

RIEMANN SURFACES AND THE THETA FUNCTION

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Introduction

The purpose of this paper is to present a clear exposition of certain theorems of Riemann, notably Theorem 8 below, which he obtained from his study of the θ function as a means of solving the Jacobi inversion problem. A perusal of Riemann's collected works, [4], shows that this was a topic of great interest to him. For this paper, one may consult [4], pp. 133–142, 212–224, 487–504, and the Supplement, pp. 1–59. Many mathematicians, in the half century after Riemann, tried to elucidate and justify his results. In this connection we may mention Christoffel, Noether, Weber, Rost, and Poincaré. Citations of the older literature may be found in the books, [2], [3], and [6].

Despite all these efforts, it is difficult for me to say whether or not complete proofs have been given to everything that has been claimed. In this paper, we hope to give correct proofs of some of these interesting results, along with some new theorems. Our method is essentially that of Riemann and his followers, although the language may be slightly more modern. The key to our method is consideration of the role of the base point, i.e., lower limit of the integrals of first kind, and its influence on the vector K of Riemann constants. The roles of the base point and K seem to have been overlooked by all, probably because of the statement of Riemann, [4], p. 133 and p. 213, that, under a suitable normalization, the vector K vanishes. Finally, having available the concept of an abstract Riemann surface gives one a distinct advantage over being tied down to a particular branched covering of the sphere.

In the first section, we prove the basic theorem concerning the zeros of certain “multiplicative functions”. On the whole, in this section, we try to conform with the

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notation of Conforto's book, [1], whose chapter on theta functions was very helpful. In the second section, we study the identical vanishing of the θ function and prove Theorem 8, the main theorem of Riemann. The third section contains a number of miscellaneous applications, and we conclude with a discussion of the hyperelliptic case, motivated by Riemann's remarks in the Supplement of [4], pp. 35–39.

We assume known most of the standard function theory on Riemann surfaces; the Riemann–Roch theorem, Abel's theorem, the Weierstrass gap theorem, the properties of Weierstrass points, and the structure of hyperelliptic surfaces. We write divisors multiplicatively and use the Riemann–Roch theorem as follows: For any divisor ζ , the dimension of the complex vector space of meromorphic functions on the surface which are multiples of ζ^{-1} , $r(\zeta^{-1})$, is given by $\text{degree}(\zeta) + i(\zeta) + 1 - g$. Here, $\text{degree}(\zeta)$ is the sum of the exponents of ζ , $i(\zeta)$ the dimension of the space of abelian differentials which are multiples of ζ , and g the genus of the surface. In particular, we use a consequence of the Riemann–Roch theorem, that given any point P on the surface S of genus g , one may choose a basis for the g dimensional space of differentials of the first kind $\varphi_1, \dots, \varphi_g$, such that φ_i has at P a zero of order $n_i - 1$, where n_1, \dots, n_g are g gaps at P . For details one may consult the book by Springer, [5; Chapter 10].

C^g denotes the space of g complex variables. A point $u \in C^g$ is considered a column vector, i.e. a g by 1 matrix. If A is a matrix, \tilde{A} denotes the transpose of A .

While preparing this paper, I was informed by Prof. D. C. Spencer that Theorem 8 below, Riemann's theorem on the vanishing of the θ function, was proved by A. Mayer in his Princeton thesis. Upon completion of this paper, I sent a copy to Prof. D. Mumford who communicated to me that he had found a proof of the abstract algebro-geometric formulation of Riemann's theorem, which is valid for arbitrary characteristic, not only characteristic zero.

Section I.

Let S be a compact Riemann surface of genus $g \geq 2$ and $a_j, b_j, 1 \leq j \leq g$, the $2g$ cycles of a canonical dissection of S . These cycles are $2g$ closed curves on S which begin and end at a common point and have the following intersection numbers:

$$a_j \times a_k = b_j \times b_k = 0, \quad a_j \times b_k = \delta_{jk},$$

the Kronecker delta. When S is cut along these cycles, one obtains a simply connected region S_0 with oriented boundary ∂S_0 , traversed in the positive direction as $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. The homology classes of these $2g$ cycles generate the first

homology group of S . Corresponding to these $2g$ cycles, there is uniquely determined a basis, $\varphi_1, \dots, \varphi_g$, of the complex g dimensional space of abelian differentials of the first kind on S , by the normalization, $\int_{a_k} \varphi_j = \delta_{jk} \pi i$, ($i = \sqrt{-1}$). Setting $t_{jk} = \int_{b_k} \varphi_j$, one obtains the $g \times 2g$ period matrix $\Omega = (\pi i I, T)$, where I is the $g \times g$ identity matrix and $T = (t_{jk})$ is a non-singular symmetric matrix with negative definite real part. It can be shown that the $2g$ columns of the period matrix are independent over the reals and generate a discrete abelian subgroup A of C^g . The Jacobian variety, $J(S)$, is the quotient group C^g/A , a compact abelian group. If u^1, u^2 are two points in C^g , then we write $u^1 \equiv u^2$ if they are congruent modulo A . Thus, $u^1 \equiv u^2$ if and only if $u^1 - u^2$ is a linear combination with integer coefficients of the columns of the period matrix, i.e., $u^1 - u^2 = \Omega m$, where m is a $2g \times 1$ vector of integers.

The significance of $J(S)$ is that there is a map $S \rightarrow J(S)$, defined as follows. Fix any point $B_0 \in S$ as base point and for each point $P \in S$ choose a path from B_0 to P .

Set

$$u_j(P) = \int_{B_0}^P \varphi_j, \quad 1 \leq j \leq g,$$

where the integral is taken along the chosen path and denote by $u(P) \in C^g$ the vector $(u_1(P), \dots, u_g(P))$. For another choice of path one may obtain a different vector $u(P)$, but it is clear that all values of $u(P)$ are congruent modulo A , hence determine in a well-defined manner an element of $J(S)$. This gives then a map of S into $J(S)$. For convenience we shall denote this map simply by $P \rightarrow u(P)$, where it is understood that $u(P)$ is any representative in C^g of the point of $J(S)$ into which P is mapped.

Of course, the map $S \xrightarrow{u} J(S)$ depends in a vital way upon the choice of the point B_0 and this will be discussed later. The map u may be extended to map the group of divisors of S into $J(S)$ by defining for any divisor $\zeta = P_1^{n_1} \dots P_k^{n_k}$,

$$u(\zeta) = \sum_{j=1}^k n_j u(P_j).$$

The *degree* of ζ is the sum of the exponents;

$$\deg(\zeta) = \sum_{j=1}^k n_j.$$

One says that two divisors are *equivalent*, $\zeta_1 \sim \zeta_2$, if the quotient $\zeta_1 \zeta_2^{-1}$ is the divisor of a function. Abel's theorem states that

$$\zeta_1 \sim \zeta_2 \text{ if and only if } \deg(\zeta_1) = \deg(\zeta_2) \text{ and } u(\zeta_1) \equiv u(\zeta_2).$$

Since there is no function on S with only one pole, $u(P) \equiv u(Q)$ if $P \neq Q$ are two points of S , so that $S \xrightarrow{u} J(S)$ is a one-one imbedding of S in $J(S)$. It is also non-singular, for the differential of the mapping at $P \in S$ is simply $du = (du_1(P), \dots, du_g(P))$, which has maximal rank, i.e. one; for, $du_j(P)$ is—in a suitable local parameter—only $\varphi_j(P)$, and it is a well-known consequence of the Riemann–Roch theorem that not all differentials of the first kind vanish at P .

Let e_k be the k th column of the $2g$ by $2g$ identity matrix; then Ωe_k is the k th column of the period matrix. We wish to consider functions $f(u)$, holomorphic in all of C^g , with the following periodicity property:

$$f(u + \Omega e_k) = \exp [2\pi i \tilde{e}_k (\tilde{\Lambda} u + \gamma)] f(u), \quad (1)$$

where $1 \leq k \leq 2g$, Λ , γ are matrices of complex numbers, of respective size g by $2g$ and $2g$ by 1 . The equality $f(u + \Omega e_k + \Omega e_n) = f(u + \Omega e_n + \Omega e_k)$, implies (cf. [1], p. 57) the relation

$$\tilde{\Omega} \Lambda - \tilde{\Lambda} \Omega = N, \quad (2)$$

where N is a $2g$ by $2g$ skew symmetric matrix of integers. N is called the characteristic matrix of f .

Such functions are not well defined on $J(S)$ but are “multiplicative functions” there. Nevertheless, since the multipliers are exponentials which can never vanish, it is clear that if $u^1 \equiv u^2$, then $f(u^1) = 0$ if and only if $f(u^2) = 0$. Hence, one may say in a meaningful way that a point of $J(S)$ is, or is not, a zero of f . By means of the map $u: S \rightarrow J(S)$, $f(u(P))$ is a multiplicative holomorphic function on S . In particular, choosing a definite value for $u(B_0)$, which is, of course, always $\equiv 0$, $f(u(P))$ is a single valued holomorphic function in the simply connected region S_0 with well-defined values on ∂S_0 . When continued over ∂S_0 , a new single valued branch of $f(u(P))$ on S_0 is obtained, which has the same zeros as the first branch. Whenever we write $f(u(P))$, we assume some such single valued branch chosen, but which particular one is irrelevant to our present purpose.

