IDEAL THEORY AND LAPLACE TRANSFORMS FOR A CLASS OF MEASURE ALGEBRAS ON A GROUP

BY

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In this paper we introduce, and undertake the study of a class of Banach algebras associated with a locally compact group G. These algebras are related to the two-sided Laplace transform in the same way that the group algebra $L^1(G)$ and the measure algebra $M(G)$ are related to the Fourier transform. In the following paragraph, we indicate the nature of some of our final results by exposing them in the simplest nontrivial case.

If A is a compact convex subset of R^n let $\mathfrak{L}(A)$ denote the space of measurable functions on R^n for which

$$\|f\|_A = \int_{R^n} |f(x)|\,\varphi_A(x)\,dx < \infty, \quad \text{where} \quad \varphi_A(x) = \sup_{y\in A} e^{-x\cdot y}.$$

Note that for $f \in \mathfrak{L}(A)$, the Laplace transform

$$f^\wedge(z) = \int_{R^n} f(x)\,e^{-z\cdot x}\,dx$$

converges absolutely for $\operatorname{Re} z = (\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n) \in A$. The following facts concerning $\mathfrak{L}(A)$ are special cases of results of this paper:

F1. (Lemma 2.2) $\mathfrak{L}(A)$ is a Banach algebra under the norm $\|\;\|_A$ and convolution multiplication;

F2. (Corollary to Theorem 6.1.) The maximal ideal space of $\mathfrak{L}(A)$ can be identified with $\{z \in \mathbf{C}^n : \operatorname{Re} z \in A\}$, and the Gelfand transform of $f \in \mathfrak{L}(A)$ can be identified with the Laplace transform $f^\wedge$ restricted to $\{z \in \mathbf{C}^n : \operatorname{Re} z \in A\}$;

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F3. (Theorem 6.3.) If J is a regular ideal of $\mathfrak{L}(A)$ for which $h(J)=\{z: f^\wedge(z)=0$ for all $f\in J\}$ is contained in the open tube $\{z: \operatorname{Re} z \in \operatorname{int} A\}$, then $h(J)$ is a finite set $\{z_1, ..., z_k\}$ and there exists a finite dimensional subspace $J^\perp$ of $C(R^n)$ such that:

(1) Each $g\in J^\perp$ is a linear combination of functions of the form $P(x)e^{-z_i\cdot x}$, where P is a polynomial in n variables, and

(2) If $f\in \mathfrak{L}(A)$, then $f\in J$ if and only if $\int fg=0$ for every $g\in J^\perp$.

For the setting described above, F1 and F2 are simple to prove; however, as far as we know, F3 requires most of the machinery of this paper. Of course, all three results are trivial in the case $n=1$.

We shall actually study a situation considerably more general than that described above. Rather than working with R^n, we shall work with a more general class of locally compact abelian groups. Also, we shall work with algebras $\mathfrak{N}(A)$ consisting of measures which satisfy a growth condition like that defining $\mathfrak{L}(A)$. When the measures in $\mathfrak{N}(A)$ are all absolutely continuous, then $\mathfrak{N}(A)=\mathfrak{L}(A)$.

We were led to the results of this paper in the process of studying another problem. Let G be a locally compact abelian group and let $M(G)$ denote the convolution algebra of finite regular Borel measures on G. We say N is an L-subalgebra of $M(G)$ if it is a closed subalgebra for which $\mu\in N$ implies $\nu\in N$ whenever ν is absolutely continuous with respect to μ. A surprisingly involved question is the following: For which L-subalgebras N of $M(G)$ is it true that the maximal ideal space of N is the dual group of G? A partial solution to this problem was obtained in [9]. The missing link for a general solution is supplied by Theorem 6.2 of this paper. The generalizations of F1–F3 obtained here can be considered spin-off from Theorem 6.2. The fitting together of Theorem 6.2 with results of [9] to obtain the general solution to the above problem is reserved for another paper, [10], in which we also study the implications of this result for the structure theory of $M(G)$ and other convolution measure algebras.

In section 1 we define the class of groups to be considered and discuss certain background information concerning Laplace transforms and analytic functions in this setting.

We define, in section 2, the class of measure algebras to be considered and develop the elementary properties of these algebras. For a given group G, we begin with an L-subalgebra N of $M(G)$ and construct an algebra $\mathfrak{N}(A)$ consisting of measures which are locally in N and satisfy a growth condition which insures that their Laplace transforms exist in a "tube" based on A. The set A plays the same role that it did in our opening discussion.

Sections 3, 4, and 5 are devoted to developing machinery to process the combinatorial problems that arise in working with the correspondence $A\to\mathfrak{N}(A)$ (these problems are not apparent in the one variable case, which is precisely why F2 and F3 are so easily proved in this case). In Section 3 we develop a cohomology theory for the correspondence $A \to \mathfrak{N}(A)$, which is essentially just the Čech cohomology of presheaves. In Section 4 we use the results of Section 3 and elementary double complex arguments to study linear equations in $\mathfrak{N}(A)$. For certain types of linear equations we obtain a measure, called the residue measure, which is locally in N and determines the solvability of the equation. In Section 5 we determine the form of this residue measure through the use of the inverse Laplace transform and the Cauchy integral theorem. We show that the residue measure is absolutely continuous and its Radon–Nikodym derivative is a linear combination of exponentials multiplied by polynomials.

Finally, in Section 6 we put together the results of the previous sections to obtain ideal theoretic results for $\mathfrak{N}(A)$. These results depend strikingly on whether or not N contains absolutely continuous measures. Because of this, we are able to give a spectral condition (Theorem 6.2) which ensures that N contains absolutely continuous measures. In the case where $N = L_e$, the algebra of absolutely continuous measures with the identity adjoined, we obtain a generalization of F3 (Theorem 6.3). Theorem 6.1 gives a generalization of F2 for general N.

Our discussion comprises only a bare beginning of the study of the algebras $\mathfrak{N}(A)$. There are many problems concerning these algebras that we have not touched on. We have not completely solved those problems that we have touched on. We list below several ways in which one might attempt to extend our results:

(1) Work with a larger class of groups.

(2) Work with algebras of distributions satisfying growth conditions rather than with measures.

(3) Attempt to characterize a larger class of ideals.

(4) Study algebras $\mathfrak{N}(A)$ for which A is allowed to be noncompact or have dimension less than that of the space X.

(5) Obtain our results by a direct application of analytic function theory, without the machinery of Sections 3, 4, and 5.

In attacking some of the above problems, one may be able to use directly the machinery of Sections 3 and 4. With this in mind, we have stated the results of these sections in slightly more generality than is necessary for our later results.

1. Preliminaries

Let G be a locally compact abelian group. Mackey, in [5], began the study of the Laplace transform in this setting. His concern was primarily with Laplace transforms of L^2-functions. Arens and Singer, in [1], considered a one-sided Laplace transform in connection with certain subalgebras of $L^1(G)$. Our initial discussion in this section will overlap Mackey's to a certain extent. We shall use freely the modern theory of harmonic analysis on groups as expounded in [3] and [6].

Definition 1.1. We denote by Ω the group of all continuous homomorphisms of G into the multiplicative group of nonzero complex numbers. By Γ we shall mean the subgroup of Ω consisting of the bounded functions in Ω; i.e., Γ is the dual group of G. The real vector space consisting of all continuous homomorphisms of G into the group of additive reals will be denoted by X.

We shall use additive notation in G and multiplicative notation in Γ and Ω. If $x \in X$ then the equation $\omega(g) = e^{x(g)}$ defines an element $\omega = e^x$ of Ω which is positive in the sense that it has positive range. Conversely, if $\omega \in \Omega$ and ω is positive, then $x(g) = \log \omega(g)$ defines an element $x = \log \omega$ of X such that $\omega = e^x$. If $\omega \in \Omega$ then $0 < |\omega| \in \Omega$ and $\omega/|\omega| \in \Gamma$. It follows that each element of Ω can be uniquely written in the form $e^x\gamma$ for some $x \in X$ and $\gamma \in \Gamma$. If $A \subset X$, $B \subset \Gamma$, we shall often use the notation $e^A B$ for $\{e^x\gamma : x \in A, \gamma \in B\}$. The map $x \to e^x$ is clearly a group isomorphism of X onto a subgroup e^X of Ω.

If $W = X + iX$ is the complex vector space of continuous homomorphisms of G into the additive complex numbers, then there is an analogous homomorphism $w \to e^w$ of W into Ω. If $w \in W$ then w is a purely imaginary element of W if and only if $e^w \in \Gamma$; in this case $w = ix$ for some $x \in X$.

Suppose f is a function defined on a subset of Ω and $\omega \in \Omega$, $x \in X$. If $f(e^{zx}\omega)$ is defined for z in a neighborhood of zero in $\mathbf{C}$ and is a holomorphic function of z in this neighborhood, then we shall say that f is holomorphic at ω in the x-direction. The derivative $f_x(\omega)$ of f in the x-direction at ω will be the derivative of $f(e^{zx}\omega)$ with respect to z at zero.

We may topologize Ω by giving it the topology of uniform convergence on compact subsets of G. With this topology, Ω is a topological group.

Definition 1.2. If U is an open subset of Ω and f is a function which is defined and continuous on U and holomorphic in every direction at each point of U, then we shall say that f is holomorphic on U. The algebra of all holomorphic functions on U will be denoted by $\mathfrak{A}(U)$.

By a measure μ on G we shall mean a complex valued set function defined on the bounded Borel sets of G, such that if K is any compact subset of G and $\mu_K(E) = \mu(E \cap K)$

for each Borel set $E \subset G$, then μ_K is a finite regular Borel measure on G. If μ is a measure on G, we define the total variation measure $|\mu|$ for μ by $|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|$, where the supremum is taken over all finite disjoint collections $\{E_i\}_{i=1}^{n}$ of bounded Borel subsets of E. It follows that $|\mu|$ is a positive, inner regular, Borel measure. If $\|\mu\| = |\mu|(G) < \infty$ then μ is called finite. If μ and ν are finite measures, then their convolution product $\mu \cdot \nu$ is the unique measure satisfying $\int f d\mu \cdot \nu = \int\int f(g_1 + g_2) d\mu(g_1) d\nu(g_2)$ for all continuous functions f with compact support on G. We denote by $M(G)$ the Banach algebra of all finite regular Borel measures on G under convolution multiplication. The subalgebra of $M(G)$ consisting of measures with compact support will be called $M_c(G)$.

Definition 1.3. If μ is a measure on G we define its Laplace transform $\mu^\wedge$ by the equation

$$\mu^\wedge(\omega) = \int \omega^{-1}(g)\, d\mu(g)$$

whenever ω is an element of Ω for which this integral converges absolutely.

Note that ω is in the domain of $\mu^\wedge$ if and only if $\int |\omega^{-1}(g)|\, d|\mu|(g) < \infty$.

LEMMA 1.1. (a) *If μ is a measure on G, then the domain of $\mu^\wedge$ has the form $e^A\Gamma$, where A is a convex subset of X.* (b) *If $\mu \in M_c(G)$ then $\mu^\wedge \in \mathfrak{A}(\Omega)$; the map $\mu \to \mu^\wedge$ is an isomorphism of $M_c(G)$ onto a subalgebra of $\mathfrak{A}(\Omega)$.*

Proof. Part (a) follows from Hölders inequality and part (b) is obvious.

The next lemma gives the form of the inversion theorem that we shall use in Section 5.

LEMMA 1.2. *If $\mu^\wedge$ exists on $e^x\Gamma$ for some $x \in X$ and $\mu^\wedge(e^x\gamma)$ is an integrable function of γ relative to Haar measure on Γ, then μ is absolutely continuous with a continuous Radon–Nikodym derivative h_μ given by*

$$h_\mu(g) = \int \mu^\wedge(\gamma e^x)\, \gamma(g)\, e^{x(g)} d\gamma,$$

where $d\gamma$ represents integration with respect to an appropriately normalized Haar measure on Γ.

Proof. If $\mu^\wedge$ exists on $e^x\Gamma$ and $d\nu(g) = e^{-x(g)} d\mu(g)$, then $\nu \in M(G)$ and $\nu^\wedge(\gamma) = \mu^\wedge(e^x\gamma)$. The lemma now follows from the inversion formula for the Fourier transform (cf. [6], Chapter 1).

Many of our later results depend heavily on the theory of holomorphic functions of several complex variables. For this reason, we impose conditions on G which ensure that

Ω is an analytic manifold. Henceforth, we shall assume that G satisfies the following conditions:

G1. The vector space X is finite dimensional.

G2. The subgroup $\Gamma_0 = e^{iX}$ of Γ is open (hence closed).

Using the structure theory for locally compact groups (cf. [6], chapter 2), one can deduce that G1 and G2 are equivalent to the following: The subgroup $H = \{g \in G : x(g) = 0$ for all $x \in X\}$ is compact and G/H has the form $R^p \times Z^q$, with R the additive group of reals and Z the additive group of integers.

If we set $\Omega_0 = \{e^{x+iy} : x, y \in X\}$, then Ω_0 consists of exactly those functions in Ω which are constant on cosets of H. Thus, we may consider a function $\omega \in \Omega_0$ to be a function on $G/H = R^p \times Z^q$. Since ω is a homomorphism, it must have the form

$$\omega(t_1, \ldots, t_p, k_1, \ldots, k_q) = e^{t_1 z_1 + \cdots + t_p z_p} \zeta_1^{k_1} \ldots \zeta_q^{k_q},$$

where $z_1, \ldots, z_p$ are arbitrary complex numbers and $\zeta_1, \ldots, \zeta_q$ are nonzero complex numbers. It follows that Ω_0 is isomorphic to the group

$$\mathbf{C}^p \times (\mathbf{C}^*)^q = \{(z_1, \ldots, z_p, \zeta_1, \ldots, \zeta_q) \in \mathbf{C}^{p+q} : \zeta_1, \ldots, \zeta_q \neq 0\},$$

where the operation is addition in the first factor and multiplication in the second factor. A glance at Definition 1.1 shows that if U is an open subset of Ω_0, then $f \in \mathfrak{A}(U)$ if and only if f is a holomorphic function of the variables $z_1, \ldots, z_p, \zeta_1, \ldots, \zeta_q$ in U. Condition G2 implies that Ω_0 is an open subgroup of Ω and, hence, Ω is the discrete union of the cosets of Ω_0. It follows from these considerations that Ω is a $p+q$ dimensional analytic manifold and the space $\mathfrak{A}(U)$ of Definition 1.1 is precisely the space of holomorphic functions on U in the ordinary sense.

If we represent G/H as $R^p \times Z^q$ and Ω_0 as $\mathbf{C}^p \times (\mathbf{C}^*)^q$, then we also have

$$\Gamma_0 = \Gamma \cap \Omega_0 = e^{iX}$$

$$= \{(z_1, \ldots, z_p, \zeta_1 \ldots, \zeta_p) \in \mathbf{C}^p \times (\mathbf{C}^*)^q : \operatorname{Re} z_1 = \ldots = \operatorname{Re} z_p = 0, |\zeta_1| = \ldots = |\zeta_q| = 1\}.$$

We may represent X as R^{p+q}, where $x = (x_1, \ldots, x_p, x_1', \ldots, x_q') \in X$ acts on $G/H = R^p \times Z^q$ as follows:

$$x(t_1, \ldots, t_p, k_1, \ldots, k_q) = x_1 t_1 + \ldots + x_p t_p + x_1' k_1 + \ldots + x_q' k_q.$$

If $\omega = (z_1, \ldots, z_p, \zeta_1, \ldots, \zeta_q) \in \Omega_0$, $x = (x_1, \ldots, x_p, x_1', \ldots, x_q') \in X$, and $y = (y_1, \ldots, y_p, y_1', \ldots, y_q') \in X$, then the equation $\omega = e^{x+iy}$ simply means $z_j = x_j + iy_j$ and $\zeta_k = e^{x_k' + iy_k'}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

We shall use the above coordinatized representations of G/H, Ω_0, and X in the remainder of this section but not in later sections.

If ε_g is the point mass at $g \in G$, then its Laplace transform $\varepsilon_g^\wedge$, restricted to Ω_0, has the form

$$\varepsilon_g^\wedge(\omega) = \omega^{-1}(g) = e^{-t_1 z_1 - \dots - t_p z_p} \zeta_1^{-k_1} \dots \zeta_q^{-k_q},$$

where g determines the point $(t_1, \dots, t_p, k_1, \dots, k_q)$ of $R^p \times Z^q = G/H$ and ω determines the $p+q$-tuple $(z_1, \dots, z_p, \zeta_1, \dots, \zeta_q)$ of $\mathbf{C}^p \times (\mathbf{C}^*)^q$. If $f \in \mathfrak{A}(\Omega_0)$ then f may be expanded in a multiple Laurent series in the variables $\zeta_1, \dots, \zeta_q$ with coefficients holomorphic in the variables $z_1, \dots, z_p$. Linear combinations of exponential functions are dense in $\mathfrak{A}(\mathbf{C}^p)$ in the topology of uniform convergence on compact subsets. It follows that the space of functions in $\mathfrak{A}(\Omega_0)$ which are linear combinations of the functions $\varepsilon_g^\wedge$ for $g \in G$, is dense in $\mathfrak{A}(\Omega_0)$ in the topology of uniform convergence on compact subsets of Ω_0.

LEMMA 1.3. *If $M_c^\wedge$ denotes the space of Laplace transforms of measures with compact support on G, then $M_c^\wedge$ is dense in $\mathfrak{A}(\Omega)$ in the topology of uniform convergence on compacta.*

Proof. Based on the above discussion, we have that $M_c^\wedge$ restricted to Ω_0, is dense in $\mathfrak{A}(\Omega_0)$. If K is a compact subset of Ω, then K is contained in a finite union $\bigcup_{i=1}^n \gamma_i \Omega_0$ of cosets of Ω_0. If $\mu \in M_c$ and ϱ is Haar measure on H, we set $d\mu_i = \gamma_i d(\mu \cdot \varrho)$. We have $\mu_i^\wedge(\gamma_i \omega) = \mu^\wedge(\omega)$ for $\omega \in \Omega_0$ and $\mu_i^\wedge = 0$ elsewhere on Ω. It follows that if $f \in \mathfrak{A}(\Omega)$ then f may be uniformly approximated on $K \cap \gamma_i \Omega_0$ by functions in $M_c^\wedge$ which are zero on $K \cap \gamma_j \Omega_0$ for $i \neq j$. Hence, f may be uniformly approximated on K by linear combinations of such functions.

The space Ω_0 is not only an analytic manifold; it is also a Stein manifold (cf. [4], Definition 5.1.3). This is easily seen from the representation of Ω_0 as $\mathbf{C}^p \times (\mathbf{C}^*)^q$. If U is an open convex subset of X, then $e^U \Gamma_0 = \{e^{x+iy} \in \Omega : x \in U, y \in X\}$ is holomorphically convex in Ω_0, since $\omega = (z_1, \dots, z_p, \zeta_1, \dots, \zeta_q) \in e^U \Gamma_0$ simply means that $(\operatorname{Re} z_1, \dots, \operatorname{Re} z_p, \log|\zeta_1|, \dots, \log|\zeta_q|)$ lies in an open convex set $U \subset R^{p+q}$. If Ω_0 does not have countable order in Ω, then Ω may not be a Stein manifold, since a Stein manifold is, by definition, countable at infinity. However, the following is true and is all we shall require:

LEMMA 1.4. *Each coset of Ω_0 in Ω is an n-dimensional Stein manifold where n is the dimension of X. Also if U is an open convex subset of X, then the intersection of $e^U \Gamma$ with each coset of Ω_0 is holomorphically convex in this coset.*

We shall apply results from the theory of holomorphic functions on Stein manifolds in several key places in the paper. The following lemmas state explicitly two such applications.

LEMMA 1.5. *If U is an open convex subset of X, then the space $M_c^\wedge$ of Laplace transforms of measures with compact support, is dense in $\mathfrak{A}(e^U \Gamma)$ in the topology of uniform convergence on compact subsets of $e^U \Gamma$.*

Proof. This follows directly from Lemmas 1.3. and 1.4 and VII.A.9 of [2], and the fact that $e^U\Gamma$ is the discrete union of its intersections with cosets of Ω_0.

LEMMA 1.6. *Let U be an open convex subset of X and let $f_1, \ldots, f_m, k$ be elements of $\mathfrak{A}(e^U\Gamma)$. If for each $\omega \in e^U\Gamma$, the equation $f_1 h_1 + \ldots + f_m h_m = k$ can be solved for functions $h_1, \ldots, h_m$ holomorphic in a neighborhood of ω, then this equation can be solved globally for functions $h_1, \ldots, h_m \in \mathfrak{A}(e^U\Gamma)$.*

Proof. Note that it is enough to prove this for the intersection of $e^U\Gamma$ with each coset of Ω_0. In view of Lemma 1.4, this is just a special case of Theorem 7.2.9 of [4].

A primary obstacle to extending the results of Section 6 to general groups is the lack of analytic function theory, including results like Lemmas 1.5 and 1.6, for Ω in the general case. It is possible that extensions can be obtained by placing more emphasis on the group theoretic aspects of the problem.

2. The algebras $\mathfrak{M}(A)$ and $\mathfrak{N}(A)$

Recall that $M(G)$ denotes the algebra of all regular Borel measures on G under convolution mutiplication. We shall work with a closed subalgebra N of $M(G)$ which satisfies the following conditions:

N1. If $\mu \in N$ and $\nu \in M(G)$ with ν absolutely continuous with respect to μ, then $\nu \in N$;

N2. N is dense in $M(G)$ in the weak-* topology of $M(G)$;

N3. N contains the identity e of $M(G)$.

Condition N1 makes N an L-subalgebra of $M(G)$ in the terminology of [8]. In the presence of N1, condition N2 is equivalent to the requirement that no open subset of G is a set of measure zero for every measure in N. Condition N3 is simply a convenience; if a subalgebra N does not contain the identity we can always adjoin it. The identity of $M(G)$ is the point mass at 0 in G.

