ALGEBRAS OF BOUNDED ANALYTIC FUNCTIONS ON RIEMANN SURFACES

BY

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Let (W, Γ) be a bordered Riemann surface. We say that (W, Γ) satisfies the AB-maximum principle if each bounded analytic function on $W \cup \Gamma$ assumes its maximum on Γ. If W has genus zero, then W is conformally equivalent to a plane domain whose boundary consists of the image of Γ and a totally disconnected perfect set E which has the property that it is "removable" for every bounded analytic function, i.e. every function bounded and analytic in $U \sim E$ for some neighborhood U of E can be extended to be analytic in U. A similar situation occurs when W has finite genus, but when W has infinite genus we cannot represent it as the complement of a set on a compact surface, and so the question arises of whether we can have some generalization of this notion of "removability".

Myrberg [4] and Selberg [6] have shown that for certain twosheeted covering surfaces of the disc each bounded analytic function on the surface is obtained by lifting a bounded analytic function from the disc to the surface, and Heins [2] has shown that, if W has a single end and that a parabolic end, then there is a mapping φ of that end into a disc so that every bounded analytic function on the end is of the form $f \circ \varphi$ where f is analytic on the disc. In the present paper we generalize these results by establishing Theorem 3 which states that, if (W, Γ) satisfies the AB-maximum principle, then there is an analytic mapping φ of $W \cup \Gamma$ into a compact Riemann surface such that each bounded analytic function on $W \cup \Gamma$ is of the form $f \circ \varphi$ where f is an analytic function in a neighborhood of the closure of $\varphi[W \cup \Gamma]$.

The proof of this theorem relies on the application of techniques from the theory of function algebras to the algebra of bounded analytic functions on $W \cup \Gamma$. We begin in Section 1 by showing that, if we have any algebra of analytic functions on a Riemann

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surface, then there is a Riemann surface on which the algebra really lives in the sense that it weakly distinguishes points on this surface and the surface is the largest one to which the algebra can be extended with this separation property. This is generalized to cover the case of a surface with border. We then show that if W is the proper Riemann surface for an algebra A of analytic functions and K a compact subset of W then the homomorphisms of A into the complex numbers which are bounded by the maximum of the functions on K are all obtained by evaluations at points of W which are either in K or in a compact component of the complement of K. This result (Theorem 1) and a generalization (Theorem 2) to enable us to handle the case of a bordered surface then lead to our result on "removability" given by Theorem 3.

In proving Theorem 1 considerable use is made of what are here called analytic characters on an algebra. These are analyzed in some detail in Section 4 and used to associate bounded homomorphisms with points on a Riemann surface. The ideas used here were first used by Wermer [7] in proving that, under certain conditions, an algebra of analytic functions on the unit circumference could be extended to a finite Riemann surface bounded by the unit circumference. This theorem of Wermer's was the origin of my investigations, and Theorems 1 and 2 of this paper can be considered to be generalizations of Wermer's theorem. In Section 7 we derive Wermer's theorem as a consequence of Theorem 2.

Errett Bishop [1] has also generalized the work of Wermer, and his Theorem 2 has virtually the same content as Theorem 1 here. His method of proof differs from that used here in that he considers the maximal ideal space of an algebra and introduces an analytic structure into it, whereas here we construct the proper Riemann surface for an algebra and then show that the maximal ideal space of the algebra can be represented on this surface. Both methods of proof, however, rely in some form on Wermer's ideas.

1. Algebras and their representations

It will be convenient to consider algebras of analytic functions not only on connected Riemann surfaces but also on not necessarily connected Riemann surfaces. In accordance with the usual terminology, we shall use the term "Riemann surface" to mean a connected Riemann surface, unless otherwise specified. If A is an algebra of analytic functions on a not necessarily connected Riemann surface, then we say that points p and q of the surface are *weakly separated* by A if there are functions f and g in A such that there are neighborhoods of p and q in which f has only isolated zeros and g/f assumes different values at p and q.

LEMMA 1. *If A separates p weakly from q, then there are neighborhoods U and V of p and q such that A separates each pair of points in $U \times V$ except $\langle p, q\rangle$.*

Proof. Let f and g be elements of the algebra such that f has only isolated zeros near p and q and g/f has different values at p and q. Since A is an algebra, we may assume that g/f is 0 at p and ∞ at q. Let U be a neighborhood of p in which f has no zeros other than p and where $|g/f| < 1$. Let V be a neighborhood of q in which f has no zeros other than q and where $|g/f| > 1$. Then f and g separate each pair of points in $U \times V$ except possibly $\langle p, q\rangle$.

LEMMA 2. *If A separates weakly on a Riemann surface W, then the subset C of $W \times W$ defined by $C = \{\langle p, q\rangle : p \neq q;\ f(p) = f(q) \text{ for all } f \in A\}$ is countable.*

Proof. Let Δ be the diagonal of $W \times W$, i.e. $\Delta = \{\langle p, p\rangle\}$. Then Lemma 1 states that $W \times W \sim \Delta$ can be covered by neighborhoods $U \times V$ each of which contains at most one element of C. Since W is separable (i.e. first countable), so is $W \times W \sim \Delta$, and so this covering by neighborhoods has a countable subcovering. Consequently, C is countable.

LEMMA 3. *Let A be a weakly separating algebra of analytic functions on a (not necessarily connected) Riemann surface W, and ψ an analytic map of W into a (not necessarily connected) Riemann surface V such that for each $f \in A$ there is an analytic function $\hat{f}$ on V with $f = \hat{f} \circ \psi$. Then ψ is one-to-one.*

Proof. Let p and q be two distinct points of W, and let f and g be elements of A with g/f taking different values at p and q and f having only isolated zeros near p and q. Then $\hat{g}/\hat{f}$ must take different values at $\psi(p)$ and at $\psi(q)$. Hence $\psi(p) \neq \psi(q)$, and ψ is one-to-one.

We say that the algebra A of analytic functions is *primitive* at the point p if there are elements f and g in A such that g/f has a simple zero at p.

LEMMA 4. *Let A be primitive at the points p and q. Then A weakly separates p and q if and only if there is an element of A whose order at p is different from its order at q.*

Proof. If A weakly separates p from q, then there are functions f and g in A with g/f having different values at p and q. Hence either $g - f$ or f has an order at p different from that at q, and the "only if" part of the lemma is established.

Suppose A is primitive at p and q and that there is a function h with different orders at p and q. Let g_1/f_1 and g_2/f_2 have simple zeros at p and q, respectively. By adding to f_1 a suitable power of f_2 and to f_2 a suitable power of f_1 we may assume also that f_1 does not vanish identically near q and f_2 does not vanish identically near p. If neither g_1/f_1 or g_2/f_2

separates p from q, then they are both zero at both p and q and so some linear combination of them will have a simple zero at both p and q. Let g/f be this linear combination. Let h be a function with different orders at p and q, and let n be the smaller order. Then hf^n/g^n separates p from q, and the lemma is proved.

Let R be a (not necessarily connected) Riemann surface and σ a homomorphism of an algebra A onto a subalgebra A^σ of the analytic functions on R. We call σ a *representation* of A on R if for each component of R there is an $f^\sigma \in A^\sigma$ which is not constant on that component. When speaking of an algebra A we always assume that it is an algebra with unit over the complex field and that it has at least one representation. This is always the case if A is an algebra of analytic functions on a Riemann surface, for the identity homomorphism is a representation. If O is an open subset of R, we define the *restriction* to O of a representation σ of A on R to be σ followed by restriction to O.

If ϱ and σ are representations of A on R and S (or more generally homomorphisms into the analytic functions on R and S), we say that ϱ and σ are conformally equivalent if there is a one-to-one analytic map ψ of R onto S such that $f^\varrho = f^\sigma \circ \psi$ for each $f \in A$. If ϱ and σ are two representations of A onto neighborhoods of 0 in the complex plane, we say that ϱ and σ are *locally equivalent* (at 0) if there is a one-to-one conformal map ψ of some neighborhood U of 0 onto a neighborhood of 0 such that $\psi(0)=0$ and such that $f^\varrho = f^\sigma \circ \psi$ on the neighborhood U. Clearly the restriction of ϱ to any smaller neighborhood of 0 is equivalent to ϱ. The following lemma gives a criterion for local equivalence.

LEMMA 5. *Let ϱ and σ be two representations of A onto neighborhoods of* 0. *Then ϱ and σ are locally equivalent at* 0 *if and only if for each f in A the functions f^ϱ and f^σ have the same order at* 0.

Proof. The "only if" statement is trivial. Let us suppose that f^ϱ and f^σ have the same order for each $f \in A$. Let f and g be elements of A such that the order of f^ϱ/g^ϱ is positive and is the smallest positive order at 0 among all elements in the field of quotients of A^ϱ. Then f^σ/g^σ has the same order and it also is minimal for the field of quotients of A^σ. Replacing ϱ and σ by locally equivalent representations we may assume that $f^\varrho/g^\varrho = f^\sigma/g^\sigma = \zeta^n$ and that the representations are on the disk $U = \{\zeta: |\zeta| < 1\}$. Thus for each h in A the function h^ϱ can be expanded in U in a power series in ζ. Since n is the minimal positive order in the field of quotients of A^ϱ and since ζ^n is in this field of quotients, it follows that this power series contains only terms in powers of ζ^n, i.e. $h^\varrho = \sum_{\nu=0}^\infty a_\nu \zeta^{n\nu}$. Similarly, $h^\sigma = \sum_{\nu=0}^\infty b_\nu \zeta^{n\nu}$. Let $P(\lambda) = \sum_{\nu=0}^N a_\nu \lambda^\nu$. Then P is the unique polynomial of order N such that $h^\varrho - P(f^\varrho/g^\varrho)$ has order greater than N at 0. But this implies that $h^\sigma - P(f^\sigma/g^\sigma)$ has order greater than N at 0, whence $a_N = b_N$. Thus $h^\sigma = h^\varrho$, and the representations are locally equivalent.

LEMMA 6. *Let σ be a representation of A on a Riemann surface W and p a point of W. Then there is an analytic map ψ of a neighborhood U of p into the plane with $\psi(p)=0$ and a representation ϱ of A on $\psi[U]$ which is primitive at 0 and for which $f^\sigma = f^\varrho \circ \psi$ on U. The representation ϱ is unique to within local equivalence at 0.*

Proof. Let f and g be elements of A such that g^σ/f^σ has the smallest positive order at p. Then we can choose a uniformizing variable z at p which maps a neighborhood U of p onto $|z|<1$ and such that $g^\sigma/f^\sigma = z^n$. Now for each $h \in A$, the function h^σ can be expanded in U in a convergent power series in z. Since n is the minimal positive order at p in the field of quotients of A^σ and since z^n is in this field of quotients, it follows that only powers of z^n occur in this expansion, i.e. $h^\sigma = \sum a_\nu z^{n\nu}$.

