Let $x_2 = \varepsilon x_1$ and $x_3 = \varepsilon^* x_1$. Since, as we proved in Chapter I, Corollary 7.2, $\varepsilon^*\varepsilon + \varepsilon\varepsilon^*(l+1)/2(l+2) = I$ on $T^*\otimes\Lambda^{l+1}T^*\otimes C^0$, we have

$$\|x_1\|^2 = \|x_2\|^2 + (l+1)/2(l+2)\,\|x_3\|^2.$$

Since $x_3 = \varepsilon^* \pi^* x = \pi^* \sigma^* x$, we have that $\|x_3\|^2 = \|\sigma^* x\|^2 = \lambda \|x\|^2$. Thus

$$\|x\|^2 = \left(1 - \frac{\lambda(l+1)}{2(l+2)}\right) \|x\|^2. \tag{2.2}$$

By commutativity, $\pi x_2 = \sigma x = 0$, so there is an $x_5 \in \Lambda^{l+1} T^* \otimes g_1^0$ such that $\delta x_5 = x_2$. Choose x_5 to have minimal norm, so $\delta^* x_5 = 0$. From Theorem 1.6 we conclude that $\|\delta x_5\|^2 \geqslant (1/l+2) \|x_5\|^2$ or since $\delta x_5 = x_2$,

$$\|x_5\|^2 \leqslant (l+2) \|x_2\|^2 \tag{2.3}$$

Let $x_6 = \varepsilon^* x_5$, and $x_7 = \delta x_6$. Then $\|x_6\|^2 = \|x_5\|^2$, and $\varepsilon(x_1 - x_7) = 0$. Since $I = \varepsilon^* \varepsilon + \varepsilon \varepsilon^* (l+1)/2(l+2)$ on $T^* \otimes \Lambda^{l+1} T^* \otimes C^0$, if $x_8 = ((l+1)/2(l+2))\, \varepsilon^*(x_1 - x_7)$, we have $\varepsilon x_8 = x_1 - x_7$. Then $\|x_8\|^2 = \|x_1 - x_7\|^2 (l+1)/2(l+2)$. Since $\langle x_7, x_1 \rangle = \langle \pi \delta x_6, x \rangle = 0$, we have that $\|x_1 - x_7\|^2 = \|x_1\|^2 + \|x_7\|^2$. Thus

$$\|x_8\|^2 = (\|x\|^2 + \|x_7\|^2)(l+1)/2(l+2). \tag{2.4}$$

Let $x_9 = \pi x_8$. Then $\|x_9\|^2 \leqslant \|x_8\|^2$ and $\sigma x_9 = \sigma \pi x_8 = \pi \varepsilon x_8 = \pi(x_1 - x_7) = x$. We have already proved that x_4 is the element of least norm in $S^2 T^* \otimes C^l$ which maps into x, so therefore $\|x_4\|^2 \leqslant \|x_9\|^2 \leqslant \|x_8\|^2$. Thus,

$$(1/\lambda) \|x\|^2 = \|x_4\|^2 \leqslant (\|x\|^2 + \|x_7\|^2)(l+1)/2(l+2),$$

or

$$\|x_7\|^2 \geqslant ((2(l+2)/\lambda(l+1)) - 1) \|x\|^2. \tag{2.5}$$

Apply Lemma 1.3 to the rectangle

$$\begin{tikzcd}
T^* \otimes \Lambda^l T^* \otimes g_1^0 \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & T^* \otimes \Lambda^{l+1} T^* \otimes C^0 \arrow[d, "\varepsilon"] \\
\Lambda^{l+1} T^* \otimes g_1^0 \arrow[r, "\delta"] & \Lambda^{l+2} T^* \otimes C^0
\end{tikzcd}$$

to obtain

$$(l+2) \varepsilon^* \delta x_5 = (l+1) \delta \varepsilon^* x_5 + (-1)^{l+1} \eta x_5.$$

Using that $\|\varepsilon^* \delta x_5\|^2 = \|x_2\|^2$, that $\|\eta x_5\|^2 = \|x_5\|^2$, that $\langle \varepsilon^* \delta x_5, \eta x_5 \rangle = (-1)^{l+1} \|x_2\|^2$, and that $\delta \varepsilon^* x_5 = x_7$, we can conclude that

$$(l+1)^2 \|x_7\|^2 = l(l+2) \|x_2\|^2 + \|x_5\|^2.$$

Combined with (2.5), this becomes

$$l(l+2) \|x_2\|^2 + \|x_5\|^2 \geqslant ((2(l+1)(l+2)/\lambda) - (l+1)^2) \|x\|^2. \tag{2.6}$$

With (2.3) and (2.2) this gives

$$\tfrac{1}{2}(2(l+1)(l+2) - \lambda(l+1)^2) \geqslant (1/\lambda)(2(l+1)(l+2) - \lambda(l+1)^2).$$

If $\lambda < 2$ we face a contradiction, because then $2(l+1)(l+2) - \lambda(l+1)^2 > 0$, so that $\frac{1}{2} \geqslant 1/\lambda$ or $\lambda \geqslant 2$. Thus we conclude that $\lambda \geqslant 2$. Q.e.d.

Lemma 2.2 shows that the δ-estimate implies a restriction on the spectrum of $\sigma_1(D^l)\sigma_1(D^l)^*$. Lemma 2.3 will show that this spectrum is essentially the spectrum of $\delta^*\delta$ on $T^*\otimes g_1^l \cap \ker \delta^*$ multiplied by 4.

LEMMA 2.3. *Assume that g_{k+1} is involutive. Let Λ be the spectrum of $\sigma_1(D^l)\sigma_1(D^l)^*$ on $T^*\otimes C^l$, and let Λ' be the spectrum of $\delta^*\delta$ on $T^*\otimes g_1^l$. Then the map $\lambda\to\lambda/4$ is a one-to-one correspondence between $\Lambda\cap(0,4)$ and $\Lambda'\cap(0,1)$, where (a,b) is the open interval between a and b.*

Proof. Observe that 4 is the maximum possible eigenvalue of $\sigma_1(D^l)\sigma_1(D^l)^*$, since $\sigma_1(D^l)$ is a restriction of δ: $S^2T^*\otimes C^0\to T^*\otimes T^*\otimes C^0$, and the eigenvalue of $\delta\delta^*$ on $S^2T^*\otimes C^0$ is 4. Similarly, 1 is the maximum possible eigenvalue of $\delta^*\delta$ on $T^*\otimes g_1^l\cap\ker\delta^*$.