Of particular importance is the algebra L consisting of all absolutely continuous measures in $M(G)$. The Radon–Nikodym derivative defines an isomorphism-isometry between L and $L^1(G)$, the algebra of Haar integrable functions on G. Clearly L satisfies N1 and N2 but not, in general, N3. Hence, we shall work with the algebra L_e consisting of all measures in $M(G)$ of the form $\mu + ze$, where $\mu \in L$ and z is any complex number.

Definition 2.1. Let N_{loc} denote the linear space of all measures μ on G such that the restriction μ_K of μ to each compact subset K of G, is in N.

Note that $M(G)_{\text{loc}}$ consists of all measures on G, with the term measure used as in Section 1. Also, $(L_e)_{\text{loc}}$ consists of all measures of the form $\mu + ze$, where $z \in \mathbf{C}$ and μ is an absolutely continuous measure (i.e., $\mu(E)=0$ whenever E is a bounded Borel set of Haar measure zero).

If A is a convex subset of X, then the convex tube in Ω based on A is the set $e^A\Gamma = \{\omega \in \Omega \colon \log|\omega| \in A\}$. Given a compact convex subset $A \subset X$ and the subalgebra N of $M(G)$, we shall define a Banach algebra $\mathfrak{N}(A)$ whose elements are in N_{loc} and have Laplace transforms defined in the tube based on A.

Definition 2.2. If A is a nonempty compact subset of X, set $\varphi_A(g) = \sup\{e^{-x(g)} \colon x \in A\}$ for each $g \in G$.

LEMMA 2.1. *If A and B are nonempty compact subsets of X, $x \in X$, and $g_1, g_2 \in G$, then:*

(a) *φ_A is continuous on G;*

(b) *$\varphi_A(g_1+g_2) \leqslant \varphi_A(g_1)\varphi_A(g_2)$ and $\varphi_A(0)=1$;*

(c) *$\varphi_A = \varphi_{\langle A\rangle}$, where $\langle A\rangle$ is the convex hull of A;*

(d) *$A \subset B$ implies $\varphi_A \leqslant \varphi_B$;*

(e) *$\varphi_{A\ \ B} = \max(\varphi_A, \varphi_B)$; and*

(f) *$\varphi_{A+x} = e^{-x}\varphi_A$.*

Proof. Part (a) follows from the compactness of A; part (b) follows from the multiplicativity of e^{-x} as a function on G; and part (c) follows from the convexity of $e^{-x(g)}$ as a function of x. The remaining parts of the lemma are immediate from the definition.

Definition 2.3. If A is a nonempty compact subset of X, then

(a) for $\mu \in M(G)_{\mathrm{loc}}$ set $\|\mu\|_A = \int \varphi_A(g)\, d|\mu|(g)$ and $\|\mu\|'_A = \int \varphi_A^{-1}(-g)\, d|\mu|(g)$;

(b) set $\mathfrak{M}(A) = \{\mu \in M(G)_{\mathrm{loc}} \colon \|\mu\|_A < \infty\}$ and $\mathfrak{M}'(A) = \{\mu \in M(G)_{\mathrm{loc}} \colon \|\mu\|'_A < \infty\}$; and

(c) for the subalgebra N of $M(G)$ set $\mathfrak{N}(A) = N_{\mathrm{loc}} \cap \mathfrak{M}(A) = \{\mu \in N_{\mathrm{loc}} \colon \|\mu\|_A < \infty\}$ and $\mathfrak{N}'(A) = N_{\mathrm{loc}} \cap \mathfrak{M}'(A) = \{\mu \in N_{\mathrm{loc}} \colon \|\mu\|'_A < \infty\}$.

We also set $\mathfrak{M}(\varnothing) = \mathfrak{N}(\varnothing) = (0)$, where $\varnothing$ is the empty set and (0) is the subalgebra of $M(G)$ consisting of the zero element.

LEMMA 2.2. *Let A be a compact subset of X. If $\mu \in \mathfrak{N}(A)$ and $\nu \in \mathfrak{N}'(A)$, then $\mu \cdot \nu$ exists, $\mu \cdot \nu \in \mathfrak{N}'(A)$, and $\|\mu \cdot \nu\|'_A \leqslant \|\mu\|_A \|\nu\|'_A$. If $\mu, \nu \in \mathfrak{N}(A)$ then $\mu \cdot \nu \in \mathfrak{N}(A)$ and $\|\mu \cdot \nu\|_A \leqslant \|\mu\|_A \|\nu\|_A$. Under convolution multiplication, $\mathfrak{N}(A)$ is a Banach algebra and $\mathfrak{N}'(A)$ is an $\mathfrak{N}(A)$-module.*

Proof. Clearly $\mathfrak{N}(A)$ and $\mathfrak{N}'_A(A)$ are Banach spaces under the norms $\|\ \|_A$ and $\|\ \|'_A$ respectively.

If N_c denotes the space of measures in N with compact support, then N_c is a dense subspace of both $\mathfrak{N}(A)$ and $\mathfrak{N}'(A)$; this follows from condition N1.

It follows from (b) of Lemma 2.1 that $\varphi_A^{-1}(-g_1-g_2) \leqslant \varphi_A(g_1)\varphi_A^{-1}(-g_2)$. Thus, if f is a continuous function on G with compact support, then

$$\left|\int f(g_1+g_2)\varphi_A^{-1}(-g_1-g_2)d\mu(g_1)\right| \leqslant \int |f(g_1+g_2)|\varphi_A(g_1)\varphi_A^{-1}(-g_2)d|\mu|(g_1)$$
$$\leqslant \varphi_A^{-1}(-g_2)\|f\|_\infty\|\mu\|_A$$

and
$$\left|\iint f(g_1+g_2)\varphi_A^{-1}(-g_1-g_2)d\mu(g_1)d\nu(g_2)\right| \leqslant \|f\|_\infty\|\mu\|_A\|\nu\|_A'.$$

It follows from the Riesz representation theorem that there is a finite measure ϱ such that

$$\int f d\varrho = \iint f(g_1+g_2)\varphi^{-1}(-g_1-g_2)d\mu(g_1)d\nu(g_2)$$

for every continuous function f with compact support, and $\|\varrho\| \leqslant \|\mu\|_A\|\nu\|_A'$. If we set $d\lambda(g) = \varphi_A(-g)d\varrho(g)$, then $\|\lambda\|_A' = \|\varrho\| \leqslant \|\mu\|_A\|\nu\|_A'$ and $\int f d\lambda = \iint f(g_1+g_2)d\mu(g_1)d\nu(g_2)$ for every continuous function f with compact support. Hence, λ is the convolution product $\mu\cdot\nu$. If μ and ν have compact support, then $\mu\cdot\nu \in N_c \subset \mathfrak{N}'(A)$, since N is an algebra. That $\mu\cdot\nu\in\mathfrak{N}'(A)$ in general now follows from the fact that N_c is dense in both $\mathfrak{N}(A)$ and $\mathfrak{N}'(A)$ and the inequality $\|\mu\cdot\nu\|_A' \leqslant \|\mu\|_A\|\nu\|_A'$.

If $\mu, \nu \in \mathfrak{N}(A)$ then we may repeat the above argument, using the inequality $\varphi_A(g_1+g_2) \leqslant \varphi_A(g_1)\varphi_A(g_2)$, and conclude that $\mu\cdot\nu\in\mathfrak{N}(A)$ and $\|\mu\cdot\nu\|_A \leqslant \|\mu\|_A\|\nu\|_A$.

Since the associative, distributive, and commutative laws hold in N_c, they hold also in $\mathfrak{N}(A)$. Hence, $\mathfrak{N}(A)$ is a commutative Banach algebra. Through convolution, $\mathfrak{N}(A)$ acts as an algebra of bounded linear operators on $\mathfrak{N}'(A)$. In this sense, $\mathfrak{N}_A'(A)$ is an $\mathfrak{N}(A)$-module.

For each compact subset A of X, let $\mathfrak{A}(e^A\Gamma)$ denote the algebra of functions on the tube $e^A\Gamma$ which are uniform limits on $e^A\Gamma$ of functions bounded and holomorphic in a neighborhood of $e^A\Gamma$. We give $\mathfrak{A}(e^A\Gamma)$ the supremum norm.

Lemma 2.3. *The Laplace transform is a norm decreasing isomorphism of* $\mathfrak{N}(A)$ *into* $\mathfrak{A}(e^A\Gamma)$.

Proof. If $\mu\in\mathfrak{N}(A)$ and $x\in A$, then $\int e^{-x}d|\mu| \leqslant \int\varphi_A d|\mu| = \|\mu\|_A < \infty$. It follows that $\mu^\wedge(\omega)$ exists at every point ω of $e^A\Gamma$ and $|\mu^\wedge(\omega)| \leqslant \|\mu\|_A$. If $\mu\in N_c$ then $\mu^\wedge\in\mathfrak{A}(\Omega)$ and $\mu^\wedge$ is bounded in a neighborhood of $e^A\Gamma$. Since N_c is dense in $\mathfrak{N}(A)$ we have $\mu^\wedge\in\mathfrak{A}(e^A\Gamma)$ for every $\mu\in\mathfrak{N}(A)$. Thus, $\mu\to\mu^\wedge$ is a homomorphism of $\mathfrak{N}(A)$ into $\mathfrak{A}(e^A\Gamma)$. If for some $x\in A$,

$\mu^\wedge(e^x\gamma)=0$ for all $\gamma \in \Gamma$, then $\nu^\wedge(\gamma)=0$ for all $\gamma \in \Gamma$, where $d\nu = e^{-x}d\mu$. However, the Fourier transform is one to one on $M(G)$ and so $\nu=0$; i.e., $\mu=0$ if $\mu^\wedge=0$ on $e^A\Gamma$. This completes the proof.

LEMMA 2.4. *If A and B are nonempty compact subsets of X and $x \in X$, then*

(a) *$B \subset A$ implies $\mathfrak{R}(A) \subset \mathfrak{R}(B) \subset \mathfrak{R}'(B) \subset \mathfrak{R}'(A)$;*

(b) *$\mathfrak{R}(A \cup B) = \mathfrak{R}(A) \cap \mathfrak{R}(B)$, and $\mathfrak{R}(A) = \mathfrak{R}(\langle A \rangle)$;*

(c) *$\mathfrak{R}(\{x\}) = \mathfrak{R}'(\{x\})$ and $\mathfrak{R}(\{0\}) = N$;*

(d) *if we set $dT_x\mu = e^x d\mu$, then T_x is an isomorphism-isometry of $\mathfrak{R}(A)$ onto $\mathfrak{R}(A+x)$;*

(e) *for fixed $\mu \in \mathfrak{R}(A)$, the map $x \to T_{-x}\mu$ is a continuous map of A into N.*

Proof. Part (a) follows from the inequality $\varphi_A^{-1}(-g) \leqslant \varphi_B^{-1}(-g) \leqslant \varphi_B(g) \leqslant \varphi_A(g)$ (cf. Lemma 2.1 (b) and (d)). Part (b) follows from (e) of Lemma 2.1. Part (c) follows from the fact that $\varphi_{\{x\}}(g) = e^{-x(g)} = (e^{x(g)})^{-1} = \varphi_{\{x\}}^{-1}(-g)$ and $\varphi_{\{0\}} \equiv 1$.

Clearly the map T_x preserves convolution and is linear. Part (f) of Lemma 2.1 shows that T_x maps $\mathfrak{R}(A)$ isometrically onto $\mathfrak{R}(A+x)$. If $x \in A$ then T_{-x} maps $\mathfrak{R}(A)$ onto $\mathfrak{R}(A-x) \subset \mathfrak{R}(\{0\}) = N$. If $x \in A$ and $\mu \in \mathfrak{R}(A)$ then $e^{-x} \leqslant \varphi_A \in L^1(\mu)$. It follows from the Lebesgue dominated convergence theorem that if $\{x_n\}_{n=1}^\infty \subset A$ and $x_n \to x \in A$, then $e^{-x_n} \to e^{-x}$ in $L^1(\mu)$ norm and, hence, $T_{-x_n}\mu \to T_{-x}\mu$ in measure norm in N. This establishes (d) and (e).

We should point out that part (a) of the above lemma refers to set theoretic containment. The norms in the spaces $\mathfrak{R}(A)$, $\mathfrak{R}(B)$, $\mathfrak{R}'(B)$, and $\mathfrak{R}'(A)$ are all different. However, the linear structures of these spaces are consistent and convolution is consistent in $\mathfrak{R}(B)$ and $\mathfrak{R}(A)$. Note that we also have the following: $\mathfrak{R}(B)$ is not only a linear subspace of $\mathfrak{R}'(A)$, but it is also closed under multiplication by elements of $\mathfrak{R}(A)$; hence $\mathfrak{R}(B)$ is a submodule of the $\mathfrak{R}(A)$-module $\mathfrak{R}'(A)$. One must, however, exercise care in using these relationships. A typical situation is the following: There may be disjoint subsets B and C of A and a measure $\mu \in \mathfrak{R}(A)$ such that μ has an inverse ν_B as an element of $\mathfrak{R}(B)$ and an inverse $\nu_C \neq \nu_B$ as an element of $\mathfrak{R}(C)$; since $\nu_B, \nu_C \in \mathfrak{R}'(A)$, the computation $\mu \cdot (\nu_B - \nu_C) = e - e = 0$ is valid; however, the computation $0 = \nu_B \cdot 0 = \nu_B \cdot \mu \cdot (\nu_B - \nu_C) = e(\nu_B - \nu_C) = \nu_B - \nu_C$ is not valid because $\nu_B \notin \mathfrak{R}(A)$ and we cannot treat $\mathfrak{R}'(A)$ as an $\mathfrak{R}(B)$-module. In the case where X has dimension one, the residue measure constructed in Section 4 has precisely the form $\nu_B - \nu_C$ as above. The situation described above cannot occur when $B \cap C \neq \varnothing$, for in this case $\mathfrak{R}(A)$, $\mathfrak{R}(B)$, and $\mathfrak{R}(C)$ are all subalgebras of $\mathfrak{R}(B \cap C)$.

The following lemma will be important when we apply the results of Section 4 to ideal theory in $\mathfrak{R}(A)$:

LEMMA 2.5. *Let A be a compact subset of X and $\mu_1, \ldots, \mu_n \in \mathfrak{N}(A)$. If $x \in A$ and the equation $\mu_1 \nu_1 + \ldots + \mu_n \nu_n = e$ has a solution for $\nu_1, \ldots, \nu_n \in \mathfrak{N}(\{x\})$, then there is a closed set E, containing x in its interior, such that this equation can be solved for $\nu_1, \ldots, \nu_n \in \mathfrak{N}(A \cap E)$.*

Proof. Choose $\nu_1, \ldots, \nu_n \in \mathfrak{N}(\{x\})$ such that $\mu_1 \nu_1 + \ldots + \mu_n \nu_n = e$. Since N_c is dense in $\mathfrak{N}(\{x\})$, we may choose $\nu_1', \ldots, \nu_n' \in N_c$ such that $\|\mu_1 \nu_1' + \ldots + \mu_n \nu_n' - e\|_{\{x\}} < 1$. For each n let U_n be the open ball of radius $1/n$ in X centered at x, and set $B_n = \bar{U}_n \cap A$. Since $\mu_1 \nu_1' + \ldots + \mu_n \nu_n' - e = \lambda \in \mathfrak{N}(A)$, the function φ_A is λ-integrable. We have $\varphi_{B_n} \to e^{-x}$ pointwise on G and $\varphi_{B_n} \leqslant \varphi_A$ for each n. Thus, $\|\lambda\|_{B_n} = \int \varphi_{B_n} d|\lambda| \to \int e^{-x} d|\lambda| = \|\lambda\|_{\{x\}} < 1$, by the Lebesgue dominated convergence theorem. Hence, we may choose an n for which $\|\lambda\|_{B_n} < 1$. This implies that $\mu_1 \nu_1' + \ldots + \mu_n \nu_n'$ has an inverse ϱ in $\mathfrak{N}(B_n)$. If we set $\varrho_i = \varrho \nu_i'$, then $\mu_1 \varrho_1 + \ldots + \mu_n \varrho_n = e$ and $\varrho_1, \ldots, \varrho_n \in \mathfrak{N}(B_n) = \mathfrak{N}(A \cap \bar{U}_n)$. This completes the proof.

The next two lemmas give the crucial facts needed in calculating the cohomology modules defined in Section 3.

LEMMA 2.6. *If A and B are compact subsets of X with $A \cup B$ convex and if $\mu \in \mathfrak{N}(A \cap B)$, then there exist $\mu_A \in \mathfrak{N}(A)$ and $\mu_B \in \mathfrak{N}(B)$ such that $\mu = \mu_A + \mu_B$.*

Proof. We first show that if $A \cup B$ is convex, then $\varphi_{A \cap B} = \min(\varphi_A, \varphi_B)$. For a fixed $g \in G$, choose $x \in A$ and $y \in B$ such that $\varphi_A(g) = e^{-x(g)}$ and $\varphi_B(g) = e^{-y(g)}$; such a choice is possible because A and B are compact. Since $A \cup B$ is convex, the line segment joining x and y lies entirely in $A \cup B$. Since A and B are closed, there must be a point $u = tx + (1-t)y$ on this line segment such that $u \in A \cap B$. We have $u(g) = tx(g) + (1-t)y(g)$ lies between $x(g)$ and $y(g)$, and so $e^{-u(g)}$ lies between $\varphi_A(g) = e^{-x(g)}$ and $\varphi_B(g) = e^{-y(g)}$. However, $e^{-u(g)} \leqslant \varphi_{A \cap B}(g) \leqslant \min(\varphi_A(g), \varphi_B(g))$. It follows that $\varphi_{A \cap B}(g) = \min(\varphi_A(g), \varphi_B(g))$.

If we set $f = \varphi_{A \cap B} \varphi_A^{-1}$, then $0 < f \leqslant 1$, $\varphi_A f = \varphi_{A \cap B}$, and $\varphi_B(1-f) \leqslant \varphi_{A \cap B}$; the last inequality follows from the fact that for each $g \in G$ either $f(g) = 1$ or $\varphi_B(g) = \varphi_{A \cap B}(g)$. We set $d\mu_A = f d\mu$ and $d\mu_B = (1-f) d\mu$. We then have

$$\|\mu_A\|_A = \int \varphi_A d|\mu_A| = \int \varphi_A f d|\mu| = \int \varphi_{A \cap B} d|\mu| = \|\mu\|_{A \cap B} < \infty,$$

and, similarly,
$$\|\mu_B\|_B \leqslant \|\mu\|_{A \cap B} < \infty.$$

Thus, $\mu_A \in \mathfrak{N}(A)$, $\mu_B \in \mathfrak{N}(B)$, and clearly $\mu = \mu_A + \mu_B$.

LEMMA 2.7. *If $A_1, \ldots, A_n$, and B are compact subsets of X, $\mu_i \in \mathfrak{N}(A_i)$ for $i = 1, \ldots, n$, and $\mu_1 + \ldots + \mu_n = \mu \in \mathfrak{N}(B)$, then there are measures $\nu_i \in \mathfrak{N}(\langle A_i, B \rangle)$ for $i = 1, \ldots, n$ such that $\nu_1 + \ldots + \nu_n = \mu$, where $\langle A_i, B \rangle$ represents the convex hull of $A_i \cup B$.*

Proof. We may assume without loss of generality that the measures $\mu_1, \ldots, \mu_n$ are absolutely continuous with respect to μ, for the μ-singular parts of the measures $\mu_1, \ldots, \mu_n$ must add up to zero. Hence, we assume $d\mu_i = f_i d\mu$ for $i = 1, \ldots, n$. We shall choose $h_1, \ldots, h_n$ such that $|h_i| \leq \min(|f_i|, 1)$ and $h_1 + \ldots + h_n = 1$. Let $E_1 = \{g \in G: |f_1(g)| > 1\}$ and $E_i = E_{i-1} \cup \{g \in G: |f_i(g)| > 1\}$ for $i = 2, \ldots, n$. We define $h_i(g) = f_i(g)$ for $g \in G \setminus E_n$, $h_i(g) = 1$ for $g \in E_i \setminus E_{i-1}$, and $h_i(g) = 0$ for all other $g \in G$. Clearly, $\{h_i\}_{i=1}^n$ has the required properties and each h_i is μ-measurable.

If we set $d\nu_i = h_i d\mu$, then $|\nu_i| \leq |\mu_i|$ and $|\nu_i| \leq |\mu|$, since $|h_i| \leq \min(|f_i|, 1)$. It follows that $\nu_i \in \mathfrak{N}(A_i) \cap \mathfrak{N}(B) = \mathfrak{N}(A_i \cup B) = \mathfrak{N}(\langle A_i, B \rangle)$. We have $\nu_1 + \ldots + \nu_n = \mu$, since $h_1 + \ldots + h_n = 1$.

Note that throughout this section we have tacitly used the fact that N satisfies condition N1. We used condition N3 in Lemma 2.5 and we shall use it extensively beginning with Section 4. Our first use of condition N2 comes in the next, and final, lemma of this section. This is an approximation result that we shall apply in Sections 5 and 6.

LEMMA 2.8. *If U is an open convex subset of X, then the space $N_c^\wedge$ of Laplace transforms of elements of N_c, is dense in $\mathfrak{A}(e^U\Gamma)$ in the topology of uniform convergence on compact subsets of $e^U\Gamma$.*

Proof. By Lemma 1.5, $M_c^\wedge$ is dense in $\mathfrak{A}(e^U\Gamma)$. Thus, it suffices to prove that $N_c^\wedge$ is dense in $M_c^\wedge$ in the topology of uniform convergence on compact subsets of $e^U\Gamma$. Let K be a compact subset of $e^U\Gamma$. If $\mu \in M_c$ and $\mu^\wedge$ is not in the uniform closure on K of $N_c^\wedge$, then there exists a finite regular Borel measure λ on K such that $\int \mu^\wedge d\lambda \neq 0$ but $\int \nu^\wedge d\lambda = 0$ for all $\nu \in N_c$. However, $\int \nu^\wedge d\lambda = \int \lambda^\wedge d\nu$, where $\lambda^\wedge(g) = \int \omega^{-1}(g) d\lambda(\omega)$ is a continuous function on G. Since N is weak-$*$ dense in $M(G)$ (condition $N2$), we have that $\lambda^\wedge = 0$. However, $\int \lambda^\wedge d\mu = \int \mu^\wedge d\lambda \neq 0$. The resulting contradiction shows that $M_c^\wedge$ is contained in the uniform closure of $N_c^\wedge$ on K. This completes the proof.

