# THE RADON TRANSFORM ON EUCLIDEAN SPACES, COMPACT TWO-POINT HOMOGENEOUS SPACES AND GRASSMANN MANIFOLDS

BY

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## § 1. Introduction

As proved by Radon [16] and John [13], a differentiable function $f$ of compact support on a Euclidean space $\mathbf{R}^n$ can be determined explicitly by means of its integrals over the hyperplanes in the space. Let $J(\omega, p)$ denote the integral of $f$ over the hyperplane $\langle x, \omega\rangle = p$ where $\omega$ is a unit vector and $\langle,\rangle$ the inner product in $\mathbf{R}^n$. If $\Delta$ denotes the Laplacian on $\mathbf{R}^n$, $d\omega$ the area element on the unit sphere $\mathbf{S}^{n-1}$ then (John [14], p. 13)

$$f(x) = \tfrac{1}{2}(2\pi i)^{1-n}(\Delta_x)^{\frac{1}{2}(n-1)}\int_{\mathbf{S}^{n-1}} J(\omega, \langle \omega, x\rangle)\, d\omega, \quad (n \text{ odd}); \tag{1}$$

$$f(x) = (2\pi i)^{-n}(\Delta_x)^{\frac{1}{2}(n-2)}\int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} \frac{dJ(\omega, p)}{p - \langle \omega, x\rangle}, \quad (n \text{ even}), \tag{2}$$

where, in the last formula, the Cauchy principal value is taken.

Considering now the simpler formula (1) we observe that it contains two dual integrations: the first over the set of points in a given hyperplane, the second over the set of hyperplanes passing through a given point. Generalizing this situation we consider the following setup:

(i) Let $X$ be a manifold and $G$ a transitive Lie transformation group of $X$. Let $\Xi$ be a family of subsets of $X$ permuted transitively by the action of $G$ on $X$, whence $\Xi$ acquires a $G$-invariant differentiable structure. Here $\Xi$ will be called the *dual* space of $X$.

(ii) Given $x \in X$, let $\check{x}$ denote the set of $\xi \in \Xi$ passing through $x$. It is assumed that each $\xi$ and each $\check{x}$ carry measures $\mu$ and $\nu$, respectively, such that the action of $G$ on $X$ and $\Xi$ permutes the measures $\mu$ and permutes the measures $\nu$.

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(iii) If $f$ and $g$ are suitably restricted functions on $X$ and $\Xi$, respectively, we can define functions $\hat{f}$ on $\Xi$, $\check{g}$ on $X$ by

$$\hat{f}(\xi) = \int_{\xi} f(x)\, d\mu(x), \quad \check{g}(x) = \int_{\check{x}} g(\xi)\, d\nu(\xi).$$

These three assumptions have not been made completely specific because they are not intended as axioms for a general theory but rather as framework for special examples. In this spirit we shall consider the following problems.

A. *Relate function spaces on $X$ and $\Xi$ by means of the transforms $f \to \hat{f}$ and $g \to \check{g}$.*

B. *Let $\mathbf{D}(X)$ and $\mathbf{D}(\Xi)$, respectively, denote the algebras of $G$-invariant differential operators on $X$ and $\Xi$. Does there exist a map $D \to \hat{D}$ of $\mathbf{D}(X)$ into $\mathbf{D}(\Xi)$ and a map $E \to \check{E}$ of $\mathbf{D}(\Xi)$ into $\mathbf{D}(X)$ such that*

$$(Df)^{\wedge} = \hat{D}\hat{f}, \quad (Eg)^{\vee} = \check{E}\check{g}$$

*for all $f$ and $g$ above?*

C. *In case the transforms $f \to \hat{f}$ and $g \to \check{g}$ are one-to-one find explicit inversion formulas. In particular, find the relationships between $f$ and $(\hat{f})^{\vee}$ and between $g$ and $(\check{g})^{\wedge}$.*

In this article we consider three examples within this framework: (1) The already mentioned example of points and hyperplanes (§ 2–§ 4); (2) points and antipodal manifolds in compact two-point homogeneous spaces (§ 5–§ 6); $p$-planes and $q$-planes in $\mathbf{R}^{p+q+1}$ (§7–§ 8). Other examples are discussed in [11] which also contains a bibliography on the Radon transform and its generalizations. See also [5].

The following notation will be used throughout. The set of integers, real and complex numbers, respectively, is denoted by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$. If $x \in \mathbf{R}^n$, $|x|$ denotes the length of the vector $x$; $\Delta$ denotes the Laplacian on $\mathbf{R}^n$. If $M$ is a manifold, $C^\infty(M)$ (respectively $\mathcal{D}(M)$) denotes the space of differentiable functions (respectively, differentiable functions with compact support) on $M$. If $L(M)$ is a space of functions on $M$, $D$ and endomorphism of $L(M)$ and $p \in M$, $f \in L(M)$ then $[Df](p)$ (and sometimes $D_p(f(p))$) denotes the value of $Df$ at $p$. The tangent space to $M$ at $p$ is denoted $M_p$. If $\tau$ is a diffeomorphism of a manifold $M$ onto a manifold $N$ and if $f \in C^\infty(M)$ then $f^\tau$ stands for the function $f \circ \tau^{-1}$ in $C^\infty(N)$. If $D$ is a differential operator on $M$ then the linear transformation of $C^\infty(N)$ given by $D^\tau$: $f \to (Df^{\tau^{-1}})^\tau$ is a differential operator on $N$. For $M = N$, $D$ is called *invariant* under $\tau$ if $D^\tau = D$.

The adjoint representation of a Lie group $G$ (respectively, Lie algebra $\mathfrak{G}$) will be denoted $\mathrm{Ad}_G$ (respectively $\mathrm{ad}_{\mathfrak{G}}$). These subscripts are omitted when no confusion is likely.

## § 2. The Radon transform in Euclidean space

Let $\mathbf{R}^n$ be a Euclidean space of arbitrary dimension $n$ and let $\Xi$ denote the manifold of hyperplanes in $\mathbf{R}^n$.

If $f$ is a function on $\mathbf{R}^n$, integrable on each hyperplane in $\mathbf{R}^n$, the *Radon transform* of $f$ is the function $\hat{f}$ on $\Xi$ given by

$$\hat{f}(\xi) = \int_{\xi} f(x)\, d\sigma(x), \quad \xi \in \Xi, \tag{1}$$

where $d\sigma$ is the Euclidean measure on the hyperplane $\xi$. In this section we shall prove the following result which shows, roughly speaking, that $f$ has compact support if and only if $\hat{f}$ does.

THEOREM 2.1. *Let $f \in C^\infty(\mathbf{R}^n)$ satisfy the following conditions:*

(i) *For each integer $k > 0$ $|x|^k f(x)$ is bounded.*

(ii) *There exists a constant $A > 0$ such that $\hat{f}(\xi) = 0$ for $d(0,\xi) > A$, $d$ denoting distance.*

*Then* $$f(x) = 0 \quad \text{for} \quad |x| > A.$$

*Proof.* Suppose first that $f$ is a radial function. Then there exists an even function $F \in C^\infty(\mathbf{R})$ such that $f(x) = F(|x|)$ for $x \in \mathbf{R}^n$. Also there exists an even function $\hat{F} \in C^\infty(\mathbf{R})$ such that $\hat{F}(d(0,\xi)) = \hat{f}(\xi)$. Because of (1) we find easily

$$\hat{F}(p) = \int_{\mathbf{R}^{n-1}} F((p^2 + |y|^2)^{\frac{1}{2}})\, dy = \Omega_{n-1} \int_0^\infty F((p^2 + t^2)^{\frac{1}{2}}) t^{n-2}\, dt, \tag{2}$$

where $\Omega_{n-1}$ is the area of the unit sphere in $\mathbf{R}^{n-1}$. Here we substitute $s = (p^2 + t^2)^{-\frac{1}{2}}$ and then put $u = p^{-1}$. Formula (2) then becomes

$$u^{n-3} \hat{F}(u^{-1}) = \Omega_{n-1} \int_0^u (F(s^{-1})\, s^{-n})\, (u^2 - s^2)^{\frac{1}{2}(n-3)}\, ds. \tag{3}$$

This formula can be inverted (see e.g. John [14], p. 83) and we obtain

$$F(s^{-1})\, s^{-n} = c\, s \left(\frac{d}{d(s^2)}\right)^{n-1} \int_0^s (s^2 - u^2)^{\frac{1}{2}(n-3)} u^{n-2} \hat{F}(u^{-1})\, du, \tag{4}$$

where $c$ is a constant. Now by (ii), $\hat{F}(u^{-1}) = 0$ for $0 < u \leqslant A^{-1}$ so by (4), $F(s^{-1}) = 0$ for $0 < s \leqslant A^{-1}$, proving the theorem for the case when $f$ is radial.

Now suppose $f \in C^\infty(\mathbf{R}^n)$ arbitrary, satisfying (i) and (ii). Let $K$ denote the orthogonal group $\mathbf{O}(n)$. For $x, y \in \mathbf{R}^n$ we consider the spherical average

$$f^*(x, y) = \int_K f(x + k \cdot y)\, dk,$$

where $dk$ is the Haar measure on $\mathbf{O}(n)$, with total measure 1. Let $R_2 f^*$ be the Radon transform of $f^*$ in the second variable. Since $(f^\tau)^\wedge = (\hat{f})^\tau$ for each rigid motion $\tau$ of $\mathbf{R}^n$ it is clear that

$$[R_2 f^*](x, \xi) = \int_K \hat{f}(x + k \cdot \xi)\, dk, \quad x \in \mathbf{R}^n,\ \xi \in \Xi, \tag{5}$$

where $x + k \cdot \xi$ is the translate of $k \cdot \xi$ by $x$. Now it is clear that the distance $d$ satisfies the inequality

$$d(0, x + k \cdot \xi) \geqslant d(0, \xi) - |x|$$

for all $x \in \mathbf{R}^n$, $k \in K$. Hence we conclude from (5)

$$[R_2 f^*](x, \xi) = 0 \quad \text{if} \quad d(0, \xi) > A + |x|. \tag{6}$$

For a fixed $x$, the function $y \to f^*(x, y)$ is a radial function in $C^\infty(\mathbf{R}^n)$ satisfying (i). Since the theorem is proved for radial functions, (6) implies that

$$\int_K f(x + k \cdot y)\, dk = 0 \quad \text{if} \quad |y| > A + |x|.$$

The theorem is now a consequence of the following lemma.

Lemma 2.2. *Let $f$ be a function in $C^\infty(\mathbf{R}^n)$ such that $|x|^k f(x)$ is bounded on $\mathbf{R}^n$ for each integer $k > 0$. Suppose $f$ has surface integral 0 over every sphere which encloses the unit sphere. Then $f(x) \equiv 0$ for $|x| > 1$.*

*Proof.* The assumption about $f$ means that

$$\int_{\mathbf{S}^{n-1}} f(x + L\omega)\, d\omega = 0 \quad \text{for} \quad L > |x| + 1. \tag{7}$$

This implies that
$$\int_{|y| \geqslant L} f(x + y)\, dy = 0 \quad \text{for} \quad L > |x| + 1. \tag{8}$$

Now fix $L > 1$. Then (8) shows that

$$\int_{|y| \leqslant L} f(x + y)\, dy$$

is constant for $0 \leqslant |x| < L - 1$. The identity

$$\int_{S^{n-1}} f(x+L\omega)\,(x_i+L\omega_i)\,d\omega = x_i \int_{S^{n-1}} f(x+L\omega)\,d\omega + L^{2-n}\frac{\partial}{\partial x_i}\int_{|y|<L} f(x+y)\,dy$$

then shows that the function $x_i f(x)$ has surface integral 0 over each sphere with radius $L$ and center $x$ $(0 \leqslant |x| < L-1)$. In other words, we can pass from $f(x)$ to $x_i f(x)$ in the identity (7). By iteration, we find that on the sphere $|y| = L$ $(L>1)$ $f(y)$ is orthogonal to all polynomials, hence $f(y) \equiv 0$ for $|y| = L$. This concludes the proof.

*Remark.* The proof of this lemma was suggested by John's solution of the problem of determining a function on $\mathbf{R}^n$ by means of its surface integrals over all spheres of radius 1 (John [14], p. 114).