Now consider the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_2^l & \xrightarrow{i} & S^2T^*\otimes C^l & \xrightarrow{\sigma_1(D^l)} & T^*\otimes C^{l+1} & \xrightarrow{\sigma(D^{l+1})} & C^{l+2}\to 0\\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \longrightarrow & T^*\otimes g_1^l & \xrightarrow{i} & T^*\otimes T^*\otimes C^l & \xrightarrow{1\otimes\sigma(D^l)} & T^*\otimes C^{l+1} & \longrightarrow & 0\\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \longrightarrow & \Lambda^2T^*\otimes C^l & \xrightarrow{i} & \Lambda^2T^*\otimes C^l & \longrightarrow & 0 & &
\end{array}$$

which restricts to

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g_2^l & \xrightarrow{i} & S^2T^*\otimes C^l & \xrightarrow{\sigma_1(D^l)} & g_1^{l+1} & \longrightarrow & 0\\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \longrightarrow & T^*\otimes g_1^l & \xrightarrow{i} & T^*\otimes T^*\otimes C^l & \xrightarrow{1\otimes\sigma(D^l)} & T^*\otimes C^{l+1} & \longrightarrow & 0\\
 & & \downarrow\delta & & \downarrow\delta & & & & \\
0 & \longrightarrow & \Lambda^2T^*\otimes C^l & \xrightarrow{i} & \Lambda^2T^*\otimes C^l & \longrightarrow & 0 & &
\end{array} \qquad (27_l)$$

The diagram is commutative and, since g_{k+1} is involutive, is exact. Also, $1\otimes\sigma(D^l)$ is a projection. To see this, recall that π: $\Lambda^{l+1}T^*\otimes C^0\to C^{l+1}$ is a projection. Then by considering the adjoint of part of diagram (2.1_{l-1})

$$\begin{array}{ccccc}
T^*\otimes\Lambda^lT^*\otimes C^0 & \xleftarrow{\pi^*} & T^*\otimes C^l & \longleftarrow & 0\\
\uparrow\varepsilon^* & & \uparrow\sigma(D^l)^* & & \\
\Lambda^{l+1}T^*\otimes C^0 & \xleftarrow{\pi^*} & C^{l+1} & \longleftarrow & 0\\
\uparrow & & \uparrow & & \\
0 & & 0 & &
\end{array}$$

we see that if $x\in C^{l+1}$, then $\|x\| = \|\pi^*x\| = \|\varepsilon^*\pi^*x\| = \|\pi^*\sigma^*x\| = \|\sigma^*x\|$. Thus $\sigma(D^l)^*$ is an isometry, so $\sigma(D^l)$ is a projection. Therefore $1\otimes\sigma(D^l)$ is a projection. We shall denote both $\sigma_1(D^l)$ and $1\otimes\sigma(D^l)$ by σ.

Let $\lambda\in\Lambda\cap(0, 4)$. Then there is a non-zero $x\in g_1^{l+1}$ such that $\sigma\sigma^*x=\lambda x$. Thus $\sigma\delta\sigma^*x = \delta\sigma\sigma^*x=\lambda\delta x$. Then $\sigma(\sigma^*\lambda\delta x-\delta\sigma^*x)=0$ since $\sigma\sigma^*$ is the identity on $T^*\otimes C^{l+1}$. Therefore $\sigma^*\lambda\delta x-\delta\sigma^*x=ii^*(\sigma^*\lambda\delta x-\delta\sigma^*x)$, or since $i^*\sigma^*=0$,

$$\lambda\sigma^*\delta x-\delta\sigma^*x = -ii^*\delta\sigma^*x. \tag{2.8}$$

Now we claim that $i^*\delta\sigma^*x$ is an eigenvector of $\delta^*\delta$. Since i^*i is the identity on $\Lambda^2T^*\otimes C^l$,

$$\delta^*\delta i^*\delta\sigma^*x = \delta^*i^*i\delta i^*\delta\sigma^*x = \delta^*i^*\delta ii^*\delta\sigma^*x.$$

By (2.8), this becomes

$$-\delta^*i^*\delta(\lambda\sigma^*\delta x-\delta\sigma^*x) = -\lambda\delta^*i^*\delta\sigma^*\delta x = -\lambda i^*\delta^*\delta\sigma^*\delta x.$$

Since on $T^*\otimes T^*\otimes C^l$, the identity map is $\delta^*\delta+\frac{1}{4}\delta\delta^*$, this becomes

$$-\lambda i^*(I-\tfrac{1}{4}\delta\delta^*)\sigma^*\delta x = (\lambda/4)\,i^*\delta\delta^*\sigma^*\delta x = (\lambda/4)\,i^*\delta\sigma^*\delta^*\delta x = (\lambda/4)\,i^*\delta\sigma^*x.$$

Therefore $i^*\delta\sigma^*x$ is an eigenvector of $\delta^*\delta$ with an eigenvalue $\lambda/4$. We must prove that $i^*\delta\sigma^*x$ is not zero. Suppose that $i^*\delta\sigma^*x=0$. Then $\sigma^*\sigma\delta\sigma^*x=\delta\sigma^*x$. But we know also that $\sigma^*\sigma\delta\sigma^*x = \sigma^*\delta\sigma\sigma^*x=\lambda\sigma^*\delta x$. Therefore $\delta\sigma^*x=\lambda\sigma^*\delta x$. This, with commutation, implies that $\lambda\delta\sigma^*x = \lambda\delta\sigma^*\delta^*\delta x=\lambda\delta\delta^*\sigma^*\delta x=\delta\delta^*\delta\sigma^*x$. But $\delta^*\delta=4I$ on $S^2T^*\otimes C^0$. so $\lambda\delta\sigma^*x=4\delta\sigma^*x$. We assumed that $x\neq 0$, so $\delta\sigma^*x\neq 0$. Therefore $\lambda=4$. This contradicts the assumption that $\lambda\in\Lambda\cap(0, 4)$. Thus $i^*\delta\sigma^*x\neq 0$, and $\lambda/4\in\Lambda'\cap(0, 1)$. A nearly identical diagram chase will show that if x is a non-zero eigenvector for $\delta^*\delta$ in $T^*\otimes g_1^l$, with an eigenvalue $\lambda/4\in\Lambda'\cap(0\ 1)$, then $\sigma\delta^*ix$ is a non-zero eigenvector of $\sigma\sigma^*$ with an eigenvalue λ.