3. Cohomology

In this section we develop a cohomology theory which greatly facilitates the study of the spaces defined in Section 2. Our theory is patterned after the Čech cohomology theory of presheaves. We shall need only the more primitive notions of this theory. Introductions to the theory of sheaves and presheaves appear in [2], Chapters IV and VI, and [4], Chapter VII. Our development will initially parallel that of [4].

Rather than working explicitly with the spaces $\mathfrak{N}(A)$ of the previous section, we shall, in this section and Section 4, work with somewhat more abstract spaces. We do this with the hope that possible future generalizations of our results might be obtained without excessive reworking of these sections.

Definition 3.1. Let $\mathcal{R}$ be a ring, A a set, and $\mathcal{S}$ a family of subsets of A such that $\mathcal{S}$ is closed under finite intersection and contains the empty set $\emptyset$. Let $\mathfrak{T}$ be a map which assigns to each $B\in\mathcal{S}$ an $\mathcal{R}$-module $\mathfrak{T}(B)$, and let θ be a map which assigns to each pair $(B, C)\subset\mathcal{S}$, with $B\subset C$, an $\mathcal{R}$-module homomorphism $\theta_{B,C}$: $\mathfrak{T}(C)\to\mathfrak{T}(B)$. We shall call $\mathfrak{T}$ a stack of $\mathcal{R}$-modules on $\mathcal{S}$ if the following conditions are satisfied:

(a) $\mathfrak{T}(\emptyset)=(0)$;

(b) $\theta_{B,C}\circ\theta_{C,D}=\theta_{B,D}$ for $B\subset C\subset D$; and

(c) $\theta_{B,B}=\mathrm{id}$: $\mathfrak{T}(B)\to\mathfrak{T}(B)$ for $B\in\mathcal{S}$.

Ordinarily, A is a topological space and $\mathcal{S}$ consists of all open subsets of A. The term presheaf, rather than stack, is more common in this situation. Swan in [7] uses the term stack. We have borrowed this term to use in our peculiar setting. In our development, A will be a compact convex set and $\mathcal{S}$ will consist of compact convex subsets of A. However, in defining our cohomology modules, we shall use only finite collections of sets in $\mathcal{S}$ and we shall not pass to the direct limit. Therefore, the topological properties of the sets in $\mathcal{S}$ will be unimportant.

Note that if A is a compact convex subset of X and $\mathcal{S}$ consists of all compact convex subsets of A, then the correspondence $B\to\mathfrak{R}(B)$ of Section 2, defines a stack $\mathfrak{R}$ of $\mathfrak{R}(A)$-modules on $\mathcal{S}$, where $\theta_{B,C}$ is the injection of $\mathfrak{R}(C)$ into $\mathfrak{R}(B)$ for $\emptyset\neq B\subset C$ and $\theta_{\emptyset,C}=0$ for all C.

Throughout this section $\mathfrak{T}$ will denote a stack of $\mathcal{R}$-modules on a collection of sets $\mathcal{S}$. If $B, C\in\mathcal{S}$, $B\subset C$, and $\mu\in\mathfrak{T}(C)$, we will often write $\mu|_B$ for $\theta_{B,C}\mu$.

Definition 3.2. (a) By a complex $\mathcal{B}$ in $\mathcal{S}$, we shall mean a finite indexed family $\{B_1, ..., B_n\}$ of elements of $\mathcal{S}$.

(b) Let I_n^{p+1} denote the collection of all ordered $(p+1)$-tuples $(s_0, ..., s_p)$ of integers between 1 and n. If $\mathcal{B}=\{B_1, ..., B_n\}$ is a complex and $p\geqslant 0$, then a p-cochain c will be a map $\sigma\to c_\sigma$ which assigns to each $\sigma=(s_0, ..., s_p)\in I_n^{p+1}$ an element $c_\sigma\in\mathfrak{T}(B_{s_0}\cap ...\cap B_{s_p})$ such that $c_\sigma=-c_{\sigma'}$ whenever σ' is an odd permutation of σ.

(c) For $p\geqslant 0$ let $C^p(\mathcal{B}, \mathfrak{T})$ denote the $\mathcal{R}$-module of all p-cochains for the complex $\mathcal{B}$ and the stack $\mathfrak{T}$, where for $b, c\in C^p(\mathcal{B}, \mathfrak{T})$ and $r\in\mathcal{R}$ we define $(b+c)_\sigma=b_\sigma+c_\sigma$ and $(rc)_\sigma=rc_\sigma$.

(d) If $c\in C^p(\mathcal{B}, \mathfrak{T})$ and $\sigma=(s_0, ..., s_{p+1})\in I_n^{p+2}$, we set

$$(\delta^p c)_\sigma=\sum_{k=0}^{p+1}(-1)^k c_{\sigma_k}|_{B_{s_0}\cap\ldots\cap B_{s_{p+1}}},$$

where $\sigma_k=(s_0, ..., \hat{s}_k, ..., s_{p+1})$ is σ with the index s_k deleted.

Note that for $p \geq 0$, the map δ^p is an $\mathcal{R}$-module homomorphism of $C^p(\mathcal{B}, \mathfrak{T})$ into $C^{p+1}(\mathcal{B}, \mathfrak{T})$. A simple calculation shows that $\delta^{p+1} \circ \delta^p = 0$ for $p \geq 0$. Hence, we have a sequence

$$0 \to C^0(\mathcal{B}, \mathfrak{T}) \xrightarrow{\delta^0} C^1(\mathcal{B}, \mathfrak{T}) \xrightarrow{\delta^1} \dots C^p(\mathcal{B}, \mathfrak{T}) \xrightarrow{\delta^p} C^{p+1}(\mathcal{B}, \mathfrak{T}) \to \dots \tag{1}$$

in which the image of each map is contained in the kernel of the succeeding map.

Note that if $\sigma = (s_0, \dots, s_p) \in I_n^{p+1}$ and $s_i = s_j$ for some $i \neq j$, then the requirement that $c_\sigma = -c_{\sigma'}$ when σ' is an odd permutation of σ, forces $c_\sigma = 0$ for $c \in C^p(\mathcal{B}, \mathfrak{T})$. Hence, if the complex $\mathcal{B}$ contains n sets, then $C^p(\mathcal{B}, \mathfrak{T}) = 0$ when $p > n-1$.

Definition 3.3. We denote the image of δ^p by $B^p(\mathcal{B}, \mathfrak{T})$ and the kernel of δ^{p+1} by $Z^p(\mathcal{B}, \mathfrak{T})$. Elements of $Z^p(\mathcal{B}, \mathfrak{T})$ will be called p-cocycles and elements of $B^p(\mathcal{B}, \mathfrak{T})$ will be called p-coboundaries. We set $H^p(\mathcal{B}, \mathfrak{T}) = Z^p(\mathcal{B}, \mathfrak{T})/B^p(\mathcal{B}, \mathfrak{T})$ for $p > 0$, and $H^0(\mathcal{B}, \mathfrak{T}) = Z^0(\mathcal{B}, \mathfrak{T})$. The $\mathcal{R}$-module $H^p(\mathcal{B}, \mathfrak{T})$ is called the pth cohomology module of the complex $\mathcal{B}$ with coefficients in $\mathfrak{T}$.

A subcollection $\mathcal{S}'$ of $\mathcal{S}$ which is closed under finite intersection will be called a subdomain of $\mathcal{S}$. If $\mathcal{S}'$ is a subdomain of $\mathcal{S}$, $\mathfrak{T}$ and $\mathfrak{T}'$ are stacks of $\mathcal{R}$-modules on $\mathcal{S}$, and α assigns to each set $B \in \mathcal{S}'$ an $\mathcal{R}$-module homomorphism $\alpha_B : \mathfrak{T}(B) \to \mathfrak{T}'(B)$, then α will be called a stack homomorphism from $\mathfrak{T}$ to $\mathfrak{T}'$ on $\mathcal{S}'$ provided α commutes with restriction; i.e., provided $\alpha_B \theta_{B,C} = \theta'_{B,C} \alpha_C$ for $B, C \in \mathcal{S}'$ with $B \subset C$. A sequence, $\dots \to \mathfrak{T} \xrightarrow{\alpha} \mathfrak{T}' \xrightarrow{\beta} \mathfrak{T}'' \to \dots$, of stack homomorphisms, each defined on the same subdomain $\mathcal{S}'$, will be called exact on $\mathcal{S}'$ if the sequence $\dots \to \mathfrak{T}(B) \xrightarrow{\alpha_B} \mathfrak{T}'(B) \xrightarrow{\beta_B} \mathfrak{T}''(B) \to \dots$ is exact for each $B \in \mathcal{S}'$. Where it will not cause confusion, we shall write simply α for each of the homomorphisms α_B making up a stack homomorphism.

If $\alpha : \mathfrak{T} \to \mathfrak{T}'$ is a stack homomorphism on a subdomain $\mathcal{S}' \subset \mathcal{S}$, and if $\mathcal{B}$ is a complex whose elements are elements of $\mathcal{S}'$, then α defines a homomorphism $\bar{\alpha} : C^p(\mathcal{B}, \mathfrak{T}) \to C^p(\mathcal{B}, \mathfrak{T}')$ for each p, by $(\bar{\alpha} c)_\sigma = \alpha(c_\sigma)$. Clearly, $\bar{\alpha}$ commutes with δ and, thus, defines a homomorphism α^*: $H^p(\mathcal{B}, \mathfrak{T}) \to H^p(\mathcal{B}, \mathfrak{T}')$. We have the following version of the standard lemma for exact sequences (cf. [4], Theorem 7.3.4):

LEMMA 3.1. *Let $\mathcal{S}'$ be a subdomain of $\mathcal{S}$ and let $\mathcal{B}$ be a complex whose elements are in $\mathcal{S}'$. If $0 \to \mathfrak{T} \xrightarrow{\alpha} \mathfrak{T}' \xrightarrow{\beta} \mathfrak{T}'' \to 0$ is an exact sequence of stack homomorphisms on $\mathcal{S}'$, then the sequence*

$$0 \to C^p(\mathcal{B}, \mathfrak{T}) \xrightarrow{\bar{\alpha}} C^p(\mathcal{B}, \mathfrak{T}') \xrightarrow{\bar{\beta}} C^p(\mathcal{B}, \mathfrak{T}'') \to 0$$

is exact for each p. Furthermore, there is a homomorphism δ^*: $H^p(\mathcal{B}, \mathfrak{T}'') \to H^{p+1}(\mathcal{B}, \mathfrak{T})$ such that the following sequence is exact:

$$0 \to H^0(\mathcal{B}, \mathfrak{T}) \to \dots \to H^p(\mathcal{B}, \mathfrak{T}) \xrightarrow{\alpha^*} H^p(\mathcal{B}, \mathfrak{T}') \xrightarrow{\beta^*} H^p(\mathcal{B}, \mathfrak{T}'') \xrightarrow{\delta^*} H^{p+1}(\mathcal{B}, \mathfrak{T}) \to \dots.$$

If $\mathcal{B}=(B_1, ..., B_n)$ and $\mathcal{C}=(C_1, ..., C_m)$ are complexes in $\mathcal{S}$, we shall say that $\mathcal{C}$ is a refinement of $\mathcal{B}$ if each set in $\mathcal{C}$ is contained in some set in $\mathcal{B}$. In such a case, we may choose a map $k: I_m \to I_n$ such that $C_i \subset B_{k(i)}$ for $i=1, ..., m$. For $c \in C^p(\mathcal{B}, \mathfrak{T})$ we set

$$(\bar{k}c)_{(s_0, \ldots, s_p)} = c_{(k(s_0), \ldots, k(s_p))} \mid C_{s_0} \cap \ldots \cap C_{s_p}.$$

Clearly, this defines a homomorphism $\bar{k}$ of $C^p(\mathcal{B}, \mathfrak{T})$ into $C^p(\mathcal{C}, \mathfrak{T})$ which commutes with δ. Thus, $\bar{k}$ induces a homomorphism k^*: $H^p(\mathcal{B}, \mathfrak{T}) \to H^p(\mathcal{C}, \mathfrak{T})$.

LEMMA 3.2. *If $\mathcal{C}$ is a refinement of $\mathcal{B}$, then the map k^*, defined above, is independent of the choice of k. If $\mathcal{B}$ is also a refinement of $\mathcal{C}$, then k^* is an isomorphism of $H^p(\mathcal{B}, \mathfrak{T})$ onto $H^p(\mathcal{C}, \mathfrak{T})$.*

Proof. The first statement is essentially Proposition 7.3.1 of [4]. The second statement follows from the first. In fact, if $\mathcal{B}$ and $\mathcal{C}$ are mutual refinements of one another, then there are maps k: $I_m \to I_n$ and l: $I_n \to I_m$ such that $C_i \subset B_{k(i)}$ and $B_j \subset C_{l(j)}$ $(i=1, ..., m; j=1, ..., n)$. Thus, we have maps $kl: I_n \to I_n$ and lk: $I_m \to I_m$, with $B_{(j)} \subset B_{kl(j)}$ and $C_i \subset C_{lk(i)}$. The first statement of the lemma implies that $l^*j^* = (kl)^* = \mathrm{id}: H^p(\mathcal{B}, \mathfrak{T}) \to H^p(\mathcal{B}, \mathfrak{T})$ and $k^*l^* = (kl)^* = \mathrm{id}$: $H^p(\mathcal{C}, \mathfrak{T}) \to H^p(\mathcal{C}, \mathfrak{T})$. In other words, k^* is an isomorphism with inverse l^*.

Let $\mathcal{S}'$ be a subdomain of $\mathcal{S}$ and C a subset of A with the property that $C \cap B \in \mathcal{S}'$ whenever $B \in \mathcal{S}'$. We may define a stack $\mathfrak{T}_C$ on $\mathcal{S}'$ as follows: Let $\mathfrak{T}_C(B) = \mathfrak{T}(B \cap C)$ for each $B \in \mathcal{S}'$, and define the restriction maps in the obvious way. The restriction map from $\mathfrak{T}(B)$ into $\mathfrak{T}(B \cap C)$ defines a stack homomorphism $\alpha_C: \mathfrak{T} \to \mathfrak{T}_C$ on $\mathcal{S}'$, and hence, a homomorphism $\bar{\alpha}_C$: $C^p(\mathcal{B}, \mathfrak{T}) \to C^p(\mathcal{B}, \mathfrak{T}_C)$ whenever $\mathcal{B}$ is a complex contained in $\mathcal{S}'$. Also, note that if $\mathcal{B}=(B_1, ..., B_n) \subset \mathcal{S}'$, then C determines a new complex $\mathcal{B}_C = (B_1 \cap C, ..., B_n \cap C)$. There is, trivially, an isomorphism between $C^p(\mathcal{B}, \mathfrak{T}_C)$ and $C^p(\mathcal{B}_C, \mathfrak{T})$ which commutes with δ. Hence, there is a homomorphism of $C^p(\mathcal{B}, \mathfrak{T})$ to $C^p(\mathcal{B}_C, \mathfrak{T})$, which we shall also call $\bar{\alpha}_C$; this clearly commutes with δ.

If C and D are both subsets of A for which $C \cap B$ and $D \cap B$ are in the subdomain $\mathcal{S}'$ whenever B is, then we have a sequence $0 \to \mathfrak{T} \xrightarrow{\alpha_C \oplus \alpha_D} \mathfrak{T}_C \oplus \mathfrak{T}_D \xrightarrow{\beta_C - \beta_D} \mathfrak{T}_{C \cap D} \to 0$ of stack homomorphisms, where $(\alpha_C \oplus \alpha_D)(\mu) = \alpha_C(\mu) \oplus \alpha_D(\mu) = \mu|_{C \cap B} \oplus \mu|_{D \cap B}$ for $\mu \in \mathfrak{T}(B)$, and $(\beta_C - \beta_D)(\mu \oplus \nu) = \mu|_{C \cap D \cap B} - \nu|_{C \cap D \cap B}$ for $\mu \in \mathfrak{T}_C(B)$, $\nu \in \mathfrak{T}_D(B)$. If this sequence is exact on $\mathcal{S}'$, then, by Lemma 3.1, it induces an exact sequence

$$0 \to H^0(\mathcal{B}, \mathfrak{T}) \to \ldots \to H^p(\mathcal{B}, \mathfrak{T}) \xrightarrow{(\alpha_C \oplus \alpha_D)^*} H^p(\mathcal{B}, \mathfrak{T}_C \oplus \mathfrak{T}_D)$$
$$\xrightarrow{(\beta_C - \beta_D)^*} H^p(\mathcal{B}, \mathfrak{T}_{C \cap D}) \xrightarrow{\delta^*} H^{p+1}(\mathcal{B}, \mathfrak{T}) \to \ldots$$

whenever $\mathcal{B}$ is a complex contained in $\mathcal{S}'$. If we note that $H^p(\mathcal{B}, \mathfrak{T}_C \oplus \mathfrak{T}_D)$ is naturally isomorphic to $H^p(\mathcal{B}, \mathfrak{T}_C) \oplus H^p(\mathcal{B}, \mathfrak{T}_D)$ and use the isomorphisms between $H^p(\mathcal{B}, \mathfrak{T}_C)$, $H^p(\mathcal{B}, \mathfrak{T}_D)$,

$H^p(\mathcal{B}, \mathfrak{T}_{C\cap D})$ and $H^p(\mathcal{B}_C, \mathfrak{T})$, $H^p(\mathcal{B}_D, \mathfrak{T})$, $H^p(\mathcal{B}_{C\cap D}, \mathfrak{T})$ respectively, then we have the exact sequence

$$0 \to H^0(\mathcal{B}, \mathfrak{T}) \to \ldots \to H^p(\mathcal{B}, \mathfrak{T}) \xrightarrow{\alpha_C^* \oplus \alpha_D^*} H^p(\mathcal{B}_C, \mathfrak{T}) \oplus H^p(\mathcal{B}_D, \mathfrak{T})$$
$$\xrightarrow{\beta_C^* - \beta_D^*} H^p(\mathcal{B}_{C\cap D}) \xrightarrow{\delta^*} H^{p+1}(\mathcal{B}, \mathfrak{T}) \to \ldots.$$

This is called the Mayer–Vietoris sequence. The next lemma summarizes this discussion. Conditions (1) and (2) of the lemma simply insure that the sequence $0 \to \mathfrak{T} \to \mathfrak{T}_C \oplus \mathfrak{T}_D \to \mathfrak{T}_{C\cap D} \to 0$ is exact

LEMMA 3.3. *Let $\mathcal{S}'$ be a subdomain of $\mathcal{S}$ and C and D subsets of A for which $C \cap B$ and $D \cap B$ are elements of $\mathcal{S}'$ whenever B is. Let $\mathfrak{T}$ be a stack on $\mathcal{S}$ satisfying the following conditions:*

(1) if $B \in \mathcal{S}'$ and $\mu \in \mathfrak{T}(B)$ then $\mu|_{C\cap B} = \mu|_{D\cap B} = 0$ implies $\mu = 0$; and

(2) if $B \in \mathcal{S}'$ and $\mu \in \mathfrak{T}(B \cap C \cap D)$, then there exist $\mu_C \in \mathfrak{T}(B \cap C)$ and $\mu_D \in \mathfrak{T}(B \cap D)$ such that $\mu = \mu_C|_{B\cap C\cap D} + \mu_D|_{B\cap C\cap D}$.

Under these circumstances, there is a Mayer-Vietoris exact sequence

$$0 \to H^0(\mathcal{B}, \mathfrak{T}) \to \ldots \to H^p(\mathcal{B}, \mathfrak{T}) \xrightarrow{\alpha_C^* \oplus \alpha_D^*} H^p(\mathcal{B}_C, \mathfrak{T}) \oplus H^p(\mathcal{B}_D, \mathfrak{T})$$
$$\xrightarrow{\beta_C^* - \beta_D^*} H^p(\mathcal{B}_{C\cap D}, \mathfrak{T}) \xrightarrow{\delta^*} H^{p+1}(\mathcal{B}, K) \to \ldots$$

for each complex $\mathcal{B} \subset \mathcal{S}'$.

At this point we impose conditions on $\mathfrak{T}$ which are satisfied by the stack $\mathfrak{N}$ of Section 2, and which allow us to calculate $H^p(\mathcal{B}, \mathfrak{T})$ in certain special cases.

Definition 3.4. Let A be a compact convex subset of a locally convex topological vector space X, and let $\mathcal{S}$ be the family of all compact convex subsets of A. Let Q be a fixed $\mathcal{R}$-module. A stack $\mathfrak{T}$ of $\mathcal{R}$-modules on $\mathcal{S}$ will be called a convex stack of submodules of Q on A if the following conditions hold:

(1) for each $B \in \mathcal{S}$, $\mathfrak{T}(B)$ is a submodule of Q;

(2) if $B, C \in \mathcal{S}$ and $\emptyset \neq B \subset C$, then $\mathfrak{T}(C) \subset \mathfrak{T}(B)$ and the restriction map $\theta_{B,C}$ is the inclusion map of $\mathfrak{T}(C)$ into $\mathfrak{T}(B)$;

(3) if $B, C \in \mathcal{S}$ and $\langle B, C\rangle$ is the convex hull of $B \cup C$, then $\mathfrak{T}(B) \cap \mathfrak{T}(C) = \mathfrak{T}(\langle B, C\rangle)$;

(4) if $B, C \in \mathcal{S}$ and $B \cup C$ is convex, then $\mathfrak{T}(B \cap C) = \mathfrak{T}(B) + \mathfrak{T}(C)$;

(5) if $B_1, \ldots, B_n, C \in \mathcal{S}$ then $[\mathfrak{T}(B_1) + \ldots + \mathfrak{T}(B_n)] \cap \mathfrak{T}(C) = [\mathfrak{T}(B_1) \cap \mathfrak{T}(C)] + \ldots + [\mathfrak{T}(B_n) \cap \mathfrak{T}(C)]$.