In the circle $|\zeta|<1$ define h^ϱ by $h^\varrho(\zeta) = \sum a_\nu \zeta^\nu$. Then $h \to h^\varrho$ is a representation of A and $g^\varrho/f^\varrho = \zeta$. Thus ϱ is primitive at 0, and the mapping ψ on U defined by $\psi = z^n$ has the desired properties.

The orders at p of functions in the field of quotients of A^σ form an additive subgroup of the integers, and hence consist of all integral multiples of n, the smallest positive element. If ϱ and ψ are any representation and mapping satisfying the conditions of the lemma, we have the order of f^ϱ at 0 equal to $1/N$ times the order of f^σ at p where N is the order of ψ at p. Since ϱ is primitive at 0, we have $N=n$, and the order of f^ϱ at 0 is determined by p and σ. Thus any two representations ϱ satisfying the lemma are locally equivalent by Lemma 5.

By a *local representation* of A we mean an equivalence class of representations of A which are locally equivalent at 0. If σ is a representation of A on R and p is a point of R, let ψ be a one-to-one analytic map of a neighborhood U of p into the complex plane so that $\psi(p)=0$. Define a representation ϱ on $\psi[U]$ by $f^\varrho = f^\sigma \circ \psi^{-1}$. Then for different choices of ψ we obtain representations locally equivalent to ϱ, and so the local representation to which ϱ belongs is uniquely determined by p (and σ). We call this local representation the local representation at p.

We say that a local representation is primitive if one (and hence all) of the representations belonging to it is primitive at 0. Denote by Rep A the set of all primitive local representations of A. For $p \in \text{Rep } A$ let ϱ be a representation belonging to p and f and g such that f^ϱ/g^ϱ has order one at 0. Let U be a neighborhood of 0 on which A^ϱ is defined and on which f^ϱ/g^ϱ is univalent. Let V be the set consisting of the local representations at points of U. Since f^ϱ/g^ϱ is univalent in U, σ is primitive at each q in U, and so $V \subset \text{Rep } A$. The collection of all such V forms a base for a Hausdorff topology on Rep A. Since f^ϱ/g^ϱ is univalent on U, A^ϱ weakly separates points of U. Thus by Lemmas 4 and 5 the local representations at different points of U are different. Thus the points of U are in one-to-one correspondence with the points of V, and this correspondence is readily seen to be a homeomorphism. Since these homeomorphisms for overlapping V's are related by conformal mappings, we have defined a one-dimensional complex analytic structure for Rep A, and so each component of Rep A is a Riemann surface. If ϱ and σ are locally equivalent, then $f^{\varrho}(0)=f^{\sigma}(0)$, and so for each $p\in\operatorname{Rep} A$ we may define $\hat{f}(p)=f^{\varrho}(0)$ for some ϱ belonging to p. Thus for each $f\in A$ we define a function $\hat{f}$ on Rep A. On a neighborhood V, constructed as above, $\hat{f}$ is carried onto f^{ϱ} by the natural correspondence between V and U. Thus each $\hat{f}$ is an analytic function on Rep A, and the map $f\rightarrow\hat{f}$ is a representation of A as an algebra $\hat{A}$ on Rep A. These and other properties of Rep A are summarized by the following proposition:

PROPOSITION 1. *The space* Rep A *has a one-dimensional complex analytic structure, and so each component of* Rep A *is a Riemann surface. The algebra* $\hat{A}$ *is a representation of* A *on* Rep A *which weakly separates any pair of points.*

If σ *is any representation of* A *on a (not necessarily connected) Riemann surface* R, *there is a unique analytic map* τ *of* R *onto an open set* $O\subset\operatorname{Rep} A$ *such that* $f^{\sigma}=\hat{f}\circ\tau$. *The map* τ *is one-to-one if and only if* A^{σ} *is weakly separating on* R.

Proof. To see that $\hat{A}$ separates weakly on Rep A, we note that $\hat{A}$ is primitive at any pair $\langle p, q\rangle$ of points in Rep A. Since p and q are different local representations of A and are each the same as the local representation at themselves, it follows from Lemmas 4 and 5 that $\hat{A}$ weakly separates p from q.

If σ is a representation of A on a Riemann surface R, define a mapping τ of R into Rep A by letting $\tau(p)$ be the primitive local representation associated with the local representation at p as in Lemma 6. The continuity of τ follows from the definition of the topology in Rep A, and we have $f^{\sigma}=\hat{f}\circ\tau$. Hence τ is analytic. The uniqueness of τ follows from the fact that any τ with the property that $f^{\sigma}=\hat{f}\circ\tau$ must take each p in R into the unique primitive local representation associated with the local representation at p. The last statement follows from Lemma 3.

As a corollary we establish the following lemma which we will find useful later.

LEMMA 7. *Let* A *be an algebra on a (not necessarily connected) Riemann surface* R. *If* A *separates weakly on a dense open subset of* R, *then* A *separates weakly on* R.

Proof. By Proposition 1 there is a map τ of R into Rep A such that $f=\hat{f}\circ\tau$. Since A is weakly separating on a dense open subset U of W, τ is one-to-one on U. But an analytic map which is univalent on a dense open subset of W is univalent on W, and so τ is univalent on W. Thus A must separate weakly the points of W, since $\hat{A}$ separates weakly the points of Rep A and therefore of $\tau[W]$.

LEMMA 8. *Let W be a component of* Rep A. *Then $\hat{A}$ restricted to W is a proper algebra for the Riemann surface W.*

Proof. Let V be a Riemann surface containing W to which each $\hat{f}$ has an extension f^{σ}. Then σ is a representation on V, and so by Proposition 1 there is an analytic mapping τ of V into Rep A, such that $f^{\sigma}=\hat{f}\circ\tau$. Since τ restricted to W gives $\hat{f}=\hat{f}\circ\tau$, and since the identity map of W into Rep A is the unique analytic map with this property, we see that the restriction of τ to W is the identity. Hence $\tau[V]\supset W$. Since V is connected, $\tau[V]\subset W$, and so $\tau[V]=W$. If we assume A^{σ} weakly separates on V, then τ is one-to-one, and so $V=W$. Therefore, $\hat{A}$ is proper for W.

The following lemma expresses the functorial character of Rep A.

LEMMA 9. *Let η be a homomorphism of the algebra A_0 into the algebra A_1. Then there is an analytic mapping ψ of* Rep A_1 *into* Rep A_0 *such that for $f\in A_0$ we have $\widehat{\eta f}=\hat{f}\circ\psi$.*

Proof. If ϱ is a representation of A_1, then $\eta\circ\varrho$ is a representation of A_0, and hence to each local representation p of A_1 there corresponds a local representation $\psi(p)$ of A_0 such that $\widehat{\eta f}=\hat{f}\circ\psi$. It follows from the definition of the topologies in Rep A_0 and Rep A_1 that ψ is continuous and hence analytic.

PROPOSITION 2. *Let A be an algebra of analytic functions on a Riemann surface W. Then there is a Riemann surface W', a proper algebra A' on W', and an analytic map τ of W into W' such that each $f\in A$ is of the form $g\circ\tau$ with $g\in A'$. The pair (A', W') is unique up to a conformal equivalence.*

Proof. By Proposition 1 there is an analytic map τ of W into Rep A such that $f=\hat{f}\circ\tau$ for each $f\in A$. Since W is connected, $\tau[W]$ is contained in a component W' of Rep A. Let A' be the restriction of $\hat{A}$ to W'. Then A' is proper for W' by Lemma 8.

If (A'', W'') is another pair satisfying the requirements of the lemma, then A'' is a representation of A, and so there is a map ψ of W'' into Rep A such that $f''=\hat{f}\circ\psi$. Since A'' separates weakly on W'', ψ is one-to-one, and since A'' is proper for W'', ψ is onto a component of Rep A. Thus (A'', W'') is conformally equivalent to $\hat{A}$ on a component of Rep A. Since the component of Rep A into which W is mapped by τ is unique, (A'', W'') is conformally equivalent to (A', W').

The mapping τ: $(W, A)\to(W', A')$ given by Proposition 2 is called the *resolution* of (W, A). The following lemma is an immediate corollary of Lemma 9.

LEMMA 10. *Let A_0 and A_1 be two algebras of analytic functions on a Riemann surface W with $A_0 \subset A_1$, and let τ_0: $(W, A_0) \to (W_0, A_0')$ and τ_1: $(W, A_1) \to (W_1, A_1')$ be the corresponding resolutions. Then there is an analytic map ψ of W_1 into W_0 such that for $f \in A_0$ we have $f^{\tau_1} = f^{\tau_0} \circ \psi$.*

If K is a set on a Riemann surface, we say that a function f defined on K is analytic on K if f can be extended to an analytic function defined on some open set containing K. A collection of functions is said to be analytic on K if each function in the collection is analytic on K. Note that we do not suppose that there is an open set containing K on which all the functions of the collection are analytic. Whenever this latter property holds, we speak of a collection of functions uniformly analytic on K.

If (W, Γ) is a bordered Riemann surface with compact border Γ, we can also consider an algebra A of functions analytic on $W \cup \Gamma$. Proposition 2 does not apply directly, for although each f in A is defined and analytic on some Riemann surface containing $W \cup \Gamma$, there is no fixed Riemann surface containing $W \cup \Gamma$ on which all functions of A are defined and analytic. The following proposition shows, however, that we can find a finitely generated subalgebra of A which separates as well as A does. With the help of this proposition we can establish Proposition 4, which generalizes Proposition 1 to the case of a bordered Riemann surface.

PROPOSITION 3. *Let K be a finite union of analytic arcs on a Riemann surface W and A an algebra of meromorphic functions on K. Then there is a finitely generated subalgebra A_0 of A with the property that A_0 separates weakly each pair of points which are weakly separated by A. Moreover, we can choose A_0 so that at each $p \in K$ each f in A is expressible in some neighborhood of p as a convergent power series in a function in the field of quotients of A_0.*

Proof. Let p and q be two points of K (not necessarily different) which are not weakly separated by A. Let f be an element in the field of quotients of A which has smallest positive order at p. Since A does not separate p and q, f also has the smallest positive order possible at q. Choose uniformizing variables ζ_p and ζ_q so that in the neighborhoods

$$U_p = \{|\zeta_p| < \varepsilon^{1/n}\}, \quad U_q = \{|\zeta_q| < \varepsilon^{1/m}\}$$

the function f has the form ζ_p^n and ζ_q^m, respectively. Now ζ_p (or ζ_q) maps each arc of K emanating from p (or q) onto an analytic arc going from the origin to the circumference $|\zeta| = \varepsilon$. Since these arcs are analytic, any two of them either coincide or have only a finite number of points in common. Thus we may take ε so small that two such arcs which do not coincide have only $\zeta = 0$ in common. We may also take ε so small that f' does not vanish at any point of U_p or U_q except possibly at p or q. Let g be any function in A.