Now we return to the proof of Theorem 2.1. We must prove that if $\mathcal{D}$ satisfies the δ-estimate, then

$$\|\delta x\|^2 \geq \tfrac{1}{2}\|x\|^2 \quad \text{for all } x\in T^*\otimes g_1^l\cap\ker\delta^*.$$

By Lemma 2.2, we know that 2 is a lower bound for the eigenvalues of $\sigma_1(D^l)\sigma_1(D^l)^*$. If there were a non-zero eigenvalue of $\delta^*\delta$ less than $\frac{1}{2}$, then by Lemma 2.3 there would be a non-zero eigenvalue of $\sigma_1(D^l)\sigma_1(D^l)^*$ less than 2. Thus the minimum positive eigenvalue of $\delta^*\delta$ is bounded below by $\frac{1}{2}$. But since the sequence

$$0\to g_2^l\to T^*\otimes g_1^l\to\Lambda^2T^*\otimes C^l$$

is exact, we know also that zero cannot be an eigenvalue for $\delta^*\delta$ on $T^*\otimes g_1^l\cap\ker\delta^*$. Therefore

$$\|\delta x\|^2 \geq \tfrac{1}{2}\|x\|^2 \quad \text{for all } x\in T^*\otimes g_1^l\cap\ker\delta^*. \quad \text{Q.e.d.}$$

We conclude this section by deriving an equivalent form of the δ-estimate which will be used in Chapter III to prove the Kohn–Nirenberg estimate.

THEOREM 2.4. *If $\mathcal{D}$ satisfies the δ-estimate, then for each $l \geqslant 1$,*

$$\langle S(1 \otimes \sigma(D^{l-1}))^* v, (1 \otimes \sigma(D^{l-1}))^* v\rangle \geqslant 0$$

for all $v \in g_1^l$, where

$$S\colon T^* \otimes T^* \otimes C^{l-1} \to T^* \otimes T^* \otimes C^{l-1}$$

is the switching operator, the linear map generated by $S(dx^i \otimes dx^j \otimes c) = dx^j \otimes dx^i \otimes c$.

Proof. By Lemma 2.2, if $\mathcal{D}$ satisfies the δ-estimate then

$$\|\sigma_1(D^{l-1})^* x\|^2 \geqslant 2\|x\|^2 \quad \text{for all } x \in g_1^l.$$

Now consider again the exact commutative diagram (2.7_{l-1}):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & g_2^{l-1} \arrow[r, "i"] \arrow[d, "\delta"] & S^2T^* \otimes C^{l-1} \arrow[r, "\sigma_1(D^{l-1})"] \arrow[d, "\delta"] & g_1^l \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & T^* \otimes g_1^{l-1} \arrow[r, "i"] \arrow[d, "\delta"] & T^* \otimes T^* \otimes C^{l-1} \arrow[r, "1 \otimes \sigma(D^{l-1})"] \arrow[d, "\delta"] & T^* \otimes C^l \arrow[r] & 0 \\
0 \arrow[r] & \Lambda^2T^* \otimes C^{l-1} \arrow[r, "i"] & \Lambda^2T^* \otimes C^{l-1} \arrow[r] \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

We claim that $S = \frac{1}{4}\delta\delta^* - \delta^*\delta$ since, by the calculations of the eigenvalues of the formal Laplacian,

$$\delta^*\delta(\xi^1 \otimes \xi^2 \otimes c) = \tfrac{1}{2}(\xi^1 \otimes \xi^2 \otimes c - \xi^2 \otimes \xi^1 \otimes c)$$

and

$$\tfrac{1}{4}\delta\delta^*(\xi^1 \otimes \xi^2 \otimes c) = \tfrac{1}{2}(\xi^1 \otimes \xi^2 \otimes c + \xi^2 \otimes \xi^1 \otimes c)$$

so

$$(\tfrac{1}{4}\delta\delta^* - \delta^*\delta)(\xi^1 \otimes \xi^2 \otimes c) = \xi^2 \otimes \xi^1 \otimes c.$$

Then
$$\begin{aligned}\langle S(1 \otimes \sigma(D^{l-1}))^* v, (1 \otimes \sigma(D^{l-1}))^* v\rangle &= \langle(\tfrac{1}{4}\delta\delta^* - \delta^*\delta)(1 \otimes \sigma(D^{l-1}))^* v, 1 \otimes \sigma(D^{l-1}))^* v\rangle \\ &= \tfrac{1}{4}\|\delta^*(1 \otimes \sigma(D^{l-1}))^* v\|^2 - \|\delta(1 \otimes \sigma(D^{l-1}))^* v\|^2.\end{aligned}$$

Since $\frac{1}{4}\delta\delta^* + \delta^*\delta = I$ on $T^* \otimes T^* \otimes C^{l-1}$,

$$\tfrac{1}{4}\|\delta^*(1 \otimes \sigma(D^{l-1}))^* v\|^2 + \|\delta(1 \otimes \sigma(D^{l-1}))^* v\|^2 = \|(1 \otimes \sigma(D^{l-1}))^* v\|^2.$$

We noted in Section II.2 that $(1 \otimes \sigma(D^{l-1}))^*$ is an isometry, so $\|(1 \otimes \sigma(D^{l-1}))^* v\|^2 = \|v\|^2$. From this we see that

$$\langle S(1\otimes\sigma(D^{l-1}))^*v,\ (1\otimes\sigma(D^{l-1}))^*v\rangle \geqslant 0$$

if and only if
$$\tfrac{1}{2}\|\delta^*(1\otimes\sigma(D^{l-1}))^*v\|^2 \geqslant \|v\|^2,$$

where $v\in g_1^l$ is considered to lie in $T^*\otimes C^l$ on the second line of the diagram.

By commutativity of the diagram, $\delta^*(1\otimes\sigma(D^{l-1}))^*v=\sigma_1(D^{l-1})^*\delta^*v$. Now $\delta^*\colon T^*\otimes C^l\to g_l^1$ is simply projection onto g_1^l, so since $v\in g_1^l$, we have $v=\delta^*v$. Thus

$$\langle S(1\otimes\sigma(D^{l-1}))^*v,\ (1\otimes\sigma(D^{l-1}))^*v\rangle \geqslant 0$$

if and only if
$$\|\sigma_1(D^{l-1})^*v\|^2 \geqslant 2\|v\|^2,$$

which was established at the beginning of the proof. Q.e.d.

