AN EXTENSION OF THE NEVANLINNA THEORY

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1. Introduction

The main purpose of this paper is to extend the theory of R. Nevanlinna [1] to the class $A^{-\infty}$ of functions $f(z)$ holomorphic in the unit disk $U=\{z: |z|<1\}$ and satisfying the condition

$$|f(z)| \leqslant C_f(1-|z|)^{-n_f} \quad (z \in U), \tag{1.1}$$

and to the corresponding class $\mathfrak{N}$ of meromorphic functions $h(z)$,

$$h(z)=\frac{g(z)}{f(z)} \quad (f, g \in A^{-\infty}). \tag{1.2}$$

For functions belonging to these classes we obtain a complete description of zeros (and poles) as well as a generalization of the notion of boundary measure. In our case the boundary measure turns out to be what we call a *premeasure of bounded $\varkappa$-variation*. Although lacking many good properties of a regular boundary measure in the classical factorization theory of R. Nevanlinna for functions of bounded characteristic, this premeasure nevertheless generates a regular measure of bounded variation on the so-called *Carleson sets*: *i.e.*, on those closed sets $F \subset \partial U$ of Lebesgue measure zero for which

$$\hat{\varkappa}(F)=\sum_{\nu} \frac{|I_\nu|}{2\pi}\left(\log \frac{2\pi}{|I_\nu|}+1\right)<\infty, \tag{1.3}$$

$|I_\nu|$ being the angular lengths of the complementary arcs of F. This regular measure is called the *$\varkappa$-singular part* of the corresponding premeasure. In another paper to follow soon we intend to show that these $\varkappa$-singular measures together with zero sets completely describe all the closed ideals (invariant subspaces for the operator of multiplication by z)

(1) AMS (MOS) subject classification (1970). Primary 30A08, 30A70, 31A10, 31A20.

of the topological algebra $A^{-\infty}$, roughly in the same manner as the invariant subspaces of the H^2 space are described in the classical theory of A. Beurling [2].

Closely related to the classes $A^{-\infty}$, $\mathfrak{N}$ is the class $\mathfrak{H}^+$ of harmonic functions $u(z)$, $u(0)=0$, such that

$$-\infty < u(z) \leqslant c_u \log \frac{1}{1-|z|} \quad (z \in U), \tag{1.4}$$

and a larger class $\mathfrak{H} = \mathfrak{H}^+ - \mathfrak{H}^+$. For functions belonging to $\mathfrak{H}$ we obtain a representation in the form of a generalized Poisson integral

$$u(z) = \frac{1}{2\pi} \int_{\partial U} P(\zeta, z)\, \mu(|d\zeta|), \tag{1.5}$$

$P(\zeta, z) = \operatorname{Re} (\zeta + z)/(\zeta - z)(\zeta \in \partial U, z \in U)$ being the Poisson kernel and $\mu(|d\zeta|)$ a *premeasure* defined *only on arcs* $I \subset \partial U$ and having *bounded* $\varkappa$*-variation* ($\varkappa$ for Carleson):

$$\varkappa \operatorname{Var}(\mu) = \sup_F \frac{\sum_\nu |\mu(I_\nu)|}{\hat{\varkappa}(F)} < \infty, \tag{1.6}$$

sup taken over all finite $F \subset \partial U$, $\{I_\nu\}$ being the complementary intervals of F. This $\varkappa$-variation plays essentially the same role as the usual variation

$$\operatorname*{Var}_{0<t<2\pi} \left\{ \int_0^t u(re^{i\theta})\, d\theta \right\} = \int_0^{2\pi} |u(re^{i\theta})|\, d\theta \quad (0 < r < 1), \tag{1.7}$$

in the classical theory for the class h^1 of harmonic functions which are differences of two positive harmonic functions. It is well known that the uniform boundedness of (1.7) is necessary and sufficient for a harmonic function $u(z)$ to belong to h^1. We get an analogous result for $\mathfrak{H}$ in terms of $\varkappa$-variations, as well as a corresponding result for meromorphic functions of the class $\mathfrak{N}$.

Note that our results concerning the distribution of zeros for the class $A^{-\infty}$ have many points in common with a study of zero sets for Bergman classes of functions conducted recently by C. A. Horowitz [3]. In particular, what we call the *standard* or *Horowitz distribution of zeros* (see no. 3.6) is essentially the same as that of the function

$$f(z) = \prod_{k=1}^{\infty} (1 + a z^{2^k}) \quad (a > 1), \tag{1.8}$$

examined by C. A. Horowitz. On the other hand, we have come to the conclusion that a very far reaching generalization of Nevanlinna's theory due to M. M. Djrbashian [4, 5] could hardly be applied to the kind of problems we are concerned with.

The author wishes to thank L. Carleson, M. M. Djrbashian, P. Duren, W. H. Fuchs, W. K. Hayman, C. A. Horowitz, J.-P. Kahane, Y. Katznelson, L. A. Rubel, H. S. Shapiro, A. L. Shields and others for valuable discussions.

2. General definitions and notations

$A^{-n}(n>0)$ is the class of functions $f(z)$ holomorphic in U and satisfying the condition

$$|f(z)| \leqslant C_f(1-|z|)^{-n} \quad (z \in U). \tag{2.1}$$

If provided with the norm $\|f\|_{-n} = \min C_f$, A^{-n} becomes a Banach space. A_0^{-n} is a subspace of A^{-n} consisting of those elements f for which

$$\lim_{|z|\to 1} \{(1-|z|)^n |f(z)|\} = 0. \tag{2.2}$$

A_0^{-n} is separable (in contrast to A^{-n}).

$A^{-\infty} = \bigcup_{n>0} A^{-n}$ consists of all the functions $f(z)$ holomorphic in U and satisfying condition (1.1); in other words, every element $f \in A^{-\infty}$ has the form

$$f(z) = \sum_0^\infty a_\nu z^\nu$$

with $a_\nu = O(\nu^n)(\nu \to \infty)$ for *some* n. $A^{-\infty}$ becomes a topological space (indeed, a topological algebra with the usual operation of multiplication) if provided with the following set of neighbourhoods of its zero element:

$$V(\{n_\nu\}, \{\varepsilon_\nu\}) = \bigcup_\nu S(n_\nu, \varepsilon_\nu),$$

$\{n_\nu\}$. $\{\varepsilon_\nu\}$ $(\nu = 1, 2, \ldots)$ being arbitrary sequences $(n_\nu \uparrow \infty, \varepsilon_\nu > 0)$ and $S(n, \varepsilon)$ the ε-ball in A^{-n}:

$$S(n, \varepsilon) = \{f\colon f \in A^{-n}, \ \|f\|_{-n} < \varepsilon\}.$$

The *sequential* convergence $f_\nu \to f$ in $A^{-\infty}$ means that all the f_ν belong *to the same* A^{-n} (with some $n>0$) and $\|f-f\|_{-n} \to 0$.

A^∞ is the dual of $A^{-\infty}$; it consists of all the functions $F(z)$ holomorphic in U and infinitely differentiable in $\bar{U}$, i.e.

$$F(z) = \sum_0^\infty b_\nu z^\nu.$$

with $b_\nu = o(\nu^{-n})(\nu\to\infty)$ for *every* $n>0$. The linear functionals in $A^{-\infty}$ are given by the formula

$$F(f)=\sum_0^\infty \bar{b}_\nu a_\nu = \frac{1}{2\pi}\lim_{r\to 1-0}\int_0^{2\pi}\bar{F}(re^{i\theta})\,f(re^{i\theta})\,d\theta, \tag{2.3}$$

(see [6]).

$\mathfrak{N}$ is the class of meromorphic functions having the form (1.2).

$\mathfrak{H}^+$ is the class of harmonic functions $u(z)(z\in U)$ satisfying the conditions

(i) $u(0)=0$;

(ii) $-\infty < u(z) \leqslant c_u \log \dfrac{1}{1-|z|}$. $\quad$ (2.4)

$\mathfrak{H}=\mathfrak{H}^+-\mathfrak{H}^+$, i.e. each $u(z)\in\mathfrak{H}$ has the form $u=u_1-u_2$ with $u_1,\ u_2\in\mathfrak{H}^+$.

$\mathfrak{K}$ is the set of all open, closed and half-closed arcs of the circumference ∂U, including all the single points, ∂U itself and $\varnothing$.

For every $I\in\mathfrak{K}$ put

$$\varkappa(I)=\frac{|I|}{2\pi}\left(\log\frac{2\pi}{|I|}+1\right), \tag{2.5}$$

$|I|$ being the angular length of I; if I is a single point or $\varnothing$ put $\varkappa(I)=0$. Obviously $0\leqslant\varkappa(I)\leqslant 1=\varkappa(\partial U)$.

A function μ: $\mathfrak{K}\to\mathbf{R}$ is called a *premeasure* if

(i) $\mu(I_1\cup I_2)=\mu(I_1)+\mu(I_2)$ for all $I_1,\ I_2\in\mathfrak{K}$ such that $I_1\cup I_2\in\mathfrak{K}$, $I_1\cap I_2=\varnothing$;

(ii) $\mu(\partial U)=0$;

(iii) $\lim_{\nu\to\infty}\mu(I_\nu)=0$ for every sequence $\{I_\nu\}_1^\infty(I_\nu\in\mathfrak{K})$ such that $I_1\supset I_2\supset \ldots,\ \cap I_\nu=\varnothing$.

With every premeasure μ a function $\hat{\mu}(\theta)=\mu(I_\theta)(0<\theta\leqslant 2\pi)$ can be associated with $I_\theta=\{\zeta\colon \zeta\in\partial U,\ 0\leqslant\arg\zeta<\theta\}$.

Thus a one-to-one correspondence is established between all premeasures and all real functions $\hat{\mu}(\theta)(0<\theta\leqslant 2\pi)$ satisfying the following conditions:

(i) $\hat{\mu}(\theta-0)(0<\theta\leqslant 2\pi)$ and $\hat{\mu}(\theta+0)(0\leqslant\theta<2\pi)$ exist;

(ii) $\hat{\mu}(\theta-0)=\hat{\mu}(\theta)(0<\theta\leqslant 2\pi)$;

(iii) $\hat{\mu}(2\pi)=0$.

Obviously, $\hat{\mu}(\theta)$ has at most a countable set of points of discontinuity, all of them of the first kind (jumps).

A premeasure μ (and the associated function) is said to be *$\varkappa$-bounded from above* if there is a $C>0$ such that

$$\mu(I) \leqslant C\varkappa(I) \quad (\forall\, I \in \mathfrak{R}). \tag{2.6}$$

A premeasure μ (and the associated function) is said to be of *bounded $\varkappa$-variation* if there is a $C>0$ such that for every finite set $\{I_\nu\}$, $I_\nu \in \mathfrak{R}$, $\bigcup_\nu I_\nu = \partial U$, $I_{\nu_1} \cap I_{\nu_2} = \emptyset$ $(\nu_1 \neq \nu_2)$

$$\sum_\nu |\mu(I_\nu)| \leqslant C \sum_\nu \varkappa(I_\nu). \tag{2.7}$$

$C_0 = \min C$ is called the *$\varkappa$-variation* of μ: $C_0 = \varkappa \operatorname{Var} \mu$.

A set $\emptyset \neq F \subset \partial U$ is called a *Carleson set* if

(i) F is closed and of Lebesgue measure 0;

(ii) $\hat{\varkappa}(F) = \sum_\nu \varkappa(I_\nu) < \infty$ $\qquad$ (2.8)

(see also (1.3)). $\hat{\varkappa}(F)$ will be called the *Carleson characteristic* of F.