## § 3. Rapidly decreasing functions on a complete Riemannian manifold

Let $M$ be a connected, complete Riemannian manifold, $\tilde{M}$ its universal covering manifold with the Riemannian structure induced by that of $M$, $\tilde{M} = \tilde{M}_1 \times \ldots \times \tilde{M}_l$ the de Rham decomposition of $\tilde{M}$ into irreducible factors ([17]) and let $M_i = \pi(\tilde{M}_i)$ $(1 \leqslant i \leqslant l)$ where $\pi$ is the covering mapping of $\tilde{M}$ onto $M$. Let $\Delta$, $\tilde{\Delta}$, $\Delta_i$, $\tilde{\Delta}_i$ denote the Laplace-Beltrami operators on $M$, $\tilde{M}$, $M_i$, $\tilde{M}_i$, respectively. It is clear that $\tilde{\Delta}_i$ can be regarded as a differential operator on $\tilde{M}$. In order to consider $\Delta_i$ as a differential operator on $M$, let $f \in C^\infty(M)$, $\tilde{f} = f \circ \pi$. Any covering transformation $\tau$ of $\tilde{M}$ is an isometry so $(\tilde{\Delta}_i(f\circ\pi))^\tau = \tilde{\Delta}_i(f\circ\pi)$; hence $\tilde{\Delta}_i(f\circ\pi) = F\circ\pi$, where $F \in C^\infty(M)$. We define $\Delta_i f = F$. Because of the decomposition of $\tilde{M}$ each $m \in M$ has a coordinate neighborhood which is a product of coordinate neighborhoods in the spaces $M_i$. In terms of these coordinates, $\Delta = \sum_i \Delta_i$; in particular $\Delta_i$ is a differential operator on $M$, and the operators $\Delta_i$ $(1 \leqslant i \leqslant l)$ commute.

Now fix a point $o \in M$ and let $r(p) = d(o, p)$. A function $f \in C^\infty(M)$ will be called *rapidly decreasing* if for each polynomial $P(\Delta_1, \ldots, \Delta_l)$ in the operators $\Delta_1, \ldots, \Delta_l$ and each integer $k \geqslant 0$

$$\sup_p |(1+r(p))^k [P(\Delta_1, \ldots, \Delta_l f](p)| < \infty. \qquad (1)$$

It is clear that condition (1) is independent of the choice of $o$. Let $\mathcal{S}(M)$ denote the set of rapidly decreasing functions on $M$.

In the case of a Euclidean space a function $f \in C^\infty(\mathbf{R}^n)$ belongs to $\mathcal{S}(\mathbf{R}^n)$ if and only if for each polynomial $P$ in $n$ variables the function $P(D_1^2, \ldots, D_n^2) f$ (where $D_i = \partial/\partial x_i$) goes to zero for $|x| \to \infty$ faster than any power of $|x|$. Then the same holds for the function $P(D_1, \ldots, D_n) f$ (so $\mathcal{S}(\mathbf{R}^n)$ coincides with the space defined by Schwartz [18], II, p. 89) as a consequence of the following lemma which will be useful later.

LEMMA 3.1. *Let f be a function in* $C^\infty(\mathbf{R}^n)$, *which for each pair of integers* $k, l \geq 0$ *satisfies*

$$\sup_{x \in \mathbf{R}^n} |(1+|x|)^k [\Delta^l f](x)| < \infty. \tag{2}$$

*Then the inequality is satisfied when* $\Delta^l$ *is replaced by an arbitrary differential operator with constant coefficients.*

This lemma is easily proved by using Fourier transforms.

LEMMA 3.2. *A function* $F \in C^\infty(\mathbf{R} \times \mathbf{S}^{n-1})$ *lies in* $\mathcal{S}(\mathbf{R} \times \mathbf{S}^{n-1})$ *if and only if for arbitrary integers* $k, l \geq 0$ *and any differential operator D on* $\mathbf{S}^{n-1}$,

$$\sup_{\omega \in \mathbf{S}^{n-1}, r \in \mathbf{R}} \left| (1+|r|)^k \frac{d^l}{dr^l} (DF)(\omega, r) \right| < \infty. \tag{3}$$

*Proof.* It is obvious that (3) implies that $F$ is rapidly decreasing. For the converse we must prove ($\mathbf{S}^{n-1}$ being irreducible) that (3) holds provided it holds when $l \geq 0$ is even and $D$ an arbitrary power $(\Delta_S)^m$ $(m \geq 0)$ of the Laplacian $\Delta_S$ on $\mathbf{S}^{n-1}$. Let $G(\omega, r) = d^l/dr^l(F(\omega, r))$. Of course it suffices to verify (3) as $\omega = (\omega_1, ..., \omega_n)$ varies in some coordinate neighborhood on $\mathbf{S}^{n-1}$. Let $x_i = |x|\omega_i$ $(1 \leq i \leq n)$ and suppose $G$ extended to a $C^\infty$ function $\tilde{G}$ in the product of an annulus $A_\varepsilon$: $\{x \in \mathbf{R}^n \mid |x_1^2 + ... + x_n^2 - 1| < \varepsilon < 1\}$ with $\mathbf{R}$. Regardless how this extension is made, (3) would follow (for even $l$) if we can prove an estimate of the form

$$\sup_{\omega \in \mathbf{S}^{n-1}, r \in \mathbf{R}} |(1+|r|)^k [D^\gamma \tilde{G}](\omega, r)| < \infty \tag{4}$$

for an arbitrary derivative $D^\gamma = \partial^{|\gamma|}/\partial x_1^{\gamma_1} ... \partial x_n^{\gamma_n}$ $(|\gamma| = \gamma_1 + ... + \gamma_n)$. Now, by Sobolev's lemma (see e.g. [3], Theorem 6′, p. 243) $[D^\gamma \tilde{G}](\omega, r)$ can be estimated by means of $L^2$ norms over $A_\varepsilon$ of finitely many derivatives $D_r^\alpha D_x^\gamma(\tilde{G}(x, r))$. But the $L^2$ norm over $A_\varepsilon$ of $D_r^\alpha D_x^\gamma(\tilde{G}(x, r))$ is estimated by the $L^2$ norm over $A_\varepsilon$ of $\Delta_x^m(\tilde{G}(x, r))$, $m$ being a suitable integer (see [12], p. 178–188). Now suppose $\tilde{G}$ was chosen such that for each $r$, the function $x \to \tilde{G}(x, r)$ is constant on each radius from 0. Then

$$\Delta_x(\tilde{G}(x, r)) = |x|^{-2}[\Delta_S G](\omega, r) \quad (x = |x|\omega)$$

and

$$\Delta_x^m(\tilde{G}(x, r)) = \sum_i f_i(|x|)\,[(\Delta_S)^i G](\omega, r),$$

where the sum is finite and each $f_i$ is bounded for $||x| - 1| < \varepsilon$. Hence the $L^2$ norm over $A_\varepsilon$ of $(\Delta^m)_x(\tilde{G}(x, r))$ is estimated by a linear combination of the $L^2$ norms over $\mathbf{S}^{n-1}$ of $[(\Delta_S)^i G](\omega, r)$. But these last derivatives satisfy (3), by assumption, so we have proved (4).

This proves (3) for $l$ even. Let $H(\omega, s)$ be the Fourier transform (with respect to $r$) of the function $(DF)(\omega, r)$. Then one proves by induction on $k$ that

$$\sup_{\omega \in \mathbf{S}^{n-1},\, s \in \mathbf{R}} \left| (1+|s|)^l \frac{d^k}{ds^k} H(\omega, s) \right| < \infty$$

for all $k, l \geqslant 0$ and now (3) follows for all $k, l \geqslant 0$ by use of the inverse Fourier transform.

## § 4. The Radon transforms of $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$

If $\omega \in \mathbf{S}^{n-1}$, $r \in \mathbf{R}$ let $\xi(\omega, r)$ denote the hyperplane $\langle x, \omega \rangle = r$ in $\mathbf{R}^n$. Then the mapping $(\omega, r) \to \xi(\omega, r)$ is a two-fold covering map of the manifold $\mathbf{S}^{n-1} \times \mathbf{R}$ onto the manifold $\Xi$ of all hyperplanes in $\mathbf{R}^n$; the (differentiable) functions on $\Xi$ will be identified with the (differentiable) functions $F$ on $\mathbf{S}^{n-1} \times \mathbf{R}$ which satisfy $F(\omega, r) = F(-\omega, -r)$. Thus $\mathcal{S}(\Xi)$ is, by definition, a subspace of $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$. We also need the linear space $\mathcal{S}_H(\Xi)$ of functions $F \in \mathcal{S}(\Xi)$ which have the property that for each integer $k \geqslant 0$ the integral $\int F(\omega, r) r^k dr$ can be written as a homogeneous $k$th degree polynomial in the components $\omega_1, \ldots, \omega_n$ of $\omega$. Such a polynomial can, since $\omega_1^2 + \ldots + \omega_n^2 = 1$, also be written as a $(k+2l)$th degree polynomial in the $\omega_i$.

We shall now consider the situation outlined in the introduction for $X = \mathbf{R}_n$, $\Xi$ as above and $G$ the group of rigid motions of $X$. If $x \in X$, $\xi \in \Xi$, the measure $\mu$ is the Euclidean measure $d\sigma$ on the hyperplane $\xi$, $\nu$ is the unique measure on $\check{x}$ invariant under all rotations around $x$, normalized by $\nu(\check{x}) = 1$. We shall now consider problems $A$, $B$, $C$ from § 1. If $f$ is a function on $X$, integrable along each hyperplane in $X$ then according to the conventions above

$$\hat{f}(\omega, r) = \int_{\langle x, \omega \rangle = r} f(x)\, d\sigma(X), \quad \omega \in \mathbf{S}^{n-1},\ r \in \mathbf{R}. \tag{1}$$

THEOREM 4.1. *The Radon transform $f \to \hat{f}$ is a linear one-to-one mapping of $\mathcal{S}(X)$ onto $\mathcal{S}_H(\Xi)$.*

*Proof.* Let $f \in \mathcal{S}(X)$ and let $\tilde{f}$ denote the Fourier transform

$$\tilde{f}(u) = \int f(x)\, e^{-i\langle x, u \rangle}\, dx, \quad u \in \mathbf{R}^n.$$

If $u \neq 0$ put $u = s\omega$, where $s \in \mathbf{R}$ and $\omega \in \mathbf{S}^{n-1}$. Then

$$\tilde{f}(s\omega) = \int_{-\infty}^{\infty} dr \int_{\langle x, \omega \rangle = r} f(x)\, e^{-i\langle x, \omega \rangle s}\, d\sigma(x)$$

so we obtain
$$\tilde{f}(s\omega)=\int_{-\infty}^{\infty}\hat{f}(\omega,r)\,e^{-isr}dr, \tag{2}$$

for $s\neq 0$ in $\mathbf{R}$, $\omega\in\mathbf{S}^{n-1}$. But (2) is obvious for $s=0$ so it holds for all $s\in\mathbf{R}$. Now according to Schwartz [18], II, p. 105, the Fourier transform $f\to\tilde{f}$ maps $\mathcal{S}(\mathbf{R}^n)$ onto itself. Since

$$\frac{d}{ds}(\tilde{f}(s\omega))=\sum_{i=1}^{n}\omega_i\frac{\partial\tilde{f}}{\partial u_i}\quad (u=(u_1,\ldots,u_n))$$

it follows from (2) that for each fixed $\omega$, the function $r\to\hat{f}(\omega,r)$ lies in $\mathcal{S}(\mathbf{R})$. For each $\omega_0\in\mathbf{S}^{n-1}$, a subset of $\{\omega_1,\ldots,\omega_n\}$ will serve as local coordinates on a neighborhood of $\omega_0$. To see that $\hat{f}\in\mathcal{S}(\Xi)$, it therefore suffices to verify (3) § 3 for $F=\hat{f}$ on an open subset $N$ of $\mathbf{S}^{n-1}$ where $\omega_n$ is bounded away from 0 and $\omega_1,\ldots,\omega_{n-1}$ serve as coordinates, in terms of which $D$ is expressed. Putting $\mathbf{R}^+=\{s\in\mathbf{R}\,|\,s>0\}$ we have on $N\times\mathbf{R}^+$

$$u_1=s\omega_1,\ldots,u_{n-1}=s\omega_{n-1},\ u_n=s(1-\omega_1^2-\ldots-\omega_{n-1}^2)^{\frac{1}{2}}, \tag{3}$$

so
$$\frac{\partial}{\partial\omega_i}(\tilde{f}(s\omega))=s\sum_{i=1}^{n-1}\frac{\partial\tilde{f}}{\partial u_i}-s\omega_i(1-\omega_1^2-\ldots-\omega_{n-1}^2)^{\frac{1}{2}}\frac{\partial\tilde{f}}{\partial u_n}.$$