3. The δ-estimate on g_{k+1}

We have defined the δ-estimate for $\mathcal{D}$ on the sequence

$$0\to g_2^0\to T^*\otimes g_1^0\to \Lambda^2T^*\otimes C^0.$$

This was convenient for obtaining the estimates on the sequences

$$0\to g_2^l\to T^*\otimes g_1^l\to \Lambda^2T^*\otimes C^l$$

which we shall use in the next chapter, but it has the fault that it requires that we construct the Spencer sequence in order to see whether $\mathcal{D}$ satisfies the δ-estimate. In this section we shall prove that the δ-estimate is equivalent to an estimate on

$$0\to g_{k+2}\to T^*\otimes g_{k+1}\to \Lambda^2T^*\otimes g_k$$

so that whether $\mathcal{D}$ satisfies the δ-estimate or not can be verified without constructing the Spencer sequence.

THEOREM 3.1. *The following estimates are equivalent:*

(i) $\|\delta x\|^2\geqslant c(k+1)^2\|x\|^2$ *for all* $x\in T^*\otimes g_{k+1}\cap\ker\delta^*$,

and

(ii) $\|\delta x\|^2\geqslant c\|x\|^2$ *for all* $x\in T^*\otimes g_1^0\cap\delta^*$.

In particular, an operator $\mathcal{D}$ of order k satisfies the δ-estimate if and only if

$$\|\delta x\|^2\geqslant \tfrac{1}{2}(k+1)^2\|x\|^2 \quad \textit{for all } x\in T^*\otimes g_{k+1}\cap\ker\delta^*.$$

Proof. Recall that g_k may be regarded as a subspace of $R_k=C^0$. Since $\sigma(D^0)\colon T^*\otimes C^0\to C^1$ is the natural projection onto $C^1=T^*\otimes C^0/\delta g_{k+1}$, we see that $g_1^0=\delta g_{k+1}$. Then $g_2^0=T^*\otimes g_1^0\cap S^2T^*\otimes g_k$.

Now consider the following diagram which is an analogue of (1.2_{k+2}):

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & S^2T^*\otimes g_k \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \cdots \\
 & 0 \arrow[r] & T^*\otimes g_{k+1} \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & T^*\otimes T^*\otimes g_k \arrow[r, "\delta"] \arrow[d, "\varepsilon"] & \cdots \\
0 \arrow[r] & g_{k+2} \arrow[r, "\delta"] & T^*\otimes g_{k+1} \arrow[r, "\delta"] \arrow[d] & \Lambda^2T^*\otimes g_k \arrow[r, "\delta"] \arrow[d] & \cdots \\
 & & 0 & 0 &
\end{tikzcd}
$$

Since $g_1^0=\delta g_{k+1}$, we can collapse this diagram to obtain

$$
\begin{tikzcd}
0 \arrow[dr] & & & & & & 0 \arrow[dl] \\
0 \arrow[r, dashed] & g_{k+2} \arrow[rrrr, dashed, "f"] \arrow[dr, "\delta"] & & & & g_2^0 \arrow[r, dashed] \arrow[dl, "\varepsilon"] & 0 \\
 & 0 \arrow[r] & T^*\otimes g_{k+1} \arrow[rr, "1\otimes\delta"] \arrow[dr, "\delta"] & & T^*\otimes g_1^0 \arrow[r] \arrow[dl, "\varepsilon"] & 0 & \\
 & & & \Lambda^2T^*\otimes g_k & & &
\end{tikzcd}
$$

A diagram chase shows that the dashed arrows may be added to make the diagram commutative and exact. Observe that the ε-sequence has become the usual δ-sequence and that by Lemma 1.1, $(1\otimes\delta)^*(1\otimes\delta)$ is $(k+1)^2I$.

If $x\in T^*\otimes g_{k+1}\cap\ker\delta^*$, then $0=\delta^*(1\otimes\delta)^*(1\otimes\delta)x=f^*\varepsilon^*(1\otimes\delta)x$. Since f^* is injective, $\varepsilon^*(1\otimes\delta)x=0$, so $1\otimes\delta\colon T^*\otimes g_{k+1}\cap\ker\delta^*\to T^*\otimes g_1^0\cap\ker\varepsilon^*$. Dimension considerations show that this must be an isomorphism.

Now we show that (ii) implies (i). If $x\in T^*\otimes g_{k+1}\cap\ker\delta^*$, then $(1\otimes\delta)x\in T^*g_1^0\cap\ker\varepsilon^*$ and $\|(1\otimes\delta)x\|^2=(k+1)^2\|x\|^2$. Then by (ii), $\|\delta x\|^2=\|\varepsilon(1\otimes\delta)x\|^2\geqslant c\|(1\otimes\delta)x\|^2$, so $\|\delta x\|^2\geqslant c(k+1)^2\|x\|^2$. The demonstration of the converse is similar. Q.e.d.

Remark 3.2. As we have observed, $g_1^0\cong g_{k+1}$, but δ acts on them differently since g_{k+1} is considered to be contained in $S^{k+1}T^*\otimes E$ and g_1^0 is considered to be contained in $T^*\otimes C^0$. It is for this reason that the δ's differ by the constant $(k+1)$.

4. Examples

Example 4.1. The gradient operator d. Define $d\colon \mathbf{C}\to T^*$, where $\mathbf{C}$ is the one dimensional complex trivial bundle over X, by $df=\sum \partial f/\partial x_i dx^i$. Then $\sigma(d)\colon T^*\otimes\mathbf{C}\to T^*$ is the identity. Therefore $g_1=0$ and the δ-estimate holds vacuously. The Spencer sequence in this case is the de Rham sequence.

Example 4.2. Covariant derivatives. A first order linear operator ∇: $\underline{E} \to \underline{T^* \otimes E}$ is a covariant derivative if $\sigma(\nabla)$: $T^* \otimes E \to T^* \otimes E$ is the identity. Again $g_1 = 0$ so the δ-estimate holds vacuously.

Example 4.3. The Cauchy–Riemann operator. If X is a complex manifold, then $T^* = H \otimes \bar{H}$ where H (resp. $\bar{H}$) is the space of holomorphic (resp. anti-holomorphic) cotangent vectors. In terms of local coordinates $\{z_i\}$, H is generated by $\{dz_i\}$ and $\bar{H}$ by $\{d\bar{z}_i\}$. Define $\bar{\partial}$: $\underline{C} \to \underline{T}^*$ by

$$\bar{\partial} f = \Sigma\, \partial f / \partial \bar{z}_i d\bar{z}_i.$$

Then $\sigma(\bar{\partial})$: $T^* \otimes C \to T^*$ is the identity on $\bar{H}$ and is zero on H. If we choose any metric on T^* such that $H \perp \bar{H}$, we have that $g_1 = H$, $T^* \otimes g_1 = H \otimes H \oplus \bar{H} \otimes H$, $g_2 = S^2 H$, and

$$\ker \delta^* = \{x \mid x \in T^* \otimes g_1 \text{ and } x \perp g_2\} = \Lambda^2 H \oplus \bar{H} \otimes H.$$

