On properties of functions with conditions on their mean oscillation over cubes

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Introduction

We consider spaces $BMO_{\varphi,p}$ of functions defined using mean oscillation over cubes in $\mathbf{R}^n$, which include the Morrey spaces $L^{(p,\lambda)}$, the John—Nirenberg space BMO and the Lipschitz spaces Λ_α. It is our purpose to give equivalent characterizations of the spaces $BMO_{\varphi,p}$ and to apply these characterizations to an extension problem for $BMO_{\varphi,p}(G)$, for certain open subsets G of $\mathbf{R}^n$. We prove that the spaces $BMO_{\varphi,p}$ are characterized by a property of the mean oscillation over a class of sets more general than the class of cubes used in their definition.

Although we are mainly interested in the case $1 \leqq p < \infty$, which include BMO $(\varphi(r)=1, p=1)$ and the Morrey spaces, we state some of our results in the more general situation $0<p<\infty$ and $\varphi(r)$ satisfying certain growth conditions. See the remark following Theorem 2.2 in section 2.1.

Let E be a bounded set in $\mathbf{R}^n$ with positive Lebesgue measure. Then the mean oscillation of f over E (in L^p-sense, $0<p\leqq\infty$) is defined by

$$\Omega_p(f, E) = \inf_C \left(|E|^{-1} \int_E |f(x)-C|^p \, dx\right)^{1/p}, \tag{0.1}$$

whenever it is finite. For any such set E there is $C=C_E$ for which the infimum in (0.1) is attained and C_E is from now on defined in this way. Note that C_E is not uniquely defined, cf. J-O. Strömberg [15].

The set function $\Omega_p(f, E)$ is a local best approximation of order zero of f in L^p in the sense of Ju. A. Brudnyi [1, p. 75].

The class of sets we consider is defined by

$$K_r = \{x \in \mathbf{R}^n;\ \text{dist}\ (x, K) \leqq r\},\ r>0, \tag{0.2}$$

where K ranges over all compact sets K with Lebesgue measure zero.

The spaces $BMO_{\varphi,p}$, $0<p<\infty$, are defined in section 1. We prove in Theorem 2.1 that if $\varphi(r)$ satisfies a mild growth condition and $f\in BMO_{\varphi,p}$ then for all sets K_r in (0.2) it holds that

$$\Omega_p(f, K_r) \leqq N\cdot \int_r^k \varphi(t)/t\,dt, \quad r>0, \tag{0.3}$$

where $N=c(\varphi, n, p)\cdot\|f\|_{\varphi,p}$ and k depends on r and the diameter of K.

We show by examples that (0.3) is in general best possible (section 7.1).

The conclusion of Theorem 2.1 is rather strong since the class of sets defined by (0.2) is very large. In particular, all closed balls $B(a, r)$ are of this type and (0.3) is simplified considerably in this case with a suitable growth condition on $\varphi(r)$.

The converse of Theorem 2.1 now follows easily. Assume that there is $N<\infty$ such that f satisfies (0.3) for all set K_r in (0.2) then $f\in BMO_{\varphi,p}$ and $\|f\|_{\varphi,p}\leqq c(\varphi, n, p)\cdot N(f)$, where $N(f)$ is the best constant N in (0.3) (Theorem 2.2).

This means that $N(f)$ is an equivalent norm on $BMO_{\varphi,p}$, $1\leqq p<\infty$. Special cases are BMO_p (Theorem 2.3).

The spaces $BMO_{\varphi,p}(G)$, where G is an open subset of $\mathbf{R}^n$, are defined in section 2.2. We consider the problem to decide when a function $f\in BMO_{\varphi,p}(G)$ has an extension $\tilde{f}$ in $BMO_{\varphi,p}(\mathbf{R}^n)=BMO_{\varphi,p}$. In particular when $\mathbf{R}^n\setminus G$ has measure zero we find necessary and sufficient conditions on f of the type (0.3) in Theorem 2.4.

This result also gives a method to generate equivalent norms on $BMO_{\varphi,p}$ that are in a sense intermediate between $\|f\|_{\varphi,p}$ and $N(f)$ above (Theorem 2.5).