It follows that if $D$ is any differential operator on $\mathbf{S}^{n-1}$ and $k$, $l$ integers $\geqslant 0$ then

$$\sup_{\omega\in N,\,s\in\mathbf{R}}\left|(1+s^{2k})\left[\frac{d^l}{ds^l}D\tilde{f}\right](\omega,s)\right|<\infty. \tag{4}$$

We can therefore apply $D$ under the integral sign in the inversion formula

$$\hat{f}(\omega,r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(s\omega)\,e^{isr}\,ds \tag{5}$$

and obtain
$$(1+r^{2k})\frac{d^l}{dr^l}(D_\omega(\hat{f}(\omega,r)))=\frac{1}{2\pi}\int\left(1+(-1)^k\frac{d^{2k}}{ds^{2k}}\right)((is)^lD_\omega(\tilde{f}(s\omega)))\,e^{isr}\,ds.$$

Now (4) shows that $\hat{f}\in\mathcal{S}(\Xi)$. Finally, if $k$ is an integer $\geqslant 0$ then

$$\int_{-\infty}^{\infty}\hat{f}(\omega,r)\,r^kdr=\int_{-\infty}^{\infty}r^kdr\int_{\langle x,\omega\rangle=r}f(x)\,d\sigma(x)=\int_{\mathbf{R}^n}f(x)\langle x,\omega\rangle^k dx \tag{6}$$

so $\hat{f}\in\mathcal{S}_H(\Xi)$. The Fourier transform being one-to-one it remains to prove that each $g\in\mathcal{S}_H(\Xi)$ has the form $g=\hat{f}$ for some $f\in\mathcal{S}(\mathbf{R}^n)$. We put

$$G(s,\omega)=\int_{-\infty}^{\infty}g(\omega,r)\,e^{-irs}dr.$$

Then $G(-s, -\omega)=G(s, \omega)$ and $G(0, \omega)$ is a homogeneous polynomial of degree 0 in $\omega$, hence independent of $\omega$. Hence there exists a function $F$ on $\mathbf{R}^n$ such that

$$F(s\omega)=\int_{-\infty}^{\infty} g(\omega, r)\, e^{-irs}dr, \quad s \in \mathbf{R}, \omega \in \mathbf{S}^{n-1}. \tag{7}$$

It is clear that $F$ is $C^\infty$ in $\mathbf{R}^n-\{0\}$. To prove that $F$ is $C^\infty$ in a neighborhood of 0 we consider the coordinate neighborhood $N$ on $\mathbf{S}^{n-1}$ as before. Let $h(u_1, ..., u_n)$ be any function of class $C^\infty$ in $\mathbf{R}^n-\{0\}$ and let $h^*(\omega_1, ..., \omega_{n-1}, s)$ be the function on $N \times \mathbf{R}^+$ obtained by means of the substitution (3). Then

$$\frac{\partial h}{\partial u_i}=\sum_{j=1}^{n-1} \frac{\partial h^*}{\partial \omega_j}\cdot\frac{\partial \omega_j}{\partial u_i}+\frac{\partial h^*}{\partial s}\cdot\frac{\partial s}{\partial u_i} \quad (1 \leqslant i \leqslant n)$$

and

$$\frac{\partial \omega_j}{\partial u_i}=\frac{1}{s}\left(\delta_{ij}-\frac{u_i u_j}{s^2}\right) \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n-1),$$

$$\frac{\partial s}{\partial u_i}=\omega_i \ (1 \leqslant i \leqslant n-1), \quad \frac{\partial s}{\partial u_n}=(1-\omega_1^2-\dots-\omega_{n-1}^2)^{\frac{1}{2}}.$$

Hence

$$\frac{\partial h}{\partial u_i}=\frac{1}{s}\frac{\partial h^*}{\partial \omega_i}+\omega_i\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial \omega_j}\right) \quad (1 \leqslant i \leqslant n-1),$$

$$\frac{\partial h}{\partial u_n}=(1-\omega_1^2-\dots-\omega_{n-1}^2)^{\frac{1}{2}}\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{dh^*}{\partial \omega_j}\right).$$

In order to use these formulas for $h=F$ we write

$$F(s\omega)=\int_{-\infty}^{\infty} g(\omega, r)\, dr+\int_{-\infty}^{\infty} g(r, \omega)\,(e^{-irs}-1)\, dr$$

and by assumption, the first integral is independent of $\omega$. Thus, for a constant $K>0$,

$$\left|\frac{1}{s}\frac{\partial}{\partial \omega_i}(F(s\omega))\right| \leqslant K\int (1+r^4)^{-1}s^{-1}\,|e^{-isr}-1|\,dr \leqslant K\int \frac{|r|}{1+r^4}\,dr.$$

This shows that all the derivatives $\partial F/\partial u_i$ $(1 \leqslant i \leqslant n)$ are bounded in a punctured ball $0<|u|<\varepsilon$ so $F$ is continuous in a neighborhood of $u=0$. More generally, let $q$ be any integer $>0$. Then we have for an arbitrary $q$th order derivative,

$$\frac{\partial^q h}{\partial u_{i_1} \dots \partial u_{i_q}}=\sum_{i+j \leqslant q} A_{i,j}(\omega, s)\frac{\partial^{i+j} h^*}{\partial \omega_{k_1} \dots \partial \omega_{k_i}\partial s^j}, \tag{8}$$

where the coefficient $A_{i,j}(\omega, s)=O(s^{j-q})$ near $s=0$. Also

$$F(s\omega)=\int_{-\infty}^{\infty} g(\omega,r)\sum_{k=0}^{q-1}\frac{(-isr)^k}{k!}\,dr+\int_{-\infty}^{\infty} g(\omega,r)\,e_q(-irs)\,dr, \tag{9}$$

where
$$e_q(t)=\frac{t^q}{q!}+\frac{t^{q+1}}{(q+1)!}+\dots.$$

Then it is clear that the first integral in (9) is a polynomial in $u_1, \ldots, u_n$ of degree $\leqslant q-1$ and is therefore annihilated by the differential operator (8). Now, if $0 \leqslant j \leqslant q$,

$$\left| s^{j-q}\frac{\partial^j}{\partial s^j}(e_q(-irs))\right| = |(-ir)^q(-irs)^{j-q}\,e_{q-j}(-irs)| \leqslant K_j\,r^q, \tag{10}$$

where $K_j$ is a constant, because the function $t \to (it)^{-p}e_p(it)$ is obviously bounded on $\mathbf{R}$ $(p \geqslant 0)$. Since $g \in S(\Xi)$ it follows from (8), (9), (10) that each $q$th order derivative of $F$ with respect to $u_1, \ldots, u_n$ is bounded in a punctured ball $0<|u|<\varepsilon$. Hence $F \in C^\infty(\mathbf{R}^n)$. That $F$ is rapidly decreasing is now clear from formula (7), Lemma 3.1 and the fact that ([8], p. 278)

$$\Delta h = \frac{\partial^2 h^*}{\partial s^2}+\frac{n-1}{s}\frac{\partial h^*}{\partial s}+\frac{1}{s^2}\Delta_S h^*,$$

where $\Delta_S$ is the Laplace–Beltrami operator on $\mathbf{S}^{n-1}$. If $f$ is the function in $S(X)$ whose Fourier transform is $F$ then $\hat{f}=g$ and the theorem is proved.

Let $S^*(X)$ denote the space of all functions $f \in S(X)$ which satisfy $\int f(x)P(x)\,dx=0$ for all polynomials $P(x)$. Similarly, let $S^*(\Xi)$ denote the space of all functions $g \in S(\Xi)$ which satisfy $\int g(\omega, r)P(r)\,dr \equiv 0$ for all polynomials $P(r)$. Note that under the Fourier transform, $S^*(X)$ corresponds to the space $S_0(\mathbf{R}^n)$ of functions in $S(\mathbf{R}^n)$ all of whose derivatives vanish at the origin.

Corollary 4.2. *The transforms $f \to \hat{f}$ and $g \to \check{g}$, respectively, are one-to-one linear maps of $S^*(X)$ onto $S^*(\Xi)$ and of $S(\Xi)$ onto $S^*(X)$.*

The first statement follows from (6) and the well-known fact that the polynomials $\langle x, \omega\rangle^k$ span the space of homogeneous polynomials of degree $k$. As for the second, we observe that for $f \in S(X)$ and $\xi_0$ a fixed plane through 0

$$(\hat{f})^\vee(x)=\int_K \hat{f}(x+k\cdot\xi_0)\,dk=\int_K\left(\int_{\xi_0} f(x+k\cdot y)\,dy\right)dk$$
$$=\int_{\xi_0} dy\int_K f(x+k\cdot y)\,dk=\Omega_{n-1}\int_0^\infty r^{n-2}\left(\frac{1}{\Omega_n}\int_{\mathbf{S}^{n-1}} f(x+r\omega)\,d\omega\right)dr,$$

so
$$(\hat{f})^\vee(x)=\frac{\Omega_{n-1}}{\Omega_n}\int_X |x-y|^{-1}f(y)\,dy. \tag{11}$$

This formula is also proved in [4]. Now the right-hand side is a tempered distribution, being the convolution of a tempered distribution and a member of $\mathcal{S}(X)$. By [18], II, p. 124, the Fourier transform is given by the product of the Fourier transforms so if $f \in \mathcal{S}^*(X)$ we see that $(\hat{f})^\vee$ has Fourier transform belonging to $\mathcal{S}_0(X)$. Hence $(\hat{f})^\vee \in \mathcal{S}^*(X)$ and the second statement of Cor. 4.2 follows.

*Remarks.* A characterization of the Radon transform of $\mathcal{S}(X)$ similar to that of Theorem 3.1 is stated in Gelfand–Graev–Vilenkin [5], p. 35. Their proof, as outlined on p. 36–39, is based on the inversion formula (1) § 1 and therefore leaves out the even-dimensional case. Corollary 4.2 was stated by Semyanistyi [19].

Now let $\mathcal{D}(X)$ and $\mathcal{D}(\Xi)$ be as defined in § 1, and put $\mathcal{D}_H(\Xi) = \mathcal{S}_H(\Xi) \cap \mathcal{D}(\Xi)$. The following result is an immediate consequence of Theorem 2.1 and 4.1.

COROLLARY 4.3. *The Radon transform $f \to \hat{f}$ is a linear one-to-one mapping of $\mathcal{D}(X)$ onto $\mathcal{D}_H(\Xi)$.*

Concerning problem $B$ in § 1 we have the following result which is a direct consequence of Lemmas 7.1 and 8.1, proved later.

PROPOSITION 4.4. *The algebra $\mathbf{D}(X)$ is generated by the Laplacian $\Delta$, the algebra $\mathbf{D}(\Xi)$ is generated by the differential operator $\square$: $g(\omega, r) \to (d^2/dr^2) g(\omega, r)$ and*

$$(\Delta f)^\wedge = \square \hat{f}, \qquad (\square g)^\vee = \Delta \check{g}$$

*for $f \in \mathcal{S}(X)$, $g \in C^\infty(\Xi)$.*

The following reformulation of the inversion formulas (1), (2) § 1 gives an answer to problem $C$.

THEOREM 4.5. (i) *If $n$ is odd,*

$$f = c\Delta^{\frac{1}{2}(n-1)}((\hat{f})^\vee), \quad f \in \mathcal{S}(X);$$
$$g = c\square^{\frac{1}{2}(n-1)}((\check{g})^\wedge), \quad g \in \mathcal{S}^*(\Xi),$$

*where $c$ is a constant, independent of $f$ and $g$.*

(ii) *If $n$ is even,*

$$f = c_1 J_1((\hat{f})^\vee), \quad f \in \mathcal{S}(X);$$
$$g = c_2 J_2((\check{g})^\wedge), \quad g \in \mathcal{S}^*(\Xi),$$

*where the operators $J_1$ and $J_2$ are given by analytic continuation*

$$J_1 : f(x) \to \underset{\alpha = 1-2n}{\text{anal. cont.}} \int_{R^n} f(y) \, |x - y|^\alpha dy,$$

$$J_2 : g(\omega, p) \to \underset{\beta=-n}{\text{anal. cont.}} \int_{\mathbf{R}} g(\omega, q)\, |p-q|^\beta \, dq,$$

*and* $c_1$, $c_2$ *are constants, independent of* $f$ *and* $g$.