It is possible that $f(u(P)) \equiv 0$ on S . Here we use “ $\equiv 0$ ” to mean that the function is identically zero for all P , not to be confused with “ \equiv ” meaning congruence modulo period vectors. In any given context only one meaning will be possible. Now, if $f(u(P)) \not\equiv 0$, then, by the compactness of S , it has only a finite number of zeros on S . We may assume that these do not lie on ∂S_0 , for the canonical cycles may be deformed slightly into homologous ones, without affecting any of the canonical

properties or the period matrix. Let us determine by the residue theorem the number, $N(f)$, of zeros of f on S .

$$N(f) = \frac{1}{2\pi i} \int_{\partial S_0} \frac{df}{f}. \quad (3)$$

Let f^+ denote the value taken by f at a point of ∂S_0 lying on a_k or b_k , while f^- denotes the value taken at the identified point on a_k^{-1} or b_k^{-1} respectively. We also write u^+, u^- with the same meaning. Then

$$N(f) = \frac{1}{2\pi i} \int_{\partial S_0} \frac{df}{f} = \frac{1}{2\pi i} \sum_{k=1}^g \int_{a_k} + \int_{b_k} \left(\frac{df^+}{f^+} - \frac{df^-}{f^-} \right).$$

We observe that if P is a point on a_k ,

$$u_j^-(P) = u_j^+(P) + \int_{b_k} \varphi_j = u_j^+(P) + t_{jk}, \quad (4)$$

while for P on b_k ,

$$u_j^+(P) = u_j^-(P) + \int_{a_k} \varphi_j = u_j^-(P) + \pi i \delta_{jk}. \quad (5)$$

Thus by (1) above, we have that on a_k ,

$$f^- = f(u^+ + \Omega e_{g+k}) = \exp [2\pi i \tilde{e}_{g+k} (\tilde{\Lambda} u + \gamma)] f^+, \quad (6)$$

and

$$df^- = \exp [2\pi i \tilde{e}_{g+k} (\tilde{\Lambda} u + \gamma)] (df^+ + f^+ 2\pi i \tilde{e}_{g+k} \tilde{\Lambda} du). \quad (7)$$

Hence,
$$\frac{1}{2\pi i} \sum_{k=1}^g \int_{a_k} \left(\frac{df^+}{f^+} - \frac{df^-}{f^-} \right) = \frac{-1}{2\pi i} \sum_{k=1}^g \int_{a_k} 2\pi i \tilde{e}_{g+k} \tilde{\Lambda} du = - \sum_{k=1}^g \tilde{e}_{g+k} \tilde{\Lambda} \Omega e_k.$$

A similar calculation, except that now it is more convenient to express f^+ in terms of f^- , shows that

$$\frac{1}{2\pi i} \sum_{k=1}^g \int_{b_k} \left(\frac{df^+}{f^+} - \frac{df^-}{f^-} \right) = \sum_{k=1}^g \tilde{e}_k \tilde{\Lambda} e_{g+k}.$$

Since a 1 by 1 matrix is symmetric, $\tilde{e}_k \tilde{\Lambda} \Omega e_{g+k} = \tilde{e}_{g+k} \tilde{\Omega} \Lambda e_k$. Thus, we have

$$N(f) = \sum_{k=1}^g \tilde{e}_{k+1} (\tilde{\Omega} \Lambda - \tilde{\Lambda} \Omega) e_k = \sum_{k=1}^g \tilde{e}_{g+k} N e_k. \quad (8)$$

If we write the skew symmetric N as $\begin{pmatrix} N_1 & -\tilde{N}_2 \\ N_2 & N_3 \end{pmatrix}$ then we have simply

$$N(f) = \text{trace } N_2. \quad (9)$$

Assuming again that $f(u(P)) \neq 0$ on S , let $P_1, \dots, P_{N(f)}$ be the zeros of f on S . These are not necessarily distinct but each is repeated according to its multiplicity. Again, by the residue theorem, we have, for a fixed index h , $1 \leq h \leq g$, that

$$\sum_{j=1}^N u_h(P_j) = \frac{1}{2\pi i} \int_{\partial S_0} u_h \frac{df}{f}, \quad (10)$$

which is, in our previous notation,

$$\frac{1}{2\pi i} \sum_{k=1}^g \int_{a_k} + \int_{b_k} \left(u_h^+ \frac{df^+}{f^+} - u_h^- \frac{df^-}{f^-} \right).$$

Using (3) to (6), we obtain,

$$\frac{1}{2\pi i} \int_{a_k} \left(u_h^+ \frac{df^+}{f^+} - u_h^- \frac{df^-}{f^-} \right) = -\frac{t_{hk}}{2\pi i} \int_{a_k} \frac{df^+}{f^+} - t_{hk} \tilde{e}_{\sigma+k} \tilde{\Lambda} \Omega e_k - \int_{a_k} u_h^+ \tilde{e}_{\sigma+k} \tilde{\Lambda} du. \quad (11)$$

By assumption, f^+ is different from zero along a_k and a single-valued branch of $\log f^+$ is defined in a neighborhood of a_k . If Q_0^k, Q_1^k are the (identified) initial and final points of a_k , respectively, then

$$\int_{a_k} \frac{df^+}{f^+} = \log f^+(u(Q_1^k)) - \log f^+(u(Q_0^k)).$$

But $f^+(u(Q_1^k)) = f^+(u(Q_0^k) + \Omega e_k) = \exp[2\pi i \tilde{e}_k (\tilde{\Lambda} u(Q_0^k) + \gamma)] f^+(u(Q_0^k))$; so that

$$\int_{a_k} \frac{df^+}{f^+} = 2\pi i \tilde{e}_k (\tilde{\Lambda} u(Q_0^k) + \gamma) + 2\pi i \nu_k, \quad (12)$$

where ν_k is an integer, independent of h .

In a similar way we find that

$$\begin{aligned} \frac{1}{2\pi i} \int_{b_k} \left(u_h^+ \frac{df^+}{f^+} - u_h^- \frac{df^-}{f^-} \right) &= \frac{1}{2\pi i} \int_{b_k} u_h^+ \left(\frac{df^-}{f^-} + 2\pi i \tilde{e}_k \tilde{\Lambda} du \right) - (u_h^+ - \pi i \delta_{hk}) \frac{df^-}{f^-} \\ &= \int_{b_k} u_h^+ \tilde{e}_k \tilde{\Lambda} du + \frac{1}{2} \delta_{hk} \int_{b_k} \frac{df^-}{f^-}. \end{aligned} \quad (13)$$

Denote the initial and final end points of b_h as \bar{Q}_0^h, \bar{Q}_1^h , respectively. The same argument which led to (12) gives,

$$\int_{b_h} \frac{df^-}{f^-} = 2\pi i \tilde{e}_{\sigma+h} (\tilde{\Lambda} u(\bar{Q}_0^h) + \gamma) + 2\pi i \mu_h, \quad (14)$$

where μ_h is an integer.

Summing (11) and (13) over k from 1 to g and adding, taking into account (12) and (14), we have,

$$\begin{aligned} \sum_{j=1}^{N(f)} u_h(P_j) &= - \sum_{k=1}^g t_{hk} (\tilde{e}_k (\tilde{\Lambda} u(Q_0^k) + \gamma) + \nu_k + \tilde{e}_{g+k} \tilde{\Lambda} \Omega e_k) - \sum_{k=1}^g \int_{a_k} u_h^+ \tilde{e}_{g+k} \tilde{\Lambda} du \\ &\quad + \sum_{k=1}^g \int_{b_k} u_h^+ \tilde{e}_k \tilde{\Lambda} du + \pi i \tilde{e}_{g+h} (\tilde{\Lambda} u(\bar{Q}_0^h) + \gamma) + \pi i \mu_h. \end{aligned} \quad (15)$$

This formula is not very useful in its full generality. Let us take $\Lambda = \left(O, -\frac{n}{\pi i} I \right)$, where O, I are, respectively, the g by g zero and identity matrix, and n is a positive integer. Let $G, H \in C^g$ be arbitrary; take

$$\gamma_j = G_j, \text{ for } 1 \leq j \leq g, \text{ and } \gamma_j = -\frac{n}{2\pi i} t_{jj} - H_j, \text{ for } g+1 \leq j \leq 2g.$$

Then $N = n \begin{pmatrix} O & -I \\ I & O \end{pmatrix}$ and $N(f) = ng$. (15) now reduces to

$$\sum_{j=1}^{N(f)} u_h(P_j) = - \sum_{k=1}^g t_{hk} (G_k + \nu_k + n) + \frac{n}{\pi i} \sum_{k=1}^g \int_{a_k} u_h^+ du_k - n - (u_h(\bar{Q}_0^h) + \frac{1}{2} t_{hh}) - \pi i H_h + \pi i \mu_h.$$

Define
$$K_h = -\frac{1}{\pi i} \sum_{k=1}^g \int_{a_k} u_h^+ du_k + u_h(\bar{Q}_0^h) + \frac{1}{2} t_{hh},$$

and let $(n), \nu, \mu, K, G, H$, be g -rowed column vectors whose k th entry is $n, \nu_k, \mu_k, K_k, G_k, H_k$, respectively. Thus the divisor $\zeta(f) = P_1 \dots P_{N(f)}$ satisfies

$$u(\zeta(f)) = T(-n) - G - \nu - nK - \pi i H + \pi i \mu. \quad (16)$$

But $-T(n) - T\nu + \pi i \mu \equiv 0$, and (16) can be written as

$$u(\zeta(f)) \equiv -TG - nK - \pi i H. \quad (16')$$

K is called the vector of Riemann constants. It is independent of G and H but depends, along with u , on the base point B_0 . This dependence will be clarified later. We summarize our results in the following theorem.