The distance $d(\zeta_1, \zeta_2)$ between two points on ∂U is determined by the shorter arc:

$$d(\zeta_1, \zeta_2) = \frac{1}{\pi} \min\left\{\arg\frac{\zeta_2}{\zeta_1}, \arg\frac{\zeta_1}{\zeta_2}\right\},$$

so that the distance between diametrically opposite points is 1. The distance between a point $\zeta \in \partial U$ and a set $F \subset \partial U$ is

$$d(\zeta, F) = \inf_{\zeta' \in F} d(\zeta, \zeta').$$

For every Carleson set F

$$\hat{\varkappa}(F) = \frac{1}{2\pi}\int_{\partial U} |\log d(\zeta, F)| \cdot |d\zeta|, \tag{2.9}$$

which is easily verified; therefore $F_1 \subset F_2$ implies $\hat{\varkappa}(F_1) \leqslant \hat{\varkappa}(F_2)$.

Let F be a Carleson set, $q \geqslant 1$, $0 < a < 1$ be some constants. Put

$$G_{F;q,a} = \left\{z : z \in \bar{U},\ 1 - |z| \geqslant a d^q\left(\frac{z}{|z|}, F\right)\right\} \cup \{0\}. \tag{2.10}$$

Let $\alpha = \{\alpha_\nu\}$ be a (finite or infinite) sequence of points in U, $0 \neq |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots < 1$, and F a Carleson set. Put

$$\sigma_\alpha(F) = \sigma_\alpha(F; q, a) = \sum_{\alpha_\nu \in G_{F;q,a}} \log\frac{1}{|\alpha_\nu|}. \tag{2.11}$$

Let $f(z)$ be meromorphic in U and $\{\alpha_\nu\}(0 \neq |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots)$ be its zeros repeated according to their multiplicities; then the sequence $\alpha = \{\alpha_\nu\}$ is called the *zero set* of f. In the same manner the *pole set* $\beta = \{\beta_\nu\}$ of a function is defined.

3. Zero sets for classes A^{-n}, $A^{-\infty}$

3.1. The main theorem

Definition 1. For $n > 0$, $q \geqslant 1$, $0 < a < 1$, $\alpha = \{\alpha_\nu\}$ put

$$m_\alpha = m_\alpha(n; q, a) = \inf_F \{n\hat{\varkappa}(F) - \sigma_\alpha(F; q, a)\}, \tag{3.1.1}$$

inf being taken over all the finite sets $\emptyset \neq F \subset \partial U$ or (what is equivalent) over all the Carleson sets F.

Definition 2. A sequence $\alpha = \{\alpha_\nu\}$ is said to satisfy condition $(T_n)(n > 0)$ if

$$m_\alpha(n; 1, a) > -\infty \tag{3.1.2}$$

with some a, $0 < a < 1$. We shall write in this case $\alpha \in (T_n)$.

Definition 3. A sequence $\alpha = \{\alpha_\nu\}$ is said to satisfy condition (T) if it satisfies condition (T_n) with *some* $n > 0$:

$$(T) = \bigcup_{n>0} (T_n).$$

Obviously, condition (T) is equivalent to

$$\sup_F \frac{\sigma_\alpha(F; 1, a)}{\hat{\varkappa}(F)} < \infty. \tag{3.1.3}$$

THEOREM 1. *Condition (T_n) is necessary for α to be the zero set of a function $f(z) \in A^{-((n/2)-\varepsilon)}$ and sufficient for it to be the zero set of a function $f(z) \in A^{-(2n+\varepsilon)}$ ($\varepsilon > 0$ arbitrary).*

COROLLARY 1. *Condition (T) is necessary and sufficient for α to be the zero set of a function $f(z) \in A^{-\infty}$.*

COROLLARY 2. *Every subset of an $A^{-\infty}$ zero set is an $A^{-\infty}$ zero set.*

Remark 1. A simple argument shows that condition (T_n) does not in fact depend on the constant a, so that for every α it holds either for all $a \in (0; 1)$ or for none (see also below, § 3.3)

Remark 2. If F consists of N equidistant points on ∂U, then $\varkappa(F) = \log N + 1$; on the other hand, in this case $G_{F;1,a}$ contains the disk $|z| < 1-(C/N)$ with some constant $C > 0$. Thus we get from (3.1.3) the following necessary condition:

$$\sum_{|\alpha_\nu|<1-\delta} \log \frac{1}{|\alpha_\nu|} = O\left(\log \frac{1}{\delta}\right) (\delta \to 0), \tag{3.1.4}$$

which in its turn implies

$$\Sigma(1-|\alpha_\nu|)^{1+\varepsilon} < \infty \quad (\forall \varepsilon > 0). \tag{3.1.5}$$

Both the conditions are known [3, 4]. They are easily derived directly from Jensen's inequality as well.

Remark 3. If all the zeros lie on a single radius, say, (0; 1) then we have to choose for F the one-point set $\{1\}$ to get the following necessary (and sufficient) condition

$$\sum \log \frac{1}{|\alpha_\nu|} < \infty$$

(see [7]).

3.2. Proof of the necessity

Let $\alpha = \{\alpha_\nu\}$ be the zero set of a $f(z) \in A^{-n}$. Take a finite set $F \subset \partial U$ and consider two domains $G_{F;1,a}$ and $G_{F;2,b}$ with some $b \leqslant a$. Obviously, $G_{F;1,a} \subset G_{F;2,b}$. It follows from (2.1) and (2.10) that

$$|f(z)| \leqslant C_{f,b}(\min_{\zeta \in F} |z-\zeta|)^{-2n} \quad (\forall z \in G_{F;2,b}). \tag{3.2.1}$$

Let $w = w(z)$ be the function that maps conformally int $G_{F;2,b}$ onto U so that $w(0) = 0$, $w'(0) > 0$. Let $z = z(w)$ be the inverse function and F_w be the image of F under $w = w(z)$ (we assume that $w = w(z)$ is extended to G_F by continuity). Applying some well-known results about the distortion under a conformal mapping [8] we easily obtain that for each $\varepsilon > 0$ a $b(0 < b < 1)$ exists such that

$$1-\varepsilon \leqslant \frac{|z-\zeta|}{|w(z)-w(\zeta)|} \leqslant 1+\varepsilon, \quad \left|\arg\left(\frac{z}{\zeta}-1\right) - \arg\left(\frac{w(z)}{w(\zeta)}-1\right)\right| < \varepsilon \quad (\forall \zeta \in F, z \in G_{F;2,b});$$

moreover, these inequalities hold for all the finite F (even for all the Carleson F). Thus the image of $G_{F;1,a}$ is contained in some $G_{F_w;1,a-\varepsilon_1}$ with $\varepsilon_1 = \varepsilon_1(\varepsilon) \to 0$ $(\varepsilon \to 0)$. If α_w is the image of α under $w = w(z)$, then we have

$$\sigma_{\alpha_w}(F_w;\, 1, a-\varepsilon_1) \geqslant (1-\varepsilon_2)\sigma_\alpha(F;\, 1, a),$$

$$\frac{1}{2\pi}\int_{\partial U} \log^+ |f(z(w))| \cdot |dw| \leqslant \log^+ C_{f,b} + 2n[\hat{\varkappa}(F) + \varepsilon_3]$$

with ε_2, $\varepsilon_3 \to 0$ $(\varepsilon \to 0)$. Applying Jensen's inequality to $f(z(w))$ and taking inf over all the finite F we get

$$\inf_{F_d} \{2n\hat{\varkappa}(F) - (1-\varepsilon_4)\,\sigma_\alpha(F;\, 1, a)\} > -\infty$$

with $\varepsilon_4 > 0$ being arbitrarily small, and this is equivalent to α satisfying condition $(T_{2n+\varepsilon})$, which proves the necessity part of our theorem.

3.3. Some auxiliary results

LEMMA 1. *If $\alpha = \{\alpha_\nu\}$ satisfies condition (T_n), then*

$$m_\alpha(qn;\, q, b) > -\infty \quad (q \geqslant 1,\ 0 < b < 1). \tag{3.3.1}$$

Proof. For each Carleson set F a larger one $F_1 \subset \partial U$ can be found so that

$$G_{F;\,q,\,b} \subset G_{F_1;\,1,\,a}, \quad \hat{\varkappa}(F_1) \leqslant q\hat{\varkappa}(F) + C \tag{3.3.2}$$

with some constant C. To do that we have to add to F a countable set of points in each complementary interval of F so that all the angular points of $\partial G_{F_1;\,1,\,a}$ fall either on ∂U or on $\partial G_{F;\,q,\,b}$. By a straightforward calculation we then verify (3.3.2), and this together with (3.1.2) yields (3.3.1).

LEMMA 2. *Let μ_1, μ_2 be two real measures of bounded variation on ∂U, and $R_{\zeta_0} = \{z\colon z \in U,\ z/|z| = \zeta_0\}$ be the radius going from 0 to a point $\zeta_0 \in \partial U$. If for every open arc $I \subset \partial U$ with ζ_0 at its center*

$$\mu_1(I) \leqslant \mu_2(I), \tag{3.3.3}$$

then

$$\int_{\partial U} P(z, \zeta)\, \mu_1(|d\zeta|) \leqslant \int_{\partial U} P(z, \zeta)\, \mu_2(|d\zeta|) \quad (z \in R_{\zeta_0}), \tag{3.3.4}$$

$P(z, \zeta)$ being the Poisson kernel

$$P(z, \zeta) = \mathrm{Re}\,\frac{\zeta + z}{\zeta - z} \quad (\zeta \in \partial U,\ z \in U).$$

Proof. The required result is easily obtained by partial integration.

LEMMA 3. *Let*

$$0 \neq \beta \in U, \quad \zeta = \frac{\beta}{|\beta|}, \quad B(z) = \frac{\beta - z}{1 - \bar{\beta} z} \cdot \frac{|\beta|}{\beta},$$

$$S(z) = -\frac{\zeta + z}{\zeta - z} \log \frac{1}{|\beta|}. \textit{ If for some } z \in U \; 1 - |\beta| \leqslant \frac{1}{4} |\zeta - z|,$$

then
$$|\log B(z) - S(z)| \leqslant C \left(\frac{1 - |\beta|}{|\zeta - z|} \right)^2, \tag{3.3.5}$$

C being an absolute constant and the value of $\log B(z)$ *being that obtained by analytic continuation from the value* $\log |\beta| < 0$ *at* $z = 0$ *along the radius to the point* z,

Proof. We have

$$\log B(z) = \log \frac{1}{|\beta|} + \log \left(1 - \frac{\zeta - \beta}{\zeta - z} \right) - \log \left(1 - \frac{\zeta - (1/\bar{\beta})}{\zeta - z} \right).$$

Using Taylor's formula with the second-order remainder term we easily obtain the required result.

LEMMA 4. *If an arc* $I \subset \partial U$ *is divided into* N *non-overlapping arcs* $I_1, I_2, \ldots, I_N$, *then*

$$\varkappa(I) \leqslant \varkappa(I_1) + \varkappa(I_2) + \ldots + \varkappa(I_N) \leqslant \varkappa(I) + \frac{|I|}{2\pi} \log N. \tag{3.3.6}$$

This follows immediately from the fact that $\varkappa(I)$ is a concave function of $|I|$.

By this lemma, if in (3.1.1) inf is taken only over those finite F that contain some fixed point $w \in \partial U$ then m_α is changed to another value m_α^w, and the following estimate holds:

$$m_\alpha(n; q, a) \leqslant m_\alpha^w(n; q, a) = \inf_{w \in F} \{ n\hat{\varkappa}(F) - \sigma_\alpha(F; q, a) \} \leqslant m_\alpha(n; q, a) + n \log 2.$$

Condition (T_n) is therefore equivalent to

$$m_\alpha^w(n; 1, a) > -\infty, \tag{3.3.7}$$

and this, according to Lemma 1, implies

$$m_\alpha^w(qn; q, b) > -\infty \quad (q \geqslant 1, \; 0 < b < 1). \tag{3.3.8}$$

3.4. The main lemma

Definition. Let $n>0$, $q\geqslant 1$, $0<a<1$, $w\in\partial U$ be fixed. For a given *finite* sequence $\alpha=\{\alpha_\nu\}$ $(\alpha_\nu\in U)$ a non-negative measure μ on ∂U will be called *w-admissible* if

(i) $\mu(\{w\})=0$;

(ii) for each open arc $I\subset\partial U$, $w\notin I$, the following inequality holds:

$$0\leqslant\mu(I)\leqslant\sum_{\alpha_\nu\in H_I}\log\frac{1}{|\alpha_\nu|}+n\varkappa(I) \tag{3.4.1}$$

with

$$H_I=\{z\colon z\in U,\, 1-|z|<ad^q\,(z/|z|,\partial U\backslash I)\}. \tag{3.4.2}$$

The set of all w-admissible measures will be denoted by $\mathfrak{M}_\alpha^w$ or simply by $\mathfrak{M}^w$. (3.4.1) implies that $\mu(\{\zeta\})=0$ for any $\zeta\in\partial U$ and not just for $\zeta=w$.