*Proof.* In (i) the first formula is just (1) § 1 and the second follows by Prop. 4.4. We shall now indicate how (ii) follows from (2) § 1. Since the Cauchy principal value is the derivative of the distribution log $|p|$ on $\mathbf{R}$ whose successive derivatives are the distributions $Pf\cdot(p^{-k})$ (see [18], I, p. 43) we have by (2) § 1

$$f(x) = (2\pi i)^{-n}(n-1)! \int_{\mathbf{S}^{n-1}} (Pf\cdot(p-\langle\omega, x\rangle^{-n}))\,(\hat{f}(\omega, p))\, d\omega. \tag{12}$$

On the other hand, if $\varphi \in C^\infty(X)$ is bounded we have by Schwartz [18], I, p. 45

$$[J_1\varphi](0) = \lim_{\varepsilon\to 0}\left[\int_{|x|\geq\varepsilon} |x|^{1-2n}\varphi(x)\,dx + \varepsilon(\varphi)\right], \tag{13}$$

where $$\varepsilon(\varphi) = \sum_k H_k[\Delta^k\varphi](0)\,\frac{\varepsilon^{1-n+2k}}{1-n+2k}, \quad H_k = \frac{\pi^{\frac{1}{2}n}}{2^{2k-1}k!\,\Gamma(\frac{1}{2}n+k)}.$$

In particular $$[J_1((\hat{f})^\vee](0) = \lim_{\varepsilon\to 0}\left[\Omega_n\int_\varepsilon^\infty r^{-n}F(r)\,dr + \varepsilon(\hat{f})^\vee)\right], \tag{14}$$

where $F(r)$ is the average of $(\hat{f})^\vee$ on the sphere $|x| = r$. In order to express (14) in terms of $\hat{f}$ we assume $f$ is a radial function and write $\hat{f}(p)$ for $\hat{f}(\omega, p)$. Then

$$F(r) = C\int_0^{\frac{1}{2}\pi} \hat{f}(r\cos\theta)\sin^{n-2}\theta\,d\theta, \quad C^{-1} = \int_0^{\frac{1}{2}\pi}\sin^{n-2}\theta\,d\theta, \tag{15}$$

$$[\Delta^k(\hat{f})^\vee](0) = \left(\frac{d^{2k}}{dp^{2k}}\hat{f}\right)(0). \tag{16}$$

If $q_n(p)$ is the Taylor series of $\hat{f}(p)$ around 0 up to order $n-2$ we get upon substituting (15) and (16) into (14),

$$(J_1((\hat{f})^\vee))(0) = C\,\Omega_n \lim_{\varepsilon\to 0}\int_0^{\frac{1}{2}\pi}\sin^{n-2}\theta\ \cos^{n-1}\theta\,d\theta\int_{\varepsilon\cos\theta}^\infty p^{-n}(\hat{f}(p)-q_n(p))\,dp,$$

which on comparison with (12) gives

$$f(0) = c_1 J_1((\hat{f})^\vee)(0), \quad c_1 = \text{const.} \tag{17}$$

Now put for $\varphi \in C^\infty(X)$, $x, y \in X$,

$$\varphi_x^*(y) = \int_K \varphi(x+ky)\, dk$$

and let us prove $\qquad [J_1 \varphi_x^*](0) = ((J_1 \varphi)_x^*)(0) \quad \text{if } \varphi \text{ is bounded.} \qquad (18)$

In view of (13) this is a consequence of the obvious formula

$$\int_{|y|\geq\varepsilon} |y|^{1-2n} \varphi_x^*(y)\, dy = \int_{|y|\geq\varepsilon} |y|^{1-2n} \varphi(x+y)\, dy$$

and the Darboux equation ([8], p. 279) $[\Delta^k \varphi_x^*](0) = [\Delta^k \varphi](x)$. Now, a direct computation shows that $((\hat{f})^\vee)_x^* = ((f_x^*)^\wedge)^\vee$ for $f \in \mathcal{S}(X)$ and since $f_x^*$ is radial we get from (17), (18)

$$f(x) = f_x^*(0) = c_1[(J_1((\hat{f})^\vee))_x^*](0) = c_1[J_1(\hat{f})^\vee](x).$$

Finally the inversion formula for $g \in \mathcal{S}^*(\Xi)$ would follow from the first one if we prove

$$(J_1 f)^\wedge = c_0 J_2 \hat{f}, \quad f \in \mathcal{S}^*(X),\ c_0 \text{ constant.} \tag{19}$$

To see this we take the one-dimensional Fourier transform on both sides. The function $J_1 f$ is the convolution of a tempered distribution with a rapidly decreasing function. Hence it is a tempered distribution (Schwartz [18], II, pp. 102, 124) whose Fourier transform is (since $f \in \mathcal{S}^*(X)$) a function in $\mathcal{S}(X)$. Hence $J_1 f \in \mathcal{S}(X)$. Similarly $(J_2 \hat{f})(\omega, p)$ is a rapidly decreasing function of $p$. Using the relation between the 1-dimensional and the $n$-dimensional Fourier transform ((2) § 4) and the formula for the Fourier transform of $Pf \cdot r^\lambda$ (Schwartz [18], II, p. 113) we find that both sides of (19) have the same Fourier transform, hence coincide. This concludes the proof.

*Remark* (*added in proof*). Alternative proofs of most of the results of § 4 have been found subsequently by D. Ludwig.

## § 5. The geometry of compact symmetric spaces of rank one

In this section and the next one we shall study problems A, B and C for the duality between points and antipodal manifolds in compact two-point homogeneous spaces. In the present section we derive the necessary geometric facts for symmetric spaces of rank one, without use of classification.

Let $X$ be a compact Riemannian globally symmetric space of rank one and dimension $>1$. Let $I(X)$ denote the group of isometries of $X$ in the compact open topology, $I_0(X)$ the identity component of $I(X)$. Let $o$ be a fixed point in $X$ and $s_o$ the geodesic symmetry of $X$ with repsect to $o$. Let $\mathfrak{u}$ denote the Lie algebra of $I(X)$ and $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ the decomposition of $\mathfrak{u}$ into eigenspaces of the involutive automorphism of $\mathfrak{a}$ which corresponds to the automorphism $u \to s_o u s_o$ of $I(X)$. Here $\mathfrak{k}$ is the Lie algebra of the subgroup $K$ of $I(X)$ which leaves $o$ fixed. Changing the distance function $d$ on $X$ by a constant factor we may, since $\mathfrak{u}$ is semisimple, assume that the differential of the mapping $u \to u \cdot o$ of $I(X)$ onto $X$ gives an isometry of $\mathfrak{p}$ (with the metric of the negative of the Killing form of $\mathfrak{u}$) onto $X_o$, the tangent space to $X$ at $o$. Let $L$ denote the diameter of $X$ and if $x \in X$ let $A_x$ denote the corresponding *antipodal manifold*, that is the set of points $y \in X$ at distance $L$ from $x$; $A_x$ is indeed a manifold, being an orbit of $K$. The geodesics in $X$ are all closed and have length $2L$ and the Exponential mapping Exp at $o$ is a diffeomorphism of the open ball in $X_o$ of center 0 and radius $L$ onto the complement $X - A_o$ (see [10], Ch. $X$, § 5).

PROPOSITION 5.1. *For each $x \in X$, the antipodal manifold $A_x$, with the Riemannian structure induced by $X$, is a symmetric space of rank one, and a totally geodesic submanifold of $X$.*

*Proof.* Let $y \in A_x$. Considering a geodesic in $X$ through $y$ and $x$ we see that $x$ is fixed under the geodesic symmetry $s_y$; hence $s_y(A_x) = A_x$. If $\sigma_y$ denotes the restriction of $s_y$ to $A_x$, then $\sigma_y$ is an involutive isometry of $A_x$ with $y$ as isolated fixed point. Thus $A_x$ is globally symmetric and $\sigma_y$ is the geodesic symmetry with respect to $y$. Let $t \to \gamma(t)$ $(t \in \mathbf{R})$ be a geodesic in the Riemannian manifold $A_x$. We shall prove that $\gamma$ is a geodesic in $X$. Consider the isometry $s_{\gamma(t)}\, s_{\gamma(0)}$ and a vector $T$ in the tangent space $X_{\gamma(0)}$. Let $\tau_r$: $X_{\gamma(0)} \to X_{\gamma(r)}$ denote the parallel translation in $X$ along the curve $\gamma(\varrho)$ $(0 \leqslant \varrho \leqslant r)$. Then the parallel field $\tau_r \cdot T$ $(0 \leqslant r \leqslant t)$ along the curve $r \to \gamma(r)$ $(0 \leqslant r \leqslant t)$ is mapped by $s_{\gamma(t)}$ onto a parallel field along the image curve $r \to s_{\gamma(t)}\, \gamma(r) = \sigma_{\gamma(t)}\, \gamma(r) = \gamma(2t - r)$ $(0 \leqslant r \leqslant t)$. Since $s_{\gamma(t)} \tau_t T = -\tau_t T$ we deduce that $s_{\gamma(t)} s_{\gamma(0)}\, T = -s_{\gamma(t)}\, T = \tau_{2t} T$. In particular, the parallel transport in $X$ along $\gamma$ maps tangent vectors to $\gamma$ into tangent vectors to $\gamma$. Hence $\gamma$ is a geodesic in $X$. Consequently, $A_x$ is a totally geodesic submanifold of $X$, and by the definition of rank, $A_x$ has rank one.

Let $Z \to \operatorname{ad}(Z)$ denote the adjoint representation of $\mathfrak{u}$. Select a vector $H \in \mathfrak{p}$ of length $L$. The space $\mathfrak{a} = \mathbf{R}H$ is a Cartan subalgebra of the symmetric space $X$ and we can select a positive restricted root $\alpha$ of $X$ such that $\frac{1}{2}\alpha$ is the only other possible positive restricted root (see [10], Exercise 8, p. 280 where $\Sigma$ is by definition the set of positive restricted roots). This means that the eigenvalues of $\operatorname{ad}(H)^2$ are 0, $\alpha(H)^2$ and possibly $(\frac{1}{2}\alpha(H))^2$ ($\alpha(H)$ is purely imaginary). Let $\mathfrak{u} = \mathfrak{u}_0 + \mathfrak{u}_\alpha + \mathfrak{u}_{\frac{1}{2}\alpha}$ be the corresponding decomposition of $\mathfrak{u}$ into eigenspaces and put $\mathfrak{k}_\beta = \mathfrak{u}_\beta \cap \mathfrak{k}$, $\mathfrak{p}_\beta = \mathfrak{u}_\beta \cap \mathfrak{p}$ for $\beta = 0$, $\alpha$, $\frac{1}{2}\alpha$. Then $\mathfrak{p}_0 = \mathfrak{u}$ and $\mathfrak{k}_\beta = \operatorname{ad} H(\mathfrak{p}_\beta)$ for $\beta \neq 0$.

PROPOSITION 5.2. *Let $S$ denote the subgroup of $K$ leaving the point* Exp$H$ *fixed, and let $\mathfrak{s}$ denote the Lie algebra of $S$. Then*

(i) $\mathfrak{s} = \mathfrak{k}_0 + \mathfrak{k}_\alpha$ *if H is conjugate to* 0;

(ii) $\mathfrak{s} = \mathfrak{k}_0$ *if H is not conjugate to* 0;

(iii) *If* $\frac{1}{2}\alpha$ *is a restricted root then H is conjugate to* 0.