By the argument used in the proof of Lemmas 5 and 6, we see that in a sufficiently small neighborhood about p and q (depending on g) the function g can be expanded in a power series in f. If J_1 and J_2 are two analytic arcs of K in U_p and U_q, we may express g on each of them as analytic functions g_1 and g_2 of f, since $f' \neq 0$ on J_1 and J_2. Near p and q, i.e. near $\zeta = 0$, we have g expressed as a single analytic function in f. Thus if $f[J_1] = f[J_2]$, we have $g_1 \equiv g_2$ near 0, and hence $g_1 \equiv g_2$ everywhere on $f[J_1]$ by analytic continuation. Thus g cannot separate any point of $U_p \cap K$ from any point of $U_q \cap K$ which is not already separated by f, and the same conclusion also holds if g is in the field of quotients of A.

If p and q are separated by an element f in the field of quotients of A, we can choose neighborhoods U_p and U_q such that $f[U_p]$ and $f[U_q]$ are disjoint. Cover $K \times K$ by neighborhoods $U_p \times U_q$ such that either U_p is separated from U_q by a function f_{pq} or else U_p and U_q are as in the preceding paragraph and take f_{pq} to be the function f defined there. By the compactness of $K \times K$ we can select a finite number of these neighborhoods which cover $K \times K$. The algebra A_0 generated by the functions f_{pq} corresponding to this finite covering is the desired algebra.

It should be noted that the hypothesis that K is a finite union of analytic arcs cannot be weakened to the assumption that K is a differentiable Jordan arc. For if K is that arc in $|z| \leq 1$ defined by $y = x^3 \sin(1/x)$ for $x \neq 0$, $y = 0$ at $x = 0$, and if A is the algebra of all functions analytic on K each of which has in some neighborhood of $z = 0$ a power series expansion in even powers of z, then A separates on K, but no finitely generated subalgebra does.

PROPOSITION 4. *Let $W \cup \Gamma$ be a bordered Riemann surface with compact border Γ, and let A be an algebra of analytic functions on $W \cup \Gamma$. Then there is an analytic map τ of $W \cup \Gamma$ into a Riemann surface W' and an algebra A' of analytic functions on a connected compact set containing $\tau[\Gamma]$ such that a finitely generated subalgebra of A' is proper for W' and such that on Γ each $f \in A$ is of the form $g \circ \tau$ where $g \in A'$.*

Proof. We may suppose $W \cup \Gamma$ embedded in some larger Riemann surface (say the double of W) so that each f in A is analytic in some neighborhood of $W \cup \Gamma$, the neighborhood depending on f. If Γ is not connected, we may join the components of Γ by analytic arcs lying in W. Let K be the union of Γ and these arcs, and choose a finitely generated subalgebra A_0 of A as in Proposition 3. Since A_0 is finitely generated, there is a Riemann surface $W_0 \supset W \cup \Gamma$ such that each f in A_0 is analytic on W_0. Let τ: $(W_0, A_0) \to (W', A_0')$ be the resolution given by Proposition 2. Since A_0 separates the points of K as well as A does, each function in A is carried by τ into a function on $\tau[K]$, and by the extra property of A_0 in Proposition 2, we see that each such function can be extended to be analytic in some neighborhood of $\tau[K]$. This proves the proposition.

Remarks on algebras of meromorphic functions. The preceding material on representations of A as an algebra of analytic functions on a Riemann surface R can easily be extended to representations by algebras of meromorphic functions. Let us call such a representation a meromorphic representation. Then we can construct a one-dimensional complex analytic structure on the set $\mathrm{Rep}_M A$ consisting of all primitive local meromorphic representations of A, and the analogue of Proposition 1 holds. Moreover, $\mathrm{Rep}\, A \subset \mathrm{Rep}_M A$, and is just that open subset of $\mathrm{Rep}_M A$ consisting of those points which have a neighborhood in which all f are analytic. Note that several components of $\mathrm{Rep}\, A$ may be contained in a single component of $\mathrm{Rep}_M A$.

An algebra A of meromorphic functions on a Riemann surface W is said to be meromorphically proper for W if A separates weakly on W and A cannot be extended to a weakly separating algebra on some Riemann surface properly containing W. Note that an algebra of analytic functions on W can be analytically proper for W without being meromorphically proper. The following proposition is proved in the same manner as Proposition 2. We include it in this paper for reference elsewhere.

PROPOSITION 2'. *Let A be an algebra of meromorphic functions on a Riemann surface W. Then there is a Riemann surface W', a meromorphically proper algebra A' on W', and an analytic map τ of W into W' such that each $f \in A$ is of the form $g \circ \tau$ with $g \in A'$. The pair (A', W') is unique up to conformal equivalence.*

2. Some topological properties of Riemann surfaces

In this section we discuss some of the topological properties of compact subsets of a Riemann surface. The following lemma is standard and we omit the proof:

LEMMA 11. *Let K be a compact subset of a Riemann surface and U an open set containing K. Then there is an open set O with compact closure such that $K \subset O$, $\bar{O} \subset U$, and the boundary of O consists of a finite number of piecewise analytic curves.*

If K is a compact subset of a Riemann surface W, $W \sim K$ is an open set, and since W is locally connected, each component of an open set is open. By an abuse of language we call an open subset of W compact if its closure is compact. Thus by a compact component of $W \sim K$ we mean a component whose closure is compact. Denote by K^* the union of K and the compact components of $W \sim K$. Equivalently, K^* is the complement of the union of the noncompact components of $W \sim K$. Thus K^* is closed, and $K^* \sim K$ is open. Let

∂E denote the boundary of the set E, i.e. $\partial E = \bar{E} \cap \overline{(W \sim E)}$. Some properties of the operation * are given by the following lemma.

LEMMA 12. *Let K be a compact subset of the Riemann surface W, and let K^* be the union of K and the compact components of $W \sim K$. Then K^* is a compact set, and*

(i) $\quad K^* \supset K$

(ii) $\quad K^{**} = K^*$

(iii) $\quad K^* = [\partial K]^*$

(iv) $\quad \partial K^* \subset K$

(v) $\quad$ If $K_1 \subset K_2$, then $K_1^* \subset K_2^*$

(vi) $\quad$ If $\Gamma^* \supset \partial K$, then $\Gamma^* \supset K^*$.

Proof. Property (i) follows directly from the definition of K^*, (ii) from the fact that $W \sim K^*$ is just the union of the noncompact components of $W \sim K$, (iii) from the fact that the compact components of $W \sim \partial K$ are the compact components of $W \sim K$ and the components of the interior of K.

As we remarked earlier, K^* is closed and $K^* \sim K$ is open. Hence

$$\partial K^* = K^* \sim \text{interior } K^* \subset K.$$

Thus (iv) holds.

Let $K_1 \subset K_2$. Then $W \sim K_2 \subset W \sim K_1$, and each component of $W \sim K_2$ is contained in a component of $W \sim K_1$. Let O_2 be a noncompact component of $W \sim K_2$, and let O_1 be the component of $W \sim K_1$ containing it. Then $\bar{O}_2$ is a closed noncompact subset of $\bar{O}_1$, and hence $\bar{O}_1$ is not compact. Since K_1^* is the complement of the union of the noncompact components of $W \sim K_1$ and K_2^* the complement of the union of the noncompact components of $W \sim K_2$, we have $K_1^* \subset K_2^*$.

If $\Gamma^* \supset \partial K$, then $\Gamma^* = \Gamma^{**} \supset [\partial K]^* = K^*$.

To prove that K^* is compact, we may, by Lemma 11, choose an open set $O \supset K$ such that $\bar{O}$ is compact and whose boundary Γ is a finite number of piecewise analytic curves. Since each boundary curve of O divides W into at most two parts, $W \sim \Gamma$ can have at most a finite number of components. Thus Γ^* is compact, since it is a closed set which is the union of a finite number of sets each with compact closure. But now $\Gamma^* = \bar{O}^* \supset K^*$, and K^* is a closed subset of the compact set Γ^*. Thus K^* is compact, proving the lemma.

The complement of the set $\bar{O}$ in the preceding paragraph has only a finite number of components. Each noncompact component of $W \sim K$ must meet a noncompact component of $W \sim \bar{O}$, and since each component of $W \sim \bar{O}$ is contained in a component of $W \sim K$, we see that $W \sim K$ can have only a finite number of noncompact components. We have thus established the following corollary:

COROLLARY. *If K is a compact subset of a Riemann surface W, then $W \sim K$ has only a finite number of noncompact components.*

LEMMA 13. *Let W_1 and W_2 be two noncompact Riemann surfaces, and ψ an analytic map of W_1 into W_2 which is one-to-one on a connected open subset U of W_1. Let K be a compact subset of W_1 whose boundary is contained in U. Then ψ is one-to-one on K.*

Proof. By Lemma 11 there is an open subset O of U which contains K and whose boundary Γ consists of a finite number of piecewise analytic curves. Since $\Gamma^* \supset K$, and $\partial\Gamma^* \subset \Gamma \subset O$, we may replace K by Γ^*, and it suffices to prove the lemma for the case that K is bounded by a finite number of piecewise analytic curves.

If p and q are any two distinct points of K, we must show $\psi(p) \neq \psi(q)$. By enlarging K slightly, we may assume that neither p nor q is on the boundary of K, and by reducing U we may assume that neither p nor q is in U.

If the boundary Γ of K has more than one component, let Γ_1 be one component and set $\Gamma_1' = \Gamma \sim \Gamma_1$. Since U is connected we may connect Γ_1 to Γ_1' by a piecewise analytic arc lying in $U \sim \Gamma$. By "thickening" this arc we can find the image R of a rectangle such that one end of R lies on Γ_1, and one end on Γ_1' with the remainder of R lying in $U \sim \Gamma$. Since this remainder of R is a connected set not meeting ∂K, it lies entirely in the interior of K or entirely in the complement of K. In the first case replace K by $K \sim R$ and in the second by $K \cup R$. Then we have a new compact set with boundary in U, containing p and q in its interior, and having one less boundary component. Continuing in this manner, we see that it suffices to consider the case that K is bounded by a simple closed piecewise analytic curve Γ.

Since ψ is one-to-one on U, $\psi[\Gamma]$ is also a simple closed curve on W_2 and so divides W_2 into at most two regions. Let Ω be the interior of K. Then $\psi[\Omega]$ is an open subset of W_2, and since $\psi[\Omega] \cup \psi[\Gamma]$ is compact, we see that the complement of $\psi[\Gamma]$ must have a compact component and that $\psi[\Omega]$ must be this component.

Let u be a function which is harmonic on W_2 except for a logarithmic pole at $\psi(p)$. Then the integral of $^*du$ along $\psi[\Gamma]$ is 2π and this is the same as the integral of $^*dv$ along Γ, where $v = u \circ \psi$. But this latter integral is 2π times the number of poles of v, and v has a pole at each point of Ω which is mapped into $\psi(p)$. Thus there is only one point mapped into $\psi(p)$, and $\psi(p) \neq \psi(q)$.