Referring to the stack $\mathfrak{N}$ of Section 2, we find that the conditions of Definition 3.4 are satisfied when $\mathcal{R} = \mathfrak{N}(A)$, $Q = \mathfrak{N}'(A)$, and $\mathfrak{T}(B) = \mathfrak{N}(B)$ for each compact convex subset

B of A. Conditions (1), (2), and (3) follow from Lemma 2.4, while conditions (4) and (5) are simply restatements of Lemma 2.6 and 2.7 respectively.

LEMMA 3.4. *Let $\mathfrak{T}$ be a convex stack of submodules of Q on A. Let $\mathcal{B}=\{B_1, \ldots, B_n\}$ be a complex, and let C and D be closed convex subsets of X with $\bigcup_{i=1}^{n} B_i \subset C \cup D$. If we define a refinement $\mathcal{B}'=\{B_1', \ldots, B_{2n}'\}$ of $\mathcal{B}$ by $B_i'=B_i \cap C$ for $i=1, \ldots, n$ and $B_i'=B_{i-n} \cap D$ for $i=n+1, \ldots, 2n$, then the map $k^*: H^p(\mathcal{B}, \mathfrak{T}) \to H^p(\mathcal{B}', \mathfrak{T})$ of Lemma 3.2 is an isomorphism for each p.*

Proof. Note that the conditions of Lemma 3.3 are satisfied for $\mathfrak{T}$, C, and D with $\mathcal{S}'=\mathcal{S}$. In fact, if B is a compact convex subset of A, then so are $B \cap C$ and $B \cap D$; and if $B \cap C = B \cap D = \emptyset$ then $B=\emptyset$. Hence, if $\mu \in \mathfrak{T}(B)$ and $\mu|_{B\cap C}=\mu|_{B \cap D}=0$, then (2) of Definition 3.4 implies that $B \cap C = B \cap D = B = \emptyset$ and $\mu=0$; this gives condition (1) of Lemma 3.3. Condition (2) of Lemma 3.3 follows from (4) of Definition 3.4.

The Mayer–Vietoris sequence of Lemma 3.3, applied to $\mathcal{B}$ and $\mathcal{B}'$, gives us the following diagram:

$$\begin{array}{ccccccccc}
0 \to \cdots \to H^{p-1}(\mathcal{B}_{C\cap D}, \mathfrak{T}) & \xrightarrow{\delta^*} & H^p(\mathcal{B}, \mathfrak{T}) & \xrightarrow{\alpha_C^* \oplus \alpha_D^*} & H^p(\mathcal{B}_C, \mathfrak{T}) \oplus H^p(\mathcal{B}_D, K) & \xrightarrow{\beta_C^* - \beta_D^*} & H^p(\mathcal{B}_{C\cap D}, \mathfrak{T}) \to \cdots \\
\downarrow{\scriptstyle k_{C\cap D}^*} & & \downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle k_C^* \oplus k_D^*} & & \downarrow{\scriptstyle k_{C\cap D}^*} \\
0 \to \cdots \to H^{p-1}(\mathcal{B}'_{C\cap D}, \mathfrak{T}) & \xrightarrow{\delta'^*} & H^p(\mathcal{B}', \mathfrak{T}) & \xrightarrow{\alpha_C'^* \oplus \alpha_D'^*} & H^p(\mathcal{B}'_C, \mathfrak{T}) \oplus H^p(\mathcal{B}'_D, \mathfrak{T}) & \xrightarrow{\beta_C'^* - \beta_D'^*} & H^p(\mathcal{B}'_{C\cap D}, \mathfrak{T}) \to \cdots
\end{array}$$

Note that the two rows of this diagram are exact and are both induced by the sequence $0 \to \mathfrak{T} \to \mathfrak{T}_C \oplus \mathfrak{T}_D \to \mathfrak{T}_{C\cap D} \to 0$. The maps $k_{C\cap D}^*$, k^*, k_C^*, and k_D^* are induced, as in Lemma 3.2, from the map $k: I_{2n} \to I_n$ defined by $k(i)=k(i+n)=i$ for $i=i, \ldots, n$. It follows that the above diagram is commutative. Furthermore, the complexes $\mathcal{B}'_C$, $\mathcal{B}'_D$, and $\mathcal{B}'_{C\cap D}$ and the complexes $\mathcal{B}_C$, $\mathcal{B}_D$, and $\mathcal{B}_{C\cap D}$ are, respectively, mutual refinements of one another. Hence, k_C^*, k_D^*, and $k_{C\cap D}^*$ are isomorphisms, by Lemma 3.2. It follows from the diagram that k^* is also an isomorphism.

A refinement $\mathcal{B}'$ of $\mathcal{B}$, of the type given in the above lemma, will be called a one-step regular refinement. Let $\mathcal{C}$ be a refinement of $\mathcal{B}$. If there exist refinements $\mathcal{B}=\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n$ of $\mathcal{B}$ such that $\mathcal{B}_i$ is a one-step regular refinement of $\mathcal{B}_{i-1}$ for $i=1, \ldots, n$ and $\mathcal{C}$ and $\mathcal{B}_n$ are mutual refinements of one another, then we shall call $\mathcal{C}$ a regular refinement of $\mathcal{B}$. The passage from $\mathcal{B}_n$ to $\mathcal{C}$ might eliminate redundancies introduced in the refinements $\mathcal{B}_1, \ldots, \mathcal{B}_n$. We have the following corollary to Lemma 3.4:

COROLLARY. *If $\mathfrak{T}$ is a convex stack on A and $\mathcal{B}_1$ and $\mathcal{B}_2$ are complexes with a common regular refinement, then $H^p(\mathcal{B}_1, \mathfrak{T})$ and $H^p(\mathcal{B}_2, \mathfrak{T})$ are isomorphic for each p.*

We now proceed to define the cohomology of a set as opposed to a complex. Let $\mathcal{B}=\{B_1, ..., B_n\}$ be a collection of compact subsets of A. If P is a hyperplane in X, then $P=C \cap D$ where C and D are closed half-spaces in X. We obtain another collection $\mathcal{B}'=\{B_1', ..., B_{2n}'\}$ by setting $B_i'=B_i \cap C$ for $i=1, ..., n$ and $B_i'=B_{i-n} \cap D$ for $i=n+1, ..., 2n$. If B is a compact subset of A and $(P_1, ..., P_n)$ is a sequence of hyperplanes in X, then iterating the above process yields a sequence $\{\mathcal{B}_0, ..., \mathcal{B}_n\}$ of collections of compact sets, with $\mathcal{B}_0=\{B\}$ and $\mathcal{B}_i$ obtained from $\mathcal{B}_{i-1}$ by decomposing relative to the hyperplane P_i for $i=1, ..., n$. Note that a given $\mathcal{B}_i$ may not be a complex because the sets composing it may not be convex. If, however, $\mathcal{B}_n$ is a complex, then we shall call it the polygonal decomposition of B relative to $(P_1, ..., P_n)$. A given compact set B may not have a polygonal decomposition; however, if it does we shall call B a semipolygonal set. We have the following lemma concerning these notions:

LEMMA 3.5. (a) *Compact convex subsets of A are semipolygonal, as are finite unions of ordinary polygons in A.*

(b) *The intersection of any finite number of semipolygonal sets is semipolygonal.*

(c) *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are polygonal decompositions of a semipolygonal set B, then $\mathcal{B}_1$ and $\mathcal{B}_2$ have a common regular refinement.*

Proof. Parts (a) and (b) are obvious. To prove part (c), we simply note that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are decompositions of B determined by $(P_1^1, \dots, P_n^1)$ and $(P_1^2, \dots, P_m^2)$ respectively, then the polygonal decomposition $\mathcal{B}_3$ of B, determined by $(P_1^1, \dots, P_n^1, P_1^2, \dots, P_m^2)$, is a regular refinement of both $\mathcal{B}_1$ and $\mathcal{B}_2$.

Definition 3.5. Let $\mathfrak{T}$ be a convex stack on A and let B be a semipolygonal subset of A. We define $H^p(B, \mathfrak{T})$ to be $H^p(\mathcal{B}, \mathfrak{T})$, where $\mathcal{B}$ is any polygonal decomposition of B.

Note that Lemma 3.5 (c) and the corollary to Lemma 3.4 imply that the definition of $H^p(B, \mathfrak{T})$ is independent of the choice of $\mathcal{B}$.

The cohomology modules $H^p(B, \mathfrak{T})$ could be defined for a larger class of sets B, by using more general kinds of decompositions or by taking direct limits; however, the class of semipolygonal sets is large enough for our purposes.

THEOREM 3.1. *If C, D, and $C \cup D$ are semipolygonal subsets of A and $\mathfrak{T}$ is a convex stack on A, then there is an exact sequence*

$$0 \to H^0(C \cup D, \mathfrak{T}) \to \dots \to H^p(C \cup D, \mathfrak{T}) \xrightarrow{\alpha_C^* \oplus \alpha_D^*} H^p(C, \mathfrak{T}) \oplus H^p(D, \mathfrak{T})$$
$$\xrightarrow{\beta_C^* - \beta_D^*} H^p(C \cap D, \mathfrak{T}) \xrightarrow{\delta^*} H^{p+1}(C \cup D, \mathfrak{T}) \to \dots$$

Proof. We can find a sequence $(P_1, \ldots, P_n)$ of hyperplanes which induces a polygonal decomposition $\mathcal{B}$ of $C \cup D$ such that $\mathcal{B}_C$, $\mathcal{B}_D$, and $\mathcal{B}_{C\cap D}$ are polygonal decompositions of C, D, and $C \cap D$ respectively. If $\mathcal{S}'$ is the collection of all convex compact subsets E of A for which $E \cap C$ and $E \cap D$ are convex, then $\mathcal{S}'$ is closed under finite intersection and $\mathcal{B} \subset \mathcal{S}'$. From this and (2) and (4) of Definition 3.4, it follows that the conditions of Lemma 3.3 are met. The theorem follows.

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a complex, let $B = \bigcup_{i=1}^n B_i$, and $\langle B\rangle$ be the convex hull of B. There is a natural injection $i: \mathfrak{T}(\langle B\rangle) \to C^0(\mathcal{B}, \mathfrak{T})$ given by $(i\mu)_{(j)} = \mu\,|_{B_j}$. Clearly, the image of i lies in $Z^0(\mathcal{B}, \mathfrak{T}) = H^0(\mathcal{B}, \mathfrak{T})$, and so i determines an injection i^*: $\mathfrak{T}(\langle B\rangle) \to H^0(\mathcal{B}, \mathfrak{T})$. If B is connected, then it follows from (3) of Definition 3.4 that i^* is onto. Hence, we have:

LEMMA 3.6. *If B is a compact connected subset of A and $\mathfrak{T}$ is a convex stack on A, then i^*: $\mathfrak{T}(\langle B\rangle) \to H^0(B, \mathfrak{T})$ is an isomorphism.*

We are now in a position to compute the cohomology modules of the sets we shall be dealing with in later sections.

THEOREM 3.2. *If B is a compact convex subset of A and $\mathfrak{T}$ is a convex stack on A, then $H^p(B, \mathfrak{T}) = 0$ for $p > 0$ and i^*: $\mathfrak{T}(B) \to H^0(B, \mathfrak{T})$ is an isomorphism.*

Proof. The complex $\mathcal{B}$, consisting of the set B alone, is a polygonal decomposition of B. For this complex, we have $C^p(\mathcal{B}, \mathfrak{T}) = 0$ for $p > 0$ and $C^0(\mathcal{B}, \mathfrak{T}) = Z^0(\mathcal{B}, \mathfrak{T}) = \mathfrak{T}(B)$.

Let $x_0, \ldots, x_n$ be the vertices of an n-simplex $S = \langle x_0, \ldots, x_n\rangle \subset A$. The collection, $\mathcal{B} = \{B_0, \ldots, B_n\}$, of all $(n-1)$-faces, $B_i = \langle x_0, \ldots, \hat{x}_i, \ldots, x_n\rangle$, of S, forms a complex with a regular refinement which is a polygonal decomposition of $\partial S = \bigcup_{i=0}^n B_i$. Hence $H^p(\partial S, \mathfrak{T}) = H^p(\mathcal{B}, \mathfrak{T})$ for any convex stack $\mathfrak{T}$ on A, by the corollary to Lemma 3.4. If $c \in C^{n-1}(\mathcal{B}, \mathfrak{T})$ then $(\delta c)_{(0,\ldots,n)} = \sum_{k=0}^n (-1)^k c_{(0,\ldots,\hat{k},\ldots,n)}\,|_{B_0 \cap \cdots \cap B_n} = 0$, since $B_0 \cap \ldots \cap B_n = \emptyset$. However, the expression $(\delta_0 c)_{(0,\ldots,n)} = \sum_{k=0}^n (-1)^k c_{(0,\ldots,\hat{k},\ldots,n)}$, where the addition takes place in the $\mathcal{R}$-module Q, may not be zero. The map $\delta_0 : C^{n-1}(\mathcal{B}, \mathfrak{T}) \to Q$ is an $\mathcal{R}$-module homomorphism and $\delta_0 \delta c = 0$ for $c \in C^{n-2}(\mathcal{B}, \mathfrak{T})$. Thus, δ_0 determines a homomorphism $\delta_0^* : H^{n-1}(\mathcal{B}, \mathfrak{T}) \to Q$. If we identify $H^{n-1}(\partial S, \mathfrak{T})$ with $H^{n-1}(\mathcal{B}, \mathfrak{T})$, then we have a homomorphism δ_0^*: $H^{n-1}(\partial S, \mathfrak{T}) \to Q$.

THEOREM 3.3. *Let S be an n-simplex in A and let $\mathfrak{T}$ be a convex stack on A. If $n > 1$ then $H^p(\partial S, \mathfrak{T}) = 0$ for $p \neq 0, n-1$, $\delta_0^* : H^{n-1}(\partial S, \mathfrak{T}) \to Q$ is one to one, and i^*: $\mathfrak{T}(S) \to H^0(\partial S, \mathfrak{T})$ is an isomorphism. If $n = 1$ then the sequence $0 \to \mathfrak{T}(S) \xrightarrow{i^*} H^0(\partial S, \mathfrak{T}) \xrightarrow{\delta_0^*} Q$ is exact and $H^p(\partial S, \mathfrak{T}) = 0$ for $p > 0$.*

Proof. If $n=1$ then $S=\langle x_0, x_1\rangle$, $\partial S=\{x_0, x_1\}$, and $H^0(\partial S, \mathfrak{T})=C^0(\mathcal{B}, \mathfrak{T})$ where $\mathcal{B}$ is the complex $(\{x_0\}, \{x_1\})$. If $c\in C^0(\mathcal{B}, \mathfrak{T})$ then $\delta_0 c=c_1-c_0$ where $c_1\in\mathfrak{T}(\{x_1\})$ and $c_0\in\mathfrak{T}(\{x_0\})$. If $\delta_0 c=0$ then

$$c_1=c_0\in\mathfrak{T}(\{x_1\})\cap\mathfrak{T}(\{x_0\})=\mathfrak{T}(\langle x_0, x_1\rangle)=\mathfrak{T}(S).$$

It follows that $0\to\mathfrak{T}(S)\to H^0(\partial S, \mathfrak{T})\to Q$ is exact. Since $\mathcal{B}$ contains only two elements and their intersection is empty, $H^p(\mathcal{B}, \mathfrak{T})=H^p(\partial S, \mathfrak{T})=0$ for $p>0$.

We now prove the theorem by induction on n. The above paragraph establishes the case $n=1$. We assume the theorem is true for any convex stack and any simplex of dimension less than $n>1$. Let $\mathfrak{T}$ be a convex stack and $S=\langle x_0, \ldots, x_n\rangle$ be an n-simplex with $(n-1)$-faces $B_0, \ldots, B_n$. We set $C=\bigcup_{i=0}^{n-1}B_i$. We first prove that $H^p(C, \mathfrak{T})=0$ for $p\neq 0, n-1$.

Define a new stack $\mathfrak{T}'$ on B_n as follows: Let $\mathfrak{T}'(E)=\mathfrak{T}(\langle E, x_n\rangle)$ for each compact convex subset E of B_n ,where x_n is the vertex of S opposite the face B_n. It can be verified directly that $\mathfrak{T}'$ is a convex stack of submodules of Q on B_n, according to Definition 3.4. If we let $E_i=B_i\cap B_n$ for $i=0, \ldots, n-1$, then the complexes $\mathcal{C}=(B_0, \ldots, B_{n-1})$ and $\mathcal{E}=(E_0, \ldots, E_{n-1})$ have regular refinements which are polygonal decompositions of C and $\partial B_n=\bigcup_{i=0}^{n-1}E_i$ respectively. Note that

$$B_i=\langle E_i, x_n\rangle \text{ for } i<n \text{ and } \bigcap_{k=0}^{p} B_{s_k}=\langle \bigcap_{k=0}^{p} E_{s_k}, x_n\rangle \text{ for}$$

$s_0, \ldots, s_p<n$ and $p<n-1$. It follows from this and the definition of $\mathfrak{T}'$ that $C^p(\mathcal{C}, \mathfrak{T})$ and $C^p(\mathcal{E}, \mathfrak{T}')$ are isomorphic for $p\neq n-1$. Since B_n is an $(n-1)$-simplex, we have $H^p(\mathcal{E}, \mathfrak{T}')=H^p(\partial B_n, \mathfrak{T}')=0$ for $p\neq 0$, $n-2$ and $\delta_0^*: H^{n-2}(\mathcal{E}, \mathfrak{T}')\to Q$ is one to one, by our induction hypothesis. Hence, $H^p(C, \mathfrak{T})=H^p(\mathcal{C}, \mathfrak{T})=H^p(\mathcal{E}, \mathfrak{T}')=0$ for $p\neq 0, n-2, n-1$. If $c\in Z^{n-2}(\mathcal{C}, \mathfrak{T})$ then c corresponds to $c'\in C^{n-2}(\mathcal{E}, \mathfrak{T}')=Z^{n-2}(\mathcal{E}, \mathfrak{T}')$ such that $\delta_0 c'=0$. It follows that $\delta b'=c'$ for some $b'\in C^{n-3}(\mathcal{E}, \mathfrak{T}')$. There is a corresponding element $b\in C^{n-3}(\mathcal{C}, \mathfrak{T})$ such that $\delta b=c$. Hence, $H^{n-2}(C, \mathfrak{T})=H^{n-2}(\mathcal{C}, \mathfrak{T})=0$. This completes the proof that $H^p(C, \mathfrak{T})=0$ for $p\neq 0, n-1$.

Since $\partial S=C\cup B_n$ and $\partial B_n=C\cap B_n$, Theorem 3.1 yields an exact sequence

$$\ldots\to H^{p-1}(\partial B_n, \mathfrak{T}) \xrightarrow{\delta^*} H^p(\partial S, \mathfrak{T}) \xrightarrow{\alpha_1^*\oplus\alpha_2^*} H^p(C, \mathfrak{T})\oplus H^p(B_n, \mathfrak{T}) \xrightarrow{\beta_1^*-\beta_2^*} H^p(\partial B_n, \mathfrak{T})\to\ldots.$$

By the induction assumption, $H^{p-1}(\partial B_n, \mathfrak{T})=0$ for $p\neq 1$, $n-1$. By the result of the above paragraph, $H^p(C, \mathfrak{T})=0$ for $p\neq 0, n-1$. Since B_n is convex, Theorem 3.2 implies that $H^p(B_n, \mathfrak{T})=0$ for $p\neq 0$.

Hence, we can conclude from the above sequence that $H^p(\partial S, \mathfrak{T})=0$ for $p\neq 0, 1, n-1$. To handle the case where $p=1$ and $n>2$, we note that $H^0(\partial S, \mathfrak{T})=\mathfrak{T}(S)$, $H^0(C, \mathfrak{T})=\mathfrak{T}(S)$,

$H^0(B_n, \mathfrak{T}) = \mathfrak{T}(B_n)$, and $H^0(\partial B_n, \mathfrak{T}) = \mathfrak{T}(B_n)$; this follows from Lemma 3.6 and the fact that each of these sets is connected when $n > 2$. The above exact sequence then yields

$$0 \to \mathfrak{T}(S) \xrightarrow{\alpha_1 \oplus \alpha_2} \mathfrak{T}(S) \oplus \mathfrak{T}(B_n) \xrightarrow{\beta_1 - \beta_2} \mathfrak{T}(B_n) \xrightarrow{\delta^*} H^1(\partial S, \mathfrak{T}) \to 0.$$

A glance at the definitions of the maps involved (cf. Lemma 3.3) shows that $\alpha_1 = \mathrm{id}$: $\mathfrak{T}(S) \to \mathfrak{T}(S)$ and $\beta_2 = \mathrm{id} : \mathfrak{T}(B_n) \to \mathfrak{T}(B_n)$. Hence, $\beta_1 - \beta_2$ is onto and $\delta^* : \mathfrak{T}(B_n) \to H^1(\partial S, \mathfrak{T})$ is the zero map. Thus $H^1(\partial S, \mathfrak{T}) = 0$.