*Proof.* If exp: $\mathfrak{u} \to I(X)$ is the usual exponential mapping then a vector $T$ in $\mathfrak{k}$ belongs to $\mathfrak{s}$ if and only if $\exp(-H)\exp(tT)\exp(H) \in K$ for all $t \in \mathbf{R}$. This reduces to

$$T \in \mathfrak{s} \quad \text{if and only if ad } H(T) + \tfrac{1}{3!}(\text{ad } H)^3(T) + \ldots = 0.$$

In particular, $\mathfrak{s}$ is the sum of its intersections with $\mathfrak{k}_0$, $\mathfrak{k}_\alpha$ and $\mathfrak{k}_{\frac{1}{2}\alpha}$. If $T \neq 0$ in $\mathfrak{k}_\beta$ ($\beta = 0, \alpha, \frac{1}{2}\alpha$) the condition above is equivalent to $\sinh(\beta(H)) = 0$. Thus (ii) is immediate ([10], Ch. VII, Prop. 3.1). To prove (i) suppose $H$ is conjugate to 0. Whether or not $\frac{1}{2}\alpha$ is a restricted root we have by the cited result, $\alpha(\text{H}) \in \pi i\mathbf{Z}$ so $\mathfrak{k}_\alpha \in \mathfrak{s}$. We have also $\mathfrak{s} \cap \mathfrak{k}_{\frac{1}{2}\alpha} = \{0\}$ because otherwise $\frac{1}{2}\alpha(H) \in \pi i\mathbf{Z}$ which would imply that $\frac{1}{2}H$ is conjugate to 0. This proves (i). For (iii) suppose $H$ were not conjugate to 0. The sphere in $X_0$ with radius $2L$ and center 0 is mapped by Exp onto $o$. It follows that the differential $d\,\text{Exp}_{2H}$ is 0 so using the formula for this differential ([10], page 251, formula (2)) it follows that $(\frac{1}{2}\alpha)(2H) \in \pi i\mathbf{Z}$ so $\alpha(H) \in \pi i\mathbf{Z}$ which is a contradiction.

PROPOSITION 5.3. *Suppose H is conjugate to* 0. *Then all the geodesics in X with tangent vectors in* $\mathfrak{a} + \mathfrak{p}_\alpha$ *at o pass through the point* Exp$H$. *The manifold* Exp$(\mathfrak{a} + \mathfrak{p}_\alpha)$, *with the Riemannian structure induced by that of X, is a sphere, totally geodesic in X.*

*Proof.* Let $\mathfrak{G}$ denote the complexification of $\mathfrak{u}$ and $B$ the Killing form of $\mathfrak{G}$. Since the various root subspaces $\mathfrak{G}^\beta$, $\mathfrak{G}^\gamma$ ($\beta + \gamma \neq 0$) are orthogonal with respect to $B$ ([10], p. 141) it follows without difficulty (cf. [10], p. 224) that

$$B([\mathfrak{k}_0, \mathfrak{p}_\alpha], \mathfrak{p}_{\frac{1}{2}\alpha}) = B([\mathfrak{k}_\alpha, \mathfrak{p}_\alpha], \mathfrak{p}_{\frac{1}{2}\alpha}) = 0.$$

Also, if $Z \in \mathfrak{u}_0$ then $B([H, Z], [H, Z]) = -B(Z, (\text{ad}\,H)^2 Z) = 0$ so $\mathfrak{u}_0$ equals the centralizer of $H$ in $\mathfrak{u}$. Thus $[\mathfrak{k}_0, \mathfrak{a}] = 0$. Also $[\mathfrak{k}_\alpha, \mathfrak{a}] = \mathfrak{p}_\alpha$. Combining these relations we get

$$[\mathfrak{s}, \mathfrak{a} + \mathfrak{p}_\alpha] \subset \mathfrak{a} + \mathfrak{p}_\alpha.$$

Let $S_o$ denote the identity component of $S$ and Ad the adjoint representation of the group $I(X)$. Then the tangent space to the orbit $\text{Ad}(S_0)H$ at the point $H$ is $[\mathfrak{s}, \mathbf{R}H]$ which equals $\mathfrak{p}_\alpha$, and by the relation above this orbit lies in the subspace $\mathfrak{a} + \mathfrak{p}_\alpha$. It follows that $\text{Ad}(S_0)H$ is the sphere in $\mathfrak{a} + \mathfrak{p}_\alpha$ of radius $L$ and center 0. But if $s \in S$ the geodesic $t \to s \cdot \text{Exp}\, tH = \text{Exp}(\text{Ad}(s)tH)$ passes through Exp $H$ so the first statement of the proposition is proved.

By consideration of the root subspaces $\mathfrak{G}^\beta$ as above, it is easy to see that the subspace $\mathfrak{a}+\mathfrak{p}_\alpha$ of $\mathfrak{p}$ is a Lie triple system. Thus the Riemannian manifold $X_\mathfrak{p}=\mathrm{Exp}(\mathfrak{a}+\mathfrak{p}_\alpha)$ is a totally geodesic submanifold of $X$ ([10], p. 189). It is homogeneous and is mapped into itself by the geodesic symmetry $s_0$ of $X$, hence it is globally symmetric, and being totally geodesic, has rank one. If $Z$ is a unit vector in $\mathfrak{p}_\alpha$, the curvature of $X_\mathfrak{p}$ along the plane section spanned by $H$ and $Z$, is (cf. [10], p. 206)

$$-L^{-2}B([H,Z],[H,Z]) = -L^{-2}\alpha(H)^2.$$

But since $X_\mathfrak{p}$ has rank one, every plane section is congruent to one containing $H$; hence $X_\mathfrak{p}$ has constant curvature. Finally, $X_\mathfrak{p}-\{\mathrm{Exp}\,H\}$ is the diffeomorphic image of an open ball, hence simply connected. Since $\dim X_\mathfrak{p}>1$ it follows that $X_\mathfrak{p}$ is also simply connected, hence a sphere.

PROPOSITION 5.4. *The antipodal manifold $A_{\mathrm{Exp}\,H}$ is given by*

$$A_{\mathrm{Exp}\,H} = \mathrm{Exp}\,(\mathfrak{p}_{\frac{1}{2}\alpha}) \quad \textit{if } H \textit{ is conjugate to } 0.$$

$$A_{\mathrm{Exp}\,H} = \mathrm{Exp}\,(\mathfrak{p}_{\alpha}) \quad \textit{if } H \textit{ is not conjugate to } 0.$$

*Proof.* The geodesics from $\mathrm{Exp}H$ to $o$ intersect $A_{\mathrm{Exp}\,H}$ in $o$ under a right angle (Gauss' lemma; see e.g. [1], p. 34 or [9], Theorem 3). By Propositions 5.2 and 5.3 we deduce that the tangent space $(A_{\mathrm{Exp}\,H})_o$ equals $\mathfrak{p}_{\frac{1}{2}\alpha}$ if $H$ is conjugate to 0 and equals $\mathfrak{p}_\alpha$ if $H$ is not conjugate to 0. Now use Prop. 5.1.

The next result shows that there is a kind of projective duality between points and antipodal manifolds.

PROPOSITION 5.5. *Let $x, y \in X$. Then*

(i) *$x \neq y$ implies $A_x \neq A_y$;*
(ii) *$x \in A_y$ if and only if $y \in A_x$.*

*Proof.* If $z \in A_x$ then the geodesics which meet $A_x$ in $z$ under a right angle all pass through a point $z^*$ at distance $L$ from $z$ (Prop. 5.3 and Prop. 5.4); among these are the geodesics joining $x$ and $z$. Hence $z^*=x$ and the result follows.

PROPOSITION 5.6. *Let $A(r)$ denote the surface area of a sphere in $X$ of radius $r$ $(0<r<L)$. Then*

$$A(r) = \Omega_{p+q+1}\lambda^{-p}(2\lambda)^{-q}\sin^p(\lambda r)\sin^q(2\lambda r),$$

*where $p=\dim\mathfrak{p}_{\frac{1}{2}\alpha}$, $q=\dim\mathfrak{p}_\alpha$, $\Omega_n$ is the area of the unit sphere in $\mathbf{R}^n$ and*

$$\lambda=\frac{1}{2L}|\alpha(H)|.$$

*Proof.* As proved in [8], p. 251, the area is given by

$$A(r)=\int_{\|Z\|=r}\det(A_Z)\,d\omega_r(Z), \tag{1}$$

where $d\omega_r$ is the Euclidean surface element of the sphere $\|Z\|=r$ in $\mathfrak{p}$, and

$$A_Z=\sum_0^\infty \frac{T_Z^n}{(2n+1)!},$$

where $T_Z$ is the restriction of $(\operatorname{ad} Z)^2$ to $\mathfrak{p}$. The integrand in (1) is a radial function so

$$A(r)=\Omega_{p+q+1}\cdot r^{p+q}\cdot\det(A_{H_r}),\quad \left(H_r=\frac{r}{L}H\right).$$

Since the nonzero eigenvalues of $T_{H_r}$ are $(\frac{1}{2}\alpha(H_r))^2$ with multiplicity $p$ and $\alpha(H_r)^2$ with multiplicity $q$ we obtain

$$A(r)=\Omega_{p+q+1}\,r^{p+q}\left(\frac{\sin\lambda r}{\lambda r}\right)^p\left(\frac{\sin 2\lambda r}{2\lambda r}\right)^q.$$

where $\lambda=\frac{1}{2}L^{-1}|\alpha(H)|$.

## § 6. Points and antipodal manifolds in two-point homogeneous spaces

Let $X$ be a compact two-point homogeneous space, or, what is the same thing (Wang [21]) a compact Riemannian globally symmetric space of rank one. We preserve the notation of the last section and assume $\dim X>1$. Let $G=I(X)$ and let $\Xi$ be the set of all antipodal manifolds in $X$, with the differentiable structure induced by the transitive action of $G$. On $\Xi$ we choose a Riemannian structure such that the diffeomorphism $\varphi:x\to A_x$ of $X$ onto $\Xi$ (see Prop. 5.5) is an isometry. Let $\Delta$ and $\hat{\Delta}$ denote the Laplace–Beltrami operators on X and $\Xi$, respectively. The measures $\mu$ and $\nu$ on the manifolds $\xi$ and $\check{x}$ (§ 1) are defined to be those induced by the Riemannian structures of $X$ and $\Xi$. If $x\in X$, then by Prop. 5.5

$$\check{x}=\{\varphi(y)\,|\,y\in\varphi(x)\}.$$

Consequently, if $g$ is a continuous function on $\Xi$,

$$\check{g}(x)=\int_{\check{x}}g(\xi)\,d\nu(\xi)=\int_{y\in\varphi(x)}g(\varphi(y))\,d\nu(\varphi(y))=\int_{\varphi(x)}(g\circ\varphi)(y)\,d\mu(y),$$

so
$$\check{g} = (g \circ \varphi)^{\wedge} \circ \varphi. \tag{1}$$

Because of this correspondence between the integral transforms $f \to \hat{f}$ and $g \to \check{g}$ it suffices to consider the first.

Problems $A$, $B$, and $C$ now have the following answer.

THEOREM 6.1.

(i) *The algebras* $\mathbf{D}(X)$ *and* $\mathbf{D}(\Xi)$ *are generated by* $\Delta$ *and* $\hat{\Delta}$ *respectively.*

(ii) *The mapping* $f \to \hat{f}$ *is a linear one-to-one mapping of* $C^\infty(X)$ *onto* $C^\infty(\Xi)$ *and*

$$(\Delta f)^{\wedge} = \hat{\Delta} \hat{f}.$$

(iii) *Except for the case when* $X$ *is an even-dimensional real projective space,*

$$f = P(\Delta)((\hat{f})^{\vee}), \quad f \in C^\infty(X),$$

*where* $P$ *is a polynomial, independent of* $f$, *explicitly given below.*

*Proof.* Part (i) is proved in [8], p. 270. Let $[M^r f](x)$ be the average of $f$ over a sphere in $X$ of radius $r$ and center $x$. Then

$$\hat{f}(\varphi(x)) = c[M^L f](x), \tag{2}$$

where $c$ is a constant. Since $\Delta$ commutes with the operator $M^r$ ([8], Theorem 16, p. 276) we have

$$(\hat{\Delta}\hat{f}) \circ \varphi = \Delta(\hat{f} \circ \varphi) = cM^L \Delta f = (\Delta f)^{\wedge} \circ \varphi,$$

proving the formula in (ii). For (iii) we have to use the following complete list of compact Riemannian globally symmetric spaces of rank 1: The spheres $\mathbf{S}^n$, $(n = 1, 2, \ldots)$, the real projective spaces $\mathbf{P}^n(\mathbf{R})$, $(n = 2, 3, \ldots)$, the complex projective spaces $\mathbf{P}^n(\mathbf{C})$, $(n = 4, 6, \ldots)$, the quaternion projective spaces $\mathbf{P}^n(\mathbf{H})$, $(n = 8, 12, \ldots)$ and the Cayley projective plane $\mathbf{P}^{16}(\mathbf{Cay})$. The superscripts denote the real dimension. The corresponding antipodal manifolds are also known ([2], pp. 437–467, [15], pp. 35 and 52) and are in the respective cases: A point, $\mathbf{P}^{n-1}(\mathbf{R})$, $\mathbf{P}^{n-2}(\mathbf{C})$, $\mathbf{P}^{n-4}(\mathbf{H})$, and $\mathbf{S}^8$. For the lowest dimensions, note that $\mathbf{P}^1(\mathbf{R}) = \mathbf{S}^1$, $\mathbf{P}^2(\mathbf{C}) = \mathbf{S}^2$, $\mathbf{P}^4(\mathbf{H}) = \mathbf{S}^4$. Let $A_1(r)$ denote the area of a sphere of radius $r$ in an antipodal manifold in $X$. Then by Prop. 5.6,

$$A_1(r) = C_1 \sin^{p_1}(\lambda_1 r) \sin^{q_1}(2\lambda_1 r),$$

where $C_1$ is a constant and $p_1, q_1, \lambda_1$ are the numbers $p, q, \lambda$ for the antipodal manifold.