Let K be a compact subset of $\mathbf{R}^n$ and consider the problem to extend every continuous function f_0 on K to a function f which is continuous in $\mathbf{R}^n$ and has arbitrary small norm in BMO. We use a construction in BMO by J. B. Garnett and P. W. Jones [4] to prove that this is possible if and only if K has Lebesgue measure zero (Theorem 2.6).

The plan of the paper is as follows. Section 1 contains notations, definitions and some preliminary results. We state our theorems in section 2 and prove them in sections 4, 5 and 6. We prove a number of lemmas in section 3 and give some examples in section 7.

1. Preliminaries

We consider the n-dimensional Euclidean space $\mathbf{R}^n$ with points $x=(x_1, x_2, \ldots, x_n)$. All sets are subsets of $\mathbf{R}^n$. Open, closed and compact sets are denoted by G, F and K respectively. Measure always means Lebesgue measure and is denoted by $|E|$. Functions are real or complex valued and Lebesgue measurable in $\mathbf{R}^n$. Integration with respect to Lebesgue measure is denoted by $\int_E f(x)\,dx$.

When no set of integration is indicated it is understood that integration is over the whole space and all the variables. We let L^p denote the usual Lebesgue space of p-th power integrable functions in $\mathbf{R}^n$ and $\|f\|_p=(\int |f|^p dx)^{1/p}$, $0<p\leqq\infty$. Functions f are usually assumed to be locally in a suitable class L^p. All cubes have sides parallel to the axes. The cube $I=I(a, r)$ has its centre at a and side length $l(I)=r$. We define $\lambda\cdot I(a, r)=I(a, \lambda\cdot r)$, $\lambda>0$.

As usual $B(a, r)$ denotes a closed ball. We let $\varphi(r)$ denote functions such that

$$\varphi\colon\,]0, \infty[\, \to\,]0, \infty[.$$

For all such functions $\varphi(r)$ and $0<p\leqq\infty$, $BMO_{\varphi,p}$ denotes the space of functions f for which

$$\|f\|_{\varphi,p} = \sup_I \Omega_p(f, I)/\varphi(r)$$

is finite, where supremum is over all cubes $I=I(a, r)$, and all $r>0$.

For computational reasons we often consider the quantity

$$(|E|^{-1}\int_E |f(x)-f(E)|^p\, dx)^{1/p},\ f(E) = |E|^{-1}\int_E f(x)\, dx$$

when E is a bounded set with positive measure, cf. Lemma 1.1 (a) and (b). We make the usual modifications when $p=\infty$. We usually drop p from the notation when $p=1$.

In the case when $\varphi(r)\equiv 1$ we use the notation BMO_p and $\|f\|_{*,p}$. $BMO_{\varphi,p}$, $1\leqq p<\infty$, modulo constants is a Banach space, which we also denote by $BMO_{\varphi,p}$. The verification of this is left to the reader. Compare U. Neri [10].

Let Λ_α, $0<\alpha\leqq 1$, be the space of functions f for which

$$\|f\|_\alpha = \sup_{x\neq y} |f(x)-f(y)|/|x-y|^\alpha < \infty.$$

Λ_α modulo constants is a Banach space.

When $\varphi(r)=r^{\lambda-n}$, $0<\lambda<n$, and $1\leqq p<\infty$, the spaces $BMO_{\varphi,p}$ coincide with the Morrey spaces, cf. [3]. Two other cases are also well known. If $\lambda=n$, $0<p<\infty$, then $BMO_{\varphi,p}=BMO_p$ and if $n<\lambda\leqq n+p$, $1\leqq p<\infty$, then $BMO_{\varphi,p}=\Lambda_\alpha$, $\alpha=\dfrac{\lambda-n}{p}$, S. Campanato [2] and N. G. Meyers [9]. A survey of these and related spaces is found in J. Peetre [11] and Ju. A. Brudnyi [1].

We often use unspecified constants only depending on certain numbers $\alpha, \beta, \gamma, \ldots$. Such constants are denoted by $c(\alpha, \beta, \gamma, \ldots)$. Constants which only depend on n are denoted by c. The same notation may denote different constants at different occurences.