THEOREM 1. *Let $f(u)$ be an entire function in C^g satisfying, for $1 \leq k \leq 2g$,*

$$f(u + \Omega e_k) = \exp [2\pi i \tilde{e}_k (\tilde{\Lambda} u + \gamma)] f(u);$$

$f(u)$ is a multiplicative function on $J(S)$. By means of the map $u: S \rightarrow J(S)$, $f(u(P))$

is a multiplicative function on S . Either $f(u(P))$ is identically zero on S or it has a divisor of $N(f)$ zeros, $\zeta = P_1 \dots P_{N(f)}$. $N(f) = \text{trace } N_2$ where

$$N = \begin{pmatrix} N_1 & -\tilde{N}_2 \\ N_2 & N_3 \end{pmatrix} = \tilde{\Omega} \Lambda - \tilde{\Lambda} \Omega.$$

In the case that $\Lambda = \left(O, -\frac{n}{\pi i} I \right)$, $\gamma = \begin{pmatrix} G \\ H \end{pmatrix}$, where $H'_j = -\frac{n}{2\pi i} t_{jj} - H_j$ and G, H are arbitrary points in C^g , the characteristic matrix

$$N = \begin{pmatrix} O & -nI \\ nI & O \end{pmatrix} \text{ and } N(f) = ng.$$

$\zeta(f)$ satisfies the congruence $u(\zeta(f)) + nK \equiv -\Omega \begin{pmatrix} H \\ G \end{pmatrix} = -TG - \pi i H$.

Functions having the particular form given above are called n th order theta functions, with characteristics G, H . Details concerning their construction and the number of linearly independent ones, over the complex field, may be found in [1], pp. 91-104. The first order theta function with characteristics G, H is uniquely determined up to a constant multiple. It is

$$\theta \begin{bmatrix} G \\ H \end{bmatrix} (u) = \exp(\tilde{G}TG + 2\tilde{G}u + 2\pi i \tilde{G}H) \theta(u + TG + \pi i H),$$

where $\theta(u) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (u) = \sum_m \exp(2\tilde{m}u + \tilde{m}Tm)$,

m running over all g rowed column vectors with integer entries. A crucial property of $\theta(u)$ is that it is even; $\theta(u) = \theta(-u)$, as is apparent from its definition. In fact, it is even in each variable separately.

Section II

In this section we investigate closely only a particular first order θ function; namely, let $e \in C^g$ be some given point and consider the function $\theta(u - e)$. This is the first order θ with $G = 0$, $H = -(\pi i)^{-1}e$. In this case, Theorem 1 tells us that either $\theta(u(P) - e)$ is identically zero on S , or it has a divisor of g zeros ζ , such that $e \equiv u(\zeta) + K$. Before proceeding we introduce some convenient notation. Let S^n , $n \geq 1$, denote the cartesian product of S with itself n times and D^n the symmetric product, i.e., the quotient space of S^n under the symmetric group of permutations. Briefly, S^n

consists of ordered n -tuples and D^n of non-ordered n -tuples. D^n is simply the set of integral divisors of degree n . A neighborhood of $\zeta = P_1 \dots P_n \in D^n$ shall mean the set of all divisors $Q_1 \dots Q_n$, where Q_j belongs to some parametric disk on S about P_j . Recall that $u: D^n \rightarrow J(S)$. Denote by $W^n \subset J(S)$ the image of $D^n: u(D^n) = W^n$. We agree to set $W^0 = 0 \in J(S)$ and $D^0 =$ the unit divisor, $\prod_{P \in S} P^0$. Since $J(S)$ is an abelian group, sets of the form $X + Y$ are defined. That is, if $X, Y \subset J(S)$, $X + Y$ denotes the set of elements of the form $x + y$, $x \in X$, $y \in Y$, and $-X$ the set of elements of the form $-x$, $x \in X$. In particular, $X + y$ is the set of elements of the form $x + y$, $x \in X$.

THEOREM 2. $\theta(W^{g-1} + K) = 0$; i.e., $\theta(u) = 0$ if $u \in W^{g-1} + K$.

Proof. It is well known that there exists a divisor $\zeta \in D^g$ consisting of distinct points on S such that $i(\zeta) = 0$, and that if ζ' belongs to a sufficiently small neighborhood of ζ it also has distinct points and $i(\zeta') = 0$. Let $\zeta = P_1 \dots P_g$ be such and set $e \equiv u(\zeta) + K$. Now, if $\theta(u(P) - e) \equiv 0$, then for $1 \leq j \leq g$,

$$0 = \theta(u(P_j) - e) = \theta(u(P_1 \dots \hat{P}_j \dots P_g) + K),$$

where \hat{P}_j denotes deletion of the point P_j . In the last step we used the evenness of $\theta(u)$. If $\theta(u(P) - e) \not\equiv 0$, then by Theorem 1 it has a divisor of g zeros ω such that $e \equiv u(\omega) + K$. By the construction of e we have then $u(\zeta) = u(\omega)$, which implies $\zeta = \omega$ (by our assumption that $i(\zeta) = 0$ and the Riemann-Roch and Abel theorems). Thus, again, we obtain $\theta(u(P_1 \dots \hat{P}_j \dots P_g) + K) = 0$. By our remarks at the start of the proof it is now clear that $\theta(u(Q_1 \dots Q_{g-1}) + K)$ vanishes identically on a full neighborhood D^{g-1} , hence is identically zero on D^{g-1} , which completes the proof.

COROLLARY: $\theta(W^r + K) = 0$, $1 \leq r \leq g-1$ and $\theta(K) = 0$. Thus, the vector of Riemann constants is always a zero of $\theta(u)$.

Proof. Although $D^r \not\subset D^t$ for $t > r$ it is clear that $W^r \subset W^t$; for, if $\zeta \in D^r$, $u(\zeta) \in W^r$, then $B_0^{t-r} \zeta \in D^t$ and $u(B_0^{t-r} \zeta) \equiv u(\zeta)$, so $u(\zeta) \in W^t$. This gives the first statement. To see that $\theta(K) = 0$, we need only observe that $0 \equiv u(B_0^r) \in W^r$, for every integer r .

THEOREM 3. Let ζ , $\omega \in D^g$ and $e \in C^g$.

- (a) If $\theta(u(P) - e) \not\equiv 0$ and has divisor of zeros ζ , then $i(\zeta) = 0$.
- (b) If $e \equiv u(\omega) + K$ and $\theta(u(P) - e) \not\equiv 0$, then ω is the divisor of zeros.
- (c) If $e \equiv u(\omega) + K$ and $i(\omega) > 0$, then $\theta(u(P) - e) \equiv 0$.

Proof. (a) Let $Q \in S$ be such that $\theta(u(Q) - e) \neq 0$. If $i(\zeta) > 0$, there is a $\zeta' \in D^g$ such that Q appears in ζ' and $u(\zeta') \equiv u(\zeta)$. Then by Theorem 1, $e \equiv u(\zeta') + K$ and, by Theorem 2, $\theta(u(Q) - e) = 0$, which contradicts our choice of Q .

(b) For, if ζ is the divisor of zeros, $e \equiv u(\zeta) + K \equiv u(\omega) + K$. But, by (a), $i(\zeta) = 0$, so that $\zeta = \omega$.

(c) Immediate from (a) and (b).

We now wish to analyze the dependence of the map u and the Riemann constants K on the base point B_0 . For clarification, if $B_0, B_1, \dots, B_j, \dots$ is a set of points on S , we shall denote by $u^0, u^1, \dots, u^j, \dots$, and $K^0, K^1, \dots, K^j, \dots$, the u and K obtained with respect to base point $B_0, B_1, \dots, B_j, \dots$. Of course, everything we have done up to this point has been true for any choice of B_0 , hence we have not written u^0, K^0 until now. Also note that the sets $W^r, r \geq 1$, depend on the base point B_0 , and when necessary we may write $W_{B_0}^r, W_{B_1}^r, \dots, W_{B_j}^r, \dots$. Our first result is

THEOREM 4. (a) $u^1(P) \equiv u^0(P) - u^0(B_1)$ for all $P \in S$.
 (b) $K^1 \equiv K^0 + u^0(B_1^{g-1})$.

Proof. (a) is trivial, for $\int_{B_1}^P \equiv \int_{B_0}^P - \int_{B_0}^{B_1}$.

(b) Since $\theta(u)$ is not identically zero in C^g , there is an $e \in C^g$ such that $\theta(e) \neq 0$. Then $\theta(u^1(P) - e) \neq 0$, for $P = B_1$ is not a zero. Let ζ be the divisor of zeros; $e \equiv u^1(\zeta) + K^1$. By (a), we see that $\theta(u^0(P) - u^0(B_1) - e)$ has the same divisor of zeros ζ ; hence $u^0(B_1) + e \equiv u^0(\zeta) + K^0$. Comparing these two congruences for e gives (b).

Actually (b) may be seen directly from the definition of K . An interesting consequence of this theorem is that $K^1 \equiv K^0$ if and only if $u^0(B_1^{g-1}) \equiv 0 \equiv u^0(B_0^{g-1})$. That is, B_0 and B_1 are Weierstrass points on S such that there is a function with a pole of order $g-1$ at B_0 and a zero of order $g-1$ at B_1 . This is, in fact, the case for any two Weierstrass points on a hyperelliptic surface of odd genus, but I do not know when else this occurs.