The multiplicities $p$ and $q$ are determined in Cartan [2], and show that $\frac{1}{2}\alpha$ is a restricted root unless $X$ is a sphere or a real projective space. Ignoring these exceptions we have by virtue of the results of § 5:

$L$ = diameter $X$ = diameter $A_x$
= distance of 0 to the nearest conjugate point in $X_0$
= smallest number $M>0$ such that $\lim_{r\to M} A(r)=0$.

We can now derive the following list:

$X=\mathbf{S}^n$: $p=0,\ q=n-1,\ \lambda=\pi/2L,\ A(r)=C\sin^{n-1}(2\lambda r),\ A_1(r)\equiv 0.$

$X=\mathbf{P}^n(\mathbf{R})$: $p=0,\ q=n-1,\ \lambda=\pi/4L,\ A(r)=C\sin^{n-1}(2\lambda r),\ A_1(r)=C_1\sin^{n-2}(2\lambda r).$

$X=\mathbf{P}^n(\mathbf{C})$: $p=n-2,\ q=1,\ \lambda=\pi/2L,\ A(r)=C\sin^{n-2}(\lambda r)\sin(2\lambda r),\ A_1(r)=C_1\sin^{n-4}(\lambda r)\sin(2\lambda r).$

$X=\mathbf{P}^n(\mathbf{H})$: $p=n-4,\ \ q=3,\ \ \lambda=\pi/2L,\ \ A(r)=C\sin^{n-4}(\lambda r)\sin^3(2\lambda r),$
$A_1(r)=C_1\sin^{n-8}(\lambda r)\ \sin^3(2\lambda r).$

$X=\mathbf{P}^{16}(\mathbf{Cay})$: $p=8,\ q=7,\ \lambda=\pi/2L,\ A(r)=C\sin^8(\lambda r)\sin^7(2\lambda r),\ A_1(r)=C_1\sin^7(2\lambda r).$

In each case, C and $C_1$ are constants, not necessarily the same for all cases. Now if $x\in X$ and $f\in C^\infty(X)$ let $[If](x)$ denote the average of the integrals of $f$ over the antipodal manifolds which pass through $x$. Then $(\hat{f})^\vee$ is a constant multiple of $If$. Fix a point $o\in X$ and let $K$ be the subgroup of $G$ leaving $o$ fixed. Let $\xi_o$ be a fixed antipodal manifold through $o$ and let $d\sigma$ be the volume element on $\xi_o$. Then

$$[If](g\cdot o)=\int_K\left(\int_{\xi_o} f(gk\cdot y)\,d\sigma(y)\right)dk=\int_{\xi_o}[M^r f](g\cdot o)\,d\sigma(y),$$

where $r$ is the distance $d(o, y)$ in the space $X$ between the points $o$ and $y$. Now if $d(o,y)<L$ there is a unique geodesic in $X$ of length $d(o,y)$ joining $o$ to $y$ and since $\xi_o$ is totally geodesic, $d(o,y)$ is also the distance between $o$ and $y$ in $\xi_o$. Hence, using geodesic polar coordinates in the last integral we find

$$[If](x)=\int_0^L A_1(r)\,[M^r f](x)\,dr. \tag{3}$$

In geodesic polar coordinates on $X$, the Laplace–Beltrami operator $\Delta$ equals $\Delta_r+\Delta'$ where $\Delta'$ is the Laplace-Beltrami operator on the sphere in $X$ of radius $r$ and ([10], p. 445)

$$\Delta_r=\frac{d^2}{dr^2}+\frac{1}{A(r)}\frac{dA}{dr}\frac{r}{dr}\quad (0<r<L).$$

The function $(x,r)\to[M^r f](x)$ satisfies

$$\Delta M^r f=\Delta_r(M^r f) \tag{4}$$

([8], p, 279 or [6]). Using Prop. 5,6, we have

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \lambda(p \cot(\lambda r) + 2q \cot(2\lambda r))\frac{\partial}{\partial r} \quad (0 < r < L) \tag{5}$$

(compare also [7], p. 302). Now (iii) can be proved on the basis of (3) (4) (5) by the method in [8], p. 285, where the case $\mathbf{P}^n(\mathbf{R})$ ($n$ odd) is settled. The case $X = \mathbf{S}^n$ being trivial we shall indicate the details for $X = \mathbf{P}^n(\mathbf{C})$, $\mathbf{P}^n(\mathbf{H})$ and $\mathbf{P}^{16}(\mathbf{Cay})$.

LEMMA 6.2. *Let $X = \mathbf{P}^n(\mathbf{C})$, $f \in C^\infty(X)$. If $m$ is an even integer, $0 \leqslant m \leqslant n-4$ then*

$$(\Delta - \lambda^2(n-m-2)(m+2)) \int_0^L \sin^m(\lambda r) \sin(2\lambda r)\, [M^r f](x)\, dr$$

$$= -\lambda^2(n-m-2)\, m \int_0^L \sin^{m-2}(\lambda r) \sin(2\lambda r)\, [M^r f](x)\, dr.$$

*For $m=0$ the right-hand side should be replaced by*

$$-2\lambda(n-2) f(x).$$

LEMMA 6.3. *Let $X = \mathbf{P}^n(\mathbf{H})$, $f \in C^\infty(X)$. Let $m$ be an even integer, $0 < m \leqslant n-8$. Then*

$$(\Delta - \lambda^2(n-m-4)(m+6)) \int_0^L \sin^m(\lambda r) \sin^3(2\lambda r)\, [M^r f](x)\, dr$$

$$= -\lambda^2(n-m-4)(m+2) \int_0^L \sin^{m-2}(\lambda r) \sin^3(2\lambda r)\, [M^r f](x)\, dr.$$

*Also*

$$(\Delta - 4\lambda^2(n-4))(\Delta - 4\lambda^2(n-2)) \int_0^L \sin^3(2\lambda r)\, [M^r f](x)\, dr = 16\lambda^3(n-2)(n-4) f(x).$$

LEMMA 6.4. *Let $X = \mathbf{P}^{16}(\mathbf{Cay})$, $f \in C^\infty(X)$. Let $m > 1$ be an integer. Then*

$$(\Delta - 4\lambda^2 m(11-m)) \int_0^L \sin^m(2\lambda r)\, [M^r f](x)\, dr$$

$$= -32\lambda^2(m-1) \int_0^L \sin^{m-2}(2\lambda r) \cos^2(\lambda r)\, [M^r f](x)$$

$$+ 4\lambda^2(m-1)(m-7) \int_0^L \sin^{m-2}(2\lambda r)\, [M^r f](x)\, dr;$$

$$(\Delta - 4\lambda^2(m+1)(10-m)) \int_0^L \sin^m(2\lambda r) \cos^2(\lambda r)\, [M^r f](x)\, dr$$

$$= 4\lambda^2(3m-5) \int_0^L \sin^m(2\lambda r)\, [M^r f](x)\, dr$$

$$+ 4\lambda^2(m-1)(m-15) \int_0^L \sin^{m-2}(2\lambda r) \cos^2(\lambda r)\, [M^r f](x)\, dr.$$

Iteration of these lemmas gives part (iii) of Theorem 6.1 where the polynomial $P(\Delta)$ has degree equal to one half the dimension of the antipodal manifold and is a constant multiple of

| | |
|---|---|
| 1 (the identity), | $X = \mathbf{S}^n$ |
| $(\Delta - \varkappa(n-2)1)(\Delta - \varkappa(n-4)3)\dots(\Delta - \varkappa 1(n-2))$, | $X = \mathbf{P}^n(\mathbf{R})$ |
| $(\Delta - \varkappa(n-2)2)(\Delta - \varkappa(n-4)3)\dots(\Delta - \varkappa 2(n-2))$, | $X = \mathbf{P}^n(\mathbf{C})$ |
| $[(\Delta - \varkappa(n-2)4)(\Delta - \varkappa(n-4)6)\dots(\Delta - \varkappa 8(n-6))][(\Delta - \varkappa 4(n-4))(\Delta - \varkappa 4(n-2))]$, | $X = \mathbf{P}^n(\mathbf{H})$ |
| $(\Delta - 112\varkappa)^2(\Delta - 120\varkappa)^2$, | $X = \mathbf{P}^{16}(\mathbf{Cay})$. |

In each case $\varkappa = (\pi/2L)^2$.

Finally, we prove part (ii). From (1) and (2) we derive

$$M^L M^L f = c^{-2}(\hat{f})^\vee$$

so, if $X$ is not an even-dimensional projective space, $f$ is a constant multiple of $M^L P(\Delta) M^L f$ which shows that $f \to \hat{f}$ is one-to-one and onto. For the even-dimensional projective space a formula relating $f$ and $(\hat{f})^\vee$ is given by Semyanistyi [20]. In particular, the mapping $f \to \hat{f}$ is one-to-one. To see that it is onto, let $(\varphi_n)$ be the eigenfunctions of $\Delta$. Then each $\varphi_n$ is an eigenfunction of $M^L$ ([10], Theorem 7.2, Ch. X). Since the eigenvalue is $\neq 0$ by the above it is clear that no measure on $X$ can annihilate all of $M^L(C^\infty(X))$. This finishes the proof of Theorem 6.1.

*Added in proof.* Theorem 6.1 shows that $\hat{f} = \text{constant}$ implies $f = \text{constant}$. For $\mathbf{P}^n(\mathbf{R})$ we thus obtain a (probably known) corollary.

*Corollary.* Let $B$ be an open set in $\mathbf{R}^{n+1}$, symmetric and starshaped with respect to 0, bounded by a hypersurface. Assume area $(B \cap P) = \text{constant}$ for all hyperplanes $P$ through 0. Then $B$ is an open ball.

## § 7. Differential operators on the space of $p$-planes

Let $p$ and $n$ be two integers such that $0 \leqslant p < n$. A *$p$-plane* $E_p$ in $\mathbf{R}^n$ is by definition a translate of a $p$-dimensional vector subspace of $\mathbf{R}^n$. The 0-planes are just the points of $\mathbf{R}^n$. The $p$-planes in $\mathbf{R}^n$ form a manifold $\mathbf{G}(p, n)$ on which the group $\mathbf{M}(n)$ of all isometries of $\mathbf{R}^n$ acts transitively. Let $\mathbf{O}(k)$ denote the orthogonal group in $\mathbf{R}^k$ and let $\mathbf{G}_{p,n}$ denote the manifold $\mathbf{O}(n)/\mathbf{O}(p) \times \mathbf{O}(n-p)$ of $p$-dimensional subspaces of $\mathbf{R}^n$. The manifold $\mathbf{G}(p, n)$ is a fibre bundle with base space $\mathbf{G}_{p,n}$, the projection $\pi$ of $\mathbf{G}(p, n)$ onto $\mathbf{G}_{p,n}$ being the mapping which to any $p$-plane $E_p \in \mathbf{G}(p, n)$ associates the parallel $p$-plane through the origin. Thus the fibre of this bundle $(\mathbf{G}(p, n), \mathbf{G}_{p,n}, \pi)$ is $\mathbf{R}^{n-p}$. If $F$ denotes an arbitrary fibre and $f \in C^\infty(\mathbf{G}(p, n))$ then the restriction of $f$ to $F$ will be denoted $f|F$. Consider now the linear transformation $\Box_p$ of $C^\infty(\mathbf{G}(p, n))$ given by

$$(\Box_p f)|F = \Delta_F(f|F), \quad f \in C^\infty(\mathbf{G}(p, n)),$$

for each fibre $F$, $\Delta_F$ denoting the Laplacian on $F$. It is clear that $\Box_p$ is a differential operator on $\mathbf{G}(p, n)$. For simplicity we usually write $\Box$ instead of $\Box_p$.