THE MAIN LEMMA. *For any finite $\alpha=\{\zeta_\nu\}$*

$$\sup_{\mu\in\mathfrak{M}^w}\mu(\partial U)=m_\alpha^w(n;q,a)+\sum_\nu\log\frac{1}{|\alpha_\nu|}, \tag{3.4.3}$$

and there is at least one "maximal w-admissible measure" μ_0 for which

$$\mu_0(\partial U)=m_\alpha^w(n;q,a)+\sum_\nu\log\frac{1}{|\alpha_\nu|}. \tag{3.4.4}$$

Proof. Define a finite set $F_0\subset\partial U$ consisting of w and all the points $\zeta\in\partial U$ for which

$$\partial G_{\{\zeta\}}\cap\{\alpha_\nu\}\neq\varnothing,$$

and let $\{I_k\}$ be the set of complementary arcs of F_0. For each $\mu\in\mathfrak{M}^w$ let $\tilde\mu$ denote measure which has the following properties:

(i) $\tilde\mu(I_k)=\mu(I_k)$ $(\forall k)$;

(ii) $\tilde\mu$ has a constant Lebesgue density in each I_k.

In view of the concavity of $\varkappa(I)$ expressed by the first inequality (3.3.6), it is easily proved that $\mu\in\mathfrak{M}^w$ implies $\tilde\mu\in\mathfrak{M}^w$ with $\mu(\partial U)=\tilde\mu(\partial U)$. So the problem of finding a maximal w-admissible measure is in fact a finite-dimensional one with as many unknown quantities (densities) as there are points in F_0. Therefore sup in (3.4.3) is attainable, and among maximal w-admissible measures there is at least one, say, μ_0 with $\tilde\mu_0=\mu_0$.

The set of all w-admissible measures μ that have the property $\tilde\mu=\mu$ is a convex body in a finite-dimensional vector space. This body is defined by inequalities (3.4.1) with I's having their end points in F_0 and not containing w. Let $\mathfrak{A}$ be the set of all such arcs.

A finite system $\{I_s\}$ ($I_s \in \mathfrak{A}$, I_s are not necessarily all different) and a corresponding system $\{\lambda_s\}$ of non-negative numbers will be called a *w-admissible covering* if $\Sigma_s \lambda_s X_s(\zeta) \geqslant 1$ ($\forall \zeta \in \partial U$), $X_s(\zeta)$ being characteristic functions of the closed arcs $\bar{I}_s$. Using some elementary facts from the theory of convex bodies we find that

$$\sup_{\mu \in \mathfrak{M}\, W} \mu(\partial U) = \inf \left\{ \sum_s \lambda_s \left(\sum_{\alpha_\nu \in H_s} \log \frac{1}{|\alpha_\nu|} \right) + n \sum_s \lambda_s \varkappa(I_s) \right\}, \tag{3.4.5}$$

inf being taken over all the w-admissible coverings ($H_s = H_{I_s}$). Infimum in (3.4.5) is not altered if only coverings with rational λ_s are admitted; therefore our lemma reduces to the following proposition:

For each system of arcs $\{I_s\}$ ($I_s \in \mathfrak{A}$) with

$$\sum_s X_s(\zeta) \geqslant N \quad (\forall \zeta \in \partial U, N \geqslant 1 \text{ entire}) \tag{3.4.6}$$

the following inequality holds:

$$\sum_s \left(\sum_{\alpha_\nu \in H_s} \log \frac{1}{|\alpha_\nu|} \right) + n \sum_s \varkappa(I_s) \geqslant N \left(m_\alpha^w + \sum_\nu \log \frac{1}{|\alpha_\nu|} \right)$$

with equality sign attained for $N = 1$ and for the I_s that are the complementary arcs of the set $F \subset F_0$, $w \in F$, for which

$$n\hat{\varkappa}(F) - \sigma_\alpha(F; q, a) = m_\alpha^w(n; q, a).$$

This proposition is trivial for $N = 1$. The general case is proved [1] by induction which is possible owing to the fact that the point w is not contained in any of the open arcs I_s, and therefore the coverings $\{I_s\}$ do not contain cycles.

3.5. Proof of the sufficiency

Let $\alpha = \{\alpha_\nu\}$ satisfy condition (T_n) and $q > 2$ be some fixed number. We have to construct a function $f(z) \in A^{-n_1}$ ($n_1 = qn$) with zeros at α_ν. Take a *finite* part $\check{\alpha}$ of α. We will show first that an analytic function $\check{f}(z)$ exists which has the following properties:

(i) $\check{f}(\alpha_\nu) = 0 \quad (\alpha_\nu \in \check{\alpha})$;

(ii) $|\check{f}(z)| \leqslant C(1 - |z|)^{-n_1} \quad (z \in U)$;

(iii) $|\check{f}(0)| \geqslant c > 0$,

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[1] We can assume that in (3.4.6) the equality sign holds a.e.

the constants c, C being independent of a particular choice of $\tilde{\alpha}\subset\alpha$. Using a standard argument (involving classical compactness theorems for analytic functions) it will then be possible to get the required function $f(z)$ as a limit of $\tilde{f}(z)$.

Choose a point $w\in\partial U$. Let μ be a maximal w-admissible measure with respect to $\tilde{\alpha}$ and to the parameters n_1; q, a. By our main lemma

$$\mu(\partial U)-\sum_{\alpha_\nu\in\tilde{\alpha}}\log\frac{1}{|\alpha_\nu|}=m^w_{\tilde{\alpha}}(n_1;q,a)\geqslant m^w(n_1;q,a)>-\infty. \tag{3.5.1}$$

Consider the function

$$\tilde{f}(z)=\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu(|d\zeta|)\right\}\prod_{\alpha_\nu\in\tilde{\alpha}}\frac{\alpha_\nu-z}{1-\bar{\alpha}_\nu z}\cdot\frac{|\alpha_\nu|}{\alpha_\nu}. \tag{3.5.2}$$

and check it for all the above conditions.

(iii) $\tilde{f}(0)=\exp\{\mu(\partial U)+\sum_{\alpha_\nu\in\tilde{\alpha}}\log|\alpha_\nu|\}=\exp\{m^w_{\tilde{\alpha}}(n_1;q,a(\}\geqslant\exp\{m^w_{\tilde{\alpha}}(n_1;q,a)\}=c.$

(ii) Take a pont $\zeta\in\partial U$ and project every $\alpha_\nu\in\alpha$ that lies outside the domain $G_{\{\zeta\};q,a}=G_\zeta$ to the circumference ∂U. Place at the point $\zeta_\nu=\alpha_\nu/|\alpha_\nu|$ thus obtained a negative mass $m_\nu=\log|\alpha_\nu|$, and let μ_1 be the resulting measure. Put

$$S(z)=\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu_1(|d\zeta|)=\exp\left\{\sum_{\alpha_\nu\notin G_\zeta}\log|\alpha_\nu|\frac{\zeta_\nu+z}{\zeta_\nu-z}\right\}.\right.$$

By Lemma 3 we have for $z\in R_\zeta$ and $a<1/4$:

$$\left|[S(z)]^{-1}\prod_{\alpha_\nu\in\tilde{\alpha}}\frac{\alpha_\nu-z}{1-\bar{\alpha}_\nu z}\right|\leqslant\exp\left\{C\sum_{\alpha_\nu\notin G_\zeta}\frac{(1-|\alpha_\nu|)^2}{|\zeta_\nu-z|^2}\right\}\leqslant\exp\{C\sum_{\alpha_\nu\in\alpha}(1-|\alpha_\nu|)^{2-(2/q)}\}=C_1$$

with $C_1<\infty$ (see (3.1.5)). Therefore

$$|\tilde{f}(z)|\leqslant C_1|S(z)\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu(|d\zeta|)=C_1\exp\left\{\int_{\partial U}P(\zeta,z)\mu_2(|d\zeta|)\right\}\right.\quad(\mu_2=\mu+\mu_1)$$

for $z\in R_\zeta$. On the other hand, for every open arc I containing neither ζ nor w we have:

$$\mu_2(I)=\mu(I)+\mu_1(I)\leqslant\mu(I)-\sum_{\alpha_\nu\in H_I}\log\frac{1}{|\alpha_\nu|}\leqslant n_1\varkappa(I),$$

according to (3.4.1). By Lemma 4 we then have for all the open arcs $I\subset\partial U$ (with no reservations):

$$\mu_2(I)\leqslant n_1\left[\varkappa(I)+\frac{|I|}{2\pi}\log 3\right].$$

Applying Lemma 2 we get the estimate:

$$|\tilde{f}(z)| \leqslant C_2(1-|z|)^{-n_1} \quad (z \in R_\zeta)$$

with C_2 dependent neither on $\zeta \in \partial U$ nor on $\alpha \subset \alpha$.

(i) follows from (3.5.2).

Thus our theorem is proved completely.

3.6. Some properties of $A^{-\infty}$ zero sets

THEOREM 2. *Let $\alpha = \{\alpha_\nu\}$, $\alpha' = \{\alpha'_\nu\}$ be two sequences of points in U, and let for some q, $0 < q < 1$, and all the ν the following inequalities hold:*

$$|\alpha_\nu - \alpha'_\nu| \leqslant q(1-|\alpha_\nu|). \tag{3.6.1}$$

Then $\alpha \in (T)$ implies $\alpha' \in (T)$, and vice versa.

Proof. We can choose two constants a, b $(0 < b < a < 1)$ so that for every finite $F \subset \partial U$

$$\alpha_\nu \in G_{F;1,a} \Rightarrow \alpha'_\nu \in G_{F;1,b}$$

and

$$\alpha'_\nu \in G_{F;1,a} \Rightarrow \alpha_\nu \in G_{F;1,b}.$$

Therefore

$$\sigma_\nu(F;1,a) \leqslant (1-q)^{-1}\sigma_{\nu'}(F;1,b), \quad \sigma_{\nu'}(F;1,a) \leqslant (1+q)\sigma_\alpha(F;1,b).$$

This, together with (3.1.1) and (3.1.2), yields:

$$\alpha \in (T_n) \Rightarrow \alpha' \in (T_{n'}) \quad (n' = (1+q)n),$$

$$\alpha' \in (T_n) \Rightarrow \alpha \in (T_{n''}) \quad (n'' = (1-q)^{-1}n)$$

which proves our theorem.

Definition 1. To each measurable set $G \subset U$ we shall assign the number

$$\varkappa S(G) = \int_G \frac{dA}{1-|z|} \leqslant \infty \quad \left(dA = \frac{1}{2\pi}dx\,dy, \quad z = x+iy\right) \tag{3.6.2}$$

and call it the $\varkappa$-area of G.