We now investigate the dependence of the identical vanishing of $\theta(u(P) - e)$ on the base point. As we have already remarked, if $\theta(e) \neq 0$, then $\theta(u(P) - e) \neq 0$ for every base point; for $P =$ base point, is not a zero. Thus, assume that $\theta(e) = 0$, B_0, B_1 are two arbitrary base points, and $\theta(u^j(P) - e) \neq 0$ for $j = 0, 1$. Let ζ_0, ζ_1 be the respective zero divisors. Clearly, B_0, B_1 , respectively, are zeros so that $\zeta_0 = B_0 \omega_0, \zeta_1 = B_1 \omega_1$, where $\omega_0, \omega_1 \in D^{g-1}$. By our previous results we have:

$$e \equiv u^0(\zeta_0) + K^0 \equiv u^0(\omega_0) + K^0$$

and
$$e \equiv u^1(\zeta_1) + K^1 \equiv u^0(\omega_1) - u^0(B_1^{g-1}) + K^0 + u^0(B_1^{g-1}) \equiv u^0(\omega_1) + K^0.$$

Thus, $u^0(\omega_0) \equiv u^0(\omega_1)$, $u^0(\zeta_0) \equiv u^0(B_0 \omega_1)$, and, by Theorem 3, we infer that $\zeta_0 = B_0 \omega_1$, hence $\omega_0 = \omega_1$. Since $\omega_0 \in D^{g-1}$, there is a differential φ with divisor $\omega_0 \gamma$, where $\gamma \in D^{g-1}$. γ is uniquely determined, for by Theorem 3, $i(B_0 \omega_0) = 0$, which implies $i(\omega_0) = 1$. Now, let B_2 be a point of γ and suppose $\theta(u^2(P) - e) \not\equiv 0$. Then it has divisor of zeros ζ_2 which, by our above argument (with B_2 in place of B_1), must be $\zeta_2 = B_2 \omega_0$ and, by Theorem 3, $i(B_2 \omega_0) = 0$. But φ is, by construction, a multiple of $B_2 \omega_0$, so that $i(B_2 \omega_0) = 1$, a contradiction. Thus, for each point B_2 of γ , $\theta(u^2(P) - e) \equiv 0$.

We now show that—with the same conditions on e , B_0 and B_1 —there are at most $g-1$ distinct points $B \in S$ such that $\theta(u^B(P) - e) \equiv 0$ (where u^B means u with base point B). For, by hypothesis, $\theta(u^0(P_0) - e) \not\equiv 0$ for any point $P_0 \in S$ not appearing in $B_0 \omega_0$. Since $u^0(B_0) \equiv 0$, we have that $\theta(u^0(P_0) - u^0(B_0) - e) \not\equiv 0$ so that, as a function of $B \in S$, with P_0 fixed, $\theta(u^0(P_0) - u^0(B) - e) \not\equiv 0$. This has then a divisor of zeros β satisfying $u^0(P_0) - e \equiv u^0(\beta) + K^0$. Now, if B_3 is not one of the g points of β , we have $\theta(u^0(P_0) - u^0(B_3) - e) \not\equiv 0$, i.e., $\theta(u^3(P_0) - e) \not\equiv 0$ and so $\theta(u^3(P) - e) \not\equiv 0$, as a function of P . Thus, if β contains fewer than g distinct points, our assertion is proved. Otherwise, let $\beta = Q_1, \dots, Q_g$ consist of g distinct points, such that for each j , $1 \leq j \leq g$, $\theta(u^{Q_j}(P) - e) \equiv 0$. Let $P_1 \in S$, $P_1 \neq P_0$, be a second point not in $B_0 \omega_0$. Carrying through as before, we obtain a congruence $u^0(P_1) - e \equiv u^0(\delta) + K^0$, where δ is the divisor of zeros of $\theta(u^0(P_1) - u^0(B) - e) \equiv 0$, and $\theta(u^3(P) - e) \equiv 0$, for every point B_3 not in δ . If our assumption on β was correct, then, necessarily, $\beta = \delta$, which implies $u^0(P_1) \equiv u^0(P_0)$, which is absurd. Hence, $\beta \neq \delta$, and our assertion is true. We summarize the above results as

THEOREM 5. *Let $e \in C^g$ satisfy $\theta(e) = 0$. Either*

- (a) $\theta(u^0(P) - e) \equiv 0$ for every choice of base point $B_0 \in S$, or
- (b) $\theta(u^{Q_j}(P) - e) \equiv 0$ for at most $g-1$ distinct points Q_1, \dots, Q_{g-1} .

In case (b), there is uniquely determined a divisor $\omega_0 \in D^{g-1}$, $i(\omega_0) = 1$, such that if $B \in S$ is not one of the points Q_1, \dots, Q_{g-1} , then $\theta(u^B(P) - e) \not\equiv 0$, and has divisor of zeros $\zeta = B\omega_0$. There is a uniquely determined differential φ of divisor $\omega_0 \gamma$ and the points of γ are included in the points Q_1, \dots, Q_{g-1} .

Recalling that the θ function is even, and by (a) of Theorem 4, we observe easily that the following four statements are equivalent:

- (1) $\theta(u^0(P) - e) \equiv 0$, for every base point $B_0 \in S$.
- (2) $\theta(u^0(P) - u^0(Q) - e) \equiv 0$, (as a function of P and Q) for every $B_0 \in S$.
- (3) $\theta(u^0(P) - u^0(Q) - e) \equiv 0$, for a particular $B_0 \in S$.
- (4) $\theta(u^0(P) + e) \equiv 0$, for every $B_0 \in S$.

Suppose that (1) above holds. Differentiating $\theta(u^0(P) - e) \equiv 0$ with respect to a local parameter at B_0 and setting $P = B_0$ gives,

$$\sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (-e) \varphi_j(B_0) = 0 \quad (\varphi_j = du_j).$$

Since by hypothesis this holds for every point $B_0 \in S$, we see that the differential

$$\varphi = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (-e) \varphi_j$$

is identically zero. The linear independence of the φ_j implies then

$$\frac{\partial \theta}{\partial u_j} (-e) = 0 \quad \text{for } 1 \leq j \leq g.$$

On the other hand, suppose that for a given $B_0 \in S$, $\theta(u^0(P) - e) \not\equiv 0$. By Theorem 5, we may assume B_0 different from the $g-1$ points in ω_0 . Then $\theta(u^0(P) - e) \not\equiv 0$ has divisor of zeros $B_0 \omega_0$ and a simple zero at B_0 . Thus, the derivative of $\theta(u^0(P) - e)$ —with respect to a local parameter at B_0 —does not vanish at B_0 . In other words,

$$\sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (-e) \varphi_j(B_0) \neq 0.$$

Thus we have proved

THEOREM 6. *Let $e \in C^g$ satisfy $\theta(e) = 0$. Then $\theta(u^0(P) - e) \equiv 0$ for every base point $B_0 \in S$ if and only if*

$$\frac{\partial \theta}{\partial u_j} (-e) = 0 \quad \text{for } 1 \leq j \leq g.$$

The four equivalent statements given above and Theorem 6 show that

$$\frac{\partial \theta}{\partial u_j} (-e) = 0 \quad \text{for } 1 \leq j \leq g \quad \text{if and only if} \quad \frac{\partial \theta}{\partial u_j} (e) = 0 \quad \text{for } 1 \leq j \leq g.$$

This fact, however, follows directly from the evenness of the θ function.

Theorem 6 is a special case of the more general theorem of Riemann presented below. To prove the general theorem we shall follow the exposition of Krazer, essentially, filling in certain gaps where necessary.

So far, we have shown that if $\theta(u^0(P) - e) \neq 0$ for a base point $B_0 \in S$, then there is a divisor of zeros $\zeta \in D^g$ such that $e \equiv u^0(\zeta) + K^0$. Thus, for points of the form $d = e - K^0 \in C^g$ we have solved the Jacobi inversion problem, i.e., found a divisor $\zeta \in D^g$ such that $u^0(\zeta) \equiv d$. That this can be done for any point $d \in C^g$ —i.e. $u^0: D^g \rightarrow J(S)$ is onto—can be proved quite easily without the θ function apparatus. ([5], p. 284). As we shall see though, the essential feature of Riemann's solution via the θ function, is that one can describe precisely the pre-image in D^g of a point $e \in C^g$ by means of the behavior of the θ function at e .

Let us first observe the following. $\theta(W^g + K) \neq 0$, i.e. θ does not vanish identically on the set $W^g + K \subset J(S)$. (Note that we have suppressed mention of the base point $B_0 \in S$ in the notation; for, until further notice, we shall assume B_0 fixed and it will not be varied.) Indeed, if $\zeta = P_1 \dots P_g \in D^g$ consists of g distinct points then $i(\zeta) = 0$ is equivalent to the statement $\det(\varphi_i(P_j)) \neq 0$. This determinant is the Jacobian of the map $u: D^g \rightarrow J(S)$, whose non-vanishing at ζ implies that u is a local homeomorphism. Thus, W^g contains an open set of $J(S)$ and since $\theta(u)$ is not identically zero on $J(S)$, it cannot vanish on W^g . Similarly, $\theta(u)$ does not vanish identically on $W^g + e$, for any $e \in C^g$.

Suppose now that $\theta(e) = 0$. Then there is an integer s such that $\theta(W^r - W^r - e) = 0$, for $0 \leq r \leq s$, while $\theta(W^s - W^s - e) \neq 0$. By our previous remarks $1 \leq s \leq g - 1$. Thus, by definition, there are $\omega'_0, \sigma_0 \in D^s$ such that $\theta(u(\omega_0) - u(\sigma_0)) \neq 0$. We may assume that ω'_0, σ_0 consist of $2s$ distinct points for, by continuity, if Q appears twice, we may move one occurrence to a neighboring Q' , still keeping the value of θ away from zero. Let $\omega'_0 = P_0 \omega_0, \omega_0 \in D^{s-1}$; then $\theta(u(P) + u(\omega_0) - u(\sigma_0) - e) \neq 0$, and has a divisor of zeros $\zeta_0 \in D^g$. But, $P =$ a point of σ_0 is a zero, for then $u(P) + u(\omega_0) - u(\sigma_0) - e$ is in $W^{s-1} - W^{s-1} - e$, where θ vanishes. Thus, $\zeta_0 = \sigma_0 \beta_0, \beta_0 \in D^{g-s}$, and we have the congruence $-u(\omega_0) + u(\sigma_0) + e \equiv u(\sigma_0 \beta_0) + K$, or $e \equiv u(\omega_0 \beta_0) + K$, where $\omega_0 \beta_0 \in D^{g-1}$. This proves the following extension of Theorem 2.

THEOREM 7. $W^{g-1} + K$ is the complete set of zeros of θ on $J(S)$.