LEMMA 7.1.

(i) *The operator $\Box_p$ is invariant under the action of $\mathbf{M}(n)$ on $\mathbf{G}(p, n)$.*

(ii) *Each differential operator on $\mathbf{G}(p, n)$ which is invariant under $\mathbf{M}(n)$ is a polynomial in $\Box_p$.*

*Proof.* We recall that if $\varphi$ is an isometry of a Riemannian manifold $M_1$ onto a Riemannian manifold $M_2$ and if $\Delta_1$, $\Delta_2$ are the corresponding Laplace–Beltrami operators then (cf. [10], p. 387)

$$(\Delta_1 f^{\varphi^{-1}}) = \Delta_2 f, \quad f \in C^\infty(M_2). \tag{1}$$

Now each isometry $g \in \mathbf{M}(n)$ induces a fibre-preserving diffeomorphism of $\mathbf{G}(p, n)$, preserving the metric on the fibres. Let $f \in C^\infty(\mathbf{G}(p, n))$ and $F$ any fibre. Writing for simplicity $\Box$ instead of $\Box_p$ we get from (1)

$$(\Box^g f)|F = (\Box f^{g^{-1}})^g|F = ((\Box f^{g^{-1}})|g^{-1}\cdot F)^g = (\Delta_{g^{-1}F}(f^{g^{-1}}|g^{-1}F))^g = \Delta_F(f|F) = (\Box f)|F,$$

so $\Box^g = \Box$, proving (i).

Let $E_p^o$ be a fixed $p$-plane in $\mathbf{R}^n$, say the one spanned by the $p$ first unit coordinate vectors, $Z_1, \ldots, Z_p$. The subgroup of $\mathbf{M}(n)$ which leaves $E_p^o$ invariant can be identified with the product group $\mathbf{M}(p) \times \mathbf{O}(n-p)$. For simplicity we put $G = \mathbf{M}(n)$, $H = \mathbf{M}(p) \times \mathbf{O}(n-p)$ and let $\mathfrak{G}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. If $\mathfrak{M}$ is any subspace of $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{M} + \mathfrak{h}$ (direct sum) and $\mathrm{Ad}_G(h)\mathfrak{M} \subset \mathfrak{M}$ for each $h \in H$ then we know from [8] Theorem 10 that the $G$-invariant differential operators on the space $G/H = \mathbf{G}(p, n)$ are directly given by the polynomials on $\mathfrak{M}$ which are invariant under the group $\mathrm{Ad}_G(H)$. Let $\mathfrak{o}(k)$ denote the Lie algebra of $\mathbf{O}(k)$. Then $\mathfrak{G}$ is the vector space direct sum of $\mathfrak{o}(n)$ and the abelian Lie algebra $\mathbf{R}^n$. Also if $T \in \mathfrak{o}(n)$, $X \in \mathbf{R}^n$ then the bracket $[T, X]$ in $\mathfrak{G}$ is $[T, X] = T \cdot X$ (the image of $X$ under the linear transformation $T$). The Lie algebra $\mathfrak{h}$ is the vector space direct sum of $\mathfrak{o}(p)$, $\mathfrak{o}(n-p)$ and $\mathbf{R}^p (= E_p^o)$; we write this in matrix-vector form

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} V \\ 0 \end{pmatrix} \middle| A \in \mathfrak{o}(p),\ B \in \mathfrak{o}(n-q),\ V \in E_p^o \right\}.$$

For $\mathfrak{M}$ we choose the subspace

$$\mathfrak{M}=\left\{\begin{pmatrix}0 & X\\ -{}^tX & 0\end{pmatrix}+\begin{pmatrix}0\\ Z\end{pmatrix}\middle|\begin{array}{l}X \text{ any } p\times(n-p) \text{ matrix, } {}^tX\\ \text{its transpose, } Z\in\mathbf{R}^{n-p}\end{array}\right\}.$$

Then it is clear that $\mathfrak{G}=\mathfrak{h}+\mathfrak{M}$, Let $a\in\mathbf{O}(p)$, $b\in\mathbf{O}(n-p)$, $V\in E_p^o$. Then

$$\mathrm{Ad}_G\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\cdot\left[\begin{pmatrix}0 & X\\ -{}^tX & 0\end{pmatrix}+\begin{pmatrix}0\\ Z\end{pmatrix}\right]=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}0 & X\\ -{}^tX & 0\end{pmatrix}\begin{pmatrix}a^{-1} & 0\\ 0 & b^{-1}\end{pmatrix}+\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}0\\ Z\end{pmatrix}$$

$$=\begin{pmatrix}0 & aXb^{-1}\\ -b\,{}^tXa & 0\end{pmatrix}+\begin{pmatrix}0\\ bZ\end{pmatrix}. \tag{2}$$

$$\mathrm{Ad}_G\begin{pmatrix}V\\ 0\end{pmatrix}\cdot\left[\begin{pmatrix}0 & X\\ -{}^tX & 0\end{pmatrix}+\begin{pmatrix}0\\ Z\end{pmatrix}\right]=\begin{pmatrix}0 & X\\ -{}^tX & 0\end{pmatrix}+\begin{pmatrix}0\\ Z+{}^tXV\end{pmatrix}. \tag{3}$$

It follows immediately that $\mathrm{Ad}_G(h)\mathfrak{M}\subset\mathfrak{M}$ for all $h\in H$. Now let as usual $E_{ij}$ denote the matrix $(\delta_{ai}\delta_{bj})_{1\leqslant a,\,b\leqslant n}$, put $X_{ij}=E_{ip+j}-E_{p+ji}$ $(1\leqslant i\leqslant p,\ 1\leqslant j\leqslant n-p)$ and let $Z_k\,(p+1\leqslant k\leqslant n)$ denote the $k$th coordinate vector in $\mathbf{R}^n$. Then $\{X_{ij}, Z_k\}$ is a basis of $\mathfrak{M}$. Any element $q$ in the symmetric algebra $S(\mathfrak{M})$ over $\mathfrak{M}$ can be written as a finite sum

$$q(X_{11},\ldots,X_{pn-p},Z_{p+1},\ldots,Z_n)=\sum_i r_i(Z_{p+1},\ldots,Z_n)\,s_i(X_{11},\ldots,X_{pn-p}),$$

where the $r_i$ and $s_i$ are polynomials. Suppose $q$ is homogeneous of degree $m$ (say) and invariant under $\mathrm{Ad}_G(H)$. From (2) and (3) for $X=0$ we see that a polynomial in $Z_{p+1},\ldots,Z_n$ is invariant under $\mathrm{Ad}_G(H)$ if and only if it is a polynomial in $|Z|^2=Z_{p+1}^2+\ldots+Z_n^2$. Hence the invariant polynomial $q$ can be written

$$q=\sum_{r=0}^{[\frac{1}{2}m]}|Z|^{2r}q_r(X_{11},\ldots,X_{pn-p}), \tag{4}$$

where $q_r$ is homogeneous of degree $m-2r$. Now, by (3), $q$ is invariant under the substitution $T(v)$: $X_{ij}\to X_{ij}+v_iZ_{p+j}$ ($v_1,\ldots,v_p$ being any real numbers, and $1\leqslant i\leqslant p$, $1\leqslant j\leqslant n-p$). We can write

$$q_r(X_{11}+v_1Z_{p+1},\ldots,X_{pn-p}+v_pZ_n)=\sum_{(s)}a_{r,s_1,\ldots,s_p}\otimes v_1^{s_1}\ldots v_p^{s_p},$$

where $\otimes$ denotes the tensor product (over $\mathbf{R}$) of the polynomial rings $\mathbf{R}[X_{11},\ldots,Z_n]$ and $\mathbf{R}[v_1,\ldots,v_p]$. Using (4) and the invariance of $q$ we obtain

$$\sum_{r,(s)}|Z|^{2r}a_{r,s_1,\ldots,s_p}\otimes v_1^{s_1}\ldots v_p^{s_p}=\sum_r a_{r,0,\ldots,0}.$$

It follows that
$$\sum_r|Z|^{2r}a_{r,s_1,\ldots,s_p}=0\quad\text{if } s_1+\ldots+s_p>0, \tag{5}$$

and since $a_{r,s_1,\ldots,s_p}$ has degree $s_1+\ldots+s_p$ in the $Z_i$ (5) implies $a_{r,s_1,\ldots,s_p}=0$ for $s_1+\ldots+s_p>0$, whence each $q_r$ is invariant under the substitution $T(v)$ above. This implies easily that each

$q_r$ is a constant. Thus the elements $q \in S(\mathfrak{M})$ invariant under $\mathrm{Ad}_G(H)$ are the polynomials in $|Z|^2$. By [8], Theorem 10, the polynomial $|Z|^2$ induces a $G$-invariant differential operator $D$ on $G/H$ such that for each $f \in C^\infty(G/H)$,

$$[Df]\,(E_p^o) = \left\{ \left( \frac{\partial^2}{\partial t_{p+1}^2} + \dots + \frac{\partial^2}{\partial t_n^2} \right) f((t_{p+1}\, Z_{p+1} + \dots + t_n\, Z_n) \cdot E_p^o) \right\}_{t=0}. \tag{6}$$

Thus $[Df](E_p^o) = [\square f](E_p^o)$ and since $D$ and $\square$ are both $G$-invariant, $D = \square$. Now (ii) follows from [8], Cor. p. 269.

## § 8. $p$-planes and $q$-planes in $\mathbf{R}^{p+q+1}$

The notation being as in the preceeding section put $q = n - p - 1$. Let $\mathbf{G}^*(p, n)$ and $\mathbf{G}^*(q, n)$, respectively, denote the sets of $p$-planes and $q$-planes in $\mathbf{R}^n$ not passing through the origin. The projective duality between points and hyperplanes in $\mathbf{R}^n$, realized by the polarity with respect to the unit sphere $\mathbf{S}^{n-1}$ generalizes to a duality between $\mathbf{G}^*(p, n)$ and $\mathbf{G}^*(q, n)$. In fact, if $a \neq 0$ in $\mathbf{R}^n$, let $E_{n-1}(a)$ denote the polar hyperplane. If $a$ runs through a $p$-plane $E_p \in \mathbf{G}^*(p, n)$ then the hyperplanes $E_{n-1}(a)$ intersect in a unique $q$-plane $E_q \in \mathbf{G}^*(q, n)$ and the mapping $E_p \to E_q$ is the stated duality.

We have now an example of the framework in § 1. Let $X = \mathbf{G}(p, n)$, put $G = \mathbf{M}(n)$, acting on $X$. Given a $q$-plane $E_q$ consider the family $\xi = \xi(E_q)$ of $p$-planes intersecting $E_q$. If $E_q' \neq E_q''$ then $\xi(E_q') \neq \xi(E_q'')$; thus the set of all families $\xi$—the dual space $\Xi$—can be identified with $\mathbf{G}(q, n)$. In accordance with this identification, if $E_p = x \in X$ then $\check{x} = \check{x}(E_p)$ is the set of $q$-planes intersecting $x$. Because of convergence difficulties we do not define the measures $\mu$ and $\nu$ (§ 1) directly but if $f$ is any function on $\mathbf{G}(p, n)$ we put

$$\hat{f}(E_q) = \int_{E_q} \left( \int_{a \in E_p} f(E_p)\, d\sigma_p(E_p) \right) d\mu_q(a),$$

whenever these integrals exist. Here $d\sigma_p$ is the invariant measure on the Grassmann manifold of $p$-planes through $a$ with total measure 1, $d\mu_q$ is the Euclidean measure on $E_q$. The transform $g \to \check{g}$ is defined by interchanging $p$ and $q$ in the definition of $\hat{f}$. It is convenient to consider the operators $M_p$ and $L_q$ defined by

$$[M_p f]\,(a) = \int_{a \in E_p} f(E_p)\, d\sigma_p(E_p), \quad f \in C^\infty(\mathbf{G}(p,n)) \tag{1}$$

$$[L_q F]\,(E_q) = \int_{E_q} F(a)\, d\mu_q(a), \qquad F \in \mathcal{S}(\mathbf{R}^n). \tag{2}$$

Then we have, formally, $\hat{f} = L_q M_p f$.