We must prove that if $x \in \Lambda^2 H \oplus \bar{H} \otimes H$, then $\|\delta x\|^2 = \langle \delta^* \delta x, x\rangle \geqslant \frac{1}{2}\|x\|$. Since $\Lambda^2 H$ and $\bar{H} \otimes H$ are invariant under $\delta^* \delta$, we may treat the two cases separately. If $x \in \Lambda^2 H$, then $\delta^* \delta x = x$, so $\|\delta x\|^2 = \|x\|^2$. If $x \in \bar{H} \otimes H$, then we may write $x = a_{ij} dz^i \otimes d\bar{z}^j$. Then $\delta^* \delta$ is the orthoprojection of $\frac{1}{2}\Sigma\, a_{ij}(dz^i \otimes d\bar{z}^j - d\bar{z}^j \otimes dz^i)$ onto $\bar{H} \otimes H$, so $\langle \delta^* \delta x, x\rangle = \frac{1}{2}\Sigma |a_{ij}|^2 = \frac{1}{2}\|x\|^2$. Therefore the δ-estimate holds.

Example 4.4. The operator $\partial\bar{\partial}$. Define ∂: $\underline{C} \to \underline{T}^*$ by $\partial f = \Sigma\, \partial f / \partial z_i dz_i$. Then $\sigma(\partial)$: $T^* \otimes C \to T^*$ is the identity on H and is zero on $\bar{H}$. If $\Lambda^{p,q}$ denotes the space of exterior vectors of type (p, q), then ∂ and $\bar{\partial}$ extend to give an operator

$$\partial\bar{\partial}: \underline{\Lambda}^{p,q} \to \underline{\Lambda}^{p+1,\, q+1},$$

and the symbol of $\partial\bar{\partial}$: $\underline{\Lambda}^{0,0} \to \underline{\Lambda}^{1,1}$ is given by the composition

$$S^2 T^* \otimes \Lambda^{0,0} \xrightarrow{\sigma_1(\bar{\partial})} T^* \otimes \Lambda^{0,1} \xrightarrow{\sigma(\partial)} \Lambda^{1,1}.$$

We claim that $\partial\bar{\partial}$ satisfies the δ-estimate.

By Theorem 3.1, the δ-estimate holds for $\partial\bar{\partial}$ if and only if

$$\|\delta x\|^2 \geqslant (9/2)\|x\|^2 \quad \text{for all } x \in T^* \otimes g_3 \cap \ker \delta^*$$

in the sequence

$$0 \to g_4 \to T^* \otimes g_3 \to \Lambda^2 T^* \otimes g_2. \tag{4.1}$$

However, by the proof of Theorem 1.4, it suffices to prove that

$$\|\delta x\|^2 \geqslant 2\|x\|^2 \quad \text{for all } x \in T^* \otimes g_2 \cap \ker \delta^* \tag{4.2}$$

in the sequence

$$0 \longrightarrow g_3 \xrightarrow{\delta} T^* \otimes g_2 \xrightarrow{\delta} \Lambda^2 T^* \otimes T^* \otimes \Lambda^{0,0} \tag{4.3}$$

since (4.1) is the δ-sequence obtained from (4.3) by prolongation.

If we let $S^{p,q}$ be the symmetric tensors of type (p,q), we have that $g_2 = \ker \sigma(\partial\bar\partial)$: $S^2T^* \to \Lambda^{1,1}$ is $S^{2,0} \oplus S^{0,2}$ since $\sigma(\partial\bar\partial)(dz_i^2) = \sigma(\partial\bar\partial)(d\bar z_j^2) = 0$, and $\sigma(\partial\bar\partial)(dz_i \circ d\bar z_j) = dz_i \wedge d\bar z_j$. Hence $g_3 = T^* \otimes g_2 \cap S^3T^* = S^{3,0} \oplus S^{0,3}$. Then the sequence (4.3) becomes the orthogonal sum of

$$\begin{array}{c}
0 \longrightarrow S^{3,0} \xrightarrow{\delta} H \otimes S^{2,0} \xrightarrow{\delta} \Lambda^{2,0} \otimes H \\
0 \longrightarrow S^{0,3} \xrightarrow{\delta} \bar H \otimes S^{0,2} \xrightarrow{\delta} \Lambda^{0,2} \otimes \bar H \\
0 \xrightarrow{\delta} H \otimes S^{0,2} \xrightarrow{\delta} \Lambda^{1,1} \otimes \bar H \\
0 \xrightarrow{\delta} \bar H \otimes S^{2,0} \xrightarrow{\delta} \Lambda^{1,1} \otimes H
\end{array} \tag{4.4}$$

Thus to verify that the estimate (4.2) holds on (4.3), it suffices to prove that it holds for each sequence in (4.4). The first two are trivial δ-sequences, and Theorem I.7.1, which states the eigenvalues of the formal Laplacian, applies to give

$$\|\delta x\|^2 = 4\|x\|^2 \quad \text{for all } x \in \ker \delta^*.$$

The third and fourth sequences are similar, so we consider only the third. Let $x = \xi \otimes \bar\zeta^2 \in H \otimes S^{0,2}$, where ξ and $\bar\zeta$ are unit vectors. Then $\|x\|^2 = 1$ and $\delta x = 2\xi \wedge \bar\zeta \otimes \bar\zeta$, and $\|\delta x\|^2 = 4\|\xi \wedge \bar\zeta\|^2 = 4\|\tfrac{1}{2}(\xi \otimes \bar\zeta - \bar\zeta \otimes \xi)\|^2 = 2$. Thus $\|\delta x\|^2 = 2\|x\|^2$. One can extend this argument to prove that for all $x \in H \otimes S^{0,2}$, $\|\delta x\|^2 = 2\|x\|^2$. Therefore, for all four sequences in (4.4), the estimate (4.2) holds. Therefore, the operator $\partial\bar\partial$ satisfies the δ-estimate.

III. The D-Neumann problem

0. Introduction

We shall use the estimates of Theorem II.2.4 to prove that the Kohn–Nirenberg estimate holds, and therefore that the D-Neumann problem is solvable. The rest of the proof of the exactness of the Spencer sequence follows quite easily.