Again, by a continuity argument, if $\omega \in D^{s-1}$ is sufficiently near ω_0 , $\theta(u(P) + u(\omega) - u(\sigma_0) - e) \neq 0$, and for a suitable $\beta \in D^{g-s}$, $e \equiv u(\omega\beta) + K$. To see the significance of this we must pause for two lemmas.

LEMMA 1. Let X be a topological space, F a field, f_1, \dots, f_N functions on X to F . Assume that no non-trivial linear combination $f = \sum_{i=1}^N \lambda_i f_i, \lambda_i \in F, \lambda_i$ not all zero, vanishes on an open set of X . Then, given any N non-empty open sets V_1, \dots, V_N of X , there are points $x_j \in V_j$ such that the determinant of $(f_i(x_j))$ is not zero.

This is a simple exercise in linear algebra and the proof may be omitted. In particular we have the following corollary: If X is a set, and f_1, \dots, f_N are functions on X to F which are linearly independent, then there are points $x_1, \dots, x_N \in X$ such that $\det (f_i(x_j)) \neq 0$. This follows from the lemma by topologizing X indiscretely, only the empty set and X are open, and taking all $V_j = X$.

LEMMA 2. *Let $\omega_0 \in D^m$, $\omega_0 = \zeta_0 \sigma_0$, $\zeta_0 \in D^s$, $\sigma_0 \in D^{m-s}$, $0 \leq s \leq m$. Suppose there is a neighborhood V of ζ_0 such that for any $\zeta \in V$ there is a $\sigma \in D^{m-s}$ with $u(\omega_0) \equiv u(\zeta\sigma)$. Then $i(\omega_0) \geq g + s - m$.*

Proof. Delete ζ_0 from V and select a smaller open set $V' \subset V - \zeta_0$ such that $\zeta \in V'$ consists of distinct points which do not appear in σ_0 . Also, we may take V' to be of the form $V_1 \times \dots \times V_s$, where each V_j is a disk on S . By hypothesis, for any $\zeta \in V'$, there is a σ such that $u(\zeta\sigma) \equiv u(\omega_0)$. By Abel's theorem, there is then a function f on S , with divisor $(f) = \zeta\sigma/\omega_0$. By our choice of V' , f has at least s zeros at ζ , which are not cancelled by any of the points of ω_0 . Now $r(\omega_0^{-1}) = m + i(\omega_0) + 1 - g$, and the space of functions which are multiples of ω_0^{-1} has a basis of $N + 1$ linearly independent functions f_1, \dots, f_{N+1} , where $N = m + i(\omega_0) - g$. By Lemma 1, if $s \geq N + 1$, we may select points $P_j \in V_j$, $1 \leq j \leq N + 1$ such that $\det (f_i(P_j)) \neq 0$. This means that no (non-trivial) linear combination, f , of the functions f_1, \dots, f_{N+1} vanishes at the points P_1, \dots, P_{N+1} . But P_1, \dots, P_{N+1} may be completed to a divisor $\zeta \in V'$ and, as constructed above, there is a non-constant function vanishing at $P_1 \dots P_N$. This contradiction proves that $s < N + 1$, from which $i(\omega_0) \geq g + s - m$.

By reversing the reasoning, one obtains a converse of the following form. If $i(\omega_0) \geq g + s - m$, then for any $\zeta \in D^s$ there is a $\sigma \in D^{m-s}$ such that $u(\omega_0) \equiv u(\zeta\sigma)$. Let us call s the number of free points of ω_0 , where s is the greatest integer such that for $\zeta \in D^s$ there is a $\sigma \in D^{m-s}$ with $u(\omega_0) \equiv u(\zeta\sigma)$. By the lemma, and the converse just stated, we have the following.

COROLLARY: $i(\omega_0) = g + s - m$, where $\omega_0 \in D^m$ and s is the number of free points of ω_0 .

In particular, if $m = g - n$, $0 \leq n \leq g - 1$, then $i(\omega_0) = n + s$.

Reverting to the discussion preceding the lemmas, we have $e \equiv u(\omega_0 \beta_0) + K$ and for ω sufficiently near ω_0 , there is a β such that $e \equiv u(\omega\beta) + K$, $u(\omega_0 \beta_0) \equiv u(\omega\beta)$. Thus, applying Lemma 2, with s of the Lemma as $s - 1$ and m as $g - 1$, we have $i(\omega_0 \beta_0) \geq g + (s - 1) - (g - 1) = s$.

We have shown then that $\theta(W^r - W^r - e) = 0$, for $0 \leq r \leq s - 1$, implies that $e \equiv u(\zeta) + K$, where $\zeta \in D^{s-1}$ and $i(\zeta) \geq s$. Conversely, this latter statement implies

$\theta(W^r - W^r - e) = 0$, for $0 \leq r \leq s-1$. For, we are given $i(\zeta) \geq g + (s-1) - (g-1)$, so that ζ has at least $(s-1)$ free points. $x \in W^{s-1} - W^{s-1} - e$ is of the form $x = u(P_1 \dots P_{s-1}) - u(Q_1 \dots Q_{s-1}) - e$, and we may write $e \equiv u(P_1 \dots P_{s-1} \delta) + K$, with $\delta \in D^{g-s}$. This gives $x \equiv -u(Q_1 \dots Q_{s-1} \delta) - K$, and, by Theorem 2, $\theta(x) = 0$. Thus, $\theta(W^{s-1} - W^{s-1} - e) = 0$, and, since $W^r \subset W^{s-1}$, for $0 \leq r \leq s-1$, our assertion is proved.

We now prove that $\theta(W^r - W^r - e) = 0$, for $0 \leq r < s$, implies that all partial derivatives of θ of order r vanish at $-e$ (hence also at e , for $\theta(u)$ being even implies $\theta(W^r - W^r + e) = 0$, for $0 \leq r < s$). In fact, more is true. Namely, if $\theta(W^{s-1} - W^{s-1} - e) = 0$ then

$$(r): \frac{\partial^r \theta}{\partial u_{j_1} \dots \partial u_{j_r}} (W^{s-1-r} - W^{s-1-r} - e) = 0 \text{ for } 0 \leq r \leq s-1 \text{ and } 1 \leq j_1, \dots, j_r \leq g.$$

If $r=0$ we have the θ function itself. Since $-e \in W^{s-1-r} - W^{s-1-r} - e$, we have that all partial derivatives of θ of order up to and including $s-1$ vanish at $-e$. Now the statement (r) above is true for $r=0$, by hypothesis. Assume it has been proved for all $r \leq n$, $0 \leq n < s-1$; we shall prove it for $n+1$. We have

$$\frac{\partial^n \theta}{\partial u_{j_1} \dots \partial u_{j_n}} (u(P_1 \dots P_{s-1-n}) - u(Q_1 \dots Q_{s-1-n}) - e) = 0,$$

for any points $P_1 \dots P_{s-1-n}$, $Q_1 \dots Q_{s-1-n}$ on S . Fix particular choices for all of these points except P_1 , which we let vary in a small neighborhood of Q_1 . We have then a function of P_1 which is identically zero, so that differentiating with respect to a local parameter at Q_1 and setting $P_1 = Q_1$, we still have zero. By the chain rule for differentiation.

$$\sum_{j=1}^g \frac{\partial^{n+1} \theta}{\partial u_{j_1} \dots \partial u_{j_n} \partial u_j} (u(P_2 \dots P_{s-1-n}) - u(Q_2 \dots Q_{s-1-n}) - e) du_j(Q_1) = 0.$$

This differential is a linear combination of the linearly independent $du_j = \varphi_j$, with coefficients independent of Q_1 , which vanishes at every point of S , as Q_1 was arbitrary. Thus, each coefficient is zero which proves (r) for $n+1$, completing the inductive proof.

We come now to the crucial point; namely to show that if $\theta(W^r - W^r - e) = 0$, for $0 \leq r \leq s-1$, but $\theta(W^s - W^s - e) \neq 0$, then at least one partial derivative of θ of order s does not vanish at $-e$. As we have already observed, $\theta(W^s - W^s - e) \neq 0$ implies there are divisors of $2s$ distinct points, $\omega_0, \sigma_0 \in D^s$ such that $\theta(u(\omega_0) - u(\sigma_0) - e) \neq 0$. By continuity, for $\tau \in D^s$ sufficiently near σ_0 , $\theta(u(\omega_0) - u(\tau) - e) \neq 0$. Also, $\theta(u(\tau) -$

$u(\sigma_0) - e$ cannot vanish for all τ near σ_0 , for, otherwise, this function of s variables would vanish on an open set, and so would be identically zero, contradicting the existence of ω_0 . Thus, there is a $\tau_0 \in D^s$ such that $\theta(u(\tau_0) - u(\sigma_0) - e) \neq 0$, $\theta(u(\omega_0) - u(\tau_0) - e) \neq 0$, and again, by continuity, τ_0 may be assumed to have distinct points, all different from those in ω_0 and σ_0 .