3. Even mappings and finite-sheeted Riemann surfaces

Let R be a (not necessarily connected) Riemann surface. An analytic function f defined on an open set $O \subset R$ is said to map O *evenly* onto the plane domain D if there is an integer $n>0$ such that each value $\zeta \in D$ is assumed by f exactly n times in O counting multiplicities, i.e. $f-\zeta$ has exactly n zeros in O, counting multiplicity. By a finite-sheeted Riemann surface over the plane domain D we mean a pair (R, z) consisting of a (not necessarily connected) Riemann surface R and an analytic function z which maps R evenly onto D. If z has valence n, we speak of an n-sheeted surface. The mapping z is called the projection of R onto D, and the points p such that $z(p)=\zeta$ are referred to as the points lying over ζ.

The following lemmas express properties of even mappings, and finite-sheeted Riemann surfaces. The proofs are fairly straightforward, and are left to the reader.

LEMMA 14. *If f maps O evenly onto the domain D, then f is a proper mapping in the sense that for each compact set $K \subset D$ the set $f^{-1}[K] \cap O$ is compact. If p is any point in O and J any closed arc in D having one endpoint at $f(p)$, then there is an arc J' in O having one endpoint at p and projecting onto J. The arc J' is unique if there are no critical points of f over J.*

LEMMA 15. *Let f be an analytic function on a Riemann surface R, and let O_1 and O_2 be two open subsets each of which is mapped evenly onto D by f. Then $O_1 \sim O_2$ is open, and f maps evenly onto D each of the sets $O_1 \cup O_2$, $O_1 \cap O_2$, $O_1 \sim O_2$ that is nonempty.*

COROLLARY. *If O_1 and O_2 are connected open subsets of a Riemann surface W, and if f maps each of them evenly onto D, then either $O_1 \cap O_2 = \emptyset$ or $O_1 = O_2$.*

LEMMA 16. *If f maps O evenly onto D, then f maps each component of O evenly onto D.*

LEMMA 17. *Let $P_\zeta(\lambda) = a_n(\zeta)\lambda^n + \dots$ be a polynomial whose coefficients are analytic functions of ζ in the domain D and whose discriminant does not vanish identically in D. Then there is an n-sheeted (not necessarily connected) Riemann surface (R, z) lying over D which belongs to $P_\zeta(\lambda)$ in the sense that there is an analytic function f on R such that*

$$P_\zeta(\lambda) = a_n(\zeta) \prod_{z(p)=\zeta} (f(p)-\lambda).$$

The Riemann surface is branched only over those ζ at which the discriminant of P_ζ vanishes. The triple (R, z, f) is unique up to conformal equivalence.

4. Algebras and their characters

Let A be an algebra over the complex field. A finite linear combination $\varphi = \sum_{i=1}^n \alpha_i \pi_i$ of homomorphisms π_i of A into the complex field is called a *character* on A. If each $\alpha_i \neq 0$, and $\pi_i \neq \pi_j$ for $i \neq j$, then we call φ a character of order n. It is clear that the sum or difference of two characters is again a character. The following proposition gives us a criterion for a linear functional φ on A to be a character.

PROPOSITION 5. *Let φ be a linear functional on the algebra A. Then φ is a character of order n if and only if*

(1) *For any two sets $\langle x_0,\dots,x_n\rangle$, $\langle y_0,\dots,y_n\rangle$ of $n+1$ elements of A we have* $\det[\varphi(x_i y_j)]=0$.

(2) *There are elements $x_1,\dots,x_n$, $y_1,\dots,y_n$, f such that the polynomial*

$$P(\lambda)=\det[\varphi(x_i(f-\lambda)y_j)]$$

has n distinct roots.

Proof. Suppose $\varphi=\sum_{i=1}^n \alpha_i\pi_i$, and set $a_{jk}=\varphi(x_j y_k)$. Then $a_{jk}=\sum\alpha_i\pi_i(x_j)\pi_i(y_k)$, and so the $(n+1)\times(n+1)$ matrix $[a_{jk}]$ is the product of $(n+1)\times n$ and $n\times(n+1)$ matrices $[\alpha_i\pi_i(x_j)]^T$ and $[\pi_i(y_k)]$. Hence $[a_{jk}]$ is singular, and property (1) holds. If $\pi_j\neq\pi_i$ for $i\neq j$, we can find $x_1,\dots,x_n$ in A such that $\pi_i(x_j)=\delta_{ij}$. Let $y_i=x_i$ and $f=\sum ix_i$. Then $\varphi(x_i y_j)=\alpha_i\delta_{ij}$, and $\varphi(x_i(f-\lambda)y_j)=\alpha_i(i-\lambda)\delta_{ij}$. Thus property (2) holds for these elements if each $\alpha_i\neq 0$. Thus the two conditions are necessary.

Assume now that φ is a linear functional satisfying (1) and (2) and that x_i, y_i, and f are chosen satisfying (2). Since $P(\lambda)$ has n distinct roots, the coefficient $\det[\varphi(x_i y_j)]$ of λ^n must be different from zero. Thus the matrix $[\varphi(x_i y_j)]$ has an inverse $B=[\beta_{ij}]$. If we replace x_i by $\sum_j\beta_{ij}x_j$, then $\varphi(x_i y_j)=\delta_{ij}$, and $P(\lambda)$ is merely multiplied by $\det B$. Thus we may suppose that x_i and y_i are chosen so that $\varphi(x_i y_j)=\delta_{ij}$.

Let g and h be any two elements of A. By property (1) with $x_0=g$ and $y_0=h$, we have

$$\varphi(gh)=\sum_k\varphi(gy_k)\,\varphi(x_k h).$$

Replacing g by $x_i g$ and h by hy_j, we have

$$\varphi(x_i g\,hy_j)=\sum_k\varphi(x_i\,gy_k)\,\varphi(x_k\,hy_j).$$

Thus the mapping T which takes each g in A into the matrix $[\varphi(x_i gy_j)]$ is a homomorphism of A into the algebra of $n\times n$ matrices. By (2) the matrix $T(f)$ has n distinct eigenvalues, and so there is a nonsingular matrix Γ such that $\Gamma\; T(f)\Gamma^{-1}$ is diagonal. If we replace x_i by $\sum\gamma_{ij}x_j$ and y_i by $\sum\alpha_{ij}y_j$, where $\sum\gamma_{ij}\alpha_{jk}=\delta_{ik}$, we preserve the fact that $\varphi(x_i y_j)=\delta_{ij}$, do not change $P(\lambda)$, and replace the representation $g\to T(g)$ by $g\to\Gamma\; T(g)\Gamma^{-1}$. Thus we may assume that $T(f)$ is diagonal. Since $T(f)$ is diagonal with distinct eigenvalues, each matrix which commutes with $T(f)$ must also be diagonal. Since A is commutative, each $T(g)$ is diagonal. If we let $\pi_i(g)$ be the ith diagonal element of $T(g)$, then $\pi_i(g)$ is a homomorphism of A into the complex numbers.

Now
$$\varphi(g) = \varphi(g \cdot 1) = \sum_k \varphi(gy_k)\,\varphi(x_k)$$
and
$$\varphi(gy_k) = \varphi(1 \cdot gy_k) = \sum_i \varphi(y_i)\,\varphi(x_i gy_k) = \sum_i \varphi(y_i)\,\pi_i(g)\,\delta_{ik} = \varphi(y_k)\,\pi_k(g).$$
Hence
$$\varphi(g) = \sum_k \varphi(x_k)\,\varphi(y_k)\,\pi_k(g),$$
and we see that φ is a character of order at most n. Since $\varphi(x_i y_j) = \delta_{ij}$, property (1) cannot hold with n replaced by any smaller number. Thus φ is a character of order n, proving the proposition.

Since the elements on the diagonal of $T(g)$ are the roots of $P(\lambda)$ and the values of $\pi_i(g)$, we have also established the following lemma:

LEMMA 18. *Let $\varphi = \sum_{i=1}^n \alpha_i \pi_i$ be a character of order n, and $x_1, \ldots, x_n, y_1, \ldots, y_n$ and f satisfy property* (2) *of Proposition 5. Then the rootsP of $(\lambda) = \det[\varphi(x_i(f-\lambda)y_j)]$ are the numbers $\pi_k(f)$, $k = 1, \ldots, n$.*

LEMMA 19. *Let $\varphi = \sum_{i=1}^n \alpha_i \pi_i$ be a character of order n on the algebra A, and suppose that φ is a bounded linear functional in terms of some norm $\| \;\|$ for A. Then each π_i has norm* 1, *i.e.* $|\pi_i(f)| \leq \|f\|$.

Proof. Since the π_j's are distinct, there is an $x_i \in A$ such that $\pi_j(x_i) = \delta_{ij}$. Thus $\pi_i(f) = \alpha_i^{-1}\varphi(x_i f)$, and so
$$|\pi_i(f)| \leq |\alpha_i|^{-1}\|\varphi\|\,\|x_i f\| \leq |\alpha_i|^{-1}\|\varphi\|\,\|x_i\|\,\|f\| \leq c\|f\|.$$
Consequently, for each ν
$$|\pi_i(f)|^\nu \leq c\|f^\nu\| \leq c\|f\|^\nu.$$
Taking νth roots and letting ν tend to infinity, we obtain $|\pi_i(f)| \leq \|f\|$.

By an analytic functional φ on a domain D in the complex plane we mean a mapping $\zeta \to \varphi_\zeta$ of D into the linear functionals on A such that for each $f \in A$ we have $\varphi_\zeta(f)$ an analytic function of ζ.

LEMMA 20. *Let φ be an analytic functional on D, and suppose that for each ζ in some open subset $U \subset D$ the linear functional φ_ζ is a character of order n. Then φ_ζ is a character of order n at each point of D with the exception of a set E having no cluster point in D.*

Proof. For any $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ in A, we have $\det[\varphi_\zeta(x_i y_j)]$ an analytic function on D. Since φ_ζ is a character for each ζ in U, this determinant vanishes in U and hence identically in D. Thus property (1) of Proposition 5 holds everywhere in D.

For some point ζ_0 in U, let $x_1, \ldots, x_n, y_1, \ldots, y_n$ and f be chosen satisfying property (2) of Proposition 5 for φ_ζ. Then each coefficient of $P_\zeta(\lambda)=\det[\varphi_\zeta(x_i(f-\lambda)y_j)]$ is an analytic function of ζ in D. So also is the discriminant of P_ζ, and since the discriminant is not zero at ζ_0 it can vanish only at a set E_0 having no cluster point in D. Thus φ_ζ is a character of order n everywhere in D except for a subset E of E_0.

An analytic functional φ on D, which is a character at each ζ except those in an isolated set E will be called an *analytic character* on D, and the set E will be called the exceptional set of φ. Thus the preceding lemma states that an analytic functional on D is an analytic character of order n on D if it is an analytic character of order n in some open subset of D.

PROPOSITION 6. *Let φ be an analytic character of order n on the domain D in the complex plane. Then there is a (not necessarily connected) n-sheeted Riemann surface (R, z) lying over D, a meromorphic function α on R which is not identically zero on any component of R, and a homomorphism σ of A onto an algebra of analytic functions on R such that for each $g \in A$*

$$\varphi_\zeta(g)=\sum_{z(p)=\zeta}\alpha(p)\,g^\sigma(p). \tag{1}$$

The Riemannian surface (R, z) and the homomorphism σ are unique up to conformal equivalence.