It remains to show that δ_0^*: $H^{n-1}(\partial S, \mathfrak{T}) \to Q$ is one to one. Let $\mathcal{B} = (B_0, \ldots, B_n)$ be the complex consisting of the $(n-1)$-faces of S. If $c \in C^{n-1}(\mathcal{B}, \mathfrak{T})$ and $\delta_0 c = 0$, then $c_{\sigma_n} = (-1)^n \sum_{k=0}^{n-1} (-1)^k c_{\sigma_k}$, where $\sigma = (0, \ldots, n)$ and $\sigma_k = (0, \ldots, \hat{k}, \ldots, n)$. Note that $c_{\sigma_k} \in \mathfrak{T}(B_0, \ldots, \hat{B}_k, \ldots, B_n) = \mathfrak{T}(\{x_k\})$, where x_k is the vertex of S opposite B_k. Condition (5) of Definition 3.4 yields elements $b_{\tau_k} \in \mathfrak{T}(\langle x_k, x_n \rangle) = \mathfrak{T}(B_0, \ldots, \hat{B}_k, \ldots, B_{n-1})$, such that $c_{\sigma_n} = \sum_{k=0}^{n-1} (-1)^k b_{\tau_k}$, where $\tau_k = (0, \ldots, \hat{k}, \ldots, n-1)$. If τ' is any $(n-1)$-tuple of integers in $[0, n]$, we set $b_{\tau'} = b_{\tau_k}$ if τ' is an even permutation of τ_k, $b_{\tau'} = -b_{\tau_k}$ if τ' is an odd permutation of τ_k, and $b_{\tau'} = 0$ if τ' is not a permutation of any τ_k. We then have an element $b \in C^{n-2}(\mathcal{B}, \mathfrak{T})$ such that $(c - \delta b)_{\sigma_n} = 0$ and $\delta_0(c - \delta b) = \delta_0 c = 0$. This process can be repeated for each of the vertices of S and finally yields $c \in B^{n-1}(\mathcal{B}, \mathfrak{T})$. Hence, $\delta_0 c = 0$ for $c \in C^{n-1}(\mathcal{B}, \mathfrak{T})$ implies $c \in B^{n-1}(\mathcal{B}, \mathfrak{T})$; i.e., $\delta_0^* : H^{n-1}(\partial S, \mathfrak{T}) \to Q$ is one to one.

We should note that our only use of condition (5) of Definition 3.4 comes in the above proof.

If we set $\mathbf{Q}(E) = Q$ for each nonempty compact convex subset E of A, then we obtain a convex stack $\mathbf{Q}$ of submodules of Q on A, which we shall call the constant stack. For any semipolygonal set $B \subset A$, the cohomology $H^p(B, \mathbf{Q})$ is just the ordinary pth Čech cohomology module with coefficients in Q. If we denote the injection of $\mathfrak{T}$ into $\mathbf{Q}$ by q, then q induces a homomorphism q^*: $H^p(B, \mathfrak{T}) \to H^p(B, \mathbf{Q})$ for each p and each semipolygon B. For $\mathbf{Q}$, Theorem 3.3 gives the well-known result that δ_0^*: $H^p(\partial S, \mathbf{Q}) \to Q$ is an isomorphism. We thus have the diagram

$$\begin{tikzcd} H^{n-1}(\partial S, \mathfrak{T}) \arrow[r, "\delta_0^*"] \arrow[d, "q^*"] & Q \\ H^{n-1}(\partial S, \mathbf{Q}) \arrow[ur, leftrightarrow] & \end{tikzcd}$$

which implies that $q^* : H^{n-1}(\partial S, \mathfrak{T}) \to H^{n-1}(\partial S, \mathbf{Q})$ is one to one. Theorem 3.3 also implies that q^*: $H^p(\partial S, \mathfrak{T}) \to H^p(\partial S, \mathbf{Q})$ is one to one for $p \neq n-1$ as well. Trivially, $q^* : H^p(B, \mathfrak{T}) \to H^p(B, \mathbf{Q})$ is one to one if B is convex. This indicates that q^* may be one to one for a fairly large class of semipolygons B.

It is not true that q^* is one to one for all semipolygons B. If x_0, x_1, x_2 are points of A which are not colinear and if $B=\langle x_0, x_2\rangle \cup \langle x_1, x_2\rangle$, then $H^1(B, Q)=0$, but there are convex stacks $\mathfrak{T}$ for which $H^1(B, \mathfrak{T}) \neq 0$. In fact, if $\mathfrak{T}$ is the stack $\mathfrak{R}$ of Section 2, then a nonzero element of $H^1(B, \mathfrak{R})$ corresponds to a measure $\mu \in \mathfrak{R}(\{x_2\})$ which cannot be written as $\mu_1 - \mu_0$ with $\mu_1 \in \mathfrak{R}(\langle x_1, x_2\rangle)$ and $\mu_0 \in \mathfrak{R}(\langle x_0, x_2\rangle)$. If we choose a line l, through x_2, which is parallel to the line containing x_0 and x_1, then it is trivial to pick a measure μ which is in $\mathfrak{R}(l')$ for each finite segment l' of l, but which is not in $\mathfrak{R}(C)$ for any set C not contained in l. If we could write μ as $\mu_1 - \mu_0$ with $\mu_0 \in \mathfrak{R}(\langle x_0, x_2\rangle)$ and $\mu_1 \in \mathfrak{R}(\langle x_1, x_2\rangle)$, then Lemma 2.7 would imply that μ_1 and μ_0 can be chosen from $\mathfrak{R}(\langle l', x_1\rangle)$ and $\mathfrak{R}(\langle l', x_0\rangle)$, respectively, for any segment $l' \subset l$. It would follow that $\mu \in \mathfrak{R}(\langle l', x_1\rangle \cap \langle l', x_0\rangle)$. This contradicts the choice of μ, since $\langle l', x_1\rangle \cap \langle l', x_0\rangle \not\subset l$.

The above example shows that the cohomology theory developed here seems to measure not only the "holes" in a set, but also the various degrees of nonconvexity of this set. This should, perhaps, be investigated further.

4. Linear equations

Let $\mathfrak{T}$ be a stack of $\mathcal{R}$-modules on a collection $\mathcal{S}$ of subsets of A. Let $r_1, \ldots, r_n \in \mathcal{R}$, $\lambda \in \mathfrak{T}(A)$ and consider the equation

$$r_1\nu_1 + \ldots + r_n\nu_n = \lambda, \tag{1}$$

where we seek a solution $\nu_1, \ldots, \nu_n \in \mathfrak{T}(A)$. If we set $\mathfrak{T}_1(B) = \sum^n \oplus \mathfrak{T}(B)$ for $B \in \mathcal{S}$, then we may think of the n-tuple $(r_1, \ldots, r_n)$ as defining a map $d: \mathfrak{T}_1 \to \mathfrak{T}$, by $d(\mu_1, \ldots, \mu_n) = r_1\mu_1 + \ldots + r_n\mu_n \in \mathfrak{T}(B)$ for $(\mu_1, \ldots, \mu_n) \in \mathfrak{T}_1(B)$. Equation (1) has a solution if $\lambda \in \operatorname{im} d$. If (1) has a solution for every $\lambda \in \mathfrak{T}(A)$, then the sequence $\mathfrak{T}_1(A) \xrightarrow{d} \mathfrak{T}(A) \to 0$ is exact.

There is a fairly well-known technique for studying the above situation. Suppose there is a sequence of stacks and stack homomorphisms,

$$\ldots \to \mathfrak{T}_n \xrightarrow{d_{n-1}} \mathfrak{T}_{n-1} \xrightarrow{d_{n-2}} \ldots \to \mathfrak{T}_2 \xrightarrow{d_1} \mathfrak{T}_1 \xrightarrow{d} \mathfrak{T} \to 0, \tag{2}$$

which is exact on a subdomain $\mathcal{S}'$ of $\mathcal{S}$. (Later in the section we shall show one way that such a sequence may be obtained.) If $\mathcal{B} = (B_1, \ldots, B_n)$ is a complex whose elements are in $\mathcal{S}'$, then (2) induces the following diagram

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathfrak{T}_n(A) & \xrightarrow{i} & C^0(\mathcal{B}, \mathfrak{T}_n) & \xrightarrow{\delta^0} & C^1(\mathcal{B}, \mathfrak{T}_n) & \to \dots \to & C^p(\mathcal{B}, \mathfrak{T}_n) & \to \dots \\
\downarrow{\scriptstyle d_{n-1}} & & \downarrow{\scriptstyle \bar{d}_{n-1}} & & \downarrow{\scriptstyle \bar{d}_{n-1}} & & \downarrow{\scriptstyle \bar{d}_{n-1}} & \\
\mathfrak{T}_{n-1}(A) & \xrightarrow{i} & C^0(\mathcal{B}, \mathfrak{T}_{n-1}) & \xrightarrow{\delta^0} & C^1(\mathcal{B}, \mathfrak{T}_{n-1}) & \to \dots \to & C^p(\mathcal{B}, \mathfrak{T}_{n-1}) & \to \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots & & \vdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathfrak{T}_1(A) & \xrightarrow{i} & C^0(\mathcal{B}, \mathfrak{T}_1) & \xrightarrow{\delta^0} & C^1(\mathcal{B}, \mathfrak{T}_1) & \to \dots \to & C^p(\mathcal{B}, \mathfrak{T}_1) & \to \dots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle \bar{d}} & & \downarrow{\scriptstyle \bar{d}} & \\
\mathfrak{T}(A) & \xrightarrow{i} & C^0(\mathcal{B}, \mathfrak{T}) & \xrightarrow{\delta^0} & C^1(\mathcal{B}, \mathfrak{T}) & \to \dots \to & C^p(\mathcal{B}, \mathfrak{T}) & \to \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 &
\end{array} \tag{3}$$

All columns of this diagram are exact except possibly the first one. It is the first column that we wish to draw conclusions about. The following discussion of this diagram is derived from standard double complex theory:

When convenient in discussing diagram (3), we shall use δ to denote any one of the maps $i, \delta^0, \dots, \delta^p, \dots$ and $\bar{d}$ to denote any one of the maps $d, d_1, \dots, d_p, \dots$. We shall also use the notation $\mathfrak{T}_0 = \mathfrak{T}$ and $C^{-1}(\mathcal{B}, \mathfrak{T}_p) = \mathfrak{T}_p(A)$.

Definition 4.1. For the diagram (3), we make the following definitions:

(a) $D^{p-1} = \operatorname{im} \delta^{p-2} + \operatorname{im} d_p \subset C^{p-1}(\mathcal{B}, \mathfrak{T}_p)$ for $p \geq 0$, where $\delta^{-2} = 0$ and $\delta^{-1} = i$;

(b) A residue sequence $a^0, a^1, \dots, a^p, \dots$ is a sequence with $a^p \in C^{p-1}(\mathcal{B}, \mathfrak{T}_p)$ and $\bar{d}a^{p+1} = \delta a^p$ for $p \geq 0$.

LEMMA 4.1. (a) *If $\lambda \in \mathfrak{T}(A)$ then there is a residue sequence, $a^0, a^1, \dots, a^p, \dots$, with $\lambda = a^0$.*

(b) *If $\lambda = a^0, a^1, \dots, a^p, \dots$ and $\lambda = b^0, b^1, \dots, b^p, \dots$ are two residue sequences starting at λ, then $a^p - b^p \in D^{p-1}$ for each p.*

(c) *If $\lambda = a^0, a^1, \dots, a^p, \dots$ is any residue sequence for which $a^q \in D^{q-1}$, then $a^p \in D^{p-1}$ for $p \geq q$.*

Proof. We construct the sequence $a^0, a^1, \dots, a^p, \dots$ by induction on p. Suppose we have $a^0, a^1, \dots, a^{p-1}$ defined with $\lambda = a^0$ and $\bar{d}a^q = \delta a^{q-1}$ for $q \leq p-1$. Since $\bar{d}\delta a^{p-1} = \delta \bar{d} a^{p-1} = \delta\delta a^{p-2} = 0$, the exactness of the columns of (3) implies that there exists $a^p \in C^{p-1}(\mathcal{B}, \mathfrak{T}_p)$ such that $\bar{d}a^p = \delta a^{p-1}$. This proves part (a).

Note that if we consider the sequence $0=a_0-b_0, a_1-b_1, \ldots, a_n-b_n, \ldots$, then part (b) follows from part (c) with $q=0$. To prove part (c), note that if $a^p \in D^{p-1}$ then $a^p=\delta b^p+\bar{d}c^{p+1}$, where $b^p \in C^{p-2}(\mathcal{B}, \mathfrak{T}_p)$ and $c^{p+1} \in C^{p-1}(\mathcal{B}, \mathfrak{T}_{p+1})$. It follows that $\bar{d}(a^{p+1}-\delta c^{p+1})=\delta a^p-\delta\bar{d}c^{p+1}=\delta\delta b^p=0$. Hence, there exists $c^{p+2} \in C^p(\mathcal{B}, \mathfrak{T}_{p+2})$ such that $a^{p+1}-\delta c^{p+1}=\bar{d}c^{p+2}$; i.e., $a^{p+1}=\delta b^{p+1}+\bar{d}c^{p+2}$ if we set $c^{p+1}=b^{p+1}$. We conclude by induction that part (c) is true.

Note that, in the above lemma, D^{-1} is the image of d in $\mathfrak{T}(A)$. Thus, part (c) of the lemma says in particular that if $\lambda \in \operatorname{im} d$, then $a^p \in D^{p-1}$ for all $p \geq 0$. Under appropriate conditions on the cohomology groups $H^p(\mathcal{B}, \mathfrak{T}_n)$, we can prove an analogue of part (c) which goes the other direction.

LEMMA 4.2. *Suppose* $H^{p-1}(\mathcal{B}, \mathfrak{T}_p)=0$ *for* $1<p<m$, *and* i^*: $\mathfrak{T}(A) \to H^0(\mathcal{B}, \mathfrak{T})$ *and* i^*: $\mathfrak{T}_1(A) \to H^0(\mathcal{B}, \mathfrak{T}_1)$ *are isomorphisms. If* $\lambda=a^0, a^1, \ldots, a^n, \ldots$ *is a residue sequence starting at* $\lambda \in \mathfrak{T}(A)$ *and* $a^m \in D^{m-1}$, *then* $a^p \in D^{p-1}$ *for* $0 \leq p \leq m$. *In particular,* $\lambda \in \operatorname{im} d$ *under these circumstances.*

Proof. If $a^m \in D^{m-1}$ then $a^m=\delta b^m+\bar{d}c^{m+1}$, where $b^m \in C^{m-2}(\mathcal{B}, \mathfrak{T}_m)$ and $c^{m+1} \in C^{m-1}(\mathcal{B}, \mathfrak{T}_{m+1})$. It follows that $\delta(a^{m-1}-\bar{d}b^m)=\bar{d}(a^m-\delta b^m)=\bar{d}\bar{d}c^{m+1}=0$. If $m \geq 2$ it follows from the hypotheses that there exists $b^{m-1} \in C^{m-3}(\mathcal{B}, \mathfrak{T}_{m-1})$ such that $a^{m-1}-\bar{d}b^m=\delta b^{m-1}$; i.e., $a^{m-1}=\delta b^{m-1}+\bar{d}c^m$ if we set $c^m=b^m$. If $m=1$, then $a^1=\delta b^1+\bar{d}c^2$ implies $i\bar{d}b^1=\bar{d}a^1=ia^0$. Since $\mathfrak{T}(A) \xrightarrow{i} C^0(\mathcal{B}, \mathfrak{T})$ is one to one, we have $\bar{d}b^1=a^0=\lambda$. Hence $\lambda \in \operatorname{im} d$, in this case. We now have that the lemma is true, by induction.

Definition 4.2. Let A be a set and $\mathcal{S}$ a collection of subsets of A which is closed under finite intersection and contains $\emptyset$. Let $\mathfrak{T}$ be a map which assigns to each $B \in \mathcal{S}$ a ring $\mathfrak{T}(B)$ with identity e_B, and let θ be a map which assigns to each pair $(B, C) \subset \mathcal{S}$, with $B \subset C$, a ring homomorphism $\theta_{B,C}: \mathfrak{T}(C) \to \mathfrak{T}(B)$ such that $\theta_{B,C}=e_B$. We shall call $\mathfrak{T}$ a stack of rings with identity on $\mathcal{S}$ if $\mathfrak{T}(\emptyset)=(0)$, θ satisfies the transitive law, and $\theta_{B,B}=\mathrm{id}$, as in Definition 3.1.

Note that if $\mathfrak{T}$ is a stack of rings with identity and $(B, C) \in \mathcal{S}$ with $B \subset C$, then we may consider $\mathfrak{T}(B)$ as an algebra over the ring $\mathfrak{T}(C)$ under the operation $\mu \cdot \nu=(\theta_{B,C}\mu) \cdot \nu$ for $\mu \in \mathfrak{T}(C)$, $\nu \in \mathfrak{T}(B)$. In particular, $\mathfrak{T}(B)$ is an algebra over $\mathfrak{T}(A)$ for every $B \in \mathcal{S}$, and the maps $\theta_{B,C}$ are $\mathfrak{T}(A)$-algebra homomorphisms. If we consider only the additive structure in $\mathfrak{T}(B)$ for each B, then $\mathfrak{T}$ may be considered a stack of $\mathfrak{T}(A)$-modules as in Definition 3.1.

Definition 4.3. Let A be a compact convex subset of a topological vector space X and let $\mathfrak{T}$ be a stack of rings with identity over the compact convex subsets of A. Let Q be a $\mathfrak{T}(A)$-module which contains $\mathfrak{T}(B)$ as a submodule for each $B \in \mathcal{S}$, in such a way that the maps $\theta_{B,C}$ are inclusion maps for $B \neq \emptyset$. If $\mathfrak{T}$, considered as a stack of $\mathfrak{T}(A)$-modules, is a convex stack of submodules of Q according to Definition 3.4, then $\mathfrak{T}$ will be called a convex stack of rings with identity in Q.

Note that with $Q=\mathfrak{R}'(A)$, the correspondence $B\to\mathfrak{R}(B)$, of Section 2, defines a convex stack of rings in Q. In fact, Definition 4.3 abstracts precisely those properties of the correspondence $A\to\mathfrak{R}(A)$ which are needed to carry out the results of this section.

If $\mathfrak{T}$ is a stack of rings with identity, there is a canonical way of constructing a sequence like (2). This is the Koszal complex, which we describe below. For each $B\in\mathcal{S}$ we set $\mathfrak{T}_1(B)=\sum^n\oplus\mathfrak{T}(B)$. For $m>1$, we let $\mathfrak{T}_m(B)=\wedge_m\mathfrak{T}_1(B)$ be the m-fold exterior product of $\mathfrak{T}_1(B)$ over the ring $\mathfrak{T}(B)$. This space may be described as follows: Let e be the identity of $\mathfrak{T}(B)$ and for $i=1, ..., n$ set $e_i=(0, ..., e, ..., 0)\in\mathfrak{T}_1(B)$, with e appearing in the ith position; the elements $e_1, ..., e_n$ form a basis for $\mathfrak{T}_1(B)$ over $\mathfrak{T}(B)$. For $m>1$, we let $\mathfrak{T}_m(B)$ be the free $\mathfrak{T}(B)$-module having as generators the symbols $e_{i_1}\wedge ... \wedge e_{i_m}$, where we make the identifications: $e_{i_1}\wedge ... \wedge e_{i_m}=(-1)^{\varepsilon}e_{j_1}\wedge ... \wedge e_{j_m}$ if $(j_1, ..., j_m)$ is a permutation of $(i_1, ..., i_m)$ with $\varepsilon=1$ if the permutation is odd and $\varepsilon=0$ if the permutation is even, and $e_{i_1}\wedge ... \wedge e_{i_m}=0$ if the subscripts $i_1, ..., i_m$ are not all distinct. Note that $\mathfrak{T}_m(B)$ is a free $\mathfrak{T}(B)$-module of dimension $\binom{n}{m}$. In particular, $\mathfrak{T}_m(B)=(0)$ for $m>n$ and $\mathfrak{T}_n(B)$ is isomorphic to $\mathfrak{T}(B)$; i.e., $\mathfrak{T}_n(B)$ is a free $\mathfrak{T}(B)$-module with a single generator $e_1\wedge ... \wedge e_n$.

For $\mu\in\mathfrak{T}_p(B)$, $\nu\in\mathfrak{T}_q(B)$ with

$$\mu=\sum_{i_1,\dots,i_p}\mu_{i_1,\dots,i_p}e_{i_1}\wedge\dots\wedge e_{i_p}$$

and

$$\nu=\sum_{j_1,\dots,j_q}\mu_{j_1,\dots,j_q}e_{j_1}\wedge\dots\wedge e_{j_q},$$

we set

$$\mu\wedge\nu=\sum_{i_1,\dots,j_q}\mu_{i_1,\dots,i_p}\cdot\nu_{j_1,\dots,j_q}e_{i_1}\wedge\dots\wedge e_{i_p}\wedge e_{j_1}\wedge\dots\wedge e_{j_q}$$

We then have $\mu\wedge\nu\in\mathfrak{T}_{p+q}(B)$ and $\mu\wedge\nu=(-1)^{pq}\nu\wedge\mu$. Under the multiplication induced by the wedge product $\mu\wedge\nu$, the space $\sum_{p=0}^{\infty}\oplus\mathfrak{T}_p(B)$ is an associative algebra over $\mathfrak{T}(B)$, where we set $\mathfrak{T}_0(B)=\mathfrak{T}(B)$ and $\mu\wedge\nu=\mu\cdot\nu$ for $\mu,\nu\in\mathfrak{T}(B)$.

If $(B, C)\subset\mathcal{S}$ with $B\subset C$, then the map $\theta_{B,C}$: $\mathfrak{T}(C)\to\mathfrak{T}(B)$ induces a homomorphism $\theta_{B,C}$: $\sum_{p=0}^{\infty}\oplus\mathfrak{T}_p(C)\to\sum_{p=0}^{\infty}\oplus\mathfrak{T}_p(B)$ which preserves wedge products and carries $\mathfrak{T}_p(C)$ into $\mathfrak{T}_p(B)$. In particular, if we consider $\mathfrak{T}_p(C)$ and $\mathfrak{T}_p(B)$ as $\mathfrak{T}(A)$-modules, then $\theta_{B,C}$ is a $\mathfrak{T}(A)$-module homomorphism of $\mathfrak{T}_p(C)$ into $\mathfrak{T}_p(B)$. In other words, we may consider $\mathfrak{T}_p$ to be a stack of $\mathfrak{T}(A)$-modules on $\mathcal{S}$.