Let $d\eta$ denote the normalized abelian differential of third kind on S , with zero periods on a_1, \dots, a_g , and with residue $+1$ at the points of τ_0 and -1 at the points of σ_0 . Thus, if $\sigma_0 = Q_1 \dots Q_s$, $\tau_0 = R_1 \dots R_s$, $d\eta = \sum_{j=1}^s d\eta_{Q_j, R_j}$, where $d\eta_{Q_j, R_j}$ is the abelian differential of third kind on S with zero periods on a_1, \dots, a_g and residue $+1$ at R_j and -1 at Q_j . Recall that the Riemann period relations for such differentials are

$$\int_{v_k} d\eta_{Q_j, R_j} = 2(u_k(R_j) - u_k(Q_j)) \quad (1 \leq k \leq g, 1 \leq j \leq s), \quad (17)$$

where $u_k(R_j) - u_k(Q_j) = \int_{Q_j}^{R_j} \varphi_k$ is taken over a path from Q_j to R_j lying completely in the simply connected region S_0 . We are assuming here, as is clearly permissible, that the points of σ_0, τ_0 lie on S_0 . Consider now the following function of s points on S ,

$$f(P_1, \dots, P_s) = \frac{\theta(u(P_1 \dots P_s) - u(\sigma_0) - e)}{\theta(u(P_1 \dots P_s) - u(\tau_0) - e)} \cdot E, \quad \text{where } E = \exp \left(\sum_{j=1}^s \int_{B_0}^{P_j} d\eta \right).$$

f is not always zero over zero, for at $P_1 \dots P_s = \omega_0$ it has a finite value. Consider for the moment $P_2 \dots P_s$ fixed, and examine f as a function of P_1 . For $P_2 \dots P_s$ fixed at values such that numerator and denominator do not vanish identically in P_1 , f is a single valued meromorphic function of P_1 on all S . This follows directly from the period properties of θ and the relations (17). We claim now that this function of P_1 is a constant. Indeed, leaving aside the quantity E for the present, $f(P_1)$ has zeros due to the zeros of θ in the numerator, which are g in number. By our hypothesis that $\theta(W^r - W^r - e) = 0$, for $0 \leq r < s$, s of these zeros are at σ_0 . Thus, the divisor of zeros for the numerator is $\sigma_0 \gamma$, $\gamma \in D^{g-s}$, and there is a congruence,

$$-u(P_2 \dots P_s) + u(\sigma_0) + e \equiv u(\sigma_0 \gamma) + K, \quad \text{or } e \equiv u(P_2 \dots P_s \gamma) + K.$$

By Theorem 3, the divisor of zeros $\sigma_0 \gamma$ has $i(\sigma_0 \gamma) = 0$. In the same way, the divisor of zeros for the denominator is $\tau_0 \delta$, $\delta \in D^{g-s}$; we again have a congruence

$$e \equiv u(P_2 \dots P_s \delta) + K, \quad \text{and } i(\tau_0 \delta) = 0.$$

Thus, $u(\gamma) \equiv u(\delta)$. If $\gamma \neq \delta$, then, by the Riemann-Roch and Abel theorems, $i(\gamma) \geq s + 1$. The Riemann-Roch theorem also shows that adding a point to a divisor decreases

the index i by at most one. Thus, $i(\sigma_0\gamma) \geq 1$, which contradicts the fact that $i(\sigma_0\gamma) = 0$. Thus, $\gamma = \delta$, and the only zeros and poles of $f(P_1)$ due to the quotient of θ 's, are the zeros at σ_0 and the poles at τ_0 . Consider now E , which, as a function of P_1 , contributes the term $\exp(\int_{B_0}^{P_1} d\eta)$. This is finite, and not zero, for P_1 not one of the points in σ_0, τ_0 . As P_1 varies in a neighborhood of a point Q of σ_0 with local parameter z , $z(Q) = 0$, $\exp(\int_{B_0}^P d\eta)$ is, up to a finite non-zero factor,

$$\exp\left(\int_{P_0=z_0}^{P_1=z_1} \frac{-1}{z} dz\right) = \exp(-\log z_1 + \log z_0).$$

Here $z_0 = z(P_0)$ is arbitrary, as long as $z_0 \neq 0$. Letting $z_1 \rightarrow 0$, we see that $E(P_1)$, as $P_1 \rightarrow Q$ of σ_0 , has a simple pole. This cancels with the zero at Q in σ_0 from the θ in the numerator. On the other hand, using a similar notation at a point R of τ_0 , we see that, as $P_1 \rightarrow R$, $E(P_1)$ behaves like $\exp(\int_{z_0}^{z_1=P_1} z^{-1} dz)$ as $z_1 \rightarrow 0$. That is, $E(P_1)$ at $P_1 = R$ has a simple zero, which cancels the pole due to the zero of the denominator at $P_1 = R$. Thus, all zeros and poles cancel, and $f(P_1)$ must be a constant C .

The constant C depends on $P_2 \dots P_s$. But, f is symmetric in $P_1 \dots P_s$, its value is not dependent on the order of the points $P_1 \dots P_s$. This implies that if f is constant in P_1 , then it is a constant in all s variables.

We have then the following equation:

$$C\theta(u(P_1 \dots P_s) - u(\tau_0) - e) = \theta(u(P_1 \dots P_s) - u(\sigma_0) - e)E. \quad (18)$$

Differentiate (18) with respect to (a local parameter z at) P_1 and set $P_1 = R_1$. This yields

$$\begin{aligned} C \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (u(P_2 \dots P_s) - u(R_2 \dots R_s) - e) du_j(R_j) \\ = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (u(R_1 P_2 \dots P_s) - u(\sigma_0) - e) du_j(R_1) E(R_1) \\ + \theta(u(R_1 P_2 \dots P_s) - u(\sigma_0) - e) dE(R_1), \end{aligned} \quad (19)$$

where

$$E(R_1) = \exp\left(\int_{B_0}^{R_1} d\eta + \sum_{k=2}^s \int_{B_0}^{P_k} d\eta\right).$$

As we have already remarked, $E(R_1)$ has a simple zero, so that the first term on the right is zero. $dE(R_1)$ is a finite non-zero quantity, essentially of the form $\exp(\sum_{k=2}^s \int_{B_0}^{P_k} d\eta)$.

Thus, if we now differentiate (19) with respect to P_2 , and set $P_2 = R_2$, we have

$$\begin{aligned} C \sum_{j_1, j_2=1}^g \frac{\partial^2 \theta}{\partial u_{j_1} \partial u_{j_2}} (u(P_3 \dots P_s) - u(R_3 \dots R_s) - e) du_{j_1}(R_1) du_{j_2}(R_2) \\ = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (u(R_1 R_2 P_3 \dots P_s) - u(\sigma_0) - e) du_j(R_2) (dE(R_1))(R_2) \\ + \theta(u(R_1 R_2 P_3 \dots P_s) - u(\sigma_0) - e) d(dE(R_1))(R_2). \end{aligned}$$

Again, the first term on the right vanishes, for it is essentially $\exp(\int^{R_2} d\eta)$ which has a simple zero at R_2 , while the second term is a non-zero quantity, essentially of the form $\exp(\sum_{k=3}^s \int_{B_0}^{P_k} d\eta)$. Continuing in this fashion, we finally obtain, after differentiating s times,

$$C \sum_{j_1, \dots, j_s=1}^g \frac{\partial^s \theta}{\partial u_{j_1} \dots \partial u_{j_s}} (-e) du_{j_1}(R_1) \dots du_{j_s}(R_s) = \theta(u(R_1 \dots R_s) - u(\sigma_0) - e) F, \quad (20)$$

where F is a finite non-zero quantity due to the factor E . But our construction assured from the outset that $\theta(u(\tau_0) - u(\sigma_0) - e) \neq 0$. Thus, not all coefficients on the left of (20) vanish, so that some partial of θ of order s does not vanish at $-e$. We collect our results in the following main theorem.

THEOREM 8. *Let $e \in C^g$. If $\theta(e) \neq 0$, then $e \equiv u(\zeta) + K$ for a unique $\zeta \in D^g$, and $i(\zeta) = 0$. If $\theta(e) = 0$, let s , $1 \leq s \leq g-1$, be the least integer such that $\theta(W^s - W^s - e) \neq 0$. Then there is a $\zeta \in D^{g-1}$, $i(\zeta) = s$, such that $e \equiv u(\zeta) + K$. All partial derivatives of θ of order less than s vanish at e while at least one partial derivative of order s does not vanish at e . The integer s is the same for both e and $-e$.*

Observe that there is no mention at all here of the base point B_0 . This is to be expected, for the order of vanishing of θ at a point in C^g is independent of the choice of B_0 . Indeed, by (a) of Theorem 4, a set of the form $W^r - W^r$ is uniquely determined in $J(S)$, independently of the point B_0 , even though W^r is not. Thus it is only in considering unsymmetric expressions of the form $\theta(u(P) - e)$, i.e., $\theta(W^1 - e)$, that the base point B_0 plays a role.

One other point needs clarification. When it is stated that $e \equiv u(\zeta) + K$ for some $\zeta \in D^{g-1}$ with $i(\zeta) = s$, then this implies that if also $e \equiv u(\omega) + K$ for $\omega \in D^{g-1}$, then $i(\omega) = s$. For completeness this is stated as a lemma.

LEMMA 3. *Let $\zeta, \omega \in D^n$, and suppose $u(\zeta) \equiv u(\omega)$. Then $i(\zeta) = i(\omega)$.*

Proof. The hypothesis implies, by Abel's theorem, that there is a function f_1 with divisor ζ/ω . Extend to a basis f_1, f_2, \dots, f_N of the space of functions which are multiples of ω . By the Riemann-Roch theorem, $N = n + i(\omega) + 1 - g$. The functions $h_1 = 1, h_2 = f_2/f_1, \dots, h_N = f_N/f_1$, are linearly independent multiples of ζ , so that $N \leq n + i(\zeta) + 1 - g$. Thus, $i(\omega) \leq i(\zeta)$, and interchanging ζ and ω gives the result.