Proof. For some $\zeta_0 \in D$ for which φ_{ζ_0} is a character of order n, we may choose $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, and f such that at ζ_0 the discriminant of $P_\zeta(\lambda)=\det[\varphi(x_i(f-\lambda)y_j)]$ is different from zero. Since this discriminant is analytic in D and nonzero at ζ_0, it vanishes only at a set E having no cluster points in D. Let R be the n-sheeted Riemann surface lying over D belonging to $P_\zeta(\lambda)$ in the sense of Lemma 17. Let R' be the subset of R lying over $D \sim E$.

Denote the function f of Lemma 17 by f^σ. Then by Lemma 18 the values of f^σ at the n points of R' over $\zeta \in D \sim E$ are just the numbers $\pi_k(f)$, $k=1, \ldots, n$, where $\varphi_\zeta=\sum \alpha_k\pi_k$. For each $p \in R'$, let π_p be that homomorphism in $\varphi_{z(p)}$ for which $\pi_p(f)=f^\sigma(p)$. Since f^σ takes distinct values at the points of R' lying over $z(p)$, the homomorphism π_p is uniquely determined.

For each $g \in A$, define a function g^σ on R' by $g^\sigma(p)=\pi_p(g)$. For the element f this definition agrees with our earlier definition of f^σ. The mapping $g \to g^\sigma$ is a homomorphism of A into the algebra of all complex-valued functions on R'. If we let $\alpha(p)$ be the weight of π_p in $\varphi_{z(p)}$, then α is a complex-valued function on R', and

$$\varphi_\zeta(g)=\sum_{z(p)=\zeta}\alpha(p)\,g^\sigma(p)$$

for $\zeta \in D \sim E$.

Thus the proposition will be established if we can show that each g^σ is analytic on

R' and admits an analytic continuation to R and that α is analytic on R' and has a meromorphic extension to R.

Since R' is an unbranched n-sheeted covering of $D \sim E$, that part of R' which lies over a disk U in $D \sim E$ consists of n components U_i each of which is mapped conformally onto U by z. Let z_i^{-1} be the inverse of z restricted to U_i, and set $f_i = f^\sigma \circ z_i^{-1}$, $\alpha_i = \alpha \circ z_i^{-1}$. Then

$$\sum \alpha_i(\zeta) f_i^\nu(\zeta) = \varphi_\zeta(f^\nu).$$

Taking $\nu = 0, 1, \ldots, n-1$, this is a system of n linear equations for $\alpha_i(\zeta)$. The determinant of the system is a van der Monde determinant and equal to $\prod_{i \neq j} (f_i - f_j)$. This is an analytic function which does not vanish in U, and since the coefficients and right hand side of our system are all analytic functions of ζ, Cramer's rule shows that each α_i is also an analytic function of ζ. Hence α is analytic on R'.

The same argument applied to the system of equations

$$\sum \alpha_i(\zeta) g_i(\zeta) f_i^\nu(\zeta) = \varphi_\zeta(g f^\nu),$$

where $g_i = g^\sigma \circ z_i^{-1}$, shows that αg^σ is analytic. Since φ_ζ is a character of order n at each point ζ in $D \sim E$, α cannot vanish on R', and so g^σ is analytic on R'.

Let ζ_0 be a point of E and take U to be a disk about ζ_0 which is so small that ζ_0 is the only point of E in $\bar{U}$. Let $U' = U \sim \{\zeta_0\}$, $V = z^{-1}[U]$, $V' = z^{-1}[U']$. Then for $\zeta \in U'$, we have

$$\sum_{z(p)=\zeta} \alpha(p)\,[f^\sigma(p)]^\nu = \varphi(f^\nu), \tag{2}$$

and for $\nu = 0, \ldots, n-1$ this gives us as before a system of n equations for the values of $\alpha(p)$ at the points p where $z(p) = \zeta$. The determinant of this system is $\prod_{p \neq q} (f^\sigma(p) - f^\sigma(q))$ which is (when multiplied by a suitable power of the leading coefficient) the discriminant $\Delta(\zeta)$ of $P_\zeta(\lambda)$. Thus $\Delta(\zeta)$ is analytic at ζ_0, and $\Delta \circ z$ is analytic at the points of R lying over ζ_0. Cramer's rule shows that in $V' = z^{-1}[U']$ we have $\Delta(z(p))\,\alpha(p)$ expressed as a polynomial in $f^\sigma(q)$ and $\varphi_{z(p)}(f^\nu)$. Since these functions are bounded in V', we have $\alpha \cdot \Delta \circ z$ bounded in V'. Since $V \sim V'$ contains only isolated points, a bounded analytic function on V' can have only a removable singularity there. Thus $\alpha \cdot \Delta \circ z$ has an analytic extension to V and α a meromorphic extension. Thus α has a meromorphic extension from R' to R.

To show that each g^σ has an analytic extension to R', take ζ_0 and U as in the preceding paragraph, and let C be the boundary of U. Define

$$\|g\| = \max_{z(p) \in C} |g^\sigma(p)|.$$

Then this is a norm on A, and for $\zeta \in C$ we have $|\varphi_\zeta(f)| \leqslant c\|f\|$ where

$$c = n \max_{z(p) \in C} \alpha(p).$$

Since $\varphi_\zeta(f)$ is analytic in U, this inequality holds in U. Thus by Lemma 19, $|g^\sigma(p)| \leqslant \|g\|$ for $p \in V'$. Consequently, g^σ is bounded on V', and so admits an analytic extension to V. This shows that we can extend the homomorphism $g \to g^\sigma$ to be a homomorphism of A onto an algebra of analytic functions on R.

The uniqueness of (R, z, f^σ) follows from Lemma 17, and since α and g^σ are defined by equations (2) and (1), the homomorphism σ is also unique.

In general the image A^σ of the homomorphism σ given in the preceding proposition need not contain the projection z of R onto D, i.e. there may not be an element g in A such that $g^\sigma = z$. The following lemma gives a criterion for this to be the case.

LEMMA 21. *Let φ, D, and R be as in Proposition* 6, *and g an element of A. Then $g^\sigma = z$ if and only if $\varphi(gf) = \zeta\varphi(f)$ for every f in A.*

Proof. The "only if" part is trivial. To prove the "if" part, let f and R' be as in the proof of Proposition 6, and p a point of R'. Since f^σ takes different values at the points q lying over $z(p)$, there is an element f_1 in A such that $f_1^\sigma(p) = 1$ and $f_1^\sigma(q) = 0$ for $z(q) = z(p)$, $q \neq p$: Then

$$\alpha(p)\,g^\sigma(p) = \varphi_{z(p)}(gf) = z(p)\,\varphi(f_1) = z(p)\,\alpha(p).$$

Thus $g^\sigma = z$ on R', and hence also on R by continuity.

The homomorphism σ of Proposition 6 need not be a representation, since A^σ may contain only constants on some component of R. The following lemma shows that if $z \in A^\sigma$, then A^σ is a representation and separates weakly on R.

LEMMA 22. *The homomorphism σ given by Proposition* 6 *is a representation of A on R if there is a g in A with $g^\sigma = z$. In this case the algebra A^σ is weakly separating on R.*

Proof. If $g^\sigma = z$, then g^σ is constant on no component of R, and σ is a representation. Let f and R' be as in the proof of Proposition 6. Then if p and q are two points of R', g^σ separates them if $z(p) \neq z(q)$ and f^σ separates them if $z(p) = z(q)$. Hence A^σ separates on R' and hence separates weakly on R by Lemma 7.

PROPOSITION 7. *Let φ be an analytic character on D and f an element in A such that $\varphi(fg) = \zeta\varphi(g)$ for all g in A. Then there is an open set $O \subset \operatorname{Rep} A$ which is mapped evenly onto D by $\hat{f}$ and a meromorphic function α defined in O such that*

$$\varphi_\zeta(g) = \sum_{\substack{\hat{f}(p)=\zeta \\ p \in O}} \alpha(p)\,\hat{g}(p). \tag{3}$$

If we require that α does not vanish identically in any component of O, then O is uniquely determined by φ.

If U is an open set contained in D, then the open set corresponding to φ restricted to U is $f^{-1}[U] \cap O$.

If φ_1 and φ_2 are analytic characters on D and $\varphi_i(fg) = \zeta\varphi_i(g)$ for all g in A, then $\varphi = \varphi_1 + \varphi_2$ is an analytic character in D with this property and for the open sets O, O_1, O_2 corresponding to φ, φ_1, φ_2 we have $O \subset O_1 \cup O_2$.

Proof. Let R, z, and σ be as in Proposition 6. Then by Lemma 21 $f^\sigma = z$, and σ is a representation by Lemma 22. By Proposition 1 there is an analytic mapping τ of R onto an open set $O \subset \operatorname{Rep} A$ such that $g^\sigma = \hat{g} \circ \tau$ for each $g \in A$. Since A^σ is weakly separating on R, τ is one-to-one, and carries the α of Proposition 6 onto an α on O such that (3) holds. The uniqueness of O follows from the uniqueness parts of Propositions 1 and 6. The statement about the restriction of φ to U follows directly from the uniqueness of O.

To prove the last statement of the proposition, we note that φ_ζ is a character at each ζ where φ_1 and φ_2 are characters. Since this happens everywhere in D except for an isolated set, φ is an analytic character in D. Clearly $\varphi(fg) = \zeta\varphi(g)$. Let O_1 and O_2 be the open sets corresponding to φ_1 and φ_2 and α_1 and α_2 the corresponding meromorphic functions. Since f is even on O_1 and on O_2, the sets $O_1 \cap O_2$, $O_1 \sim O_2$, $O_2 \sim O_1$ are disjoint open sets by Lemma 15. Define α on $O_1 \cup O_2$ by setting α equal to $\alpha_1 + \alpha_2$, α_1, α_2, respectively, on these three sets. Now f is even on $O_1 \cup O_2$ by Lemma 15, and hence by Lemmas 15 and 16 f is also even on any open set O which is a union of components of $O_1 \cup O_2$. Let O be the union of those components of $O_1 \cup O_2$ on which α is not identically zero. Adding the corresponding formulae for φ_1 and φ_2, we see that formula (3) holds for this O. Thus this is the O corresponding to φ and $O \subset O_1 \cup O_2$.

We shall also find the following lemma useful.

LEMMA 23. *Let φ_ζ be an analytic character in the domain D, and let R be the finite-sheeted surface over D belonging to φ_ζ in the sense of Proposition 6. Suppose that for some ζ_0 in D the character φ_{ζ_0} is a nonzero homomorphism π. Then there is a point p in R lying over ζ_0 such that $\pi(g) = g(p)$ for all $g \in A$.*

Proof. Suppose not. Then for each p_i over ζ_0 we can find a g_i in A such that $g_i(p_i) = 0$, $\pi(g) = 1$. By taking a suitable power of the product of the g_i we obtain a g in A with $\pi(g) = 1$, and g vanishing at each p_i to an order so great that αg vanishes at p_i. Thus

$$\varphi_{\zeta_0}(g) = \sum_i \alpha(p_i)\, g(p_i) = 0 \neq \pi(g).$$

The lemma follows by contraposition.