In all our theorems the function $\varphi(r)$ is such that

$$A \cdot \inf_{r \leqq t \leqq 2r} \varphi(t) \geqq \varphi(r), \quad r > 0 \tag{$*$}$$

holds for some real number A independent of r, $A \geqq 1$.

We also need the following property for $\varphi(r)$. We say that $\varphi(r)$ has property $(*, *)$ if

$$\sup_{r \leqq t \leqq 2r} \varphi(t) \leqq B \cdot \varphi(r), \quad r > 0, \tag{$*, *$}$$

holds for some real number B independent of r, $B \geqq 1$.

From now on the numbers A and B are always defined as in $(*)$ and $(*, *)$ respectively.

If $\varphi(r)$ is non-decreasing, then $(*)$ holds trivially and $(*, *)$ is equivalent to a Δ_2-condition [1, p. 79].

A positive function $g(x)$ on $\mathbf{R}^1$ is called almost decreasing if there is M, $0 < M < \infty$, such that

$$g(x) \leqq M \cdot g(y), \quad \text{for all } x \geqq y.$$

Basic properties of BMO_φ are found in S. Spanne [14]. See also [5] and [10]. We collect some of them, generalized to $1 \leqq p < \infty$ or $0 < p < \infty$, in a lemma.

Lemma 1.1. *Assume that E is a bounded set with positive measure, $1 \leqq p < \infty$ and let f be a function.*

(a) $$\Omega_p(f, E) \leqq \left(|E|^{-1} \int_E |f(x) - f(E)|^p \, dx\right)^{1/p} \leqq 2 \cdot \Omega_p(f, E)$$

(b) $$\Omega_p(f, E) \leqq \left(|E|^{-2} \int_E \int_E |f(x) - f(y)|^p \, dx \, dy\right)^{1/p} \leqq 2 \cdot \Omega_p(f, E).$$

(c) *Let I and J be two cubes with side lengths r and s respectively. Then if $I \subset J$, $s \leqq 2 \cdot r$, $f \in BMO_{\varphi, p}$, and $\varphi(r)$ satisfies $(*)$ we get for $0 < p < \infty$*

$$|C_I - C_J| \leqq 2^{\frac{n+1}{p}} \cdot \|f\|_{\varphi, p} \cdot (\varphi(r) + \varphi(s)).$$

(d) *Let $F: \mathbf{R}^1 \to \mathbf{R}^1$ be such that $|F(x) - F(y)| \leqq M \cdot |x - y|$, all $x, y \in \mathbf{R}^1$. Then for $0 < p < \infty$,*

$$\|F \circ f\|_{\varphi, p} \leqq M \cdot \|f\|_{\varphi, p},$$

where $F \circ f(x) = F(f(x))$.

(e) *Let $\varphi(r)$ be non-decreasing and $\varphi(r)/r$ almost decreasing. Then*

$$g(x) = \int_{|x|}^1 \varphi(t)/t \, dt$$

belongs to BMO_φ.

Proof of Lemma 1.1. The statements (a) and (b) follow from Minkowski inequality and a proof of (d) only uses the definitions. The parts (c), $1 \leqq p < \infty$

and (e) are proved in S. Spanne [14]. We prove (c) for $0<p<1$. Integrating the inequality

$$|C_I-C_J|^p \leqq |f(x)-C_I|^p+|f(x)-C_J|^p$$

over I we get

$$\begin{aligned}|C_I-C_J|^p &\leqq |I|^{-1}\cdot \int_I |f(x)-C_I|^p\,dx \\ &\quad +2^n\cdot |J|^{-1}\int_J |f(x)-C_J|^p\,dx \\ &\leqq \|f\|^p_{\varphi,p}\cdot(\varphi(r)^p+2^n\cdot\varphi(s)^p) \\ &\leqq 2^{n+1-p}\cdot\|f\|^p_{\varphi,p}\cdot(\varphi(r)+\varphi(s))^p\end{aligned}$$

which completes the proof of Lemma 1.1.