An easy calculation shows that for domains G of the type $G_{F;1,a}$ ($F \subset \partial U$ finite)

$$|\varkappa S(G) - \hat{\varkappa}(F)| \leqslant C < \infty \quad (G = G_{F;1,a}),$$

C being independent of F. Therefore condition (T) is equivalent to the following:

$$(T') \qquad \sup_G \frac{\tilde{\sigma}_\alpha(G)}{\varkappa S(G)} < \infty, \tag{3.6.3}$$

sup being taken over all the domains G of the form $G_{F;1,a}$ ($F \subset \partial U$ finite) and

$$\tilde{\sigma}_\alpha(G) = \sum_{\alpha_\nu \in G} \log \frac{1}{|\alpha_\nu|}. \tag{3.6.4}$$

We obtain yet another form of condition (T) if we choose domains G in (3.6.3) and (3.6.4) by means of the following construction. Divide the disk U into a countable set of "cells":

1 cell of rank 0:

$$\mathfrak{C} = \{z: |z| < \tfrac{1}{2}\};$$

2 cells of rank 1:

$$\mathfrak{C}^{(0)} = \{z: \tfrac{1}{2} \leqslant |z| < \tfrac{3}{4}, 0 \leqslant \operatorname{Arg} z < \pi\}, \quad \mathfrak{C}^{(1)} = \{z: \tfrac{1}{2} \leqslant |z| < \tfrac{3}{4}, \pi \leqslant \operatorname{Arg} z < 2\};$$

4 cells of rank 2:

$$\mathfrak{C}^{(00)} = \{z: \tfrac{3}{4} \leqslant |z| < \tfrac{7}{8}, 0 \leqslant \operatorname{Arg} z < \tfrac{1}{2}\pi\}, \quad \mathfrak{C}^{(01)} = \{z: \tfrac{3}{4} \leqslant |z| < \tfrac{7}{8}, \tfrac{1}{2}\pi \leqslant \operatorname{Arg} z < \pi\},$$

$$\mathfrak{C}^{(10)} = \{z: \tfrac{3}{4} \leqslant |z| < \tfrac{7}{8}, \pi \leqslant \operatorname{Arg} z < \tfrac{3}{2}\pi\} \quad \mathfrak{C}^{(11)} = \{z: \tfrac{3}{4} \leqslant |z| < \tfrac{7}{8}, \tfrac{3}{2}\pi \leqslant \operatorname{Arg} z < 2\pi\},$$

and so on, so that $\mathfrak{C}^{(\gamma_1\gamma_2 \dots \gamma_r 0)}$, $\mathfrak{C}^{(\gamma_1\gamma_2 \dots \gamma_r 1)}$ are the two cells of rank $r+1$ $(1-2^{-r-1} \leqslant |z| < 1-2^{-r-2})$ adjacent to the cell $\mathfrak{C}^{(\gamma_1\gamma_2\dots\gamma_r)}$ of rank r. All the cells (except $\mathfrak{C}$) are thus enumerated by means of finite binary sequences $\gamma = (\gamma_1\gamma_2 \dots \gamma_r)$, $r = 1, 2, \dots$

Take an arbitrary set of cells having the same rank, and consider the smallest starlike domain composed of cells and containing the initial ones; we shall call all domains thus obtained the *canonical* ones. It is easily shown that condition (T) can be put in the following equivalent form:

$$(T'') \qquad \sup_G \frac{\tilde{\sigma}_\alpha(G)}{\varkappa S(G)} < \infty, \tag{3.6.5}$$

$\tilde{\alpha}_\alpha(G)$ being defined by (3.6.4) and sup taken over all the canonical domains.

According to Theorem 2 what matters for a sequence $\alpha = \{\alpha_\nu\}$ to satisfy (or otherwise) condition (T) is the number of zeros in each cell:

$$n_\gamma = \sum_{\alpha_\nu \in \mathfrak{C}^{(\gamma)}} 1 \quad (\gamma = (\gamma_1\gamma_2 \dots \gamma_r)). \tag{3.6.6}$$

Definition 2. A table of the form

$$\begin{array}{ccccccc} & n_{(0)} & & & & n_{(1)} & \\ & / \quad \backslash & & & & / \quad \backslash & \\ n_{(00)} & & n_{(01)} & & n_{(10)} & & n_{(11)} \\ \wedge & & \wedge & \cdots & \wedge & & \wedge \end{array}$$

with n_γ defined by (3.6.6) is called an *α-array*; each place in the table is called a *node* and will be identified with the corresponding subscript $\gamma = (\gamma_1 \dots \gamma_r)$; the number n_γ is called the *nodal number of rank* r, r being also the *rank of the node* γ: $r = r(\gamma)$.

Definition 3. *A branch* is a set of nodes of the type:

$$\mathfrak{B} = \{(\gamma_1), (\gamma_1\gamma_2), (\gamma_1\gamma_2\gamma_3), \dots, (\gamma_1\gamma_2\gamma_3 \dots \gamma_r)\}.$$

Every branch is uniquely determined by its node of the highest rank r; r is called *the rank* of the branch $\mathfrak{B}$.

Definition 4. *A tree* $\mathfrak{T}$ is the union of a set of branches having the same rank r which is called the rank of the three: $r = r(\mathfrak{T})$.

Definition 5. Let $\alpha = \{\alpha_\nu\}$ be a sequence $(\alpha_\nu \in U)$ and n_γ be its array. To every tree $\mathfrak{T}$ a number is assigned:

$$v_\alpha(\mathfrak{T}) = \sum_{\gamma \in \mathcal{T}} n_\gamma \cdot 2^{-r(\gamma)} \tag{3.6.7}$$

which is called *the α-value of the tree* $\mathfrak{T}$.

Definition 6. A sequence $\alpha = \{\alpha_\nu\}$ is said to have a *standard* or *Horowitz distribution* if all the nodal numbers n_γ of its array are equal to 1. In this case the α-value of a tree $\mathfrak{T}$ is called its *standard value*:

$$h(\mathfrak{T}) = \sum_{\gamma \in \mathfrak{T}} 2^{-r(\gamma)} = \sum_{k=1}^{r(\mathfrak{T})} 2^{-k} b_k, \tag{3.6.8}$$

b_k being the number of nodes of rank k.

The numbers b_k have the following property:

$$b_k \leqslant b_{k+1} \leqslant 2b_k \quad (k = 1, 2, \dots, r(T) - 1).$$

Definition 7. An α-array is called *bounded* if for all the trees $\mathfrak{T}$ the inequality holds

$$v_\alpha(\mathfrak{T}) \leqslant Ch(\mathfrak{T}) \tag{3.6.9}$$

with some constant C.

THEOREM 3. *A sequence $\alpha=\{\alpha_\nu\}$ $(\alpha_\nu \in U)$ is an $A^{-\infty}$ zero set iff its α-array is bounded.*

Proof. It is clear that there is a one-to-one correspondence between all the trees and all the canonical domains composed of those cells whose indices belong to the tree:

$$G_{\mathfrak{T}} = \mathfrak{C} \cup \Big(\bigcup_{\gamma \in \mathfrak{T}} \mathfrak{C}^{\gamma} \Big).$$

We have

$$ch(\mathfrak{T}) \leqslant \varkappa S(G_{\mathfrak{T}}) \leqslant Ch(\mathfrak{T}),$$

$$v_\alpha(\mathfrak{T}) \leqslant \tilde{\sigma}_\alpha(G_{\mathfrak{T}}) \leqslant C v_\alpha(\mathfrak{T})$$

for all $\mathfrak{T}$ and all α, with c, C being some absolute positive constants. Therefore the boundedness of an α-array is equivalent to α satisfying condition (T'') (see (3.6.5)), and the theorem is thus proved.

The function

$$H(z) = \prod_{k=1}^{\infty} (1 + e z^{2^k}) \tag{3.6.10}$$

examined by C. A. Horowitz [3] is the example of an $A^{-\infty}$ function with the standard distribution of zeros. Now we will consider more general functions

$$\tilde{H}(z) = \prod_{k=1}^{\infty} (1 + e z^{2^k})^{s_k} \quad (s_k \geqslant 0) \tag{3.6.11}$$

and use them to prove the following

THEOREM 4. *In order that an $A^{-\infty}$ zero set $\alpha=\{\alpha_\nu\}$ exists with prescribed moduli of the zeros,*

$$|\alpha_\nu| = \varrho_\nu \quad (0 < \varrho_1 \leqslant \varrho_2 \leqslant \ldots),$$

it is necessary and sufficient that

$$\sum_{\nu=1}^{N} \log \frac{1}{\varrho_\nu} = O\left(\log \frac{1}{1-\varrho_N}\right), \tag{3.6.12}$$

or (what is equivalent)

$$\sum_{\nu=1}^{N} \log \frac{1}{\varrho_\nu} = O(\log N). \tag{3.6.13}$$

Proof. The necessity of (3.6.12) is already proved (see (3.1.4.)). To prove the sufficiency observe first that (3.6.12) is equivalent to

$$S_r = \sum_{k=1}^{r} N_k 2^{-k} \leqslant Cr \quad (r = 1, 2, \ldots) \tag{3.6.14}$$

with

$$N_r = \sum_{1-2^{-r} \leqslant \varrho_\nu < 1-2^{-r-1}} 1.$$

Now, consider the array

$$n_\gamma = [2^{-r(\gamma)} N_{r(\gamma)}] + 1. \tag{3.6.15}$$

To prove the theorem it is sufficient to show that this array is bounded, because the total number of its zeros in each annulus $1-2^{-r} \leqslant |z| - 1 - 2^{-r-1}$ exceeds N_r. This could be done directly by checking (3.6.9), but we prefer to prove this result by actually constructing a function which has the required array of zeros. It is easily seen that the function

$$\tilde{H}(z) = \prod_{k=1}^{\infty} (1 + ez^{2^k})^{[N_k 2^{-k}]+1} \tag{3.6.16}$$

has exactly n_γ zeros in each cell of C^γ. What remains to be proved is that $\tilde{H}(z) \in A^{-\infty}$. Obviously, $|\tilde{H}(z)| \leqslant \tilde{H}(|z|)$; therefore we have to estimate $\tilde{H}(z)$ on the radius $z = \varrho (0 < \varrho < 1)$. Using partial summation and bearing in mind (3.6.14) we get

$$\begin{aligned}
\log \tilde{H}(\varrho) &\leqslant \sum_{k=1}^{\infty} (N_k 2^{-k} + 1) \log (1 + e\varrho^{2^k}) = \sum_{k=1}^{\infty} (S_k + k) [\log (1 + e\varrho^{2^k}) - \log (1 + e\varrho^{2^{k+1}})] \\
&\leqslant (C+1) \sum_{k=1}^{\infty} k [\log (1 + e\varrho^{2^k}) - \log (1 + e\varrho^{2^{k+1}})] = (C+1) \sum_{k=1}^{\infty} \log (1 + e\varrho^{2^k}) \\
&= (C+1) \log H(\varrho),
\end{aligned}$$

$H(z)$ being the Horowitz function (see (3.6.10)). Now,

$$H(\varrho) = \prod_{k=1}^{\infty} (1 + e\varrho^{2^k}) = 1 + \sum_{k=1}^{\infty} e^{s(k)} \varrho^k,$$

$s(k)$ being the sum of the digits in the binary expression of k. Clearly, $s(k) \leqslant \log_2 k + 1$, and therefore $H(z) \in A^{-\infty}$ (see als [3]).

To prove the equivalence of (3.6.12) and (3.6.13), observe that (3.6.12) implies $(\varrho_1 \varrho_2 \dots \varrho_k)^c \geqslant 1 - \varrho_k$, $k = 1, 2, \dots,$ with some $c > 0$. Therefore

$$(\varrho_1 \varrho_2 \dots \varrho_{k+1})^{-c} - (\varrho_1 \varrho_2 \dots \varrho_k)^{-c} = (1 - \varrho_{k+1}^c)(\varrho_1 \varrho_2 \dots \varrho_{k+1})^{-c} = O(1).$$

Summing up these relations from $k=1$ to $k = N-1$ we get $(\varrho_1 \varrho_2 \dots \varrho_N)^{-c} = O(N)$ which is equivalent to (3.6.13). Conversely, if (3.6.13) holds, then we have

$$N \log \frac{1}{\varrho_N} = O(\log N),\ 1 - \varrho_N = O\left(\frac{\log N}{N}\right),\ \frac{N}{\log N} = O\left(\frac{1}{1-\varrho_N}\right)$$

and, finally, $\log N = O(\log 1/(1-\varrho_N))$. This together with (3.6.13) yields (3.6.12).