PROPOSITION 8. *Let W be a Riemann surface, A a proper algebra of analytic functions on W, and f a nonconstant function in A. Let $p_1, \ldots, p_n$ be points of W at which f is real, and let V_i be a neighborhood of p_i which is mapped one-to-one by f onto an open set U_i. Let the plane domains $D_0, D_1, \ldots, D_n$ be such that the intersection of each D_i with the real axis is an interval and such that $U_i \subset D_i \cap D_{i-1}$. Let φ_i be an analytic linear functional of A on D_i with $\varphi_0 \equiv 0$ and $(\varphi_i - \varphi_{i-1})(g) = \beta_i(\zeta) g(e_i(\zeta))$ on U_i, where e_i is the inverse of f restricted to V_1, and β_i is an analytic function on V_i.*

Then for each i, $1 \leq i \leq n$, there is an open set O_i on W and a meromorphic function α_i on O_i such that O_i is a finite-sheeted Riemann surface over D_i with projection f, and

$$\varphi_i(g) = \sum_{\substack{f(p)=\zeta \\ p \in O_i}} \alpha_i(p) g(p). \tag{4}$$

Moreover, each $p \in O_i$ with $f(p)$ real can be joined to one of the p_j by an arc on which f is real.

Proof. Replacing V_1 by a smaller neighborhood, we may suppose that $U_1 = f[V_1]$ intersects the real axis in an interval and our hypotheses are unchanged if we take $D_0 = f[V_1]$. Then the conclusion of the proposition is true for $i=0$ if we take $O_0 = V_1$ and $\alpha_0 \equiv 0$. We prove the proposition by induction on i, and hence assume the conclusion for $i = \nu - 1$, and will show that this implies the conclusion for $i = \nu$.

Thus $\varphi_{\nu-1}$ is an analytic character in $D_{\nu-1}$, and hence in $f[V_\nu]$. Consequently, φ_ν must also be an analytic character in $f[V_\nu]$, since it is the sum there of the analytic character $\varphi_{\nu-1}$ and the analytic character ψ defined by $\psi_\zeta(g) = \beta_i(\zeta) g(e_i(\zeta))$. By Lemma 20, φ_ν is then an analytic character on all of D_ν. By our induction hypothesis, f maps an open set $O_{\nu-1}$ in W evenly onto D so that (4) holds. Hence by Lemma 21 $\varphi_{\nu-1}(fg) = \zeta\varphi_{\nu-1}(g)$ in $D_{\nu-1}$ and hence U_ν. Since

$$\psi(fg) = \beta_i(\zeta) f(e_i(\zeta)) g(\zeta)) = \beta_i(\zeta) \zeta g(e_i(\zeta)) = \zeta\psi(g),$$

we have $\varphi_\nu(fg) = \zeta\varphi_\nu(g)$ in U_ν and hence in D_ν. Thus by Proposition 7 there is an open set $O_\nu \subset \operatorname{Rep} A$ and a meromorphic function α_ν on O_ν such that (4) holds.

Since A is proper for W, W is isomorphic with a component of $\operatorname{Rep} A$ which we identify with W. By Proposition 7 the open subsets of $\operatorname{Rep} A$ corresponding to $\varphi_{\nu-1}$ and φ_ν restricted to U_ν are $O_{\nu-1} \cap f^{-1}[U_\nu]$ and $O_\nu \cap f^{-1}[U_\nu]$. Since the open set corresponding to ψ is V_ν, the last statement of Proposition 7 asserts that $O_\nu \cap f^{-1}[U_\nu] \subset O_{\nu-1} \cup V_\nu \subset W$. Since f maps each component of O_ν onto D, each component of O_ν contains points in $f^{-1}[U_\nu]$ and hence of W. Thus each component of O_ν is contained in W, and so $O_\nu \subset W$.

If $f \in O_\nu$ and $f(p)$ is real, then, since D_ν intersects the real axis in an interval and f is even on O_ν, p can be joined to a point q with $f(q) = f(p_\nu)$ by a curve in O_ν lying over the interval from $f(p)$ to $f(p_\nu)$. Now $q \in O_{\nu-1} \cup V_\nu$, and the only point of V_ν lying over $f(p_\nu)$

is p_ν. Thus either $q=p_\nu$ or $q\in O_{\nu-1}$. In the latter case we can join q to some p_i by an arc along which f is real by our induction hypothesis. This completes the proof of the proposition.

5. The bounded homomorphisms of A

Let A be a proper algebra of analytic functions on the Riemann surface W, and K a compact set in W. We define the K-norm $\|g\|_K$ of an element g in A by

$$\|g\|_K=\sup_K |g|.$$

This is clearly a multiplicative norm for A. We say that a linear functional φ on A is K-bounded if it is bounded in the K-norm, i.e. if for some c, $|\varphi(g)|\leqslant c\|g\|_K$ for all $g\in A$.

For each p in W, the map $g\to g(p)$ is a homomorphism of A into the complex numbers, and we call this homomorphism the evaluation at p. If the evaluation at p is K-bounded we say that A is K-bounded at p, and say that A is K-bounded on a set E if A is K-bounded at each point of E. The set H consisting of all p on W for which A is K-bounded is called the hull of K. The purpose of this section is to characterize the hull of K and the K-bounded homomorphisms of A.

PROPOSITION 9. *Let A be a proper algebra on W, K a compact subset of W, and f a nonconstant function in A which has no zero on K. Let $0<\delta<\inf|f[K]|$. Then there is an open subset O of W with $\bar{O}$ compact such that A is K-bounded on O and such that each K-bounded homomorphism π of A into the complex numbers with $|\pi(f)|<\delta$ is evaluation at some point of O, i.e. there is a $p\in O$ with $\pi(g)=g(p)$ for all $g\in A$.*

Proof. Let M be the open set where $|f|>\delta$. Then we may choose an open set M_1 whose boundary Γ consists of piecewise analytic arcs so that $K\subset M_1$, $\bar{M}_1\subset M$. Since f has only a countable number of critical points, and since f can be multiplied by any number of modulus one, we may assume that f is not real at any of its critical points (except possibly where $f=0$) and that there are only a finite number of points $p_1, \ldots, p_n$ on Γ at which f is negative. We may further assume that Γ is analytic and nonsingular at each p_i. By varying Γ slightly we can insure that the values $f(p_i)$ are distinct. Let $f(p_i)=\xi_i$, and suppose that the numbering is such that $\xi_1<\ldots<\xi_n<0$. Then each interval (ξ_i, ξ_{i+1}) is contained in a component Δ_i of the complement in plane of $f[\Gamma]$. Let Δ_0 be the component of the complement of $f[\Gamma]$ which is unbounded, and Δ_n the component containing 0. Then ξ_i is a common boundary point of Δ_{i-1} and Δ_i for $i=1, \ldots, n$.

Let π be any homomorphism satisfying the hypothesis of the proposition. Then by the Hahn-Banach theorem we can extend π to a bounded linear functional on the space of all continuous functions on K. Such a linear functional is represented by a finite complex-valued measure μ on K. Thus $\pi(g)=\int g\,d\mu$ for all $g\in A$. Let $G(p,q)$ be the Green's function of M (or of an open subset of M which contains M_1), and define u by $u(p)=\int_K G(p,q)\,d\mu(q)$. Then u is a positive harmonic function in $M\sim K$, and if we let θ be the differential $(\pi i)^{-1}\partial u$, i.e. that differential which in terms of a uniformizer z has the form

$$\frac{1}{\pi i}\frac{\partial u}{\partial z}\,dz,$$

then θ is an analytic differential in $M\sim K$, and $\int_\Gamma g\theta=\int_K g\,d\mu$ for each function g analytic in M.

Let $\pi(f)=\zeta_0$, and for each complex number ζ not in $f[\Gamma]$ define

$$\varphi_\zeta(g)=\int_\Gamma \frac{f-\zeta_0}{f-\zeta}\,g\,\theta.$$

Then φ_ζ is analytic in each component of the complement of $f[\Gamma]$, and $\varphi_{\zeta_0}(g)=\int_\Gamma g\theta=\pi(g)$ for $g\in A$.

If $|\zeta|<\delta$, the function $(f-\zeta_0)(f-\zeta)^{-1}g$ is analytic in M, and so for such ζ we have

$$\varphi_\zeta(g)=\int\frac{f-\zeta_0}{f-\zeta}\,g\,d\mu,$$

whence
$$|\varphi_\zeta(g)|\leq\|g\|_K\cdot\max_K\frac{f-\zeta_0}{f-\zeta}\cdot\mu(K).$$

Thus φ_ζ is K-bounded for each ζ in $|\zeta|<\delta$.

If $|\zeta|>\|f\|_\Gamma$, then $(f-\zeta)^{-1}$ can be uniformly approximated on K by polynomials P_n in f. Hence for such ζ and for g in A we have

$$\varphi_\zeta(g)=\lim\int(f-\zeta_0)P_n(f)\,g\theta=\lim\pi[(f-\zeta_0)P_n(f)g]=0$$

since $\pi(f-\zeta_0)=0$ and $P_n(f)$ belongs to A. Thus $\varphi_\zeta(g)$ vanishes for all sufficiently large ζ, and hence everywhere in Δ_0, the unbounded component of the complement of $f[\Gamma]$. Therefore, $\varphi_\zeta\equiv 0$ in Δ_0.

Let Δ_i, $i=1,\ldots,n-1$, be that component of the complement of $f[\Gamma]$ which contains (ξ_i,ξ_{i+1}), and let Δ_n be the component containing 0. Denote by φ_i the restriction of φ to Δ_i. Then $\varphi_0\equiv 0$. We shall show that each φ_i can be extended to be analytic in a neighborhood of ξ_i. Let ε be a positive number such that the intersection of the disk U about ξ_i of radius ε with $f(\Gamma)$ is a simple analytic arc, and take ε so small that there is a neighborhood V of p_i which is mapped one-to-one by f onto an open set containing $\bar{U}$. Let

$U' = U \cap \Delta_i$, and set $V' = V \cap f^{-1}[U_0]$. Let Γ_0 be the boundary of V', and set $\Gamma' = \Gamma - \Gamma_0$, in the usual combinatorial sense. Then $U \cup \Delta_i$ is contained in the complement of $f[\Gamma']$, and the functional ψ on A defined by

$$\psi_\zeta = \int_{\Gamma'} \frac{f - \zeta_0}{f - \zeta} g\theta$$

is an analytic functional in $U \cup \Delta_i$. Now

$$\varphi_\zeta - \psi_\zeta = \int_{\Gamma} \frac{f-\zeta_0}{f-\zeta} g\theta - \int_{\Gamma'} \frac{f-\zeta_0}{f-\zeta} g\theta = \int_{\Gamma_0} \frac{f-\zeta_0}{f-\zeta} g\theta.$$

If $\zeta \notin \bar{U}'$, this last integral is zero, since the integrand is analytic on $\bar{V}'$. Thus $\psi_\zeta = \varphi_i$ for $\zeta \in \Delta_i$. Thus φ_i can be extended to be analytic in a neighborhood U of ξ_i. We can similarly extend φ_{i-1} to be analytic in U.