If $\mu_1, ..., \mu_n\in\mathfrak{T}(A)$, then we may define a stack homomorphism d_p: $\mathfrak{T}_{p+1}\to\mathfrak{T}_p$ as follows: For

$$\nu=\sum_{j_1,\dots,j_{p+1}}\nu_{j_1,\dots,j_{p+1}}e_{j_1}\wedge\dots\wedge e_{j_{p+1}}\in\mathfrak{T}_{p+1}(B),$$

we set $$d_p \nu = \sum_{j_1, \ldots, j_{p+1}} \sum_{k=1}^{p+1} (-1)^{k+1} \mu_{j_k} \cdot \nu_{j_1, \ldots, j_{p+1}} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_k} \wedge \ldots \wedge e_{j_{p+1}}.$$

A simple calculation shows that $d_{p-1} \circ d_p = 0$ and $d_{p+q+1}(\nu \wedge \varrho) = (d_p \nu) \wedge \varrho + (-1)^p \nu \wedge (d_q \varrho)$ for $\nu \in \mathfrak{T}_{p+1}(B)$ and $\varrho \in \mathfrak{T}_{q+1}(B)$.

LEMMA 4.3. *If the equation $\mu_1 \nu_1 + \ldots + \mu_n \nu_n = e$ can be solved for $\nu_1, \ldots, \nu_n \in \mathfrak{T}(B)$, then the sequence $0 \to \mathfrak{T}_n(B) \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} \mathfrak{T}_1(B) \xrightarrow{d_0} \mathfrak{T}(B) \to 0$ is exact.*

Proof. The hypothesis says that d_0 is onto, since $d_0(\nu_1 e_1 + \ldots + \nu_n e_n) = \mu_1 \nu_1 + \ldots + \mu_n \nu_n$. Since $d_{p-1} d_p = 0$, we need only show that if $d_{p-1} \varrho = 0$ then $\varrho = d_p \lambda$ for some $\lambda \in \mathfrak{T}(B)$. We set $\lambda = \nu \wedge \varrho$, where $\nu = \nu_1 e_1 + \ldots + \nu_n e_n$ and $\mu_1 \nu_1 + \ldots + \mu_n \nu_n = e$. If $d_{p-1} \varrho = 0$, then $d_p \lambda = (d_0 \nu) \wedge \varrho + \nu \wedge (d_{p-1} \varrho) = e \wedge \varrho + \nu \wedge 0 = \varrho$.

Thus, if $\mathcal{S}'$ is a subcollection of $\mathcal{S}$ which is closed under finite intersection, $\mathcal{B}$ is a complex whose elements are in $\mathcal{S}'$, and the equation

$$\mu_1 \nu_1 + \ldots + \mu_n \nu_n = e \quad (\mu_1, \ldots, \mu_n \in \mathfrak{T}(A)) \tag{4}$$

is solvable in $\mathfrak{T}(B)$ for each $B \in \mathcal{S}'$, then we have a sequence $0 \to \mathfrak{T}_n \xrightarrow{d_{n-1}} \ldots \to \mathfrak{T}_1 \xrightarrow{d_0} \mathfrak{T} \to 0$ which is exact on $\mathcal{S}'$, and the results of Lemmas 4.1 and 4.2 apply for the complex $\mathcal{B}$.

If $\mathfrak{T}$ is a convex stack of rings with identity in Q, then we define $Q_p = \{\sum_{j_1, \ldots, j_p} \nu_{j_1, \ldots, j_p} e_{j_1} \wedge \ldots \wedge e_{j_p} : \nu_{j_1, \ldots, j_p} \in Q\}$, with the appropriate identifications among the symbols $e_{j_1} \wedge \ldots \wedge e_{j_p}$. We may consider Q_p to be a $\mathfrak{T}(A)$-module containing $\mathfrak{T}_p(B)$ as a submodule for each $B \in \mathcal{S}$. Since $\mathfrak{T}_p$ is just the $\binom{n}{p}$-fold direct sum of copies of the stack $\mathfrak{T}$, it follows that $\mathfrak{T}_p$ is a convex stack of submodules of Q_p for each p. We may now apply the results of the previous section and Lemmas 4.1 and 4.2 to obtain specific theorems concerning equation (4) for convex stacks.

Let $\mathfrak{T}$ be a convex stack of rings in Q, on a compact convex set $A \subset X$. If $x \in X$ and $\mu_1, \ldots, \mu_n \in \mathfrak{T}(A)$, we shall say that equation (4) is solvable locally at x if there is a convex neighborhood U of x such that (4) can be solved for $\nu_1, \ldots, \nu_n \in \mathfrak{T}(B)$, where $B = \bar{U} \cap A$. If $\mathfrak{T}$ is the stack $\mathfrak{R}$ of Section 2, then Lemma 2.5 says exactly that (4) is solvable locally at $x \in A$ if (4) is solvable in $\mathfrak{R}(\{x\})$.

THEOREM 4.1. *Let $\mathfrak{T}$ be a convex stack of rings on A and $\mu_1, \ldots, \mu_n \in \mathfrak{T}(A)$. If (4) is solvable locally at each point of A, then (4) is solvable in $\mathfrak{T}(A)$.*

Proof. Since A is compact, we have that the topology of A is the weak topology generated by the family of linear functionals on X. It follows that if (4) is locally solvable at each point of A, then we may choose hyperplanes $P_1, \ldots, P_k$ such that if $\mathcal{B} = \{B_1, \ldots, B_l\}$ is the polygonal decomposition of A induced by $P_1, \ldots, P_k$, then (4) is solvable in $\mathfrak{T}(B_i)$

for $i=1, \ldots, l$. Also, if $\mathcal{S}'$ is the collection of all finite intersections of elements of $\mathcal{B}$, then (4) is solvable in $\mathfrak{T}(B)$ for every $B \in \mathcal{S}'$. Hence, by Lemma 4.3, we have a sequence of the form (2) which is exact on $\mathcal{S}'$, and Lemma 4.1 applies. This gives us a sequence $e=a^0, \ldots, a^p, \ldots$, with $a^p \in C^{p-1}(\mathcal{B}, \mathfrak{T}_p)$ and $\delta a^p = \bar{d} a^{p+1}$.

By Theorem 3.2, $H^p(\mathcal{B}, \mathfrak{T}_q)=0$ for $p>0$, and $i: \mathfrak{T}_q(A) \to H^0(\mathcal{B}, \mathfrak{T}_q)$ is an isomorphism for all q. Hence, the hypothesis of Lemma 4.2 is satisfied for all m. Thus, if $a^m \in D^{m-1}$ for some m, then $e \in D^{-1} = \operatorname{im} d_0$, and (4) is solvable in $\mathfrak{T}(A)$. However, $C^{m-1}(\mathcal{B}, \mathfrak{T}_m)=0=D^{m-1}$ if $m>n$ or $m>l+1$. This completes the proof.

The above theorem shows that local solvability implies global solvability in $\mathfrak{T}(A)$ for the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = e$. It would be useful to have a similar theorem for the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = \varrho$ with $e \neq \varrho \in \mathfrak{T}(A)$. To prove such a theorem by our present methods would require constructing a sequence $\ldots \to \mathfrak{T}_p \to \ldots \mathfrak{T}_1 \to \mathfrak{T}_0 \to 0$ which is exact in sufficiently small neighborhoods of each point of A, where $\mathfrak{T}_p$ is a convex stack for $p \geqslant 1$ and $\mathfrak{T}_0(B)$ is the submodule $\mu_1\mathfrak{T}(B) + \ldots + \mu_n\mathfrak{T}(B)$ of $\mathfrak{T}(B)$ for each $B \in \mathcal{S}$. We have not been able to do this in the case of the stack $\mathfrak{R}$ if Section 2. A solution to an analogous problem for the sheaf of germs of analytic functions is presented in IV.F.5. of [2]. This solution is quite involved and requires a great deal of information concerning the local structure of the sheaf.

In the next two theorems we will be concerned with the case where (4) is locally solvable on a subset of A. Using Theorem 3.3, we obtain a particularly useful result in the case where (4) is locally solvable on the boundary of an m-simplex $S \subset A$.

THEOREM 4.2. *Let $\mathfrak{T}$ be a convex stack of rings on A, S an m-simplex in A, and $\mu_1, \ldots, \mu_n \in \mathfrak{T}(A)$. If $\mu_1\nu_1 + \ldots + \mu_n\nu_n = e$ is locally solvable at each point of ∂S, then*

(a) *if $n<m$ the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = e$ is solvable in $\mathfrak{T}(S)$;*

(b) *if $n=m$ there is an element $\varrho \in Q$ such that, for each $\lambda \in \mathfrak{T}(A)$, the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = \lambda$ is solvable in $\mathfrak{T}(S)$ if and only if $\lambda\varrho = 0$.*

Proof. Let $\mathcal{B} = \{B_0, \ldots, B_m\}$ be the complex consisting of the $(m-1)$-faces of S. Since (4) is locally solvable at each point of ∂S, Theorem 4.1 implies that (4) is solvable in $\mathfrak{T}(B)$ for each compact convex set B which is a subset of some B_i. Hence, by Lemma 4.3, we have that Lenma 4.1 applies for the complex $\mathcal{B}$. By Theorem 3.3, $H^p(\mathcal{B}, \mathfrak{T}_q)=0$ for $0<\varrho<m-1$ and $H^0(\mathcal{B}, \mathfrak{T}_q) = \mathfrak{T}_q(S)$ for all q, provided $m>1$. Hence, for $m>1$, Lemma 4.2 also applies.

Let $e=a^0, a^1, \ldots, a^p, \ldots$ be the residue sequence guaranteed by Lemma 4.1. If $n<m$ then $\mathfrak{T}_m=0$ and $a^m=0 \in D^{m-1}$. Hence, by Lemma 4.2, $e \in \operatorname{im} d$ and $\mu_1\nu_1 + \ldots + \mu_n\nu_n = e$ is solvable in $\mathfrak{T}(S)$.

If $n=m>1$, then Theorem 3.4 implies that $\delta_0^*: H^{n-1}(\mathcal{B}, \mathfrak{T}_n)\to Q_n=Q$ is one to one. If $\lambda\in\mathfrak{T}(A)$ then $\lambda=\lambda a^0, \lambda a^1, \ldots, \lambda a^p, \ldots$ is a residue sequence, since δ and $\bar{d}$ are $\mathfrak{T}(A)$-module homomorphisms. It follows from Lemma 4.2 that $\mu_1\nu_1+\ldots+\mu_n\nu_n=\lambda$ is solvable in $\mathfrak{T}(S)$ if and only if $\lambda a^n\in D^{n-1}$. However, $C^{n-1}(\mathcal{B}, \mathfrak{T}_{n+1})=0$, since $\mathfrak{T}_{n+1}=0$, and so $D^{n-1}=B^{n-1}(\mathcal{B},\mathfrak{T}_n)$. It follows that $\mu_1\nu_1+\ldots+\mu_n\nu_n=\lambda$ is solvable in $\mathfrak{T}(S)$ if and only if $\lambda\delta_0 a^n=\delta_0\lambda a^n=0$. The proof is complete for $m>1$ if we set $\varrho=\delta_0 a^n$.

For $m=n=1$, we use the last statement of Theorem 3.3. This gives us the diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \to & \mathfrak{T}_1(S) & \xrightarrow{i} & C^0(\mathcal{B}, \mathfrak{T}_1) & \xrightarrow{\delta_0} & Q \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \to & \mathfrak{T}(S) & \xrightarrow{i} & C^0(\mathcal{B}, \mathfrak{T}) & \xrightarrow{\delta_0} & Q, \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

with exact rows and exact second column. If $\lambda\delta_0 a^1=0$ then $\lambda=db$, where $ib=\lambda a^1$. Thus, $\varrho=\delta_0 a^1$ is the required element of Q in the case $m=n=1$.

Definition 4.3. The element $\varrho\in Q$, given by Theorem 4.2, will be called the residue of the system $\mu_1, \ldots, \mu_n$ on the n-simplex S.

An important feature of the residue ϱ is that it is an element of the $\mathfrak{T}(A)$-module Q. In working with $\mathfrak{R}(A)$, Q will be $\mathfrak{R}'(A)$, and so ϱ will be a measure which is locally in our original subalgebra N of $M(G)$. We will show, in the next section, that ϱ must be a very smooth absolutely continuous measure. Hence, Theorem 4.2 gives a strong connection between spectral properties in $\mathfrak{R}(A)$ and the existence of absolutely continuous measures in N.

THEOREM 4.3. *Let $\mathfrak{T}$ be a convex stack of rings on A. Let B and C be compact semipolygonal subsets of A such that $A=B\cup C$. Suppose that $\mu_1, \ldots, \mu_n, \lambda\in\mathfrak{T}(A)$ and B and C satisfy the following conditions:*

(1) *B is connected and $\langle B\rangle=A$;*

(2) *the equation $\mu_1\nu_1+\ldots+\mu_n\nu_n=e$ is locally solvable at each point of B;*

(3) *the equation $\mu_1\nu_1+\ldots+\mu_n\nu_n=\lambda$ is solvable in $\mathfrak{T}(\langle E\rangle)$ for each connected component E of $B\cap C$:*

Then the equation $\mu_1\nu_1+\ldots+\mu_n\nu_n=\lambda$ is solvable in $\mathfrak{T}(A)$.

Proof. Since B and C are semipolygonal sets, we may choose a polygonal decomposition $\mathcal{A}$ of A such that $\mathcal{A}_B$, $\mathcal{A}_C$, and $\mathcal{A}_{B\cap C}$ are polygonal decompositions of B, C, and $B\cap C$ respectively. As a first step in the proof, we reinterpret conditions (1), (2), and (3) in terms of the complexes $\mathcal{A}_B$, $\mathcal{A}_C$, and $\mathcal{A}_{B\cap C}$.

In view of Lemma 3.6, condition (1) is equivalent to

(1′) i^*: $\mathfrak{T}(A)\to H^0(\mathcal{A}_B, \mathfrak{T})=Z^0(\mathcal{A}_B, \mathfrak{T})$ is an isomorphism.

By Lemma 4.3, condition (2) implies that $0\to\mathfrak{T}_n\xrightarrow{d_{n-1}}\dots\to\mathfrak{T}_1\xrightarrow{d}\mathfrak{T}\to 0$ is an exact sequence on S', where S' is the collection of all compact convex subsets of B. In view of Lemma 4.1, we have that condition (2) implies

(2′) there are residue sequences $\lambda=b^0,\dots,b^p,\dots$ and $\lambda=a^0,\dots,a^p,\dots$ with $b^p\in C^{p-1}(\mathcal{A}_B,\mathfrak{T}_p)$ and $a^p\in C^{p-1}(\mathcal{A}_{B\cap C},\mathfrak{T}_p)$.

If $E_1,\dots,E_k$ are the components of $B\cap C$, then i: $\mathfrak{T}_p(\langle E_1\rangle)\oplus\dots\oplus\mathfrak{T}_p(\langle E_k\rangle)\to H^0(\mathcal{A}_{B\cap C},\mathfrak{T}_p)=Z^0(\mathcal{A}_{B\cap C},\mathfrak{T}_p)$ is an isomorphism for each p. Condition (3) implies that d: $\mathfrak{T}_1(\langle E_1\rangle)\oplus\dots\oplus\mathfrak{T}_1(\langle E_k\rangle)\to\mathfrak{T}(\langle E_1\rangle)\oplus\dots\oplus\mathfrak{T}\langle E_k\rangle$ has the element $\lambda|_{\langle E_1\rangle}\oplus\dots\oplus\lambda|_{\langle E_k\rangle}$ in its image. In other words, the element $a^1\in C^0(\mathcal{A}_{B\cap C},\mathfrak{T}_1)$, of the residue sequence $\lambda=a^0, a^1,\dots,a^p,\dots$, may be chosen from $Z^0(\mathcal{A}_{B\cap C},\mathfrak{T}_1)$. Thus $\delta a^1=0$ and we may choose $a^2=0$. Hence, condition (3) implies

(3′) there is a residue sequence $\lambda=a^0, a^1,\dots,a^p,\dots$ for the complex $\mathcal{A}_{C\cap B}$ such that $a^2=0$.

Recall, from the proof of Theorem 3.1, that the sequence

$$0\to\mathfrak{T}\xrightarrow{\alpha_B\oplus\alpha_C}\mathfrak{T}_B\oplus\mathfrak{T}_C\xrightarrow{\beta_B-\beta_C}\mathfrak{T}_{B\cap C}\to 0$$

is exact on a subdomain containing $\mathcal{A}$, $\mathcal{A}_B$, $\mathcal{A}_C$, and $\mathcal{A}_{B\cap C}$, and it induces the exact sequence

$$0\to C^p(\mathcal{A},\mathfrak{T})\xrightarrow{\bar\alpha_B\oplus\bar\alpha_C}C^p(\mathcal{A}_B,\mathfrak{T})\oplus C^p(\mathcal{A}_C,\mathfrak{T})\xrightarrow{\bar\beta_B-\bar\beta_C}C^p(\mathcal{A}_{B\cap C},\mathfrak{T})\to 0.$$

This, in turn, induces the exact sequence

$$\dots\to H^p(\mathcal{A},\mathfrak{T})\xrightarrow{\alpha_B^*\oplus\alpha_C^*}H^p(\mathcal{A}_B,\mathfrak{T})\oplus H^p(\mathcal{A}_C,\mathfrak{T})\xrightarrow{\beta_B^*-\beta_C^*}H^p(\mathcal{A}_{B\cap C},\mathfrak{T})\xrightarrow{\delta^*}H^{p+1}(\mathcal{A},\mathfrak{T})\to\dots.$$

Since $H^p(\mathcal{A},\mathfrak{T})=0$ for $p>0$, we have that $\beta_B^*-\beta_C^*$ is an isomorphism for $p>0$. This implies that $\bar\beta_B-\bar\beta_C$ carries $B^p(\mathcal{A}_B,\mathfrak{T})\oplus B^p(\mathcal{A}_C,\mathfrak{T})$ onto $B^p(\mathcal{A}_{B\cap C},\mathfrak{T})$ for each $p>0$. These considerations hold for each $\mathfrak{T}_q$ as well as for $\mathfrak{T}$. In particular, for $p=1$ and $q=2,3$ this yields the following commutative diagram:

$$\begin{array}{ccc} C^1(\mathcal{A}_B, \mathfrak{T}_3) \oplus C^1(\mathcal{A}_C, \mathfrak{T}_3) & \xrightarrow{\bar\beta_B - \bar\beta_C} & C^1(\mathcal{A}_{B\cap C}, \mathfrak{T}_3) \to 0 \\ \downarrow \bar d \oplus \bar d & & \downarrow \bar d \\ C^1(\mathcal{A}_B, \mathfrak{T}_2) \oplus C^1(\mathcal{A}_C, \mathfrak{T}_2) & \xrightarrow{\bar\beta_B - \bar\beta_C} & C^1(\mathcal{A}_{B\cap C}, \mathfrak{T}_2) \to 0. \end{array} \tag{5}$$

In (5) the rows are exact and $\bar\beta_B - \bar\beta_C$ maps $B'(\mathcal{A}_B, \mathfrak{T}_q) \oplus B'(\mathcal{A}_C, \mathfrak{T}_q)$ onto $B'(\mathcal{A}_{B\cap C}, \mathfrak{T}_q)$ for $q = 2, 3$.

If $\lambda = b^0, b^1, \dots, b^p, \dots$ is the residue sequence for $\mathcal{A}_B$ given by condition (2′), then $\lambda = \bar\beta_C b^0$, $\bar\beta_C b^1, \dots, \bar\beta_C b^p, \dots$ is a residue sequence for the complex $\mathcal{A}_{B,C}$. However, it follows from condition (3′) and Lemma 4.1 (b), that $\bar\beta_C b^2 \in D^1(\mathcal{A}_{B\cap C}) = B^1(\mathcal{A}_{B\cap C}, \mathfrak{T}_2) + \operatorname{im}(d\colon C^1(\mathcal{A}_{B\cap C}, \mathfrak{T}_3) \to C^1(\mathcal{A}_{B\cap C}\mathfrak{T}_2))$. Diagram (5) then gives us that $b^2 \in D^1(\mathcal{A}_B)$. Condition (1′) implies that the hypothesis of Lemma 4.2 is satisfied with $m = 1$. Hence, $\lambda \in \operatorname{im}(d\colon \mathfrak{T}_1(A) \to \mathfrak{T}(A))$, i.e., the equation $\mu_1\nu_1 + \dots + \mu_n\nu_n = \lambda$ is solvable in $\mathfrak{T}(A)$.

We shall apply the above result in Section 6. In this application, C will be a finite union of disjoint simplices in A, and B will be $A \setminus \operatorname{int} C$.

5. The residue for $\mathfrak{R}$(A)

In this section and the next, we return to the study of the algebra $\mathfrak{R}(A)$ of Section 2. The results of Section 4, concerning convex stacks of rings, apply with X as defined in Definition 1.1, A a compact convex subset of X, $\mathfrak{T} = \mathfrak{R}$, and $Q = \mathfrak{R}'(A)$. Our purpose in this section is to determine the form of the residue measure ϱ constructed in Theorem 4.2.

We assume throughout this section that X has dimension n, int A contains an n-simplex S, and $\mu_1, \dots, \mu_n \in \mathfrak{R}(A)$ have the property that $|\hat\mu_1| \geq \varepsilon > 0$ in a neighborhood of infinity in $e^A\Gamma$, and the equation

$$\mu_1\nu_1 + \dots + \mu_n\nu_n = e \tag{4}$$

is locally solvable at each point of ∂S.

Theorem 4.2 yields a measure $\varrho \in \mathfrak{R}'(A)$ such that if $\lambda \in \mathfrak{R}(A)$, then the equation $\mu_1\nu_1 + \dots + \mu_n\nu_n = \lambda$ has a solution in $\mathfrak{R}(S)$ if and only if $\lambda\varrho = 0$. If $\mathcal{B} = \{B_0, \dots, B_n\}$ is the complex consisting of the $(n-1)$-faces of S, then according to the proof of Theorem 4.2, there exists $a^n \in C^{n-1}(\mathcal{B}, \mathfrak{R}_n) = C^{n-1}(\mathcal{B}, \mathfrak{R})$ such that $\varrho = \delta_0 a^n$.