4. Premeasures of bounded ϰ-variation

4.1. General properties

Definition. A sequence of premeasures $\{\mu_k\}_1^\infty$ is said to be $\varkappa$-weakly convergent to a premeasure μ ($\mu_k \xrightarrow{\varkappa w} \mu$) if

(i) μ_k have uniformly bounded $\varkappa$-variations,

$$\varkappa \operatorname{Var} \mu_k \leqslant C < \infty \quad (k = 1, 2, \ldots);$$

(ii) $\lim\limits_{k\to\infty} \hat{\mu}_k(\theta) = \hat{\mu}(\theta) \quad (0 < \theta \leqslant 2\pi)$

at every point of continuity of the associated function $\hat{\mu}(\theta)$ (for the definition of the associated function see Ch. 2).

Of course, in this case the limit premeasure μ is of bounded $\varkappa$-variation too, $\varkappa \operatorname{Var} \mu \leqslant C$.

THEOREM 1. (*Helly-type selection theorem*). *Let $\{\mu_k\}_1^\infty$ be a sequence of premeasures having uniformly bounded $\varkappa$-variations. Then there exists a subsequence $\{\mu_{k_\nu}\}$ $(k_1 < k_2 < \ldots)$ which is* $\varkappa$-weakly convergent to a premeasure μ.

We omit the proof because it runs on the same lines as that of the classical Helly selection theorem.

THEOREM 2. *If a premeasure μ is $\varkappa$-bounded from above,*

$$\mu(I) \leqslant C\varkappa(I) \quad (\forall\, I \in \mathfrak{K}),$$

then it is of bounded $\varkappa$-variation and $\varkappa \operatorname{Var} \mu \leqslant 2C$.

Proof. Let $\{I_\nu\}$ $(I_\nu \in \mathfrak{K})$ be a finite set of arcs, $\bigcup_\nu I_\nu = \partial U$, $I_{\nu_1} \cap I_{\nu_2} = \emptyset (\nu_1 \neq \nu_2)$. We have

$$\sum_\nu |\mu(I_\nu)| = \sum_\nu \max(\mu(I_\nu), 0) + \sum_\nu \max(-\mu(I_\nu), 0) = S_1 + S_2.$$

Obviously, $S_1 - S_2 = 0$ and $0 \leqslant S_1 \leqslant C\Sigma_\nu \varkappa(I_\nu)$, so $S_1 + S_2 \leqslant 2C\Sigma_\nu \varkappa(I_\nu)$, and the theorem is thus proved.

THEOREM 3. *Let μ be a premeasure, $I_0 = \{\zeta : |\zeta| = 1,\ \alpha \leqslant \operatorname{Arg} \zeta < \beta\}$ be an arc, $I_1 = \partial U \setminus I_0$. Define the premeasure σ as follows:*

$$\sigma(I) = \mu(I \cup I_1) + \frac{|I \cup I_0|}{|I_0|}\mu(I_0), \tag{4.1.1}$$

so that the associated function $\hat{\sigma}(\theta)$ is linear in the closed interval $\alpha \leqslant \theta \leqslant \beta$ and coincides with $\hat{\mu}(\theta)$ outside the open interval $\alpha < \theta < \beta$. Then

$$\varkappa \operatorname{Var} \sigma \leqslant \varkappa \operatorname{Var} \mu. \tag{4.1.2}$$

Proof. Let $C = \varkappa \operatorname{Var} \mu < \infty$, and let $\{\zeta_\nu\}_0^N$, $\zeta_N = \zeta_0$, be some points on ∂U arranged counterclockwise with first k of them belonging to $\bar{I}_0$ (and all the others lying outside $\bar{I}_0$):

$$\alpha \leqslant \theta_0 = \operatorname{Arg} \zeta_0 \leqslant \theta_1 = \operatorname{Arg} \zeta_1 \leqslant \ldots \leqslant \theta_{k-1} = \operatorname{Arg} \zeta_{k-1} \leqslant \beta.$$

Fix all the points $\{\zeta_\nu\}_k^{N-1}$ and consider the function

$$f(\theta_0, \theta_1, \ldots, \theta_{k-1}) = \sum_{\nu=0}^{N-1} |\hat{\sigma}(\zeta_{\nu+1}) - \hat{\sigma}(\zeta_\nu)| \, {}^{(1)} \tag{4.1.3}$$

in the $\varkappa$-dimensional simplex $\mathfrak{S}$:

$$\alpha \leqslant \theta_0 \leqslant \theta_1 \leqslant \ldots \leqslant \theta_{k-1} \leqslant \beta.$$

It is easily seen that this function is convex in $\mathfrak{S}$, because $\hat{\sigma}(\zeta)$ is linear in I_0. On the other hand, the function

$$\hat{\varkappa}(\theta_0, \theta_1, \ldots, \theta_{k-1}) = \hat{\varkappa}(\{\zeta_0, \zeta_1, \ldots, \zeta_N\})$$

is concave. At the vertices of $\mathfrak{S}$ (where θ_j, $j = 0, 1, \ldots, k-1$, are equal either to α or to β) we have

$$f(\theta_0, \theta_1, \ldots, \theta_{k-1}) = \sum_{j=0}^{N-1} |\hat{\mu}(\zeta_{j+1}) - \hat{\mu}(\zeta_j)| \leqslant C\hat{\varkappa}(\{\zeta_\nu\}),$$

so this inequality must hold in $\mathfrak{S}$ as well:

$$\sum_{j=1}^{N-1} |\hat{\sigma}(\zeta_{j+1}) - \hat{\sigma}(\zeta_j)| \leqslant C\hat{\varkappa}(\{\zeta_\nu\}),$$

which proves our theorem.

COROLLARY. *If μ is $\varkappa$-bounded from above,*

$$\mu(I) \leqslant C\varkappa(I) \quad (\forall I \in \mathfrak{R}),$$

then the same inequality holds for σ.

---

(1) We will write sometimes $\hat{\sigma}(\zeta) = \hat{\sigma}(e^{i\theta})$ instead of $\hat{\sigma}(\theta)$.

THEOREM 4. *Let $F=\{\zeta_\nu\}_1^N$ be a finite set of points on ∂U, $\{I_\nu\}_1^\infty$ be the complementary arcs of F and μ be a premeasure of $\varkappa$-bounded variation. Let μ_l be the measure whose associated function $\hat{\mu}_l(\zeta)$ is linear in each I_ν and coincides with $\hat{\mu}(\zeta)$ for $\zeta \in F$. Then*

$$\varkappa \operatorname{Var} \mu_l \leqslant \varkappa \operatorname{Var} \mu, \tag{4.1.4}$$

and

$$\mu_l \xrightarrow{\varkappa w} \mu \quad \text{as} \quad \max_\nu |I_\nu| \to 0.$$

This theorem is a direct consequence of Theorem 3.

4.2. The decomposition theorem

THEOREM 5. *Every premeasure μ of $\varkappa$-bounded variation,*

$$\varkappa \operatorname{Var} \mu = C < \infty,$$

is the difference of two premeasures that are $\varkappa$-bounded from above: (1)

$$\mu = \mu_1 - \mu_2, \quad \mu_j(I) \leqslant (1+\log 2)C\varkappa(I) \quad (\forall\, I \in \mathfrak{K},\, j=1,2). \tag{4.2.1}$$

Proof. Take a finite set $F \subset \partial U$ containing some fixed point w, and let μ_l be the corresponding piecewise linear measure constructed as in Theorem 4 (that is, having a constant density in each of the complementary open arcs $\{I_\nu\}$ of the set F). Now, let us first show that $\mu_l = \mu_l^{(1)} - \mu_l^{(2)}$, $\mu_l^{(1)}$ and $\mu_l^{(2)}$ having the same structure and satisfying the inequalities

$$\mu_l^{(j)}(I) \leqslant C\varkappa(I) \quad (j=1,2;\ \forall\, I \in \mathfrak{K},\, w \notin I). \tag{4.2.2}$$

Using the concavity of $\varkappa(I)$ and the piecewise linearity of $\hat{\mu}_l^{(j)}(\theta)$ we easily come to the conclusion that to ensure the inequalities (4.2.2) for all I, $w \notin I$, it is sufficient to do this only for the I's whose end points are in F. Thus the problem becomes a finite-dimensional one with a finite system of ineqequalities 4.2.2) and a system of equalities expressing the requirements that $\mu_l^{(j)}$ be additive, that $\mu_l^{(j)}(\partial U)=0$ and $\mu_l(I_\nu) = \eta_l^{(1)}(I_\nu) - \mu_l^{(2)}(I_\nu)$ $(\forall \nu)$. Applying the method already used in § 3.4, we can easily prove that this system of inequalities and equations has a solution for every F. Letting max $|I_y|$ tend to 0 and using the Helly-type selection theorem, we obtain a decomposition

$$\mu = \mu_1 - \mu_2, \quad \mu_j(I) \leqslant C\varkappa(I) \quad (j-1,2;\ \forall I \in \mathfrak{K},\, w \notin I).$$

(1) In fact, a somewhat sharper result $\mu_j(I) \leqslant C\varkappa(I)$ holds.

For those arcs I that contain w, we get the following estimate, according to Lemma 4, §3.3:

$$\mu_j(I) \leqslant C\left[\varkappa(I) + \frac{\log 2}{2\pi}|I|\right] \leqslant C(1+\log 2)\,\varkappa(I) \quad (j=1,2).$$

4.3. The $\varkappa$-singular part of a premeasure

THEOREM 6. *Let μ be a premeasure of bounded $\varkappa$-variation. Define for every Carleson set F*

$$\mu_s(F) = -\sum_\nu \mu(I_\nu), \tag{4.3.1}$$

$\{I_\nu\}$ being the set of complementary arcs of F. There exists a unique countably additive finite measure on the σ-ring generated by all Carleson sets $F \subset F_0$(1) *which coincides with μ_s for those sets (F_0 being an arbitrary fixed Carleson set).*

Proof. Fix a F_0, and let $\{I_\nu^0\}$ be the set of complementary arcs of F_0. Let $\hat{\mu}(\theta)$ be the function associated with the premeasure μ. Define a function $\hat{\mu}_l(\theta)$ as follows:

(i) for $e^{i\theta} \in F_0$ $\quad \hat{\mu}_l(\theta) = \hat{\mu}(\theta)$;

(ii) for $e^{i\theta} \in I_\nu^0$, $I_\nu^0 = \{\zeta\colon\ |\zeta| = 1,\ \alpha_\nu < \arg\zeta < \beta_\nu\}$, $\hat{\mu}_l(\theta)$ is linear between $\hat{\mu}(\alpha_\nu + 0)$ and $\hat{\mu}(\beta_\nu - 0) = \hat{\mu}(\beta_\nu)$.

Prove that $\hat{\mu}_\nu(\theta)$ is of (classical) bounded variation. Let $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_k = \zeta_0$ be a finite set of points on ∂U arranged counterclockwise. Writing $\hat{\mu}_l(\zeta)$, $\hat{\mu}(\zeta)$ instead of $\hat{\mu}_l(\theta)$, $\hat{\mu}(\theta)$ ($\theta = \operatorname{Arg}\zeta$, $|\zeta| = 1$), we have to prove the boundedness of the sum

$$S = \sum_{j=0}^{k-1} |\hat{\mu}_l(\zeta_{j+1}) - \hat{\mu}(\zeta_j)| \leqslant C < \infty \tag{4.3.2}$$

for all sets $\{\zeta_\nu\}$. Without loss of generality we can assume that none of the ζ_ν belongs to F_0. Let $\{I_j'\}$ be the set of those (open) arcs among I_ν^0 which contain at least one point ζ_ν, $\{I_j''\}$ be the set of closed arcs which lie between I_j', and $F_1 \subset F_0$ be the (finite) set of all the end points of the arcs I_j'. Taking into account the linearity of $\hat{\mu}_l$ in every I_ν, we get the following estimate for the sum (4.3.2):

$$S \leqslant \sum_j |\mu(I_j')| + \sum_j |\mu(I_j'')| \leqslant \varkappa \operatorname{Var}\mu \cdot \hat{\varkappa}(F_1) \leqslant \varkappa \operatorname{Var}\mu \cdot \hat{\varkappa}(F_0).$$

---

(1) This σ-ring is the ring of all Borel sets $B \subset F_0$.