Let $\lambda\in\mathbf{R}^1$ and $F(x)=\max(x,\lambda)$, $x\in\mathbf{R}^1$, then

$$F\circ f(x)=\max\bigl(f(x),\lambda\bigr),$$

for any real valued function f. By Lemma 1.1 (d) we get for $0<p<\infty$

$$\|F\circ f\|_{\varphi,p}\leqq\|f\|_{\varphi,p}.$$

The function $F\circ f$ is f truncated from below at the level λ. Truncation from above gives an analogous result.

Combining this with Lemma 1.1 (e), we get that $\ln|x|$ and $\ln^+|x|=\max(\ln|x|,0)$ are functions in BMO. These facts are used in some of our examples in section 7.

2. Statements of the theorems

In this section we state our main results and some of their consequences. The proofs are found in sections 4, 5 and 6.

2.1. We start with our basic estimate of $\Omega_p(f,K_r)$, when K_r is of the type (0.2).

Theorem 2.1. *Let $0<p<\infty$ and let $\varphi(r)$ satisfy $(*)$. Let K be a compact set with measure zero. Then if $f\in BMO_{\varphi,p}$ we have*

$$\Omega_p(f,K_r)\leqq N\cdot\int_r^k\varphi(t)/t\,dt \tag{2.1}$$

where $N=c^{1+\frac{1}{p}}\cdot A\cdot\|f\|_{\varphi,p}$, $k=4\cdot\bigl(d+2r(\sqrt{n}+1)\bigr)$ and d is the diameter of K.

Corollary. *Let $f\in BMO_p$, $0<p<\infty$, then*

$$\Omega_p(f,K_r)\leqq N\cdot\ln\left(2+\frac{d}{r}\right),\quad r>0 \tag{2.2}$$

where $N=c^{1+\frac{1}{p}}\cdot\|f\|_{*,p}$.

The corollary follows by taking $\varphi(r)\equiv 1$ in Theorem 2.1. We show by examples in section 7.1. that in general (2.1) is best possible when $p=1$. In particular (2.2) is best possible in the sense that there is $f\in BMO$ and K such that

$$\Omega(f, K_r) \geqq c\cdot \|f\|_* \cdot \ln\left(2+\frac{d}{r}\right), \quad r>0,$$

where $d=\operatorname{diam} K$.

This raises the question for which sets K, $|K|=0$, there is $f\in BMO$ such that (2.2) holds with the inequality reversed. It is clearly necessary that K has at least two points. It is an open question if this is a sufficient condition. We give some examples in section 7.1.

Theorem 2.1 is also a theorem on the best local approximation of order zero in the L^p-norm over r-neighbourhoods K_r of compact sets K with measure zero. See [1] for definitions and a survey of results on local approximation.

Remark. Theorems 2.1—2.5 can be proved also for $p=\infty$ by simple arguments from measure theory. Let $f\in BMO_{\varphi,p}$, $p=\infty$, and let $\varphi(r)$ be non-decreasing, then

$$|f(x)-f(y)| \leqq c\cdot \|f\|_{\varphi,p}\cdot \varphi(|x-y|),$$

for a.e. $x, y\in \mathbf{R}^n$. This implies that $BMO_{\varphi,\infty}\subset \Lambda_\varphi$, with continuous imbedding. The converse inclusion holds if $\varphi(r)$ satisfies $(**)$, cf. S. Jansson [5]. We do not go any further into this case.

The converse of Theorem 2.1 is easily proved when $\varphi(r)$ satisfies a suitable growth condition.

Theorem 2.2. *Let $0<p<\infty$ and let $\varphi(r)$ satisfy $(**)$. Assume that there is $N<\infty$, only depending on f, φ, p and n such that*

$$\Omega_p(f, K_r) \leqq N\cdot \int_r^k \varphi(t)/t\,dt, \quad r>0, \tag{2.3}$$

for all compact sets K of measure zero, where $k=c(r+d)$, $c>1$, and d is the diameter of K. Then $f\in BMO_{\varphi,p}$ and

$$\|f\|_{\varphi,p} \leqq c^{\frac{1}{p}}\cdot c(B, n)\cdot N.$$

Remark. Actually we only need to assume that (2.3) holds for $d=0$ and $r>0$. Theorem 2.2 becomes the easy part of our next theorem which gives a new characterization of $BMO_{\varphi,p}$.