Let $x_0, \dots, x_n$ be the vertices of S, where $\{x_i\} = \bigcap_{j \neq i} B_j$; i.e., x_i is the vertex opposite the face B_i. We set

$$\varrho_i = (a^n)_{(0,\dots,\hat\imath,\dots,n)} \in \mathfrak{R}\Big(\bigcap_{j\neq i} B_j\Big) = \mathfrak{R}(\{x_i\}).$$

Thus
$$\varrho=\delta_0 a^n=\sum_{j=0}^{n}(-1)^j\varrho_j.$$

Note that for each i, the Laplace transform $\varrho_i^\wedge$ of ϱ_i exists on $e^{x_i}\Gamma$.

Let C be any closed subset of Γ. If E is a compact convex subset of A, we let $\mathfrak{A}_C(E)$ be the algebra of functions on $e^E C$ which are uniform limits of functions bounded and analytic in a neighborhood of $e^E C$. The correspondence $E\to\mathfrak{A}_C(E)$ defines a stack $\mathfrak{A}_C$ of rings with identity on A, where for $\emptyset\neq E\subset F$ the map $\theta_{E,F}\colon\mathfrak{A}_C(F)\to\mathfrak{A}_C(E)$ is the restriction map $f\to f|e^E C$.

The Laplace transform, $\mu\to\mu^\wedge$, defines a homomorphism of $\mathfrak{R}(E)$ into $\mathfrak{A}_C(E)$ for each $E\subset A$, and, hence, a homomorphism of the stack $\mathfrak{R}$ into the stack $\mathfrak{A}_C$ (cf. Section 1 and Lemma 2.3).

LEMMA 5.1. *There is a closed set $C\subset\Gamma$ and a set $\{f_{ij}\}$ of functions, with $f_{ij}\in\mathfrak{A}_C(\langle x_i, x_j\rangle)$, such that $U=\Gamma\setminus C$ has compact closure in Γ, $f_{ij}=-f_{ji}$, and $\varrho_i^\wedge=\sum_{j=1}^n(-1)^j f_{ij}$ on $e^{x_i}C$ for $i=1, \ldots, n$.*

Proof. Let U be an open set with compact closure in Γ such that $|\mu_1^\wedge|\geqslant\varepsilon>0$ on $\Gamma\setminus U$. If we set $C=\Gamma\setminus U$, then $(\mu_1^\wedge)^{-1}$ is bounded on $e^A C$. Also, $\mu_i^\wedge$ can be uniformly approximated on $e^A\Gamma$ by elements of $N_c^\wedge$. It follows that $(\mu_1^\wedge)^{-1}\in\mathfrak{A}_C(A)$.

Let $e=a^0, a^1, \ldots, a^n, \ldots$ be the residue sequence appearing in the proof of Theorem 4.2. This determines a residue sequence $1=(a^0)^\wedge, (a^1)^\wedge, \ldots, (a^n)^\wedge, \ldots$ for the stack $\mathfrak{A}_C$ and the complex $\mathcal{B}$. However, the equation $\mu_1^\wedge f_1+\ldots+\mu_n^\wedge f_n=1$ is trivially solvable in $\mathfrak{A}_C(A)$, since $(\mu_1^\wedge)^{-1}\in\mathfrak{A}_C(A)$. It follows from Lemma 4.1 (c) that $(a^n)^\wedge=\delta\beta+\bar{d}\gamma$, where $\beta\in C^{n-2}(\mathcal{B}, (\mathfrak{A}_C)_n)$ and $\gamma\in C^{n-1}(\mathcal{B}, (\mathfrak{A}_C)_{n+1})=(0)$; i.e., $(a^n)^\wedge=\delta\beta$. If we set $f_{ij}=\beta_{(0,\ldots,\hat{j},\ldots,\hat{i},\ldots,n)}$ for $j<i$, $f_{ij}=-f_{ji}$ for $i<j$, and $f_{ii}=0$, then $f_{ij}\in\mathfrak{A}_C(\bigcap_{k\neq i,j}B_k)=\mathfrak{A}_C(\langle x_i, x_j\rangle)$ and

$$\begin{aligned}\varrho_i^\wedge&=(a^n)^\wedge_{(0,1,\ldots,\hat{i},\ldots,n)}=(\delta\beta)_{(0,1,\ldots,\hat{i},\ldots,n)}\\&=\sum_{j=0}^{i-1}(-1)^j\beta_{(0,\ldots,\hat{j},\ldots,\hat{i},\ldots,n)}-\sum_{j=i+1}^{n}(-1)^j\beta_{(0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n)}=\sum_{j=0}^{n}(-1)^j f_{ij}.\end{aligned}$$

This completes the proof.

If h is a continuous function with compact support on G, then the convolution product $\varrho_i*h(g)=\int h(g-g')d\varrho_i(g')$ defines a continuous function ϱ_i*h on G, with Laplace transform $(\varrho_i*h)^\wedge=\varrho_i^\wedge h^\wedge$. If $h^\wedge(e^x\gamma)$ is integrable with respect to Haar measure on Γ for each fixed $x\in A$, then $(\varrho_i^\wedge h^\wedge)(e^{x_i}\gamma)$ is integrable for each i, and we may recover the function $\varrho*h=\sum_{i=0}^n(-1)^i\varrho_i*h$ from the inversion formula. This leads to:

LEMMA 5.2. *There is a finite regular Borel measure τ, with compact support on $e^A\Gamma$, such that*

$$\varrho * h(g) = \int h^\wedge(\omega)\, \omega(g)\, d\tau(\omega)$$

for every continuous function h, with compact support on G, for which $h^\wedge(e^x\gamma)$ is γ-integrable for each $x \in A$.

Proof. Let h be continuous, with compact support on G, such that $h^\wedge(e^x\gamma)$ is γ-integrable for each $x \in A$. From the inversion formula (Lemma 1.1), we have

$$\varrho * h(g) = \sum_{i=0}^{n} (-1)^i \varrho_i * h(g) = \sum_{i=0}^{n-1} (-1)^i \int \varrho_i^\wedge(e^{x_i}\gamma)\, h^\wedge(e^{x_i}\gamma)\, e^{x_i(g)} \gamma(g)\, d\gamma.$$

By Lemma 5.1, we may write $\varrho_i^\wedge = \sum_{j=0}^{n} (-1)^j f_{ij}$ on $e^{x_i}C$, for some $C = \Gamma \setminus U$, where U is an open set with compact closure and $f_{ij} \in \mathfrak{A}_C(\langle x_i, x_j \rangle)$. Hence

$$\varrho * h(g) = \sum_{i=0}^{n} (-1)^i \int_U \varrho_i^\wedge(e^{x_i}\gamma)\, h^\wedge(e^{x_i}\gamma)\, e^{x_i(g)} \gamma(g)\, d\gamma$$
$$+ \sum_{i=0}^{n} \left[\sum_{j=0}^{n} (-1)^{i+j} \int_C f_{ij}(e^{x_i}\gamma)\, h^\wedge(e^{x_i}\gamma)\, e^{x_i(g)}\, \gamma(g)\, d\gamma \right].$$

The first sum in the above expression represents an integral of $h^\wedge(\omega)\omega(g) = h^\wedge(e^x\gamma) e^{x(g)}\gamma(g)$ with respect to a finite measure τ' concentrated on the bounded set $\bigcup_{i=0}^{n} e^{x_i}U$. We use the fact that $f_{ij} = -f_{ji}$ in the expression in brackets above and simplify, obtaining

$$\varrho * h(g) = \int h^\wedge(\omega)\, \omega(g)\, d\tau'(\omega)$$
$$+ \sum_{i=0}^{n} \sum_{j<i} (-1)^{i+j} \int_C [f_{ij}(e^{x_i}\gamma)\, h^\wedge(e^{x_i}\gamma)\, e^{x_i(g)} - f_{ij}(e^{x_j}\gamma)\, h^\wedge(e^{x_j}\gamma)\, e^{x_j(g)}]\, \gamma(g)\, d\gamma.$$

It suffices to prove that each of the integrals in the double sum can be reduced to the proper form. To do this, we fix i and j and set $f(\omega) = f_{ij}(\omega)\, h^\wedge(\omega)\, \omega(g)$. Thus, the integral we are interested in is just $\int_C [f(e^{x_i}\gamma) - f(e^{x_j}\gamma)]\, d\gamma$. Furthermore, $f \in \mathfrak{A}_C(\langle x_i, x_j \rangle)$.

We may assume without loss of generality that $x_j = 0$ and set $x_i = y$. We then have $f \in \mathfrak{A}_C(\langle 0, y \rangle)$ and

$$\int_C [f(e^{x_i}\gamma) - f(e^{x_j}\gamma)]\, d\gamma = \int_C [f(e^y\gamma) - f(\gamma)]\, d\gamma.$$

To finish the proof, we shall show that there is a measure τ'' on $e^{\langle 0, y\rangle}\Gamma$, with compact support, such that this integral is $\int f(\omega)\,d\tau''(\omega)$.

We choose an $(n-1)$-dimensional subspace X_1 of X such that $X=(y)+X_1$, and set $\Gamma_1=e^{iX_1}$. We choose a set $F\subset\Gamma$ such that F contains exactly one element from each coset of $\Gamma_0=e^{iX}$ in Γ. We set $d\sigma(\gamma')=\sum_{\gamma\in F} d\sigma_1(\gamma+\gamma')$, where σ_1 is Haar measure on Γ_1. If σ_1 is properly normalized, we can write

$$\int_C [f(\gamma)-f(e^x\gamma)]\,d\gamma=\int_{F\Gamma_1}\int_{K_{\gamma'}}[f(e^{ity}\gamma')-f(e^{(1+it)y}\gamma')]\,dt\,d\sigma(\gamma'),$$

where $K_{\gamma'}=\{t\in(-m,m)\colon e^{ity}\gamma'\in C\}$ and $2m$ is the period of the map $t\to e^{ity}$ ($m=\infty$ if $t\to e^{ity}$ is one to one). Note that since $\Gamma\setminus C$ is bounded, $(-m,m)\setminus K_{\gamma'}$ is bounded for each γ' and $K_{\gamma'}=(-m,m)$ except for γ' in a bounded subset J of $F\Gamma_1$.

If $\gamma'\notin J$ then

$$\int_{K_{\gamma'}}[f(e^{ity}\gamma')-f(e^{(1+it)y}\gamma')]\,dt=\frac{1}{i}\int_{-im}^{im} f(e^{zy}\gamma')\,dz-\int_{1-im}^{1+im} f(e^{zy}\gamma')\,dz=0,$$

since $t\to e^{ity}$ has period $2m$ and $f\in\mathfrak{A}_C(\langle 0,y\rangle)$ implies $z\to f(e^z\gamma')$ is bounded and holomorphic for $\operatorname{Im} z\in(-m,m)$ and $\operatorname{Re} z\in(0,1)$. Hence, we can write

$$\int_C [f(\gamma)-f(e^y\gamma)]\,d\gamma=\int_J\int_{K_{\gamma'}}[f(e^{ity}\gamma')-f(e^{(1+it)y}\gamma')]\,dt\,d\sigma(\gamma').$$

If $m<\infty$ then $K_{\gamma'}$ is bounded and we are through. If $m=\infty$ then $K_{\gamma'}=(-\infty,a(\gamma')]\cup L_{\gamma'}\cup[b(\gamma'),\infty)$ for numbers $a(\gamma')$, $b(\gamma')$, and a bounded set $L_{\gamma'}$. Note that the Cauchy integral formula yields

$$\begin{aligned}\int_{b(\gamma')}^{\infty}[f(e^{ity}\gamma')-f(e^{(1+it)y}\gamma')]\,dt&=\frac{1}{i}\left[\int_{ib(\gamma')}^{i\infty} f(e^{zy}\gamma')\,dz-\int_{1+ib(\gamma')}^{1+i\infty} f(e^{zy}\gamma')\,dz\right]\\&=\frac{1}{i}\int_{ib(\gamma')}^{1+ib(\gamma')} f(e^{zy}\gamma')\,dz,\end{aligned}$$

and a similar formula for the integral over $(-\infty,a(\gamma')]$. Putting all of this together, one can see that we have reduced $\int_C[f(\gamma)-f(e^y\gamma)]\,d\gamma$ to an integral of $f(\omega)$ with respect to a measure τ'' concentrated on a compact subset of $e^{\langle 0,y\rangle}\Gamma$. This completes the proof.

LEMMA 5.3. *The residue measure ϱ is absolutely continuous, with a Radon–Nikodym derivative h_ϱ given by*

$$h_\varrho(g)=\int\omega(g)\,d\tau(\omega),$$

for some finite measure τ with compact support in $e^S\Gamma$.

Proof. We choose a net $\{h_\alpha\}$ of continuous functions, with compact support on G, such that: (1) for each α, $h_\alpha^\wedge(e^x\gamma)$ is an integrable function of γ for each $x \in A$; and (2) $\{h_\alpha^\wedge\}$ converges uniformly to 1 on each compact subset of $e^A\Gamma$. In fact, $h_\alpha = \chi_{U_\alpha} * \chi_{U_\alpha}$ defines such a net if $\{U_\alpha\}$ is a neighborhood basis at 0 in G with $\bar{U}_\alpha$ compact for each α. Let τ be the measure of Lemma 5.2. We have,

$$\varrho * h_\alpha(g) = \int h_\alpha^\wedge(\omega)\, \omega(g)\, d\tau(\omega) \to \int \omega(g)\, d\tau(\omega),$$

since τ has compact support. It follows that for each continuous k with compact support on G, $\int k(g)\varrho * h_\alpha(g)\,dg \to \int k(g) h_\varrho(g)\,dg$, where $h_\varrho(g) = \int \omega(g)\,d\tau(\omega)$. From this it follows that $d\varrho(g) = h_\varrho(g)\,dg$.

THEOREM 5.1. *If $\lambda \in \mathfrak{N}(A)$ and the equation $\mu_1^\wedge f_1 + \dots + \mu_n^\wedge f_n = \lambda^\wedge$ has a solution in the class of functions holomorphic in a neighborhood of $e^S\Gamma$, then $\lambda\varrho = 0$.*

Proof. It follows from Lemma 5.3 that $\lambda * h_\varrho(g) = \int \lambda^\wedge(\omega)\omega(g)\,d\tau(\omega)$. By Lemma 2.8, each of the functions f_i is the uniform limit on compact subsets of $e^S\Gamma$ of a sequence $\{\nu_{ij}^\wedge\}_{j=1}^\infty$, with $\nu_{ij} \in N_c$. If we set $\lambda_j = \mu_1\nu_{1j} + \dots + \mu_n\nu_{nj}$, then $\lambda_j^\wedge \to \lambda^\wedge$ on compact subsets of $e^S\Gamma$. Furthermore, we have $\lambda_j \cdot \varrho = 0$ by Theorem 4.2. Hence,

$$\lambda * h_\varrho(g) = \int \lambda^\wedge(\omega)\, \omega(g)\, d\tau(g) = \lim_j \int \lambda_j^\wedge(\omega)\, \omega(g)\, d\tau(g) = \lim_j \lambda_j * h_\varrho(g) = 0.$$

Thus $\lambda \cdot \varrho = 0$.

Since we have assumed that $\mu_1^\wedge$ is bounded away from zero in a neighborhood of infinity on $e^A\Gamma$, the set V of common zeros of the functions $\mu_1^\wedge, \dots, \mu_n^\wedge$ is a compact subset of $e^A\Gamma$. Under the assumptions of this section, $V \cap e^{\partial S}\Gamma = \emptyset$ and so $V \cap (e^{\operatorname{int} S}\Gamma)$ is a compact subset of $e^{\operatorname{int} S}\Gamma$. However, $V \cap (e^{\operatorname{int} S}\Gamma)$ is also an analytic variety. By III.B.17. of [2], $V \cap (e^S\Gamma)$ must consist of a finite set of points $\omega_1, \dots, \omega_k$.

THEOREM 5.2. *Let $\omega_1, \dots, \omega_k$ be the common zeros of $\mu_1^\wedge, \dots, \mu_n^\wedge$ in $e^S\Gamma$, as above. If $x_1, \dots, x_n$ is a basis for X, then there are polynomials $P_1, \dots, P_k$, in $x_1, \dots, x_n$, such that*

$$h_\varrho(g) = \sum_{i=1}^k P_i(g)\, \omega_i(g).$$

Proof. If $\mu \in \mathfrak{N}(A)$, consider the function $\varphi(z) = \mu^\wedge(e^{zx}\omega)$ for fixed $x \in X$, $\omega \in \Omega$. We have

$$\varphi'(z) = \int \frac{d}{dz} e^{-zx(g)} \omega^{-1}(g)\, d\mu(g) = -\int e^{-zx(g)} \omega^{-1}(g)\, x(g)\, d\mu(g).$$

Hence, $\varphi'(0) = \int \omega(-g)\, x(-g)\, d\mu(g) = \mu * h(0)$, where $h(g) = x(g)\,\omega(g)$.

We choose a basis $x_1, \ldots, x_n$ for X, and for each m and each i we let $\mathfrak{J}_{m,i}$ be the linear span of the functions $x_1^{j_1}, \ldots, x_n^{j_n}\omega_i$ for $j_1 + \ldots + j_n \leqslant m$. We may consider each $h \in \mathfrak{J}_{m,i}$ to be an element of $\mathfrak{R}(A)^*$ if we set $h(\mu) = \int h(-g)\,d\mu(g) = \mu * h(0)$ for $\mu \in \mathfrak{R}(A)$. Likewise, h_ϱ can be considered an element of $\mathfrak{R}(A)^*$ if we set $h_\varrho(\mu) = \mu * h_\varrho(0)$. We shall show that there exists an m such that $h_\varrho(\mu) = 0$, whenever $\mu \in \mathfrak{R}(A)$ and $h(\mu) = 0$ for each $h \in \sum_{i=1}^k \mathfrak{J}_{m,i}$. From this and the Hahn–Banach Theorem it will follow that $h_\varrho \in \sum_{i=1}^k \mathfrak{J}_{m,i}$.

Let O denote the ring of germs of functions holomorphic in a neighborhood of zero in $\mathbf{C}^n$. The ring of germs of functions holomorphic in a neighborhood of ω_i in Ω is isomorphic to O through the map $f \to \varphi_f^i$, where $\varphi_f^i(z_1, \ldots, z_n) = f(e^{z_1x_1 + \ldots + z_nx_n}\omega_i)$. We let $\mathfrak{I}_m$ be the ideal in O consisting of functions all of whose derivatives of order $\leqslant m$ are zero at zero. We fix i and let $\mathfrak{I}$ be the ideal in O generated by the functions $\varphi_{\mu_1}^{i\wedge}, \ldots, \varphi_{\mu_n}^{i\wedge}$. Since $\mu_1^\wedge, \ldots, \mu_n^\wedge$ have an isolated common zero at ω_i, the ideal $\mathfrak{I}$ is contained in a unique prime ideal of O; in fact, this unique prime ideal is the maximal ideal $\mathfrak{I}_0 = \{f \in O : f(0) = 0\}$. It follows that $\mathfrak{I}_m = (\mathfrak{I}_0)^m \subset \mathfrak{I}$ for some m (cf. [2], Ch. II). This implies that the equation $\mu_1^\wedge f_1 + \ldots + \mu_n^\wedge f_n = \lambda^\wedge$ is solvable in $\mathfrak{A}(U)$, for some neighborhood U of ω_i, provided $(\partial/\partial z_1^{j_1} \ldots \partial z_n^{j_n})\,\lambda^\wedge(e^{z_1x_1 + \ldots + z_nx_n}\omega_i) = 0$ whenever $j_1 + \ldots + j_n \leqslant m$. By the result of the first paragraph, this last condition is equivalent to $h(\lambda) = \lambda * h(0) = 0$ for all $h \in \mathfrak{J}_{m,i}$. It follows that if $h(\lambda) = 0$ for every $h \in \sum_{i=1}^k \mathfrak{J}_{m,i}$, then the equation $\mu_1^\wedge f_1 + \ldots + \mu_n^\wedge f_n = \lambda^\wedge$ has a holomorphic solution in a neighborhood of ω_i for each i. Since $\omega_1, \ldots, \omega_k$ are the only common zeros of $f_1, \ldots, f_n$ in $e^S\Gamma$, we conclude that $\mu_1^\wedge f_1 + \ldots + \mu_n^\wedge f_n = \lambda^\wedge$ has a holomorphic solution in a neighborhood of each point of $e^S\Gamma$.

Since $S \subset \operatorname{int} A$, we may choose an open convex set V such that $S \subset V \subset A$. If $\lambda \in \mathfrak{R}(A)$ and $h(\lambda) = 0$ for every $h \in \sum_{i=1}^k \mathfrak{J}_{m,i}$, then we may choose V in such a way that $\mu_1^\wedge f_1^\wedge + \ldots + \mu f_n^\wedge = \lambda^\wedge$ has a holomorphic solution in a neighborhood of each point of $e^V\Gamma$. By Lemma 1.6, the equation $\mu^\wedge f_1 + \ldots + \mu_n^\wedge f_n = \lambda^\wedge$ has a global holomorphic solution in $e^V\Gamma$. By Theorem 5.1, it follows that $\lambda \cdot \varrho = 0$. Hence, $\lambda(h) = 0$ for every $h \in \sum_{i=0}^k \mathfrak{J}_{m,i}$ implies $\lambda(h_\varrho) = \lambda * h_\varrho(0) = 0$. This shows that $h_\varrho \in \sum_{i=0}^k \mathfrak{J}_{m,i}$ and completes the proof.

We should point out that the blanket assumption that $|\mu_1^\wedge| \geqslant \varepsilon > 0$ in a neighborhood of infinity, which was used throughout this section, is probably not necessary and should be removed. Also, we have assumed that $\dim S = \dim X$. It should be possible to say something about a residue measure ϱ for a simplex S with $\dim S < \dim X$.

6. Ideal theory in $\mathfrak{A}(A)$

We are now in a position to generalize the facts F2, F3 of Section 1. In the process, we obtain a fairly surprising connection between spectral theory in $\mathfrak{R}(A)$ and the smoothness of the measures in N.