Thus the function $\hat{\mu}_l(\theta)$ is of bounded variation, and therefore it generates a countably additive finite measure defined for all the Borel sets; let μ_l denote this measure. As $\mu_l(\partial U)=0$, we get from (4.3.1) the following conclusion ($F\subset F_0$ being an arbitrary Carleson set):

$$\mu_l(F) = -\sum_\nu \mu_l(I_\nu) = -\sum_\nu \mu(I_\nu) = \mu_s(F).$$

Thus our theorem is proved.

Definition. μ_s will be called *the $\varkappa$-singular part* of the premeasure μ.

COROLLARY. *The $\varkappa$-singular part μ_s of a premeasure μ is non-positive if μ is $\varkappa$-bounded from above.*

Proof. We have to prove that for every Carleson set F $\mu_s(F)\leqslant 0$. This is trivial if F is finite. In fact,

$$\mu_s(F) = -\sum_\nu \mu(I_\nu) = \sum_{\zeta\in F} \mu(\{\zeta\}) \leqslant 0,$$

because a premeasure which is $\varkappa$-bounded from above assumes non-positive values on single-point sets. If F is infinite, then we first consider a partial sum (4.3.1):

$$-\sum_{\nu=1}^{N} \mu(I_\nu) = \sum_{\nu=1}^{N} \mu(J_\nu) \leqslant C\sum_{\nu=1}^{N} \varkappa(J_\nu) = C\left[\hat{\varkappa}(F_1) - \sum_{\nu=1}^{N} \varkappa(I_\nu)\right],$$

J_ν being the (closed) arcs which lie between $I_\nu(\nu=1, 2, ..., N)$ and F_1 the set of end points of these I_ν. If $N\to\infty$ then

$$\hat{\varkappa}(F_1)\to\hat{\varkappa}(F), \sum_{\nu=1}^{N} \varkappa(I_\nu)\to\hat{\varkappa}(F),$$

and consequently $\mu_s(F)\leqslant 0$.

From (4.3.1) the following inequality is easily derived which holds for all Carleson sets F:

$$|\mu_s(F)| \leqslant \varkappa \operatorname{Var}\mu\cdot\hat{\varkappa}(F). \tag{4.3.3}$$

Remark. It can be proved that the $\varkappa$-singular part of a premeasure of $\varkappa$-bounded variation is concentrated on a $\varkappa F_\sigma$-set. More precisely, if μ is a premeasure of $\varkappa$-bounded variation, then a sequence $F_1\subset F_2\subset F_3\subset ...$ of Carleson sets exists such that

$$\mu_s(F) = \lim_{\nu\to\infty} \mu_s(F\cap F_\nu)$$

holds for all Carleson sets F, hence for all Borel sets F contained in a Carleson set.

5. Classes $\mathfrak{H}^+$, $\mathfrak{H}$ of harmonic functions

5.1. Boundary premeasures and generalized Poisson integrals

THEOREM 1. *Let $u(z)$ belong to the class $\mathfrak{H}^+$; in other words, $u(z)$ is harmonic in U, $u(0)=0$ and*

$$u(z) \leqslant c \log \frac{1}{1-|z|} \tag{5.1.1}$$

with some $c>0$. Let

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} a_\nu r^{|\nu|} e^{i\nu\theta} \quad (a_0=0,\, a_{-\nu}=\bar{a}_\nu). \tag{5.1.2}$$

Then

$$\text{(i)}\ \ |a_\nu| \leqslant C_1 c \log(1+|\nu|), \tag{5.1.3}$$

C_1 being an absolute constant;

(ii) *for every arc $I \subset \partial U$ the following limit exists*

$$\lim_{r\to 1-0} \frac{1}{2\pi} \int_I u(r\zeta)\,|d\zeta| = \tilde{\sigma}(I); \tag{5.1.4}$$

(iii) *for each $\varepsilon>0$ there is a C_ε (dependent only on ε) such that for all $I\subset\partial U$*

$$\tilde{\sigma}(I) \leqslant [(2+\varepsilon)\varkappa(I) + C_\varepsilon |I|]c; \tag{5.1.5}$$

(iv) *there is an absolute constant C_2 such that*

$$\tilde{\sigma}(I) \leqslant C_2 c \varkappa(I) \quad (\forall\, I \subset \partial U). \tag{5.1.6}$$

Proof. We have

$$a_\nu = \frac{r^{-|\nu|}}{2\pi} \int_0^{2\pi} u(re^{i\theta})\, e^{-i\nu\theta}\, d\theta (0<r<1).$$

Therefore

$$|a_\nu| \leqslant \frac{r^{-|\nu|}}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|\, d\theta = \frac{r^{-|\nu|}}{\pi} \int_0^{2\pi} u^+(re^{i\theta})\, d\theta,$$

because

$$\int_0^{2\pi} u^+(re^{i\theta})\, d\theta = \int_0^{2\pi} u^-(re^{i\theta})\, d\theta = \frac{1}{2}\int_0^{2\pi} |u(re^{i\theta})|\, d\theta.$$

Using (5.1.1) we get

$$|a_\nu| \leqslant 2cr^{-|\nu|} \log \frac{1}{1-r}.$$

Putting $r = 1-1/(|\nu|+1)(|\nu|>0)$ we obtain (5.1.3). Thus (i) is proved.

To prove (ii) and (iii) show first that the integral (5.1.4) is bounded for $0<r<1$. Let $I=\{\zeta:\ |\zeta|=1,\ \alpha\leqslant \operatorname{Arg}\zeta\leqslant\beta\}$. Put

$$\tau=\beta-\alpha,\ t=t(\theta)=\min(\theta-\alpha,\beta-\theta),\ \varrho=\varrho(\theta)=\frac{1}{\tau}(\theta-\alpha)(\beta-\theta)\ (\alpha\leqslant\theta\leqslant\beta).$$

We have for $\alpha\leqslant\theta\leqslant\beta$:

$$\tfrac{1}{2}t(\theta)\leqslant\varrho\leqslant t(\theta),\ \varrho'(\theta)\leqslant 1,\ \varrho''(\theta)=-\frac{2}{\tau}.$$

Therefore for the function $q(\theta)=1-[\varrho(\theta)]^p$ $(p>2)$ the following estimates hold:

$$|q'(\theta)|\geqslant p[t(\theta)]^{p-1},\ |q''(\theta)|\leqslant p(p-1)[t(\theta)]^{p-2}+\frac{2p}{\tau}[t(\theta)]^{p-1}\leqslant p^2[t(\theta)]^{p-2}.$$

Using these estimates and integrating by parts we get for $|\nu|>1$, $\tau<1$:

$$\left|\int_\alpha^\beta\{1-[q(\theta)]^{|\nu|}\}e^{i\nu\theta}d\theta\right|=\left|\int_\alpha^\beta[q(\theta)]^{|\nu|-1}q'(\theta)e^{i\nu\theta}d\theta\right|$$

$$\leqslant\frac{|\nu|-1}{|\nu|}\int_\alpha^\beta[q(\theta)]^{|\nu|-2}|q'(\theta)|^2d\theta+\frac{1}{|\nu|}\int_\alpha^\beta[q(\theta)]^{|\nu|-1}|q''(\theta)|d\theta$$

$$\leqslant\frac{(|\nu|-1)p^2}{|\nu|}\int_\alpha^\beta\left\{1-\left[\frac{t(\theta)}{2}\right]^p\right\}^{|\nu|-2}[t(\theta)]^{2p-2}d\theta$$

$$+\frac{p^2}{|\nu|}\int_\alpha^\beta\left\{1-\left[\frac{t(\theta)}{2}\right]^p\right\}^{|\nu|-1}[t(\theta)]^{p-2}d\theta$$

$$\leqslant 2p^2\left[\int_0^{\tau/2}e^{-(t/2)p(|\nu|-2)}t^{2p-2}dt+\frac{1}{|\nu|}\int_0^{\tau/2}e^{-(t/2)p(|\nu|-1)}t^{p-2}dt\right]$$

$$\leqslant p^2\tau\max_{0<t<\infty}\left\{e^{-(t/2)p(|\nu|-2)}t^{2p-2}+\frac{1}{|\nu|}e^{-(t/2)p(|\nu|-1)}t^{p-2}\right\}.$$

A simple calculation yields:

$$\left|\int_\alpha^\beta\{1-[q(\theta)]^{|\nu|}\}e^{i\nu\theta}d\theta\right|\leqslant C_p\tau|\nu|^{-2(1-(1/p))}(|\nu|=1,2\ldots;\ \tau<1)\tag{5.1.7}$$

with $C_p\geqslant 1$ dependent only on p. Now, for $0<r<1$ we have

$$\frac{1}{2\pi}\int_I u(r\zeta)|d\zeta|=\frac{1}{2\pi}\int_\alpha^\beta u(re^{i\theta})d\theta=\frac{1}{2\pi}\int_\alpha^\beta u[rq(\theta)e^{i\theta}]d\theta+\frac{1}{2\pi}\int_\alpha^\beta\{u(re^{i\theta})-u[rq(\theta)e^{i\theta}]\}d\theta.\tag{5.1.8}$$

For the first of these integrals we get an upper bound using (5.1.1):

$$\left|\frac{1}{2\pi}\int_\alpha^\beta\{u[rq(\theta)e^{i\theta}]d\theta\leqslant\frac{c}{2\pi}\int_\alpha^\beta\log\frac{1}{1-q(\theta)}d\theta\leqslant\frac{cp}{2\pi}\int_\alpha^\beta|\log\varrho(\theta)|d\theta\leqslant cp[\varkappa(I)+C\tau],\tag{5.1.9}$$

C being an absolute constant. Using (5.1.2), (5.1.3) and (5.1.7) we obtain:

$$\left|\frac{1}{2\pi}\int_\alpha^\beta \{u(re^{i\theta}) - u[rq(\theta)\,e^{i\theta}]\}\,d\theta\right| \leqslant \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left|\alpha_\nu\int_\alpha^\beta r^{|\nu|}\{1-[q(\theta)]^{|\nu|}\}\,e^{i\nu\theta}d\theta\right.$$

$$\leqslant \frac{C_1 C_p c\tau}{2\pi}\sum_{-\infty}^{\infty}\frac{\log(1+|\nu|)}{|\nu|^{2(1-(1/p))}} = C_p' c\tau \qquad (5.1.10)$$

with $C_p' < \infty$ dependent only on $p>2$ and $\tau=\beta-\alpha<1$.