Recall that the Kohn–Nirenberg estimate states that there exists a constant such that

$${}^{\partial\Omega}\|u\|^2 \leqslant c\{{}^{\Omega}\|(D^{l-1})^* u\|^2 + {}^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|u\|^2\}$$

holds if u is in the domain of $(D^{l-1})^*$; that is, if $u \in \Gamma(\Omega, C^l)$ and ${}^{\Omega}\langle D^{l-1}v, u\rangle = {}^{\Omega}\langle v, (D^{l-1})^* u\rangle$ for all $v \in \Gamma(\Omega, C^{l-1})$. The only obstacle to proving this is that the integral of a certain bilinear form may be negative. The estimate of Theorem II.2.4,

$$\langle S(1 \otimes \sigma(D^{l-1}))^* v, \ (1 \otimes \sigma(D^{l-1}))^* v\rangle \geqslant 0$$

for all $v \in g_1^l$, however, is sufficient to guarantee that this will not happen.

In Section 1 we discuss the D-Neumann problem. In Section 2, we state the theorem which says that the Kohn–Nirenberg estimate and Sweeney's estimate hold on certain domains for elliptic operators which satisfy the δ-estimate, and we prove the theorem in the special case of constant coefficients. Finally, in Section 3 we prove the main proposition, Theorem III.3.1, which asserts that the Spencer sequence is exact.

1. The D-Neumann problem (see Sweeney [8])

For notational convenience we shall consider the graded bundle $C = \oplus C^l (l \geqslant 0)$ and we shall write D for the graded operator which is D^l on $\Gamma(\Omega, C^l)$ and D^* for the graded operator defined by $(D^*)^l = (D^{l-1})^*$, where $(D^{l-1})^*$ is the formal adjoint of D^{l-1}. Then D is graded with degree 1 while D^* is graded with degree -1. Since D^* is not defined on C^0 it is convenient to consider $C' = \oplus C^l (l \geqslant 1)$.

Let Ω be a compact manifold-with-boundary contained in X. Define the Neumann space $\mathbf{N} = \oplus \mathbf{N}^l (l \geqslant 1)$ to be the graded space of all sections $u \in \Gamma(\Omega, C')$ satisfying the boundary conditions

$$^{\Omega}\langle Dv, u\rangle = {}^{\Omega}\langle v, D^*u\rangle \quad \text{for all } v \in \Gamma(\Omega, C) \tag{1.1}$$

$$^{\Omega}\langle Dv, Du\rangle = {}^{\Omega}\langle v, D^*Du\rangle \quad \text{for all } v \in \Gamma(\Omega, C'). \tag{1.2}$$

Thus $u \in \mathbf{N}$ if and only if both u and Du are in the domain of D^*. Define the harmonic space $\mathbf{H} = \oplus H^l (l \geqslant 1)$ to be the kernel in $\mathbf{N}$ of $DD^* + D^*D$. Since ${}^{\Omega}\langle (DD^* + D^*D)u, u\rangle = {}^{\Omega}\| D^*u\|^2 + {}^{\Omega}\| Du\|^2$ for $u \in \mathbf{N}$, we have

$$\mathbf{H} = \{u \in \mathbf{N}: \quad Du = D^*u = 0\}.$$

Definition 1.1. We say that the D-Neumann problem is solvable for $\mathcal{D}$ on Ω if $\mathbf{H}$ is closed in $L_2(\Omega, C')$, and if there exists a bounded graded operator of degree zero $N: L_2(\Omega, C') \to L_2(\Omega, C')$ mapping $\Gamma(\Omega, C')$ into $\mathbf{N}$ such that

(i) $NH = HN = 0$, where $H: L_2(\Omega, C') \to \mathbf{H}$ is the orthogonal projection;

(ii) each $u \in \Gamma(\Omega, C')$ can be written

$$u = DD^*Nu + D^*DNu + Hu, \tag{1.3}$$

where the terms are mutually orthogonal by (1.1) and (1.2); and

(iii) $DN = ND$.

One reason for wanting to solve the D-Neumann problem is that the decomposition (1.3) gives a cochain homotopy $1 - H = D(D^*N) + (D^*N)D$, which says that if $Du = 0$, then u is cohomologous to Hu. Therefore the cohomology of the sequence

$$\Gamma(\Omega, C^0) \xrightarrow{D^0} \Gamma(\Omega, C^1) \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} \Gamma(\Omega, C^n) \longrightarrow 0$$

is isomorphic to $\mathbf{H} = \oplus \mathbf{H}^l (l \geq 1)$.

If we complete $\mathbf{N}$ abstractly to a graded Hilbert space $\mathbf{B}$ with the Dirichlet inner product $Q(u, v) = {}^{\Omega}\langle Du, Dv\rangle + {}^{\Omega}\langle D^*u, D^*v\rangle + {}^{\Omega}\langle u, v\rangle$, we may show that $\mathbf{B}$ may be considered a subset of $L_2(\Omega, C')$. Let L be the Friedrichs extension to B of $DD^* + D^*D$. Then we have

THEOREM 1.2. *Assume that* $\mathcal{D}$ *is elliptic and that the inclusion* $\mathbf{B} \to L_2(\Omega, C')$ *is compact. Then we have:*

(i) $u \in \Gamma(\Omega, C')$ *whenever* $Lu \in \Gamma(\Omega, C')$;

(ii) $\mathbf{H}$ *is finite dimensional, and the range of* L *on* $\Gamma(\Omega, C')$ *is closed; and*

(iii) *the D-Neumann problem is solvable for* $\mathcal{D}$ *on* Ω.

Proof. (See Sweeney [8].)

Remark. That (i) is true is a result of Kohn and Nirenberg [3, Theorem 3]. Our assumption that $\mathcal{D}$ is elliptic comes in via Quillen's theorem, Theorem I.5.1, which implies that $DD^* + D^*D$ is then elliptic. The Kohn–Nirenberg theorem requires that the boundary be non-characteristic for $DD^* + D^*D$.

To show that the inclusion $\mathbf{B} \to L_2(\Omega, C')$ is compact, it is sufficient to prove the Kohn–Nirenberg estimate [3], which we shall do in the next section.

2. Integration by parts

Let r be a smooth function on X such that

(i) $r(x) = 0$ if and only if $x \in \partial\Omega$

(ii) $r(x) \leq 0$ for $x \in \Omega$

(iii) $|dr| = 1$ on $\partial\Omega$.

Then $*dr$ is the volume element on $\partial\Omega$. From Stokes' theorem we get for all smooth functions f and one-forms α,

$${}^{\Omega}\langle df, \alpha\rangle = {}^{\Omega}\langle f, d^*\alpha\rangle + {}^{\partial\Omega}\langle f dr, \alpha\rangle.$$

This result extends to sections of bundles and first order operators D: $\underline{E} \to \underline{F}$ to give

$${}^{\Omega}\langle De, f\rangle = {}^{\Omega}\langle e, D^*f\rangle + {}^{\partial\Omega}\langle \sigma_{dr}(D)e, f\rangle$$

for all $e \in \Gamma(\Omega, E)$ and $f \in \Gamma(\Omega, F)$. Consequently, the condition that $u \in \Gamma(\Omega, C^l)$ be in the domain of D^* is equivalent to $\sigma_{dr}(D)^* u \equiv 0$ on $\partial\Omega$.