LEMMA 8.1.

(i) *$M_p$ maps $C^\infty(\mathbf{G}(p, n))$ into $C^\infty(\mathbf{R}^n)$ and $M_p \Box_p = \Delta M_p$.*

(ii) *$L_q$ maps $\mathcal{S}(\mathbf{R}^n)$ into $C^\infty(\mathbf{G}(q, n))$ and $L_q \Delta = \Box_q L_q$.*

*Proof.* (i) Put $K = \mathbf{O}(n) \subset \mathbf{M}(n) = G$. For $f \in C^\infty(\mathbf{G}(p, n))$ let $f^* \in C^\infty(G)$ be determined by $f^*(g) = f(g \cdot E_p^o)$, $(g \in G)$. Then for a suitably normalized Haar measure $dk$ on $K$ we have

$$\int_K f^*(gk)\, dk = [M_p f]\,(g \cdot 0),$$

which shows that $M_p f \in C^\infty(\mathbf{R}^n)$.

For each $X \in \mathfrak{G}$, let $\tilde{X}$ denote the left invariant vector field on $G$ satisfying $\tilde{X}_e = X$. Since $\mathbf{R}^n \subset \mathfrak{G}$ we can consider the left invariant differential operator $\tilde{\Delta} = \sum_{i=1}^n \tilde{Z}_i^2$ on $G$. If $k \in K$, $\mathrm{Ad}_G(k)$ leaves the subspace $\mathbf{R}^n \subset \mathfrak{G}$ and the polynomial $\sum_{i=1}^n Z_i^2$ invariant. Hence, if $R(k)$ denotes the right translation $g \to gk$ on $G$,

$$(\tilde{\Delta})^{R(k)} = \sum_{i=1}^n ((\tilde{Z}_i)^{R(k)})^2 = \sum_{i=1}^n ((\mathrm{Ad}_G(k^{-1})\, Z_i)^{\sim})^2 = \sum_{i=1}^n \tilde{Z}_i^2$$

so $\tilde{\Delta}$ is invariant under $R(k)$. If $F \in C^\infty(\mathbf{R}^n)$ let $\tilde{F} \in C^\infty(G)$ be determined by $\tilde{F}(g) = F(g \cdot 0)$ for $g \in G$. Then (cf. [10], p. 392, equation (16))

$$\begin{aligned}[\tilde{\Delta}\, \tilde{F}]\,(g) &= \left\{\frac{\partial^2}{\partial t_1^2} + \dots + \frac{\partial^2}{\partial t_n^2}\; \tilde{F}(g \exp\,(t_1 Z_1 + \dots + t_n Z_n))\right\}_{t=0} \\ &= \left\{\frac{\partial^2}{\partial t_1^2} + \dots + \frac{\partial^2}{\partial t_n^2}\; F(g \cdot (t_1 Z_1 + \dots + t_n Z_n))\right\}_{t=0} \\ &= [\Delta F^{g^{-1}}]\,(0) = [\Delta F]\,(g \cdot 0)\end{aligned}$$

by (1) § 7, that is

$$\tilde{\Delta}\, \tilde{F} = (\Delta F)^{\sim}, \quad F \in C^\infty(\mathbf{R}^n). \tag{3}$$

Since

$$(M_p f)^{\sim} = \int_K (f^*)^{R(k)} dk$$

and $(\tilde{\Delta})^{R(k)} = \tilde{\Delta}$ it follows from (3) that

$$(\Delta\, M_p f)^{\sim} = \int_K (\tilde{\Delta} f^*)^{R(k)} dk$$

so

$$\begin{aligned}[\Delta\, M_p f]\,(g \cdot 0) &= \int_K \left\{\left(\frac{\partial^2}{\partial t_1^2} + \dots + \frac{\partial^2}{\partial t_n^2}\right) (f^*(gk \exp\,(t_1 Z_1 + \dots + t_n Z_n)))\right\}_{t=0} dk \\ &= \int_K \left\{\left(\frac{\partial^2}{\partial t_{p+1}^2} + \dots + \frac{\partial^2}{\partial t_n^2}\right) f(gk \exp\,(t_{p+1} Z_{p+1} + \dots + t_n Z_n) \cdot E_p^o)\right\}_{t=0} dk.\end{aligned}$$

This shows that

$$[\Delta M_p f](g\cdot 0)=\int_K [\Box_p f](gk\cdot E_p^o)\,dk=\int_K (\Box_p f)^*(gk)\,dk=[M_p\Box_p f](g\cdot 0)$$

proving (i). For (ii) let $V_q$ denote the $q$-plane through 0, parallel to $E_q$, and let $X_1, \dots, X_q, \dots, X_n$ be an orthogonal basis of $\mathbf{R}^n$ such that $X_i \in V_q$ $(1\leqslant i\leqslant q)$. The orthogonal projection of 0 onto $E_q$ has the form $s_{q+1}X_{q+1}+\dots+s_nX_n$ and

$$[L_q F](E_q)=\int F(t_1X_1+\dots+t_qX_q+\dots+s_nX_n)\,dt_1\dots dt_q$$

so

$$[\Box_q L_q F](E_q)=\left\{\frac{\partial^2}{\partial t_{q+1}^2}+\dots+\frac{\partial^2}{\partial t_n^2}(L_qF((t_{q+1}X_{q+1}+\dots+t_nX_n)\cdot E_q))\right\}_{t=0}$$

$$=\int\left(\frac{\partial^2}{\partial s_{q+1}^2}+\dots+\frac{\partial^2}{\partial s_n^2}\right)(F(t_1X_1+\dots+s_nX_n)\,dt_1\dots dt_q=\int_{E_q}[\Delta F](x)\,d\mu_q(x)$$

since $\partial^2F/\partial t_i^2$ $(1\leqslant i\leqslant q)$ gives no contribution. This proves (ii).

Let $\mathcal{S}^*(\mathbf{R}^n)$ be as in § 4 and let $\mathcal{L}_{p,n}$ be the subspace $L_p(\mathcal{S}^*(\mathbf{R}^n))$ of $C^\infty(\mathbf{G}(p,n))$.

Theorem 8.2. *Suppose $n$ odd. The transform $f\to\hat f$ is a linear one-to-one mapping of $\mathcal{L}(\mathbf{G}(p,n))$ onto $\mathcal{L}(\mathbf{G}(q,n))$ such that*

$$(\Box_p f)^\wedge=\Box_q\hat f$$
$$(\Box_p)^{\frac12(n-1)}(\hat f)^\vee=cf,\quad f\in\mathcal{L}(\mathbf{G}(p,n)),$$

*where $c$ is a constant $\neq 0$, independent of $f$.*

*Proof.* Let $r=(x_1^2+\dots+x_n^2)^{1/2}$ and $\lambda$ a complex number whose real part Re $\lambda$ is $>-n$. Then the function $r^\lambda$ is a tempered distribution on $\mathbf{R}^n$ and so is its Fourier transform, say $R_\lambda$. If $\varphi\in\mathcal{S}(\mathbf{R}^n)$ the convolution $R_\lambda*\varphi$ is a tempered distribution ([18], II, p. 102) whose Fourier transform is the product of the Fourier transforms of $\varphi$ and $R_\lambda$. If $\varphi\in\mathcal{S}^*(\mathbf{R}^n)$ then this product lies in $\mathcal{S}_0(\mathbf{R}^n)$ so the operator $\Lambda_\lambda$: $\varphi\to R_\lambda*\varphi$ maps the space $\mathcal{S}^*(\mathbf{R}^n)$ into itself. Also if $\lambda,\mu$ are complex numbers such that Re $\lambda$, Re $\mu$ and Re$(\lambda+\mu)$ all are $>-n$ then $\Lambda_{\lambda+\mu}=\Lambda_\lambda\Lambda_\mu$. In particular, $(\Lambda_2\varphi)^\sim=(2\pi)^nr^2\tilde\varphi=-(2\pi)^n(\Delta\varphi)^\sim$ so

$$\Lambda_2=-(2\pi)^n\Delta,\quad \Lambda_0=I.$$

We shall now verify that

$$M_dL_dF=\gamma_dR_{-d}*F,\quad F\in\mathcal{S}(\mathbf{R}^n),\ 0\leqslant d<n, \tag{4}$$

where $d$ is an integer and $\gamma_d$ is a constant $\neq 0$. For this let $d\omega_k$ be the surface element of the unit sphere in $\mathbf{R}^k$ and put $\Omega_k=\int d\omega_k$. Let $g\in G$ and $x=g\cdot 0$. If $d=0$, (4) is obvious so assume $0<d<n$. Then for a fixed $d$-plane $E_d$ through 0

$$[M_d L_d F](x) = \int_K dk \int_{E_d} F(gk\cdot z)\,dz = \int_{E_d} dz \int_K F(gk\cdot z)\,dk$$
$$= \int_0^\infty \Omega_d\, r^{d-1}\,dr \left\{\frac{1}{\Omega_n}\int_{|y|=1} F(x+ry)\,d\omega_{n-1}(y)\right\} = \frac{\Omega_d}{\Omega_n}\int F(y)\,|x-y|^{d-n}\,dy$$

and since $R_{-d}$ is a constant multiple of $r^{d-n}$ ([18], II, p. 113) (4) follows. As an immediate consequence of (4) we have

$$\Lambda_d M_d L_d \varphi = M_d L_d \Lambda_d \varphi = \gamma_d \varphi, \quad \varphi \in \mathcal{S}^*(\mathbf{R}^n), \quad (0 \leqslant d < n). \tag{5}$$

Now let $f \in \mathfrak{L}_{p,n}$. Then $f = L_p \varphi$ for $\varphi \in \mathcal{S}^*(\mathbf{R}^n)$ and $\hat{f} = L_q M_p f = L_q M_p L_p \varphi \in \mathfrak{L}_{q,n}$ since $M_p L_p \varphi \in \mathcal{S}^*(\mathbf{R}^n)$. If $\hat{f} = 0$ then $0 = M_q \hat{f} = M_q L_q M_p L_p \varphi = \Lambda_{-q-p}\varphi$ so $f = 0$. Similarly, if $F \in \mathfrak{L}_{q,n}$ then $F = L_q \Phi$ for $\Phi \in \mathcal{S}^*(\mathbf{R}^n)$ and by (5), $F = L_q M_p L_p \varphi$ for $\varphi \in \mathcal{S}^*(\mathbf{R}^n)$ so $F = (L_p\varphi)^\wedge$. This shows that $f \to \hat{f}$ is an isomorphism of $\mathfrak{L}_{p,n}$ onto $\mathfrak{L}_{q,n}$. Also, by Lemma 8.1,

$$(\Box_p f)^\wedge = L_q M_p \Box_p f = L_q \Delta M_p f = \Box_q L_q M_p f = \Box_q \hat{f}.$$

Since $p+q = n-1$ is even we have

$$\Lambda_p \Lambda_q = (\Lambda_2)^{\frac{1}{2}(n-1)} = ((-2\pi)^n)^{\frac{1}{2}(n-1)}\,\Delta^{\frac{1}{2}(n-1)} = c_n\,\Delta^{\frac{1}{2}(n-1)}$$

the last equation defining $c_n$. Let $f \in \mathfrak{L}_{p,n}$, $f = L_p\varphi$, $\varphi \in \mathcal{S}^*(\mathbf{R}^n)$. Then, using Lemma 8.1, and (5)

$$(\hat{f})^\vee = L_p M_q \hat{f} = L_p M_q L_q M_p L_p \varphi,$$
$$(\Box_p)^{\frac{1}{2}(n-1)}(\hat{f})^\vee = L_p\,\Delta^{\frac{1}{2}(n-1)} M_q \hat{f} = c_n^{-1} L_p\,\Lambda_p \Lambda_q M_q L_q M_p L_p \varphi$$
$$= c_n^{-1}\gamma_q L_p \Lambda_p M_p L_p \varphi = c_n^{-1}\gamma_q\gamma_p L_p\varphi = c_n^{-1}\gamma_p\gamma_q f.$$

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