Now, (5.1.8), (5.1.9) and (5.1.10) yield

$$\frac{1}{2\pi}\int_I u(r\zeta)\,|d\zeta| \leqslant [p\varkappa(I) + C_p|I|]\,c \qquad (5.1.11)$$

for $0<r<1$, $p>2$ and $|I|<1$, the last restriction being unessential owing to Lemma 4 (§ 3.3). So we have proved that the measures

$$\sigma_r(I) = \frac{1}{2\pi}\int_I u(r\zeta)\,|d\zeta| \quad (0<r<1)$$

are uniformly $\varkappa$-bounded from above. Using the Helly-type selection theorem (Theorem 1, § 4.1) we can find a sequence $r_1<r_2<\dots$, $r_\nu\to 1$, such that $\sigma \xrightarrow{\varkappa w} \sigma(\nu\to\infty)$, σ being a premeasure satisfying (5.1.5). Now, for $|z|<r<1$ we can write

$$u(z) = \int_{\partial U} P\left(\zeta, \frac{z}{r}\right)\sigma_r(|d\zeta|). \qquad (5.1.12)$$

Letting r tend to 1 and taking into consideration the definition of $\varkappa$-weak convergence of measures, as well as the smoothness of the Poisson kernel, we obtain the representation of $u(z)$ in the form of a generalized Poisson integral:

$$u(z) = \int_{\partial U} P(\zeta, z)\,\sigma(|d\zeta|) \quad z\in U), \qquad (5.1.13)$$

the integral being understood in the following sense:

$$\int_{\partial U} P(\zeta,z)\,\sigma(|d\zeta|) = -\int_0^{2\pi}\hat\sigma(\theta)\left[\frac{d}{d\theta}P(e^{i\theta},z)\right]d\theta \qquad (5.1.14)$$

with $\hat\sigma(\theta)=\sigma(I_\theta)$, $I_\theta=\{\zeta\colon |\nu|=1,\ 0\leqslant \operatorname{Arg}\zeta<\theta\}$. The boundary measure σ in (5.1.13) satisfies (5.1.5), and consequently (5.1.6), too. What remains to be proved is the existence of the limit in (5.1.4), and this is the consequence of the following theorem which is analogous to the classical Fatou theorem about the limit values of a Poisson integral:

THEOREM 2. *Let*

$$u(z) = \int_{\partial U} P(\zeta, z)\,\mu(|d\zeta|) = -\int_0^{2\pi} \hat{\mu}(\theta)\left[\frac{d}{d\theta} P(e^{i\theta}, z)\right] d\theta \quad (z \in U) \tag{5.1.15}$$

be a generalized Poisson integral with the premeasure μ *of bounded* $\varkappa$*-variation,* $\hat{\mu}(\theta) = \mu(I_\theta)$, $I_\theta = \{\zeta\colon |\zeta| = 1, 0 \leqslant \operatorname{Arg} \zeta < \theta\}$. *Then for each open arc* $I \subset \partial U$ *the following limit exists:*

$$\lim_{r \to 1-0} \frac{1}{2\pi} \int_I u(r\zeta)\,|d\zeta| = \tfrac{1}{2}[\mu(\bar{I}) + \mu(I)], \tag{5.1.16}$$

$\bar{I}$ *being the closure of* I.

Proof. Let $I = \{\zeta\colon |\zeta| = 1, \alpha < \operatorname{Arg} \zeta < \beta\}$. Integrating (5.1.15) and using some elementary properties of the Poisson kernel we get:

$$\begin{aligned}
\frac{1}{2\pi} \int_I u(r\zeta)\,|d\zeta| &= -\frac{1}{2\pi} \int_0^{2\pi} \hat{\mu}(\theta)\left\{\int_\alpha^\beta \left[\frac{d}{d\theta} P(e^{i\theta}, re^{i\phi})\right] d\phi\right\} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \hat{\mu}(\theta)\left\{\int_\alpha^\beta \left[\frac{d}{d\phi} P(e^{i\theta}, re^{i\phi})\right] d\phi\right\} d\theta \\
&= \frac{1}{2\pi}\left\{\int_0^{2\pi} \hat{\mu}(\theta)\, P(^{i\theta}, re^{i\beta})\, d\theta - \int_0^{2\pi} \hat{\mu}(\theta)\, P(e^{i\theta}, re^{i\alpha})\, d\theta\right\}.
\end{aligned}$$

Using the classical Fatou theorem we obtain

$$\lim_{r \to 1-0} \frac{1}{2\pi} \int_I u(r\zeta)\,|d\zeta| = \frac{\hat{\mu}(\beta + 0) + \hat{\mu}(\beta)}{2} - \frac{\hat{\mu}(\alpha + 0) + \hat{\mu}(\alpha)}{2},$$

which is equivalent to (5.1.16).

COROLLARY. *Premeasure* μ *of bounded* $\varkappa$*-variation in the representation* (5.1.15) *is uniquely determined by the harmonic function* $u(z)$.

5.2. Harmonic functions and their representation by generalized poisson integrals

THEOREM 3. *Every harmonic function* $u(z)$ *belonging to the class* $\mathfrak{H}$ *can be represented by a generalized Poisson integral of the form*

$$u(z) = \int_{\partial U} P(\zeta, z)\,\mu(|d\zeta|) = -\int_0^{2\pi} \left[\frac{d}{d\theta} P(e^{i\theta}, z)\right] \hat{\mu}(\theta)\, d\theta, \tag{5.2.1}$$

μ being a premeasure of bounded $\varkappa$-variation which is uniquely determined by $u(z)$; moreover,

$$\lim_{r\to 1-0} \frac{1}{2\pi}\int_I u(r\zeta)\,|d\zeta| = \frac{\mu(\bar{I})+\mu(I)}{2} \tag{5.2.2}$$

for every open arc $I\subset\partial U$. Conversely, every premeasure μ of bounded $\varkappa$-variation determines a harmonic function $u(z)\subset\mathfrak{H}$ by means of (5.2.1).

If $u(z)\in\mathfrak{H}^+$ and satisfies (5.1.1), *then μ is $\varkappa$-bounded from above; moreover, the following inequality holds:*

$$\mu(I) \leqslant [2+\varepsilon)\varkappa(I) + C_\varepsilon|I|]c \quad (\forall I\in\mathfrak{K},\ \varepsilon>0), \tag{5.2.3}$$

which implies

$$\mu(I) \leqslant C c\varkappa(I) \quad (\forall I\in\mathfrak{K}), \tag{5.2.4}$$

C being an absolute constant. Conversely, if $\mu(I)\leqslant c\varkappa(I)$ $(\forall I\in\mathfrak{K})$ with some $c>0$, then for the function $u(z)$ the inequality holds:

$$u(z) \leqslant c\left(\log\frac{1}{1-|z|}+a\right) \quad (z\in U), \tag{5.2.5}$$

with an absolute constant $a>0$.

Proof. Let μ in (5.2.1) be $\varkappa$-bounded from above: $\mu(I)\leqslant c\varkappa(I)$ $(\forall I\in\mathfrak{K})$. A straightforward computation then shows that the function $u(z)$ satisfies (5.2.5). All the other statements of the theorem follow from Theorems 1 and 2 (Ch. 5) and Theorem 5 (Ch. 4).

THEOREM 4. *Let $u(z)$ be harmonic in U, $u(0)=0$. The necessary and sufficient condition for $u(z)$ to belong to $\mathfrak{H}$ is*

$$\sup_{0<r<1} \varkappa \operatorname{Var} \mu_r < \infty, \tag{5.2.6}$$

μ_r being defined as follows:

$$\mu_r(I) = \frac{1}{2\pi}\int_I u(r\zeta)\,|d\zeta| \quad (\forall I\in\mathfrak{K}). \tag{5.2.7}$$

Proof. Let $u(z)\in\mathfrak{H}$. By Theorem 3, (5.2.1) holds. Consider the Banach space $V_\varkappa$ of all premeasures μ of bounded $\varkappa$-variation with the norm $\|\mu\| = \varkappa \operatorname{Var}\mu$. This norm is invariant under rotations $T_\zeta(\zeta\in\partial U)$:

$$\|T_\zeta\| = \|\mu\|, \quad (T_\zeta\mu)(I) = \mu(\{\zeta I\}) \quad (\forall I\in\mathfrak{K}).$$

Using (5.2.1) and (5.2.7) we readily obtain the following representation of the premeasure μ_r in the form of an abstract integral in the space $V_\varkappa$:

$$\mu_r = \frac{1}{2\pi}\int_{\partial U} P(\zeta, r)\, T_\zeta \mu\, |d\zeta|,$$

and this yields the required estimate:

$$\|\mu_r\| \leqslant \frac{1}{2\pi}\int_{\partial U} P(\zeta, r)\|T_\zeta \mu\| \cdot |d\zeta| = \|\mu\|.$$

Conversely, let (5.2.6) hold. We have for $0<r<1$:

$$u(rz) - \frac{1}{2\pi}\int_{\partial U} P(\zeta, z)\, u(r\zeta)|d\zeta| - \int_{\partial U} P(\zeta, z)\,\mu_r(|d\zeta|). \tag{5.2.8}$$

By Theorem 1 (Ch. 4), we can choose in view of (5.2.6) a $\varkappa$-weakly convergent sequence $\{\mu_{r_\nu}\}$:

$$\mu_{r_\nu} \xrightarrow{\varkappa w} \mu \quad (r_\nu \uparrow 1),$$

μ being a premeasure of bounded $\varkappa$-variation. This justifies the transition to the limit in (5.2.8) which yields

$$u(z) = \int_{\partial U} P(\zeta, z)\,\mu(|d\zeta|);$$

thus $u(z) \in \mathfrak{H}$.

6. Meromorphic functions of the class $\mathfrak{N}$ and their factorization

6.1. Generalized Blaschke products

Definition 1. Let $\alpha = \{\alpha_\nu\}$ be a (finite or infinite) sequence of complex numbers, $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots < 1$, and let

$$\sum_\nu (1 - |\alpha_\nu|)^2 < \infty. \tag{6.1.1}$$

The following product

$$\tilde{B}_\alpha(z) = \prod_\nu \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z} \cdot \frac{|\alpha_\nu|}{\alpha_\nu} \cdot \exp\left\{ \frac{\dfrac{\alpha_\nu}{|\alpha_\nu|} + z}{\dfrac{\alpha_\nu}{|\alpha_\nu|} - z} \cdot \log \frac{1}{|\alpha_\nu|} \right\}, \tag{6.1.2}$$

which converges in view of (6.1.1), will be called the generalized Blaschke product with the zero set α. If $\alpha = \varnothing$ we put $\tilde{B}_\alpha(z) \equiv 1$.

THEOREM 1. *Let $f(z) \in A^{-\infty}$, $f(0) \neq 0$, and let $\alpha = \{\alpha_\nu\}$ be the zero set of $f(z)$ or its subset Then*

$$F(z) = f(z)[B_\alpha(z)]^{-1} \in A^{-\infty}. \tag{6.1.3}$$

Proof. Take an arbitrary point $\zeta_0 \in \partial U$ and estimate the modulus of $F(z)$ along the radius $R_{\zeta_0} = \{z: r\zeta_0, 0<r<1\}$. Consider two domains: $G_1 = G_{\{\zeta_0\}; q, a}$ and $G_{\{\zeta_0\}; 2q, a}$ with some $q>2$ and $0<a<\frac{1}{4}$. Obviously, $G_1 \subset G_2$. Now prove the following

LEMMA. *For $z_1 \in G_1$, $z_2 \in U \backslash G_2$ the following inequality holds:*

$$\left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| \geqslant 1 - C(1 - |z_1|), \tag{6.1.4}$$

with $C>0$ dependent only on q and a.

Proof of the lemma. (6.1.4) is trivial if at least one of the points z_1, z_2 lies outside a fixed neighbourhood $V_\varepsilon = \{z: |z - \zeta_0| < \varepsilon\}$ of the point ζ_0. We can therefore assume that $z_1, z_2 \in V_\varepsilon$. Mapping conformally U onto the halfplane $\operatorname{Im} w > 0$ with $w(\zeta_0) = 0$ we therefore reduce (6.1.4) to the following inequality:

$$\left| \frac{w_1 - w_2}{w_1 - w_2} \right| \geqslant 1 - C \operatorname{Im} w_1 (|w_1| < 1, |w_2| < 1, \operatorname{Im} w_1 \geqslant a |\operatorname{Re} w_1|^q, 0 < \operatorname{Im} w_2 \leqslant a |\operatorname{Re} w_2|^{2q}).$$

Put $w_1 = x + iy$, $w_2 = u + iv$, so that $y \geqslant a|x|^q$, $0 < v \leqslant a|u|^{2q}$. We have

$$\left| \frac{w_1 - w_2}{w_1 - w_2} \right|^2 = \frac{(u-x)^2 + (v-y)^2}{(u-x)^2 + (z+y)^2} = 1 - 4y \frac{v}{(u-x)^2 + (v+y)^2} \geqslant 1 - 4y \frac{a|u|^{2q}}{(u-x)^2 + a^2|u|^{2q}}.$$

An easy computation shows that

$$\max_{\substack{-1 \leqslant x \leqslant 1 \\ -1 \leqslant u \leqslant 1}} \frac{a|u|^{2q}}{(u-v)^2 + a^2|x|^{2q}} = C_{a,q} < \infty,$$

which proves the lemma.