THEOREM 2.1. *If Ω is a sufficiently small ball in terms of some local coordinates, and if $\mathcal{D}$ is an elliptic operator which satisfies the δ-estimate, then*

(i)
$$^{\partial\Omega}\|u\|^2 \leqslant c\{^{\Omega}\|D^*u\|^2 + {}^{\Omega}\|Du\|^2 + {}^{\Omega}\|u\|\}$$

and

(ii)
$$^{\Omega}\|u\|_{\frac{1}{2}}^2 \leqslant c\{^{\Omega}\|D^*u\|_1^2 + {}^{\Omega}\|Du\|_1^2\}$$

for all $u \in \Gamma(\Omega, C')$ in the domain of D^*; i.e., for all u such that $\sigma_{dr}(D)^*u \equiv 0$ on $\partial\Omega$. In particular, if p is a point in X with a neighborhood N, there is a compact manifold-with-boundary Ω such that $p \in \Omega \subset N$ and (i) *and* (ii) *hold on Ω. Here ${}^{\Omega}\|\ \|_s$ is the Sobolev s-norm on Ω.*

Proof. This theorem is proved by Sweeney [10]. Since our concern is the function served by the δ-estimate, we shall prove only (i) under some restrictive conditions; the role of the δ-estimate in the more general case is the same. We assume that in some local coordinates $\mathcal{D}$ has constant coefficients, but we do not need to assume that Ω is small or that $\mathcal{D}$ is elliptic. In the general case, ellipticity is used to bound the derivatives of the coefficients of D^*.

Clearly we can assume that u is concentrated in $\Gamma(\Omega, C^l)$ for $l \geqslant 1$, so that $Du = D^l u$ and $D^*u = (D^{l-1})^*u$. We assume that in terms of our local coordinates,

$$D^{l-1} = \sum_{j=1}^{n} A_j \partial_j + A_0,$$

where $\partial_j = \partial/\partial x_j$, and
$$D^l = \sum B_j \partial_j + B_0,$$

where all the coefficients are constant. Then

$$(D^{l-1})^* = -\sum A_j^* \partial_j + A_0^*.$$

By Theorem II.2.4, the δ-estimate implies that

$$\langle S(1 \otimes \sigma(D^{l-1}))^* v,\ (1 \otimes \sigma(D^{l-1}))^* v\rangle \geqslant 0$$

for all $v \in g_1^l = \ker \sigma(D^l)$. Therefore we have that $\sum \langle A_j^* v_i, A_i^* v_j\rangle \geqslant 0$ for all $v_1, \dots, v_n$ in C^l satisfying $\sum B_i v_i = 0$. Hence we obtain the estimate

$$-\sum \langle A_j^* v_i, A_i^* v_j\rangle \leqslant c\|\sum B_i v_i\|^2$$

for all $v_1, \dots, v_n$ in C^l. But clearly

$$^{\Omega}\|\sum B_i \partial_i u\|^2 \leqslant 2\{^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|B_0 u\|^2\},$$

and
$$^{\Omega}\|B_0 u\|^2 \leqslant c\,^{\Omega}\|u\|^2,$$

so
$$-\sum{}^{\Omega}\langle A_j^* \partial_i u, A_i^* \partial_j u\rangle \leqslant c\{^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|u\|^2\}. \tag{2.1}$$

If u satisfies the boundary condition $\sigma_{dr}(D^{l-1})^* u \equiv 0$ on $\partial\Omega$, then u is in the domain of the adjoint of $\sum A_i \partial_i$, since this operator has the same symbol as D^{l-1}. Therefore,

$$\begin{aligned}^{\Omega}\|\sum A_i^* \partial_i u\|^2 &= -\sum{}^{\Omega}\langle A_j A_i^* \partial_j \partial_i u, u\rangle = -\sum{}^{\Omega}\langle \partial_i A_i^* \partial_j u, A_j^* u\rangle \\ &= \sum{}^{\Omega}\langle A_i^* \partial_j u, A_j^* \partial_i u\rangle - \sum{}^{\partial\Omega}\langle A_i^* \partial_j u, (\partial_i r)\, A_j^* u\rangle.\end{aligned}$$

Since
$$\sum{}^{\Omega}\|A_i^* \partial_i u\|^2 \leqslant c\{^{\Omega}\|(D^{l-1})^* u\|^2 + {}^{\Omega}\|u\|^2\},$$

by (2.1) we obtain

$$-\sum{}^{\partial\Omega}\langle A_i^* \partial_j u, (\partial_i r)\, A_j^* u\rangle \leqslant c\{^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|(D^{l-1})^* u\|^2 + {}^{\Omega}\|u\|\}. \tag{2.2}$$

The left-hand side of (2.2) is

$$-\sum{}^{\partial\Omega}\langle \partial_j((\partial_i r)\, A_i^* u), A_j^* u\rangle + \sum{}^{\partial\Omega}\langle (\partial_i \partial_j r)\, A_i^* u, A_j^* u\rangle.$$

The first term is zero, since by the boundary condition on u, we have

$$\sigma_{dr}(D^{l-1})^* u = \sum (\partial_i r)\, A_i^* u = r(x)\, K(x)$$

on a neighborhood of $\partial\Omega$ in Ω where $K(x)$ is differentiable. Differentiation yields

$$\sum \partial_j((\partial_i r)\, A_i^* u) = \sum (\partial_j r)\, K(x) + \sum r(x)\, (\partial_j K)$$

which is $\sum (\partial_j r)\, K(x)$ on $\partial\Omega$, where $r=0$. Therefore,

$$\sum{}^{\partial\Omega}\langle \partial_j((\partial_i r)\, A_i^* u), A_j^* u\rangle = \sum{}^{\partial\Omega}\langle (\partial_j r)\, K(x), A_j^* u\rangle = \sum{}^{\partial\Omega}\langle K(x), (\partial_j r)\, A_j^* u\rangle = 0$$

since $\sum (\partial_j r)\, A_j^* u \equiv 0$ on $\partial\Omega$.