Section III

The integer s of Theorem 8 actually has $\frac{1}{2}(g+1)$ as an upper bound. For, if $e \equiv u(\zeta) + K, \zeta \in D^{g-1}$ and $i(\zeta) = s$, choose the $s-1$ free points of ζ at B_0 , the base point of u . We may assume that B_0 is not a Weierstrass point, as Theorem 8 is independent of the base point. Thus, $e \equiv u(B_0^{s-1}\omega) + K$, where $\omega \in D^{g-s}$. Now, $i(B_0^{s-1}\omega) = g - (s-1)$, the number of gaps greater than $s-1$ at B_0 , and certainly then, $i(B_0^{s-1}\omega) \leq g - (s-1)$. But $i(B_0^{s-1}\omega) = i(\zeta) = s$, so that $s \leq g - (s-1)$, or $s \leq \frac{1}{2}(g+1)$. Combining this fact with Theorem 8 yields the following:

THEOREM 9. *Let T be a $g \times g$ ($g \geq 2$) matrix of complex numbers, symmetric, with negative definite real part, and $\theta(u)$ the associated θ function for T . Then if θ has order $> \frac{1}{2}(g+1)$ at some point $e \in C^g$, i.e., θ and all partial derivatives of order $\leq \frac{1}{2}(g+1)$ vanish at e , then T is not (the second half of) a normalized period matrix of a Riemann surface of genus g .*

We return now to the considerations of the first part of Section II to consider once again the role of the base point B_0 . Therefore, we write u^0, K^0 etc., as before. Assume $\theta(u^0(P) - e) \equiv 0$. Let s be the least integer such that $\theta(W^{s+1} - W^s - e) \not\equiv 0$; here, of course, s depends on B_0 . Then, there are $\sigma_0, \tau_0 \in D^s$, which we may assume to consist of distinct points, such that $\theta(u^0(P) + u^0(\sigma_0) - u^0(\tau_0) - e) \equiv 0$. This has a divisor ζ of g zeros, s of which are at the points of τ_0 , so that $\zeta = \tau_0 \beta_0, \beta_0 \in D^{g-s}$, and $e \equiv u^0(\sigma_0 \beta_0) + K^0$. The same holds for σ sufficiently near σ_0 . For every such σ there is a β such that $e \equiv u^0(\sigma \beta) + K^0$. Thus, $\sigma_0 \beta_0$ has at least s free points, and by Lemma 2 then, $i(\sigma_0 \beta_0) \geq s$. This shows that $\theta(u^0(P) - e) \equiv 0$ implies a congruence $e \equiv u^0(\zeta) + K^0, \zeta \in D^g$, and $i(\zeta) \geq s \geq 1$. We can now complete Theorem 3 to cover all cases by adding the following: if $e \equiv u^0(\zeta) + K^0, \zeta \in D^g$ and $i(\zeta) = 0$, then $\theta(u^0(P) - e) \equiv 0$. Also, it is now clear that in the second case of Theorem 5 the $g-1$ points, at most, for which $\theta(u^j(P) - e) \equiv 0$, are precisely the points of γ .

The above results enable us to obtain a characterization of the Weierstrass points on S in terms of the θ function and the Riemann constants.

THEOREM 10. *Let $B_0 \in S$ be arbitrary, u^0, K^0 , the map u and Riemann constants K with base point B_0 . Then, B_0 is a Weierstrass point if and only if $\theta(u^0(P) - K^0) \equiv 0$.*

Proof. $K^0 \equiv u^0(B_0^g) + K^0$. By our preceding remarks, $\theta(u^0(P) - K^0) \equiv 0$ if and only if $i(B_0^g) \geq 1$, which is the condition for a Weierstrass point. Note, that if B_0 is a Weierstrass point, we needn't have $\theta(u^1(P) - K^0) \equiv 0$ for every base point B_1 . This, in fact, by Theorem 6 (or Theorem 8), occurs if and only if

$$\frac{\partial \theta}{\partial u_j}(K^0) = 0 \text{ for } 1 \leq j \leq g.$$

By Theorem 8, this is if and only if $K^0 \equiv u^0(\zeta) + K^0$ for $\zeta \in D^{g-1}$ and $i(\zeta) \geq 2$. But $K^0 \equiv u^0(B_0^{g-1}) + K^0$, and $i(B_0^{g-1}) \geq 2$ if and only if there are two gaps greater than $g-1$, which is not true for every Weierstrass point. For example, on a "general" surface with $g(g^2-1)$ Weierstrass points, at which the gaps are $1, 2, \dots, g-1, g+1$, we have $i(B_0^{g-1}) = 1$.

Let us now prove the following classical result.

THEOREM 11. *For any $B_0 \in S$, if $\Delta \in D^{2g-2}$, then $u^0(\Delta) \equiv -2K^0$ if and only if Δ is the divisor of zeros of a differential on S .*

Proof. Let $\zeta \in D^{g-1}$ be arbitrary and set $e \equiv u^0(\zeta) + K^0$. Then, by Theorem 7, $\theta(e) = 0$ and $\theta(-e) = \theta(e) = 0$. Thus, for some $\zeta' \in D^{g-1}$, $-e \equiv u^0(\zeta') + K^0$. Adding, we have $-2K^0 \equiv u^0(\zeta\zeta')$, where $\zeta\zeta' \in D^{2g-2}$. Since $\zeta\zeta'$ has $g-1$ free points, we have $i(\zeta\zeta') = g + (g-1) - (2g-2) = 1$, and $\zeta\zeta'$ is the divisor of a differential. If $u^0(\Delta) \equiv -2K^0$, then $u^0(\Delta) \equiv u^0(\zeta\zeta')$ and the theorem follows from Lemma 3, $i(\zeta\zeta') = i(\Delta)$.

The θ function enables one, in certain cases, to obtain explicitly the linear combination of the normalized differentials $\varphi_1, \dots, \varphi_g$, which vanishes at given points. For example, let $\zeta \in D^{g-1}$ satisfy $i(\zeta) = 1$; there is then a uniquely determined $\zeta' \in D^{g-1}$ for which $i(\zeta\zeta') = 1$. Set $e \equiv u^0(\zeta) + K^0$, which, by Theorem 4, determines $e \in C^g$ (modulo Ω , of course) independently of B_0 . Consider

$$\psi = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j}(-e) \varphi_j.$$

By Theorem 8, since $i(\zeta) = 1$, s for e is 1, and ψ is not the zero differential. Let B_1 be a point in ζ , $e \equiv u^1(\zeta) + K^1 \equiv u^1(B_1\zeta) + K^1$. If $i(B_1\zeta) = 1$, then, by Theorem 3, $\theta(u^1(P) - e) \equiv 0$. Differentiating and setting $P = B_1$ gives

$$\psi(B_1) = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (-e) du_j(B_1) = 0,$$

so that ψ vanishes at B_1 . If $i(B_1 \zeta) = 0$, then $\theta(u^1(P) - e)$ has its g zeros at $B_1 \zeta$. As B_1 is in ζ , it is a double zero, and again by differentiating we have $\psi(B_1) = 0$. Since $u^0(\zeta \zeta') \equiv -u^0(\zeta) - K^0 \equiv u^0(\zeta) - K^0 \equiv u^0(\zeta') + K^0$, and, by the same argument,

$$\psi' = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (e) \varphi_j$$

vanishes at ζ' . But, by the evenness of θ ,

$$\frac{\partial \theta}{\partial u_j} (e) = \frac{\partial \theta}{\partial u_j} (-e), \quad \text{and} \quad \psi' = -\psi.$$

Thus, ψ vanishes at the points of $\zeta \zeta'$. Also, if B is not in $\zeta \zeta'$, then $\psi(B) \neq 0$. For, $e \equiv u^B(B \zeta) + K^B$, $i(B \zeta) = 0$, and $\theta(u^B(P) - e)$ has a simple zero only at B , hence $\psi(B) \neq 0$. Note that we have not proved that ψ has $\zeta \zeta'$ as its divisor of zeros; but only that ψ has a zero at each of the distinct points of $\zeta \zeta'$. However, we have proved the following particular case:

THEOREM 12. *Let $\Delta \in D^{2g-2}$ be the divisor of a differential ψ . Assume that Δ contains a divisor ζ of $g-1$ distinct points, satisfying $i(\zeta) = 1$. Then, up to a constant multiple,*

$$\psi = \sum_{j=1}^g \frac{\partial \theta}{\partial u_j} (e) \varphi_j,$$

where $e \equiv u^0(\zeta) + K^0$, for any base point B^0 .

A point $e \in C^g$ which has the property $2e \equiv 0$ may be called a half period. Any half period is necessarily of the form $e = \pi i \varepsilon' / 2 + T \varepsilon / 2$ where $\varepsilon, \varepsilon'$ are integral vectors in C^g . Modulo Ω , there are 2^{2g} distinct half periods, obtained by letting the entries in ε and ε' be 0 or 1 in all possible ways. Riemann calls a half period even if $\tilde{\varepsilon} \varepsilon' \equiv 0 \pmod{2}$, odd if $\tilde{\varepsilon} \varepsilon' \equiv 1 \pmod{2}$. An easy calculation shows that there are $2^{g-1}(2^g - 1)$ odd and $2^{g-1}(2^g + 1)$ even half periods, (see [4], p. 8 of the Supplement). The motivation for this even-odd terminology is the following. Recall the definition of $\theta \begin{bmatrix} G \\ H \end{bmatrix} (u)$ given at the end of Section I; for the half period $e = \pi i \varepsilon' / 2 + T \varepsilon / 2$ consider the function $\theta \begin{bmatrix} \varepsilon/2 \\ \varepsilon'/2 \end{bmatrix} (u)$. Then this function is an even or odd function of u according as $\tilde{\varepsilon} \varepsilon' \equiv 0$ or $1 \pmod{2}$; see [1], p. 103-4. Since $\theta \begin{bmatrix} \varepsilon/2 \\ \varepsilon'/2 \end{bmatrix} (0)$ and $\theta(\pi i \varepsilon' / 2 + T \varepsilon / 2)$ dif-

fer by a non zero, exponential, factor, the order of the θ function at a half period is the same as the order of the corresponding θ with characteristics at the origin. We recall that an odd function always vanishes at the origin and that a partial derivative of an even (odd) function is odd (even). In particular, an odd function must have odd order at the origin; if all partial derivatives of order $< s$ vanish at the origin, while some partial derivative of order s does not, then s is odd. An even function has even order at the origin.