If $f \in A^{-n}$, then by Theorem 1, Ch. 3, and Lemma 1, § 3.3,

$$\sum_{\alpha_\nu \in G_1} (1 - |\alpha_\nu|) \leqslant C_1 < \infty,$$

with C_1 dependent only on n, $\|f\|_{-n}$, q, a. Using (6.1.4) we obtain that the Blaschke product

$$B_1(z) = \prod_{\alpha_\nu \in G_1} \frac{\alpha_\nu - z}{1 - \bar{\alpha}_\nu z} \cdot \frac{|\alpha_\nu|}{\alpha_\nu} \tag{6.1.5}$$

satisfies the inequality:

$$|B_1(z)| \geqslant C_2 > 0 \quad (z \in \partial G_2).$$

Therefore the function

$$F_1(z) = f(z)[B_1(z)]^{-1}$$

has the property

$$|F_1(z)| \leqslant C_3\|f\|_{-n}|z-\zeta_0|^{-2nq} \quad (z\in\partial G_2),$$

and by the Phragmen-Lindelöf principle a similar inequality holds in G_2:

$$|F_1(z)| \leqslant C_4\|f\|_{-n}|z-\zeta_0|^{-2nq} \quad (z\in G_2). \tag{6.1.6}$$

On the other hand, applying Lemma 3, § 3.3, we can evaluate from below the modulus of that part of the generalized Blaschke product $\tilde{B}_\alpha(z)$ which is determined by the zeros $\alpha_\nu \notin G_1$:

$$\tilde{B}_2(z) = \prod_{\alpha_\nu\notin G_1} \frac{\alpha_\nu - z}{1-\bar{\alpha}_\nu z}\cdot\frac{|\alpha_\nu|}{\alpha_\nu}\cdot \exp\left\{\frac{\dfrac{\alpha_\nu}{|\alpha_\nu|}+z}{\dfrac{\alpha_\nu}{|\alpha_\nu|}-z}\cdot\log\frac{1}{|\alpha_\nu|}\right\}.$$

We have for $z\in R_{\zeta_0}$:

$$|\tilde{B}_2(z)| \geqslant \exp\left\{-C_5\sum_{\alpha_\nu\notin G_1}\frac{(1-|\alpha_\nu|)^2}{\left|\dfrac{\alpha_\nu}{|\alpha_\nu|}-z\right|^2}\right\} \geqslant \exp\{-C_6\sum_{\alpha_\nu\notin G_0}(1-|\alpha_\nu|)^{2-(2/q)}\} = C_7 < 0, \tag{6.1.7}$$

with C_7 dependent only on n, $\|f\|_{-n}$, q, a.

Now, taking into account (6.1.6) and (6.1.7) we obtain for $z\in R_{\zeta_0}$:

$$|F(z)| \leqslant |F_1(z)|\cdot|\tilde{B}_2(z)|^{-1} \leqslant C_4\|f\|_{-n}C_7^{-1}|z-\zeta_0|^{-2nq} = C_8|z-\zeta_0|^{-2nq}, \tag{6.1.8}$$

with C_8 dependent only on n, $\|f\|_{-n}$, q, a. Thus our theorem is proved, since this estimate holds for all the radii R_{ζ_0}. In fact a sharper result holds true:

COROLLARY 1. *If $f\in A^{-n}$, $\|f\|_{-n}\leqslant b$, then $F\in A^{-4n-\varepsilon}$ and*

$$\|F\|_{-4n-\varepsilon} \leqslant C < \infty, \tag{6.1.9}$$

with C dependent only on n, b and ε.

COROLLARY 2. *A generalized Blaschke product belongs to the class $\mathfrak{N}$ iff its zero set α satisfies condition (T).*

Proof. The zero set α of function $f\in\mathfrak{N}$, $f=g/h$ ($g, h\in A^{-\infty}$), is a subset of the zero set of g and therefore, by the Corollary 2 of Theorem 1, Ch. 3, an $A^{-\infty}$-zero set itself. Conversely, if $\alpha\in(T)$, then by Theorem 1, Ch. 3, a function $f\in A^{-\infty}$ exists for which α is the zero set. By the theorem we have just proved, the function

$$F(z) = f(z)[B_\alpha(z)]^{-1}$$

belongs to $A^{-\infty}$ as well, and consequently $B_\alpha(z) = f(z)/B(z)$ belongs to the class $\mathfrak{N}$.

6.2. Meromorphic functions

THEOREM 2. *Every meromorphic function $f(z)$, $f(0)\neq 0$, $f(0)\neq\infty$, belonging to the class $\mathfrak{N}$, admits a unique representation in the form*

$$f(z)=\lambda\frac{\tilde{B}_\alpha(z)}{\tilde{B}_\beta(z)}\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu(|d\zeta|)\right\},\tag{6.2.1}$$

$\lambda\neq 0$ being a complex number, α and β being two disjoint sequences satisfying condition (T) and μ a premeasure of bounded $\varkappa$-variation; in fact, $\lambda=f(0)$, α is the zero set and β is the pole set of $f(z)$. Conversely, every function of the form (6.2.10) *under above restrictions belongs to the class $\mathfrak{N}$.*

This theorem is a direct consequence of Theorem 1, Ch. 6, and Theorem 3, Ch. 5.

THEOREM 3. *Let $f(z)$ be a meromorphic function in the unit disk U, $f(0)=1$, $\alpha=\{\alpha_\nu\}$ be the zero set and $\beta=\{\beta_\nu\}$ the pole set of $f(z)$. The following conditions are necessary and sufficient for $f(z)$ to belong to the class $\mathfrak{N}$:*

(i) *α and β satisfy condition (T);*

(ii) $\sup\limits_{0<r<1}\varkappa\operatorname{Var}\mu_r<\infty$ (6.2.2)

where

$$\mu_r(I)=\frac{1}{2\pi}\int_I\log|f(r\zeta)|\cdot|d\zeta|-\sum_{\substack{|\alpha_\nu|<r\\(\alpha_\nu/|\alpha_\nu|)\in I}}\log\frac{r}{|\alpha_\nu|}+\sum_{\substack{|\beta_\nu|<r\\(\beta_\nu/|\beta_\nu|)\in I}}\log\frac{r}{|\beta_\nu|}\ (\forall I\in\mathfrak{K}).$$

Proof. Let $f(z)=g(z)/h(z)$, $g(0)=h(0)=1$, $g(z)$, $h(z)\in A^{-n}$. We can assume that the zero set of $g(z)$ is α and that of $h(z)$ is β; otherwise we could, by Theorem 1, Ch. 6, divide both $g(z)$ and $h(z)$ by $\tilde{B}_\gamma(z)$ with $\gamma=\{\gamma_\nu\}$ consisting of the common zeros of $g(z)$ and $h(z)$. It is evident that the functions $g_r(z)=g(rz)$ and $h_r(z)=h(rz)$ have uniformly bounded norms in A^{-n} for $0<r<1$. Therefore if the functions $f_r=g_r/h_r$ $(0<r<1)$ are factorized according to the formula (6.2.1), then the corresponding premeasures μ_r must have uniformly bounded $\varkappa$-variations. Thus we have proved the necessity of (i) and (ii). To prove the sufficiency we first factorize f_r:

$$f_r(z)=\frac{\tilde{B}_{\alpha'}(z)}{\tilde{B}_{\beta'}(z)}\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu_r(|d\zeta|)\right\},$$

with $\alpha'=\{\alpha_\nu/r\}$ $(|\alpha_\nu|<r)$, $\beta'=\{\beta_\nu/r\}$ $(|\beta_\nu|<r)$. To carry out the transition to the limit for $r\to 1-0$ we have to use the Helly-type selection theorem (Theorem 1, Ch. 4) which yields

$$f(z)=\frac{\tilde{B}_\alpha(z)}{B_\beta(z)}\exp\left\{\int_{\partial U}\frac{\zeta+z}{\zeta-z}\mu(|d\zeta|\right),$$

μ being some premeasure of bounded $\varkappa$-variation. In view of Theorem 2, Ch. 6, this implies $f(z)\in\mathfrak{N}$.

Now we can introduce the notion of the $\varkappa$-singular measure associated with a function $f\in\mathfrak{N}$. For convenience, we assume that $f(0)\neq 0$, $f(0)\neq\infty$.

THEOREM 4. *Let $f(z)\in\mathfrak{N}$ and let (6.2.1) be the factorization of $f(z)$. Define for every Carleson set $F\subset\partial U$*

$$\mu_s^{(f)}(F) = -\sum_\nu \mu(I_\nu) + \sum_{(\alpha_\nu/|\alpha_\nu|)\in F}\log\frac{1}{|\alpha_\nu|} - \sum_{(\beta_\nu/|\beta_\nu|)\in F}\log\frac{1}{|\beta_\nu|}, \tag{6.2.3}$$

$\{I_\nu\}$ being the set of complementary arcs of F. There exists a unique countably additive finite measure on the σ-ring of all Borel sets B contained in a fixed Carleson set F_0 which coincides with $\mu_s^{(f)}(F)$ for all the Carleson sets $F\subset F_0$.

Proof. Let F_0 be a fixed Carleson set. In view of the condition (T) which is satisfied by both $\alpha=\{\alpha_\nu\}$ and $\beta=\{\beta_\nu\}$, we have

$$\sum_{(\alpha_\nu/|\alpha_\nu|)\in F_0}\log\frac{1}{|\alpha_\nu|}<\infty, \quad \sum_{(\beta_\nu/|\beta_\nu|)\in F_0}\log\frac{1}{|\beta_\nu|}<\infty.$$

Therefore

$$\hat{\alpha}(B) = \sum_{(\alpha_\nu/|\alpha_\nu|)\in B}\log\frac{1}{|\alpha_\nu|}, \quad \hat{\beta}(B) = \sum_{(\beta_\nu/|\beta_\nu|)\in B}\log\frac{1}{|\beta_\nu|},$$

are countably additive measures defined for all the Borel sets $B\subset F_0$, and so is the $\varkappa$-singular part μ_s of the premeasure μ, in accordance with Theorem 6, Ch. 4. For Carleson sets $F\subset F_0$ we have

$$\mu_s^{(f)}(F) = \mu_s(F) + \hat{\alpha}(F) - \hat{\beta}(F). \tag{6.2.4}$$

which proves the theorem.

Definition. $\mu_s^{(f)} = \mu_s + \hat{\alpha} - \hat{\beta}$ will be called the $\varkappa$-singular measure associated with the function $f\in\mathfrak{N}$.

This notion seems to be very useful, perhaps even indispensable, for the description of closed ideals (invariant subspaces) of the topological algebra $A^{-\infty}$.

THEOREM 5. *The $\varkappa$-singular measure associated with a function $f\in A^{-\infty}$ is non-positive.*

Proof. If $f\in A^{-\infty}$, then the premeasure μ in the factorization (6.2.1) is $\varkappa$-bounded from above. Therefore its $\varkappa$-singular part μ_s is non-positive (see Theorem 6, Ch. 6, and the Corollary). Using (6.2.4) we find for every Carleson set F:

$$\mu_s^{(f)}(F) \leqslant \hat{\alpha}(F) = \sum_{(\alpha_\nu/|\alpha_\nu|)\in F}\log\frac{1}{|\alpha_\nu|}. \tag{6.2.5}$$

On the other hand, if we divide $f(z)$ by

$$B_r(z) = \prod_{|\alpha_\nu|<r} \frac{\alpha_\nu - z}{1-\bar{\alpha}_\nu z} \cdot \frac{|\alpha_\nu|}{\alpha_\nu} \quad (0<r<1)$$

then the function

$$f_r(z) = f(z) \quad [B_r(z)]^{-1}$$

has the same singular measure:

$$\mu_s^{(f_r)} = \mu_s^{(f)},$$

and (6.2.5) yields:

$$\mu_s^{(f)}(F) = \mu_s^{(f_r)}(F) \leqslant \sum_{(\alpha_\nu/|\alpha_\nu|)\in F,\, |\alpha_\nu|\geqslant r} \log \frac{1}{|\alpha_\nu|} \to 0 \quad (r\to 1-0),$$

which proves the theorem.

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Received January 27, 1975.