Let z be a uniformizing variable in V such that $f(z) = z$, and let $\theta = \gamma(z)\,dz$. Then for $\zeta \in U_0$, we have

$$\varphi_\zeta - \psi_\zeta = \int_{\Gamma_0} \frac{f-\zeta_0}{f-\zeta} g\theta,$$

and the integrand has a pole at the point p where $z = \zeta$ with residue $(\zeta - \zeta_0) g(p) \gamma(\zeta)$. Hence the integral is equal to $\beta(\zeta) g(p)$, where we have set $\beta(\zeta) = 2\pi i(\zeta - \zeta_0)\gamma(\zeta)$. If e_i is the inverse of f restricted to V, $p = e_i(\zeta)$, and we have

$$(\varphi_i - \varphi_{i-1})(g) = \beta(\zeta) g(e_i(\zeta)) \tag{5}$$

in U' and hence also in U.

Repeating this process for each i from 1 to n we obtain neighborhoods U_i of ξ_i which are one-to-one images by f of neighborhoods V_i of p_i such that each φ_i is analytic in the region Δ_i', where $\Delta_i' = \Delta_i \cup U_i \cup U_{i+1}$, $i = 1, \dots, n-1$, $\Delta_n' = \Delta n \cup U_n$, and $\Delta_0 = U_1$. Moreover, the difference $\varphi_i - \varphi_{i-1}$ is given by (5) in U_i. For each i, $i = 0, \dots, n$, let D_i be a subdomain of Δ_i' whose intersection with the real axis is an interval and such that D_i contains $[\xi_i, \xi_{i+1}]$ for $1 \leqslant i \leqslant n-1$, D_0 contains ξ_1, and D_n contains ξ_n and the closed disk $|\zeta| \leqslant \delta$. Replacing U_i by $U_i \cap D_i \cap D_{i-1}$, and V_i by $V_i \cap f^{-1}[U_i]$, we have the hypotheses of Proposition 10 fulfilled.

Let O_n be the open subset of W given by this proposition. Then f maps O_n evenly onto D_n. Let Ω be the subset of O_n where $|f| < \delta$. Since f is even on O_n, the subset of O_n where $|f| \leqslant \delta$ is compact by Lemma 14. Thus $\bar{\Omega}$ is compact. Now f maps Ω evenly onto the disk $|\zeta| < \delta$, and (Ω, f) is the finite-sheeted Riemann surface over $|\zeta| < \delta$ which belongs to φ in the sense of Proposition 6. Since φ is K-bounded at each ζ in $|\zeta| < \delta$, it follows from Lemma 19 that each g in A is K-bounded in Ω. Since $\varphi_{\zeta_0} = \pi$, it follows from Lemma

23 that there is a point $q \in \Omega$ with $g(q) = \pi(g)$ for all $g \in A$. Let O_π be the component of Ω which contains this point q.

Then O_π satisfies all the requirements of our proposition, except that it depends on π. Let $\{O_\pi\}$ be the collection of all such O_π. By Lemma 16 f maps each O_π evenly onto $|\zeta| < \delta$, and so by the corollary to Lemma 15, any two sets in $\{O_\pi\}$ either coincide or are disjoint. By Proposition 8, each point p of O_π at which $f=0$ can be connected to one of the p_i by an arc in W along which f is real. Since f has no real critical values (other than zero), there is at most one such p for each p_i and so at most n such points altogether. Thus the collection $\{O_\pi\}$ is a finite collection, and if we set $O = \bigcup O_\pi$, then O satisfies the requirements of the proposition.

LEMMA 24. *Let K be the union of a finite number of analytic arcs on W, and let π be a homomorphism of A into the complex numbers which is not evaluation at some point of K. Then there is an f in A such that f is never zero on K, and $\pi f = 0$.*

Proof. Let g be a nonconstant function in A such that $\pi g = 0$. Then there are only a finite number of points of K at which g vanishes. Since π is not evaluation at any of these points, there is an h in A vanishing at these points for which $\pi h = 1$. Now g maps K onto a collection of analytic arcs in the plane, and there are only a finite number of tangent directions at the origin. Let ε have an argument which is not one of these tangent directions. Then for sufficiently small ε the function $g + \varepsilon h$ takes the value ε at π and is different from ε on K. Set $f = g + \varepsilon h - \varepsilon$. Then f is the required function.

THEOREM 1. *Let A be a proper algebra for the Riemann surface W, and let K be a compact subset of W. Let K^* be the union of K and those components of $W \sim K$ whose closures are compact, and let*

$$\Delta = \{p \colon p \notin K^*, \exists\, q \in K^*, f(p) = f(q) \quad all \quad f \in A\}.$$

Then K^* is compact, Δ is an isolated set, and the following hold:

(i) *The hull of K is $K^* \cup \Delta$, i.e. $K^* \cup \Delta = \{p \in W \colon |f(p)| \leqslant \sup_K |f|\}$.*

(ii) *If π is a homomorphism of A into the complex numbers with $|\pi f| \leqslant \sup_K |f|$, then there is a $p \in K^*$ with $\pi f = f(p)$.*

(iii) *If ϱ is a homomorphism of A into the algebra of analytic functions on a disk D such that $\sup_D |\varrho f| \leqslant \sup_K |f|$, then there is a unique analytic map ψ of D into the interior of K^* such that $\varrho f = f \circ \psi$.*

Proof. Denote by H the hull of K, that is $H = \{p \colon |f(p)| \leqslant \|f\|_K$ for all $f \in A\}$, and let us assume for the moment that K is the union of a finite collection of analytic arcs. If π

is any K-bounded homomorphism of A into the complex numbers which is not evaluation at some point of K, then by Lemma 24 there is an $f \in A$, $\pi f = 0$, f never zero on K. Hence by Proposition 9, there is an open set $O \subset W$ on which A is K-bounded and such that π is evaluation at some point of O. Thus each K-bounded homomorphism of A which is not evaluation at a point of K is evaluation at some interior point of $H \sim K$.

Let Σ be the set consisting of those points of W which are separated by A from every other point of W, i.e. let

$$\Sigma = \{p \in W: \text{for all } q \in W,\ q \neq p,\ \exists f \in A,\ f(p) \neq f(q)\}.$$

Then $W \sim \Sigma$ is countable by Lemma 2, and so the components of $\Sigma \sim K$ are just the intersections of Σ with the components of $W \sim K$. Let S be the subset of Σ consisting of those points at which A is K-bounded, i.e. $S = \Sigma \cap H$. Then S is clearly closed in Σ. Since evaluation at a point p of S is a K-bounded homomorphism which is not evaluation at any other point of W, the point p must be an interior point of $H \sim K$, and so $S \sim K$ is open relative to Σ. Hence each component of $S \sim K$ is a component of $\Sigma \sim K$.

We now show that the closure of S in W is compact. Since W is metrizable, it suffices to show that each sequence $\langle p_n \rangle$ from S has a cluster point in W. If $\langle p_n \rangle$ has no cluster point on K, then we can find a neighborhood U of K with $\bar{U}$ compact such that only a finite number of the p_n are in U, and we may choose U so that its boundary Γ consists of a finite number of analytic arcs. Let f be a nonconstant function in A. Since $|f(p_n)| \leq \|f\|_K$, we can find a subsequence (which we again call $\langle p_n \rangle$) such that $f(p_n)$ converges. Subtracting a constant from f, we may suppose $f(p_n) \to 0$, and by varying Γ slightly we may insure that f is never 0 on Γ. Since $\|g\|_K \leq \|g\|_\Gamma$, the evaluations at p_n are all Γ-bounded, and from some n_0 on we have $|f(p_n)| < \delta < \inf_\Gamma |f|$. Thus by Proposition 9, there is an open set $O \subset W$ with compact closure such that $p_n \in O$, for $n \geq n_0$. Thus $\langle p_n \rangle$ has a cluster point in $\bar{O}$ and hence in W. This shows that $\bar{S}$ is compact.

Thus each component S_i of $S \sim K$ has compact closure. Now each component S_i of $S \sim K$ is a component of $\Sigma \sim K$ and thus the intersection of Σ with a component O_i of $W \sim K$. Since O_i is open and Σ is dense in W, $\bar{S}_i = \bar{O}_i$, and O_i has compact closure. Consequently, S is contained in the union of K and the components of $W \sim K$ with compact closure, i.e. in K^*.

Since S is dense in the interior H^0 of the hull H of K, we have $H^0 \subset K^*$. Since each K-bounded homomorphism π not an evaluation on K is evaluation at some point of H^0, we see that each K-bounded homomorphism π is evaluation at some point of K^*. This proves statement (ii) of the theorem for the case that K is the union of a finite collection of analytic arcs.

To establish this statement for an arbitrary compact set K, we note that, if Γ is a finite union of analytic arcs and bounds a relatively compact open set containing K, then each K-bounded homomorphism π must also be Γ-bounded. Thus π is evaluation at some point of Γ^*, the union of Γ and the compact components of $W \sim \Gamma$. Since $K^* = \bigcap \Gamma^*$ for all such choices of Γ, we have $\pi \in K^*$, and statement (ii) holds for an arbitrary compact set K.

Since K^* is compact and has boundary contained in K, the algebra A is K-bounded on K^* and $K^* \subset H$. On the other hand, evaluation at a point of H is a K-bounded homomorphism and so must be equal to evaluation at some point of K^*. Thus H contains in addition to K^* only the set

$$\Delta = \{p \notin K^* : (\exists q \in K^*)(f(p) = f(q) \quad \text{all} \quad f \in A)\}.$$

Since Δ is clearly contained in H, we have $H = \Delta \cup K^*$. To see that Δ is isolated, we note that if $p \in \Delta$ were the limit of a sequence $\langle p_n \rangle$ in Δ, the points $q_n \in K^*$ would have a cluster point $q \in K^*$. But by Lemma 1, p and q have neighborhoods U and V such that A separates each point of U from each point of V except q.

The preceding argument also shows that the set of $p \in K^*$ for which there is a $q \in K^*$ with $f(p) = f(q)$ for all f in A is a finite set. Thus, with a finite number of exceptions, a K-homomorphism π is evaluation at a unique point in K^*. If ϱ is the representation in statement (iii), then evaluation at each $\zeta \in D$ is a K-bounded homomorphism. Hence, except for those ζ corresponding to the exceptional homomorphisms, there is a unique point $\psi(\zeta)$ in K^* where evaluation is the same as at ζ. Thus ψ is defined on a dense subset of D and $\varrho f = f \circ \psi$ for each f in A. Let $\langle \zeta_n \rangle$ be a sequence in the domain of ψ which converges to ζ_0 in D. Then $\langle \psi(\zeta_n) \rangle$ has a cluster point in K^*, since K^* is compact. If it had two cluster points p and q, there would be f and g in A with f/g taking different values at p and q. But this is impossible since

$$(f/g)(\psi(\zeta_n)) = \varrho f(\zeta_n)/\varrho g(\zeta_n) \to \varrho f(\zeta)/\varrho g(\zeta).$$

Thus it is possible to extend ψ to all of D such that $\varrho f = f \circ \psi$. The above argument shows that ψ is continuous, and since it carries analytic functions into analytic functions, it must be analytic.