Our first result is a characterization, analogous to F2, of the maximal ideal space of the Banach algebra $\mathfrak{N}(A)$. We seek a compact Hausdorff space $\Delta(A)$ and a homomorphism $\mu \to \mu^{\vee}$ of $\mathfrak{N}(A)$ into $C(\Delta(A))$ with the following properties: (1) $e^{\vee}=1$; (2) $\mathfrak{N}(A)^{\vee}=\{\mu^{\vee}: \mu \in \mathfrak{N}(A)\}$ separates points in $\Delta(A)$; (3) every maximal ideal of $\mathfrak{N}(A)$ has the form $\{\mu \in \mathfrak{N}(A): \mu^{\vee}(m)=0\}$ for some $m \in \Delta(A)$. In the presence of condition (1), condition (3) is equivalent to the following: (3′) If $\mu_1, ..., \mu_n \in \mathfrak{N}(A)$ and the functions $\mu_1^{\vee}, ..., \mu_n^{\vee}$ do not vanish simultaneously at any point of $\Delta(A)$, then the equation $\mu_1\nu_1+...+\mu_n\nu_n=e$ has a solution in $\mathfrak{N}(A)$.

For the algebra $N=\mathfrak{N}(\{0\})$ let the maximal ideal space be Δ and the Gelfand transform be $\mu \to \mu^{\vee}$. For $\mu \in \mathfrak{N}(A)$ and $x \in A$ we set $dT_{-x}\mu = e^{-x}d\mu$. By Lemma 2.4, T_{-x} is a homomorphism of $\mathfrak{N}(A)$ into N, and for fixed $\mu \in \mathfrak{N}(A)$ the map $x \to T_{-x}\mu$ is continuous from A into N. It follows that if we set $\mu^{\sim}(x, m)=(T_{-x}\mu)^{\vee}(m)$ for $x \in A$, $m \in \Delta$, and $\mu \in \mathfrak{N}(A)$, then $\mu \to \mu^{\sim}$ is a homomorphism of $\mathfrak{N}(A)$ into $C(A \times \Delta)$. Since $T_{-x}e=e$, condition (1) is satisfied for $\mu \to \mu^{\sim}$. We shall show that (3′) is also satisfied.

If $\mu_1, ..., \mu_n \in \mathfrak{N}(A)$ and $\mu_1^{\sim}, ..., \mu_n^{\sim}$ do not vanish simultaneously on $A \times \Delta$, then for each $x \in A$ the functions $(T_{-x}\mu_1)^{\vee}, ..., (T_{-x}\mu_n)^{\vee}$ do not vanish simultaneously on Δ. Since Δ is the maximal ideal space of N, the equation $(T_{-x}\mu_1)\nu_1+...+(T_{-x}\mu_n)\nu_n=e$ has a solution in N. If we apply T_x to such a solution, we find that $\mu_1\nu_1+...+\mu_n\nu_n=e$ has a solution in $\mathfrak{N}(\{x\})$ for each $x \in A$. By Lemma 2.5, this equation can be solved locally on A, and by Theorem 4.1, it can be solved in $\mathfrak{N}(A)$. Hence, (3′) is satisfied for the map $\mu \to \mu^{\sim}$.

Unfortunately, condition (2) is not generally satisfied for $\mu \to \mu^{\sim}$. For some $m \in \Delta$, $x, y \in A$ it may be that $\mu^{\sim}(x, m)=\mu^{\sim}(y, m)$ for all $\mu \in \mathfrak{N}(A)$. However, if $m_1 \neq m_2$ then (x, m_1) and (x, m_2) can be separated for each $x \in A$; this follows from the fact that T_{-x} carries $\mathfrak{N}(A)$ onto a dense subalgebra of N.

We set $(x, m_1) \sim (y, m_2)$ if $\mu^{\sim}(x, m_1)=\mu^{\sim}(x, m_2)$ for all $\mu \in \mathfrak{N}(A)$ and let $\Delta(A)$ be the factor space of $A \times \Delta$ modulo the equivalence relation $\sim$. Let α: $A \times \Delta \to \Delta(A)$ be the natural map. If we set $\mu^{\vee}(\alpha(x, m))=\mu^{\sim}(x, m)$, then $\Delta(A)$ and $\mu \to \mu^{\vee}$ satisfy conditions 1–3. This gives us the following theorem:

THEOREM 6.1. *If $\Delta(A)$ is the maximal ideal space of $\mathfrak{N}(A)$ and $\mu \to \mu^{\vee}$ is the Gelfand transform, then there is a continuous function α from $A \times \Delta$ onto $\Delta(A)$ such that $\mu^{\vee}(\alpha(x, m))=(T_{-x}\mu)^{\vee}(m)$ for $\mu \in \mathfrak{N}(A)$, where $(T_{-x}\mu)^{\vee}$ is the Gelfand transform of $T_{-x}\mu$ as an element of N. The function α is one to one on $\{x\} \times \Delta$ for each $x \in A$.*

The Fourier transform $\mu \to \mu^{\wedge}$, is a homomorphism of N into $C(\Gamma)$. It follows that Γ may be continuously embedded in Δ in such a way that $\mu^{\wedge}$ is the restriction to Γ of the Gelfand transform $\mu^{\vee}$. In the case where $N=L_e$, the space of absolutely continuous meassures with the identity adjoined, we have that this embedding is just the embedding of Γ in its one-point compactification if Γ is noncompact. If Γ is compact then $\Gamma = \Delta$ in this case. We let Γ' denote the one-point compactification, $\Gamma \cup \{\infty\}$, of Γ if Γ is noncompact and set $\Gamma' = \Gamma$ if Γ is compact.

COROLLARY. *If the maximal ideal space of N is Γ', then the maximal ideal space of $\mathfrak{N}(A)$ is $(e^A\Gamma)'$, where $(e^A\Gamma)'$ is $e^A\Gamma$ if Γ is compact and the one-point compactification of $e^A\Gamma$ otherwise. The Gelfand transform $\mu^\vee$ for $\mu \in \mathfrak{N}(A)$ is the natural extension of the Laplace transform $\mu^\wedge$ to $(e^A\Gamma)'$.*

Proof. If we set $\alpha(x, \gamma) = e^x\gamma$ for $x \in A$, $\gamma \in \Gamma$ and $\alpha(x, \infty) = \infty$ when $\Gamma' = \Gamma \cup \{\infty\}$, then α is a continuous map of $A \times \Gamma'$ onto $(e^A\Gamma)'$. Also, $\mu^\wedge(\alpha(x, \gamma)) = \mu^\wedge(e^x\gamma) = (T_{-x}\mu)^\wedge(\gamma)$ for $x \in A$, $\gamma \in \Gamma$. Since $\mathfrak{N}(A)^\wedge$ separates points in $e^A\Gamma$, the corollary will follow from Theorem 6.1 if we can show that $\mu^\wedge(\infty) = \lim_{\gamma\to\infty} (T_{-x}\mu)^\wedge(\gamma)$ exists and is independent of x (in case Γ is not compact). However, since Γ' is the maximal ideal space of N, we have $\mu^\wedge(\infty) = \lim_{\gamma\to\infty}\mu^\wedge(\gamma)$ exists for each $\mu \in N$. This implies that each $\mu \in N$ has the form $\mu = \mu_1 + ze$, where μ_1 is a continuous measure and $\lim_{\gamma\to\infty} \mu_1^\wedge(\gamma) = 0$. It follows that each $\mu \in \mathfrak{N}(A)$ also has this form and $\lim_{\gamma\to\infty} (T_{-x}\mu)^\wedge(\gamma) = \lim_{\gamma\to\infty} (T_{-x}\mu_1)^\wedge + z = z$ for every $x \in A$. This completes the proof.

Unfortunately for the significance of the above result, the case of greatest interest is the case where $N = L_e$, and in this case we may trivially prove that $\Delta(A) = (e^A\Gamma)'$ by using the fact that the adjoint space of $\mathfrak{L}_e(A)$ is a space of measurable functions on G.

More surprising results may be obtained by applying the results of Section 5.

THEOREM 6.2. *Let N be any subalgebra of $M(G)$ satisfying N1, N2, and N3 of Section 2. Let the dimension of X be n. If, for some n-simplex $S \subset X$ and some collection $\mu_1, \ldots, \mu_n \in \mathfrak{N}(S)$, the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = e$ is solvable in $\mathfrak{N}(\{x\})$ for each $x \in \partial S$ but is not solvable in $\mathfrak{N}(S)$, and if $|\mu_1^\wedge|$ is bounded away from zero in a neighborhood of ∞ in $e^S\Gamma$, then $L_e \subset N$.*

Proof. The results of Section 5 apply. Hence, there is an absolutely continuous measure $\varrho \in \mathfrak{N}'(A)$ with the property that the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = \lambda$ is solvable in $\mathfrak{N}(A)$ if and only if $\lambda \cdot \varrho = 0$. Since, by hypothesis, the equation $\mu_1\nu_1 + \ldots + \mu_n\nu_n = e$ is not solvable in $\mathfrak{N}(A)$, we have $e \cdot \varrho = \varrho \neq 0$. Since the restriction of ϱ to any compact subset of G is an element of N, it follows that $L_e \cap N \neq (0)$. Conditions N_1, N_2, and N_3 now imply that $L_e \subset N$ (cf. [9]).

COROLLARY. *If the maximal ideal space of N is Γ', then $L_e \subset N$.*

Proof. We chose measures $\mu_1, \ldots, \mu_n \in N_c$ such that $\mu_1^\wedge, \ldots, \mu_n^\wedge$ have a nonempty discrete set of common zeros in $e^U\Gamma$ for some neighborhood U in X. We choose μ_1 such that $|\mu_1^\wedge|$

is bounded away from zero in a neighborhood of infinity in $e^U\Gamma$. That we can so choose these measures follows from Lemma 2.8. and the fact that Ω is an n-dimensional complex manifold. We now choose a simplex $S\subset U$ such that $e^S\Gamma$ contains a common zero of $\hat{\mu}_1, \dots, \hat{\mu}_n$, but $e^{\partial S}\Gamma$ does not. The corollary now follows from Theorem 6.2 and the corollary to Theorem 6.1.

If a subalgebra N of $M(G)$, without identity, satisfies N1 and N2 and has Γ as its maximal ideal space (e.g.: $N=L$) then we may adjoin the identity to obtain an algebra N_e to which the above corollary applies. We conclude that $L\subset N$ in this case. In combination with results of [9], this leads to a complete characterization of all such algebras N. The result is this: For any l.c.a. group G, a closed sulabgebra N of $M(G)$, satisfying N1, has Γ as its maximal ideal space if and only if $L\subset N\subset \sqrt{L}$, where $\sqrt{L}$ is the intersection of all maximal ideals of $M(G)$ containing L. A discussion of this result and its implications for the structure theory of $M(G)$ will be found in [10].

Unfortunately, one aspect of Theorem 6.2 is this: It shows that the situation described in Section 5 can lead to only the trivial residue measure $\varrho=0$, unless $L_e\subset N$. This suggests that, once Theorem 6.2 has been proved, the residue theory has further significance only for the algebra L_e. Thus, we shall restrict attention to this algebra from here on.

If f is a measurable function on G for which $f\varphi_A^{-1}$ is bounded (cf. Definition 2.2), then f is μ-integrable for every $\mu\in\mathfrak{L}_e(A)$ and f may be considered an element of $\mathfrak{L}_e(A)^*$ if we set $f(\mu)=\int f(-g)\,d\mu(g)$. In fact, every element of $\mathfrak{L}_e(A)^*$ clearly has this form. Hence, we may identify $\mathfrak{L}_e(A)^*$ with the space $\mathfrak{L}_e^\infty(A)$ of all equivalence classes of measurable functions f for which $f\varphi_A^{-1}$ is bounded. Two functions f, g are equivalent in this space if $f(0)=g(0)$ and $f=g$ almost everywhere with respect to Haar measure.

Definition 6.1. (a) A function $f\in\mathfrak{L}_e^\infty(A)$ will be called exponential if $f(g)=P(g)\,\omega(g)$ for some $\omega\in e^A\Gamma$ and some polynomial $P(g)=\sum_{j_1+\dots+j_n\leqslant m} a_{j_1,\dots,j_n} x_1^{j_1}(g), \dots, x_n^{j_n}(g)$ with $x_1, \dots, x_n\in X$.

(b) If J is an ideal in $\mathfrak{L}_e(A)$, we set $J^\perp=\{f\in\mathfrak{L}_e^\infty(A): f(\mu)=0 \text{ for every } \mu\in J\}$.

The correspondence $J\to J^\perp$ is one to one and onto from the class of closed ideals of $\mathfrak{L}_e(A)$ to the class of weak-* closed, translation invariant linear subspaces of $\mathfrak{L}_e^\infty(A)$. Since $J^\perp$ is translation invariant, we have that $f\in J^\perp$ if and only if $f_g(\mu)=\int f(g-g')\,d\mu(g')=\mu * f(g)=0$ for each $\mu\in J$ and $g\in G$, where $f_g(g')=f(g'-g)$. The obvious analogue of the spectral synthesis problem for $L^1(G)$ (cf. [6], Ch. 7) is the following: Under what conditions on an ideal $J\subset\mathfrak{L}_e(A)$ is it true that $J^\perp$ is the weak-* closed linear span of its exponential elements? Note that F3 gives a partial result on this problem in the case $G=R^n$. We now prove this result in our more general situation.

By the corollary to Theorem 6.1, the maximal ideal space of $\mathfrak{L}_e(A)$ may be identified with $(e^A\Gamma)'$, where for $\mu \in \mathfrak{L}_e(A)$ the Gelfand transform $\mu^\vee$ is the Laplace transform $\mu^\wedge$ extended to $(e^A\Gamma)' = e^A\Gamma \cup \{\infty\}$ ($(e^A\Gamma)' = e^A\Gamma$ if Γ is compact). If J is an ideal of $\mathfrak{L}_e(A)$, then we set $h(J) = \{\omega \in (e^A\Gamma)' : \mu^\wedge(\omega) = 0 \text{ for all } \mu \in J\}$.

LEMMA 6.1. *If the dimension of X is n, and J is an ideal of $\mathfrak{L}_e(A)$ such that $h(J) \cap (e^{\partial A}\Gamma)' = \varnothing$, then*

(1) *$h(J)$ is a finite set $\{\omega_1, ..., \omega_k\}$; and*

(2) *if K is any compact subset of* int A, *then there exist n elements $\mu_1, ..., \mu_n$ of J such that there are only finitely many common zeros of $\mu_1^\wedge, ..., \mu_n^\wedge$ in $e^K\Gamma$ and $\mu_1^\wedge$ does not vanish at infinity.*

Proof. Since $(e^{\partial A}\Gamma)'$ is compact in $(e^A\Gamma)'$, we can choose $\nu_1, ..., \nu_m \in J$ such that no common zero of $\nu_1^\wedge, ..., \nu_m^\wedge$ lies on $(e^{\partial A}\Gamma)'$. However, the set of common zeros of $\nu_1^\wedge, ..., \nu_m^\wedge$ is then a compact subvariety of $e^{\operatorname{int} A}\Gamma$. It follows that this set is finite (cf. [2], III.B.17).

To prove part (2), we choose any measure $\mu_1 \in J$ such that $\mu_1^\wedge$ does not vanish at infinity. If $V = \{\omega \in e^{\operatorname{int} A}\Gamma : \mu^\wedge(\omega) = 0\}$, then V is an analytic space of pure dimension $n-1$ (cf. [2], Ch. V). It is a direct application of V.D.4 of [2], that we can choose $\nu_2, ..., \nu_n$ from the closure of J such that $\nu_2^\wedge, ..., \nu_n^\wedge$ have a discrete set of common zeros on V. Hence, if K is a compact subset of int A, we may choose $\mu_2, ..., \mu_n \in J$ close enough to $\nu_2, ..., \nu_n$ so that $\mu_1^\wedge, ..., \mu_n^\wedge$ have a finite set of common zeros on $e^K\Gamma$. This follows from III.B.17 of [2] and the fact that $e^K\Gamma \cap V$ is compact.

THEOREM 6.3. *If J is an ideal of $\mathfrak{L}_e(A)$ such that $h(J) \cap (e^{\partial A}\Gamma)' = \varnothing$, then J is closed and $J^\perp$ is finite dimensional. Furthermore, $J^\perp$ is the linear span of exponential elements of the form $P(g)\omega(g)$, with $\omega \in h(J)$ and $P\omega \in J^\perp$. In other words, there are finitely many exponential elements $P_1\omega_1, ..., P_k\omega_k \in J^\perp$ such that $\omega_i \in h(J)$ for each i and $J = \{\lambda \in \mathfrak{L}_e(A) : \lambda^*(P_i\omega_i) = 0$ for $i = 1, ..., k\}$.*

Proof. Note that if $\omega \in e^A\Gamma$, then $\{P\omega : P \text{ is a polynomial in } x_1, ..., x_n \text{ of degree } \leqslant m\}$ is a finite dimensional translation invariant subspace of $\mathfrak{L}_e^\infty(A)$. It follows that if $\mu \in \mathfrak{L}_e(A)$, $\omega \in e^A\Gamma$, and P is any polynomial in $x_1, ..., x_n$, then $\mu * (P\omega) = P_1\omega$ for some other polynomial P_1. Hence, if $h = \sum P_i\omega_i \in \mathfrak{L}_e^\infty(A)$ with $\omega_i \neq \omega_j$ for $i \neq j$ and $\mu * h = 0$, then $\mu * (P_i\omega_i) = 0$ for each i. Thus, if J is an ideal and $h = \sum P_i\omega_i \in J^\perp$ with $\omega_i \neq \omega_j$ for $i \neq j$, then $P_i\omega_i \in J^\perp$ for each i.

If J contains an ideal J_1, and if the conclusions of the theorem hold for J_1, then they also hold for J. In fact, if J_1 is closed and $J_1^\perp$ is finite dimensional, then $\mathfrak{L}_e(A)/J_1$ is finite dimensional and contains J/J_1 as an ideal. It follows that J is closed and $J^\perp$ is finite dimensional. Since $J^{\perp} \subset J_1^{\perp}$, we have that $J^{\perp}$ is generated by exponential elements of $J_1^{\perp}$. However, the first paragraph then implies that $J^{\perp}$ is generated by exponential elements of $J^{\perp}$. Necessarily these exponential elements have the form $P\omega$, with $\omega \in h(J)$.

By Lemma 6.1, $h(J) = \{\omega_1, ..., \omega_k\}$ – a finite subset of $e^{\text{int } A}\Gamma$. We choose a compact convex set $K \subset \text{int } A$ such that $h(J) \subset e^{\text{int } K}\Gamma$. In accordance with Lemma 6.1. (b), we choose $\mu_1, ..., \mu_n \in J$ such that $\mu_1^\wedge(\infty) \neq 0$ and $\mu_1^\wedge, ..., \mu_n^\wedge$ have a finite set of common zeros in $e^K\Gamma$. This implies that the points $\omega_1, ..., \omega_k$ are isolated common zeros of the set $\{\mu_1^\wedge, ..., \mu_n^\wedge\}$. Hence, we may choose a pairwise disjoint collection $S_1, ..., S_l$ of n-simplices contained in A, such that the common zeros of $\mu_1^\wedge, ..., \mu_n^\wedge$ lie in $\bigcup_{i=1}^{l}(e^{S_i}\Gamma)$. Let J_1 be the ideal of $\mathfrak{L}_e(A)$ generated by $\mu_1, ..., \mu_n$. We have $J_1 \subset J$, and $\lambda \in J_1$ if and only if $\lambda \in \mathfrak{L}_e(A)$ and the equation $\mu_1\nu_1 + ... + \mu_n\nu_n = \lambda$ has a solution in $\mathfrak{L}_e(A)$.

Consider the case where $n > 1$. This implies that the set $B = A \setminus \bigcup_{i=1}^{l} (\text{int } S_i)$ is connected. Note that the functions $\mu_1^\wedge, ..., \mu_n^\wedge$ have no common zeros on $(e^B\Gamma)'$. Applying Theorem 4.3 with $C = \bigcup_{i=1}^{l} S_i$ and B as above, we obtain: the equation $\mu_1\nu_1 + ... + \mu_n\nu_n$ has a solution in $\mathfrak{L}_e(A)$ if and only if it has a solution in $\mathfrak{L}_e(S_i)$ for each i. By Theorem 4.2 and Theorem 5.2, there are exponential elements $P_1\omega_1, ..., P_k\omega_k$, such that $\mu_1\nu_1 + ... + \mu_n\nu_n = \lambda$ is solvable in $\mathfrak{L}_e(S_i)$ for each i whenever $\lambda * (P_j\,\omega_j) = 0$ for each j. Hence, if $J_2 = \{\lambda \in \mathfrak{L}_e(A)\colon\ \lambda * (P_j\,\omega_j) = 0 \text{ for } j = 1, ..., k\}$, then $J_2 \subset J_1 \subset J$, J_2 is closed, and $J_2^{\perp}$ is generated by exponential elements of the form $P\omega$ with $\omega \in h(J_2) = h(J_1)$. In view of our previous remark, the proof is complete in the case $n > 1$. In the case $n = 1$, A itself is a 1-simplex (an interval) and we can apply Theorem 4.2 directly without using Theorem 4.3.

The above result is not very surprising and seems small reward for all of the effort of the previous sections. We would be very interested in a simple proof of this result. It may be possible to obtain such a proof directly from the theory of several complex variables, without resorting to the special sheaf theoretic constructions of Sections 3 and 4.

The study of ideals J of $\mathfrak{L}_e(A)$, for which $h(J) \cap (e^{\partial A}\Gamma)' \neq \emptyset$, would seem to be a far more difficult task.

We conclude with the following remark concerning Theorem 6.2: In the case of a general algebra $N \subset M(G)$, satisfying N_1, N_2, and N_3, it may be possible to find further connections between spectral properties of $\mathfrak{N}(A)$ and measure theoretic properties of N, by investigating the situation described by Theorem 4.2 in the case $n > m$.

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