Let $e^{(1)}, \dots, e^{(N)}$, $N = 2^{\sigma-1}(2^{\sigma} - 1)$, be an enumeration of the odd half periods. By our above remarks, the θ function vanishes at each $e^{(j)}$. By Theorem 8, $e^{(j)} \equiv u(\zeta^{(j)}) + K$, where $\zeta^{(j)} \in D^{\sigma-1}$, $i(\zeta^{(j)}) = s_j \geq 1$, and s_j is odd. In particular, $0 \equiv 2e^{(j)} \equiv u(\zeta^{(j)}\zeta^{(j)}) + 2K$, and, by Theorem 11, there is a differential $\varphi^{(j)}$ with divisor $(\zeta^{(j)})^2$. Such differentials—or actually square roots of them—Riemann, [4] p. 488, called abelian functions.

On the other hand, $\theta(u)$ need not vanish at an even half period. If this occurs, it means that the surface S has some special property. For example, Riemann ([4], p. 54 of the Supplement) states that for $g=3$, S is hyperelliptic if and only if θ vanishes at some even half period. To see this, let us suppose first g arbitrary and e an even half period such that $\theta(e)=0$. By Theorem 8,

$$e \equiv u(\zeta) + K, \quad i(\zeta) = s \geq 1, \quad \zeta \in D^{\sigma-1}.$$

As s must be even, $s \geq 2$. Since $0 \equiv u(\zeta^2) + 2K$, there is a differential φ with divisor $\zeta^2 \in D^{2\sigma-2}$, and since $i(\zeta) \geq 2$, there is a second differential ψ , with divisor $\zeta\omega$, $\omega \in D^{\sigma-1}$, $\zeta \neq \omega$. φ/ψ is a function with divisor ζ/ω ; by Abel's theorem, $u(\zeta) \equiv u(\omega)$. Hence, $e \equiv u(\omega) + K$, which implies that there is a differential ξ with divisor ω^2 . Clearly, the differentials φ , ψ , ξ are linearly independent, while the quadratic differentials ψ^2 and $\varphi\xi$ both have the same divisor of zeros $\zeta^2\omega^2$. Thus, $\psi^2 = \lambda(\varphi\xi)$, for some constant λ . We see that θ vanishing at an even half period leads to a linear relation between quadratic differentials which are products of (abelian) differentials. Suppose now $g=3$ and $\theta(e)=0$, e an even half period. Then a relation of the form $\psi^2 = \lambda(\varphi\xi)$, where ψ, φ, ξ are linearly independent, by a well-known result called Noether's theorem, implies S is hyperelliptic. However, we do not have to appeal to Noether's theorem. Simply observe that $f = \varphi/\psi$ is a function with divisor ζ/ω ; as ω has only two points when $g=3$, f is a function with two poles on S , and S is hyperelliptic. The converse, that S hyperelliptic and $g=3$ imply $\theta(e)=0$ for an even half period, will follow below from our general discussion of hyperelliptic surfaces.

Let S be hyperelliptic and $B_0 \in S$ a Weierstrass point. Since $2g-1$ is a gap at B_0 , there is a differential on S having all its zeros at B_0 , i.e., having divisor B_0^{2g-2} .

By Theorem 11, $-2K^0 \equiv u^0(B_0^{2g-2}) \equiv 0$, and so K^0 is a half period. By our remarks after Theorem 4, we see that if g is odd, the $2g+2$ Weierstrass points on S give rise to only one half period K^0 , while if g is even, there are $2g+2$ distinct half period vectors K^0 . Is K^0 even or odd half period? This is answered by

THEOREM 13. *Let S hyperelliptic with genus $g=4k+m$, $k \geq 0$, $0 \leq m \leq 3$. Let K^0 be the vector of Riemann constants with respect to a Weierstrass point $B_0 \in S$. Then K^0 is a half period, even if $m=0$ or $m=3$, and odd if $m=1$ or $m=2$. When $m=1$ or 2 , the θ function has order $2k+1$ at K^0 , while when $m=0$ it has order $2k$ and when $m=3$ it has order $2k+2$ at K^0 .*

Proof. $K^0 \equiv u^0(B_0^{g-1}) + K^0$, and, by Theorem 8, the order of θ at K^0 is $i(B_0^{g-1})$. But $i(B_0^{g-1})$ is the number of gaps greater than $g-1$ at B_0 , which is the number of odd numbers in the sequence $g, g+1, \dots, 2g-1$. When $m=0$, $g=4k$, the gaps are $g+1, g+3, \dots, 2g-1$, and $i(B_0^{g-1}) = \frac{1}{2}g = 2k$, which is even. Since $\theta(u)$ has even order at K^0 , K^0 is an even half period. Similar considerations for the cases $m=1, 2$ or 3 give the rest of the theorem.

In the hyperelliptic case we may catalogue all even and odd half periods which zeros of $\theta(u)$ in the following way. Let A_1, \dots, A_{2g+2} be the $2g+2$ Weierstrass points on the hyperelliptic surface S . Consider all divisors of degree $g-1$ of the form:

$$\zeta_{n,(j)} = A_1^{2n} A_{j_1} \dots A_{j_{g-1-2n}}, \quad (21)$$

where $0 \leq 2n \leq g-1$, $1 \leq j_k \leq 2g+2$, $1 \leq k \leq g-1-2n$, and $j_k \neq j_m$ if $k \neq m$. We have already seen that any half period e , such that $\theta(e) = 0$, gives a divisor ζ of degree $g-1$ with $e \equiv u(\zeta) + K$, and ζ^2 is the divisor ζ of a differential. Let us call such a ζ a half period divisor. We now prove the following

THEOREM 14.

- (a) *Every half period divisor ζ is equivalent to a divisor $\zeta_{n,(j)}$ of the form (21).*
- (b) *$i(\zeta_{n,(j)}) = n+1$.*
- (c) *If $\zeta_{n,(j)} \neq \zeta_{n',(j')}$, then these divisors are also not equivalent.*

Proof. (a) It is well known that on any hyperelliptic S there is a unique involution (automorphism of order 2) which leaves the $2g+2$ Weierstrass points fixed. Also, if \bar{P} is the image of the point P , not a Weierstrass point, under this involution, then the order of any differential at P equals the order of that differential at \bar{P} , and $P\bar{P} \sim A_1^2$. Since ζ^2 is the divisor of a differential, for every appearance of P ,

not a Weierstrass point, in ζ , \bar{P} appears also; replacing each $P\bar{P}$ in ζ by A_1^2 gives an equivalent divisor. Finally, if a Weierstrass point A_j appears in ζ with some multiplicity, using the fact that $A_j^2 \sim A_1^2$, we obtain a divisor equivalent to ζ of the form $\zeta_{n,(j)}$.

(b) Since a differential cannot have a simple zero at a Weierstrass point, we have that

$$i(\zeta_{n,(j)}) = i(A_1^{2n} A_{j_1}^2 \dots A_{j_{g-1-2n}}^2) = i(A_1^{2g-2-2n}),$$

for $A_j^2 \sim A_1^2$. But $i(A_1^{2g-2-2n})$ is the number of gaps at A_1 greater than $2g-2-2n$, which is $n+1$.

(c) Suppose $\zeta_{n,(j)} \sim \zeta_{n',(j')}$. By Lemma 3 and (b) above, we must have $n=n'$. If these divisors are not equal, then there is a function, not a constant, having poles at most at $A_{j_1} \dots A_{j_{g-1-2n}}$. In other words,

$$r\left(\frac{1}{A_{j_1} \dots A_{j_{g-1-2n}}}\right) \geq 2.$$

But by the Riemann-Roch theorem

$$r = g - 1 - 2n + i(A_{j_1} \dots A_{j_{g-1-2n}}) + 1 - g.$$

Again, since a differential cannot have a simple zero at a Weierstrass point, we have that

$$i(A_{j_1} \dots A_{j_{g-1-2n}}) = i(A_{j_1}^2 \dots A_{j_{g-1-2n}}^2) = i(A_1^{2g-2-4n}) = 2n + 1,$$

which gives $r=1$, a contradiction. Thus $\zeta_{n,(j)} = \zeta_{n',(j')}$, which completes the proof of the theorem.

On the other hand, it is clear that each $\zeta_{n,(j)}^2$ is the divisor of a differential. Setting

$$e_{n,(j)} \equiv u(\zeta_{n,(j)}) + K$$

defines a half period, which is a zero of θ . By (a) and (c) of the theorem above, all half periods which are zeros of θ are obtained precisely in this way. By Theorem 8 and (b) above, θ has order $n+1$ at $e_{n,(j)}$, so that $e_{n,(j)}$ is an even half period if n is odd and an odd half period if n is even. Also, for a given n , $0 \leq 2n \leq g-1$, there are $\binom{2g+2}{g-1-2n}$ half periods at which θ has order $n+1$. We summarize this as

THEOREM 15. *Every half period which is a zero of θ is of the form*

$$e_{n,(j)} \equiv u(\zeta_{n,(j)}) + K.$$

θ has order $n+1$ at $e_{n,(j)}$ and $e_{n,(j)}$ is even if n is odd and odd if n is even. The number of even half periods at which θ vanishes is

$$\sum_{0 \leq 2k+1 \leq \frac{1}{2}(g-1)} \binom{2g+2}{g-1-2(2k+1)}.$$

Since θ vanishes at every odd half period, we have that

$$2^{g-1}(2^g-1) = \sum_{0 \leq k \leq \frac{1}{2}(g-1)} \binom{2g+2}{g-1-4k}.$$

Finally, we see that, as claimed above, if S is hyperelliptic and $g=3$, then θ vanishes at precisely one even half period. For by the above theorem, it vanishes at $\binom{8}{0} = 1$ even half period. In fact, by Theorem 13, it is the even half period of Riemann constants, K^0 , taken with base point at a Weierstrass point.

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