THEOREM 2. *Let A_0 be a proper algebra of analytic functions on the Riemann surface W, let K be a compact connected subset of W, and let A be an algebra of analytic functions on K with $A \supset A_0$. Let K^+ be the union of K and those relatively compact components of $W \sim K$ to which each function in A has an analytic extension. Then K^+ is a compact set for which the following hold:*

(i) *If π is a homomorphism of A into the complex numbers with $|\pi f| \leqslant \sup_K |f|$, then there is a $p \in K^+$ with $\pi f = f(p)$.*

(ii) *If ϱ is a homomorphism of A into the algebra of analytic functions on a disk D so that $\sup_D |\varrho f| \leqslant \sup_K |f|$, then there is a unique analytic map ψ of D into the interior of K^+ such that $\varrho f = f \circ \psi$.*

Proof. Let A_1 be any subalgebra of A which is finitely generated over A_0. Since A_1 is generated over A_0 by a finite number of functions in A, there is a neighborhood U of K on which each function of A_1 is analytic. Since K is connected, we may take U to be connected. Let $\tau: (U, A_1) \rightarrow (W_1, A_1')$ be the resolution of (U, A_1) given by Proposition 2. Since A_0 is weakly separating on U, so is A_1, and τ is one-to-one on U. Now (A_0, W) is its own resolution since A_0 is proper for W, and so there is by Lemma 10 an analytic mapping ψ of W_1 into W such that for $f \in A_0$, $f \circ \psi = f'$. On $\tau[U]$ we also have $f \circ \psi = f'$ for $f \in A_1$. Since A_1 separates weakly on U, the function f is one-to-one on $\tau[U]$.

Let $K_1 = \tau[K]$, and let K_1^* be the union of K_1 and the compact components of $W_1 \sim K_1$. Since $\partial K_1^* \subset K_1 \subset \tau[U]$, and $\tau[U]$ is a connected set on which ψ is one-to-one, it follows by Lemma 13 that ψ is one-to-one on K_1^*. For $f \in A_1$, let f' be the representative of f in A_1', and define f^* on $\psi[K_1^*]$ by $f^* = f' \circ \psi^{-1}$. Then f^* is analytic on $\psi[K_1^*]$, and $f^* = f$ on K. Hence each $f \in A_1$ admits an analytic extension to $\psi[K_1^*]$.

Since $K_1^* \sim K$ is open, so is its image under ψ. Since the union of that image and K is the image of K_1^* and consequently compact, we see that $\psi[K_1^*]$ is the union of K and a number of compact components of $W \sim K$ on which each $f \in A_1$ has an analytic extension.

Let π be any K-bounded homomorphism of A into the complex numbers. Then by Theorem 1 there is a point $p_1 \in K_1^*$ such that for $f \in A_1$, $f(p_1) = \pi f$. Setting $p = \psi(p_1)$, we have a point p in $L(A_1)$ at which $f(p) = \pi f$ for all $f \in A_1$, where $L(A_1)$ is the union of K and those compact components of $W \sim K$ to which each function in A_1 has an analytic extension.

Now there are only a finite number of points $p_1, \ldots, p_n$ in K^* such that $\pi f = f(p_i)$ for all $f \in A_0$. For each such p_i which is not in a component of $W \sim K$ to which every function of A can be analytically extended, choose $f_i \in A$ which does not extend to that component. For each of the p_i's in K_1^+ with $\pi f \neq f(p_i)$ for some $f \in A$, choose f_i to be such an f. Let A_1 be the algebra over A_0 generated by the functions f_i thus chosen. Since π restricted to A_1 is evaluation at some point p in $L(A_1)$, we see that there must have been one of the p_i's such that $p_i \in K^+$ and $\pi f = f(p_i)$ for each $f \in A$. This proves (i) and the proof of (ii) is similar to that of (iii) in Theorem 1.

6. Applications to algebras of bounded analytic functions on a Riemann surface

In this section we consider some applications of the preceding theorems to a class of Riemann surfaces. We begin by considering an example due to P. J. Myrberg [4]. Somewhat similar observations had previously been made by H. L. Selberg [6].

Let $\langle a_n \rangle$ be a sequence of points in $0<|z|<1$ with $\lim a_n = 0$, and let W be the two-sheeted Riemann surface over $0<|z|<1$ which has the points a_n as branch points. Then every bounded analytic function on W takes the same values on the two sheets of W. For the square of the difference of the values on the two sheets is a bounded analytic function of z in $0<|z|<1$, and hence also in $|z|<1$. Since this function vanishes at the points a_n it must vanish identically.

In this example we see that each bounded analytic function on W is the composition $g \circ \tau$ of an analytic function in the disk $|z|<1$ and the projection τ of W into the z-plane. Heins [2] has generalized this result to showing that, if W is a parabolic Riemann surface with precisely one ideal boundary component, then some end Ω of W can be mapped onto $0<|z|<1$ by an analytic function τ so that each bounded analytic function f on Ω is of the form $g \circ \tau$ where g is a bounded analytic function in the disk $|z|<1$.

Actually, a result of this nature holds under much weaker assumptions on W. Let (W, Γ) be a bordered Riemann surface with compact border Γ. Then W is said to satisfy the AB-maximum principle if every bounded analytic function on $W \cup \Gamma$ assumes its maximum on Γ. Then the following theorem asserts that there is an analytic mapping τ of $W \cup \Gamma$ into a compact subset C of some Riemann surface such that every bounded analytic function f on $W \cup \Gamma$ is the composition $g \circ \tau$ of τ with some function g defined and analytic in a neighborhood of C. The theorem is slightly more general than this in that it establishes the corresponding conclusion for functions in any algebra of analytic functions on $W \cup \Gamma$ which assume their maxima on Γ.

THEOREM 3. *Let (W, Γ) be a bordered Riemann surface with compact border, and A an algebra of bounded analytic functions on $W \cup \Gamma$ such that each $f \in A$ assumes its maximum on Γ. Then there is an analytic mapping τ of $W \cup \Gamma$ into a Riemann surface W' such that $\tau[W \cup \Gamma]$ has compact closure and each $f \in A$ is of the form $g \circ \tau$ where g is analytic in some neighborhood of the closure of $\tau[W \cup \Gamma]$.*

Proof. By Proposition 4, there is an analytic map τ of $W \cup \Gamma$ into a Riemann surface W' and an algebra A' defined on a connected compact set K containing $\tau[\Gamma]$ such that each f in A is of the form $g \circ \tau$ on Γ. Moreover, there is a subalgebra A_0 of A' which is proper for W'. Let K^+ be the union of K and those compact components of $W \sim K$ to which each function in A' has an analytic continuation. Then by Theorem 2, each parametric disk in W is mapped analytically into the interior of K^+ by a map ψ such that for each $f \in A$, $f = f' \circ \psi$ for this disk. Since the mappings ψ are unique, they combine to give a global map of W into K^+, and since this map coincides with τ on a neighborhood of Γ, it coincides with τ everywhere in W. Thus τ maps $W \cup \Gamma$ into K^+, and so $\tau[W \cup \Gamma]$ has compact closure in W' and each $f \in A$ has the form $g \circ \tau$ for some $g \in A'$, i.e. for some g analytic in a neighborhood of $\tau[W \cup \Gamma]$.

If (W, Γ) satisfies the AB-maximum principle, we may take the algebra A in Theorem 3 to be the algebra of all bounded analytic functions. In this case each f on W of the form $g \circ \tau$ with g analytic on the closure of $\tau[W \cup \Gamma]$ is a bounded analytic function on $W \cup \Gamma$, and so the class of bounded analytic functions consists precisely of those f which are lifted from analytic functions on $\overline{\tau[W \cup \Gamma]}$.

For many applications in function theory, one would like to know that the mapping τ in Theorem 3 has bounded valence. Examples show, however, that τ may have infinite valence even for A the algebra of all bounded analytic functions on a surface with the AB-maximum principle. If we assume that W has an absolute AB-maximum principle in the sense that each bordered Riemann surface with compact border contained in W has the AB-maximum principle, then it is possible to assert that τ has bounded valence, and that the set of W' where the valence of τ is less than the index of $\tau[\Gamma]$ is a Painlevé null-set. Details are published in [5].

7. Wermer's theorem

In [7] Wermer proved the following theorem under the assumption of certain technical hypotheses which we see are unnecessary.

THEOREM (Wermer). *Let A be an algebra of analytic functions on the unit circumference $\Gamma = \{z: |z| = 1\}$, and suppose that A separates points of Γ. Then either A is dense in the algebra of all continuous functions on Γ, or else there is a finite Riemann surface Ω with border Γ such that every f in A has an analytic extension to Ω.*

Proof. By Proposition 3 there is a finitely generated subalgebra A_0 which separates weakly on Γ. Since A_0 is finitely generated, there is some annulus R on which all the functions of A_0 are analytic. Let (W, A_0') be the resolution of (R, A_0) given by Proposition 2. Since A_0 separates weakly on Γ, Γ is mapped into a simple closed piecewise analytic curve Γ' on W. (The curve Γ' may have cusps, but that is irrelevant.) If we choose A_0 to have the final property given in Proposition 3, then each f in A can be expressed as an analytic function on Γ', so that (Γ', A') and (Γ, A) are isomorphic. Identify Γ with Γ'. If there is a bounded homomorphism π on A which is not evaluation on Γ, then by Theorem 2 $W \sim \Gamma$ must have a compact component Ω to which each $f \in A$ has an analytic extension. Since Γ is a simple closed curve on W, there can be at most one such component and so $\Gamma = \partial\Omega$. Thus Ω is a finite Riemann surface with border Γ, and each $f \in A$ has an analytic extension to Ω.

If there are no bounded homomorphisms π other than evaluation on Γ, then Γ is the maximal ideal space of $\bar{A}$, the closure of A in the uniform topology. Since $\bar{A}$ is a Banach algebra, each f which does not vanish on Γ has an inverse in $\bar{A}$, and hence $R[f] \in A$ for each rational function R with no poles on $f[\Gamma]$. Since every continuous function φ on $f[\Gamma]$ can be approximated by such a rational function, every continuous function of f is in $\bar{A}$, in particular $\mathrm{Re}\, f$. Thus the subalgebra of $\bar{A}$ consisting of real functions separates points and hence is dense in $C(\Gamma)$ by the Stone-Weierstrass theorem. Thus $\bar{A} = C(\Gamma)$, (cf. Theorem 1 of [3]).

It should be noted that the hypothesis of Wermer's theorem could be weakened to requiring only that A separates weakly on Γ, if we weaken the first alternative to the statement $\bar{A}$ has finite co-dimension in $C(\Gamma)$.

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