If we let $Lu = \sum \langle (\partial_i \partial_j r)\, A_i^* u, A_j^* u\rangle$ be the Levi form for the problem, we have that

$$\int_{\partial\Omega} Lu \leqslant c\{^{\Omega}\|(D^{l-1})^* u\|^2 + {}^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|u\|^2\}.$$

If the Levi form is positive definite, then $\int_{\partial\Omega} Lu \geqslant c^{\partial\Omega}\|u\|^2$ and we obtain the Kohn–Nirenberg estimate

$$^{\partial\Omega}\|u\|^2 \leqslant c\{^{\Omega}\|(D^{l-1})^*\|^2 + {}^{\Omega}\|D^l u\|^2 + {}^{\Omega}\|u\|^2\}.$$

To determine under what conditions the Levi form is positive definite, consider the Hessian of Ω, a bilinear form on $T^* \otimes C^l|_{\partial\Omega}$ given by $H(u,v) = \sum \langle (\partial_i \partial_j r)\, u_i, v_j\rangle$

where $u = \sum u_i dx^i$ and $v = \sum v_i dx^i$. In terms of the Hessian, the Levi form is given by

$$Lu = H(\sigma(D^l)^* u, \sigma(D^l)^* u).$$

It is well known that the Hessian of a strictly convex domain Ω is positive definite on $T^*(\partial\Omega)$, the cotangent bundle of the boundary of Ω. Since $\sigma_{dr}(D^l)^* u = 0$, we have that $\sigma(D^l)^* u$ lies in $T^*(\partial\Omega)$, and therefore that

$$Lu = H(\sigma(D^l)^* u, \sigma(D^l)^* u) \geqslant c\|\sigma(D^l)^* u\|^2.$$

Finally since $\sigma(D^l)$: $T^* \otimes C^l \to C^{l+1}$ is surjective, the adjoint is injective, so $Lu \geqslant c\|u\|^2$ if Ω is strictly convex. Therefore the Kohn–Nirenberg estimate holds on strictly convex domains for constant coefficient operators satisfying the δ-estimate. Q.e.d.

In general, with variable coefficients, the Levi convexity condition is more complicated, as in the example of the Cauchy–Riemann operator where the required convexity is strong pseudo-convexity. Sweeney's proof of the above theorem, however, shows that in any coordinate system a sufficiently small sphere satisfies the necessary convexity conditions.

3. The exactness of the Spencer sequence

THEOREM 3.1. *If $\mathcal{D}$: $\underline{E} \to \underline{F}$ is a formally integrable elliptic differential operator which satisfies the δ-estimate, then:*

(i) *The Spencer sequence*

$$0 \longrightarrow \theta \longrightarrow \underline{C}^0 \xrightarrow{D^0} \underline{C}^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} \underline{C}^n \longrightarrow 0$$

is exact, and is a fine resolution of the sheaf θ of germs of solutions of the homogeneous equation $\mathcal{D}u = 0$. Consequently the cohomology of

$$0 \longrightarrow \Gamma(X, C^0) \xrightarrow{D^0} \Gamma(X, C^1) \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} \Gamma(X, C^n) \longrightarrow 0$$

is isomorphic to the cohomology of the manifold X with coefficients in θ.

(ii) *There exists an operator $\mathcal{D}'$: $\underline{F} \to \underline{G}$ such that the sequence*

$$0 \longrightarrow \theta \longrightarrow \underline{E} \xrightarrow{\mathcal{D}} \underline{F} \xrightarrow{\mathcal{D}'} \underline{G}$$

is exact.

Proof. Since $\mathcal{D}$ satisfies the δ-estimate, we know that g_{k+1} is involutive. Since $\mathcal{D}$ is formally integrable, we can define the Spencer sequence and the operators D^l, and we know that C^l is a bundle for $l \geqslant 0$.

To prove (i) suppose that $\underset{\sim}{f}$ is a germ of a section $\underset{\sim}{C}^l$ such that $D^l\underset{\sim}{f}=0$. Then there exists a local section f on a neighborhood N such that $D^lf=0$. Let $\Omega\subset N$ be the convex neighborhood described in Theorem 2.1. The Dirichlet norm is compact on Ω, and by Theorem 1.2, the D-Neumann problem can be solved on Ω. Therefore, an isomorphism from the cohomology of

$$\Gamma(\Omega, C^0)\xrightarrow{D^0}\Gamma(\Omega, C^1)\xrightarrow{D^1}\dots\xrightarrow{D^{n-1}}\Gamma(\Omega, C^n)\longrightarrow 0 \tag{3.1}$$

to $\mathbf{H}=\sum \mathbf{H}^l$ $(l\geqslant 1)$ is given by the homotopy operator $\mathbf{1}-\mathbf{H}=D^*ND+DD^*N$, which shows that each $u\in\Gamma(\Omega, C^l)$ satisfying $D^lu=0$ is cohomologous to Hu, and the harmonic space is finite dimensional. But by the estimate

$$^{\Omega}\|u\|^2_{\frac{1}{2}}\leqslant c\{^{\Omega}\|(D^{l-1})^*u\|^2_1+{}^{\Omega}\|D^lu\|^2_1\}$$

for all $u\in\Gamma(\Omega, C^l)$ in the domain of $(D^{l-1})^*$, it follows that the harmonic space is zero, for if $(D^{l-1})^*u=D^lu=0$, then $^{\Omega}\|u\|_{\frac{1}{2}}=0$ so $u\equiv 0$. Therefore (3.1) is exact, and there exists a section $g\in\Gamma(\Omega, C^{l-1})$ such that $D^{l-1}g=f$ on Ω. If g is the corresponding germ, $D^{l-1}\underset{\sim}{g}=\underset{\sim}{f}$, so the Spencer sequence is exact.

The sheaves $\underset{\sim}{C}^l$ are all fine sheaves, since they are sheaves of germs of differentiable sections of a vector bundle. That the cohomology of

$$0\longrightarrow\Gamma(X, C^0)\xrightarrow{D^0}\Gamma(X, C^1)\xrightarrow{D^1}\dots\xrightarrow{D^{n-1}}\Gamma(X, C^n)\longrightarrow 0$$

is the cohomology of X with coefficients in θ follows from standard sheaf theoretical arguments.

Part (ii) follows from several diagram chases which prove that there exists an operator $\mathcal{D}'$: $\Gamma(\Omega, F)\to\Gamma(\Omega, G)$ such that the cohomology of

$$\Gamma(\Omega, E)\xrightarrow{\mathcal{D}}\Gamma(\Omega, F)\xrightarrow{\mathcal{D}'}\Gamma(\Omega, G)$$

is isomorphic to $\mathbf{H}^1$. See Sweeney [8]. Q.e.d.

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