Remark. Here we want to make a general remark on the case $0<p<1$. In the following, we state and prove our theorems (Theorems 2.3—2.5) only for $1\leqq p<\infty$, since we are mainly interested in the case when $\|f\|_{\varphi,p}$ is a norm on the Banach space $BMO_{\varphi,p}$. However, Theorems 2.3—2.5 hold also for $0<p<1$, with only minor changes in their statements ($\|f\|_{\varphi,p}$ is no longer a norm) and proof. We leave the details to the interested reader. See also J-O. Strömberg [15] about results for BMO_p, $0<p<1$, and related spaces.

Let $N(f)$ be the best constant N in (2.3), $1 \leqq p<\infty$. It is easily seen that $N(f)$ is a seminorm on the space of functions f for which $N(f)<\infty$ and that $N(f)=0$ if and only if f is constant a.e. Theorems 2.1 and 2.2 now show that $N(f)$ is a norm on $BMO_{\varphi,p}$ equivalent to $\|f\|_{\varphi,p}$, $1 \leqq p<\infty$.

Theorem 2.3. *Let $1 \leqq p<\infty$ and assume that $\varphi(r)$ satisfies $(*)$ and $(**)$. Let f be a function defined in $\mathbf{R}^n$ and let $N(f)$ be the best constant N in* (2.3). *Then $N(f)$ is a norm on $BMO_{\varphi,p}$ equivalent to $\|f\|_{\varphi,p}$ and*

$$c(A,n)\cdot\|f\|_{\varphi,p} \leqq N(f) \leqq c(B,n)\cdot\|f\|_{\varphi,p}.$$

Theorem 2.3 is an immediate consequence of the Theorems 2.1 and 2.2. We have the following special case of Theorem 2.3.

Corollary. *A function f belongs to BMO_p, $1 \leqq p<\infty$, if and only if* (2.2) *holds for some real number N only depending on f, p and n. Then*

$$c_1\cdot\|f\|_{*,p} \leqq N(f) \leqq c_2\cdot\|f\|_{*,p},$$

where $N(f)$ is the best constant N in (2.2).

2.2. Let G be an open subset of $\mathbf{R}^n$ and $0<p<\infty$. We define $BMO_{\varphi,p}(G)$ as the space of functions f for which

$$\|f\|_{\varphi,p,G} = \sup_I \Omega_p(f,I)/\varphi(r)$$

is finite, where supremum is over all cubes $I=I(a,r)\subset G$. We write $BMO_{\varphi,p}$ instead of $BMO_{\varphi,p}(G)$, when $G=\mathbf{R}^n$. When $\varphi(r)\equiv 1$ we adopt a notation analogous to the case $G=\mathbf{R}^n$ in section 1. We consider $BMO_{\varphi,p}(G)$, $1 \leqq p<\infty$, as linear spaces with semi-norms $\|f\|_{\varphi,p,G}$. If G is a connected set then $BMO_{\varphi,p}(G)$, $1 \leqq p<\infty$, is a Banach space modulo constants.

We study the problem when a function $f\in BMO_{\varphi,p}(G)$ can be extended to a function $\tilde{f}$ in $BMO_{\varphi,p}$. When $\mathbf{R}^n\setminus G$ has measure zero we must have $\tilde{f}(x)=f(x)$ a.e. In this case we can apply Theorems 2.1 and 2.2 to get necessary and sufficient conditions on $f\in BMO_{\varphi,p}(G)$ so that $\tilde{f}=f\in BMO_{\varphi,p}$.

Theorem 2.4. *Let $1 \leqq p<\infty$ and let G be an open connected set in $\mathbf{R}^n$ whose complement $F=\mathbf{R}^n\setminus G$ is a non-empty set of measure zero Let $\varphi(r)$ satisfy $(*)$ and $(**)$ and let f be a function defined in $\mathbf{R}^n$. Then $f\in BMO_{\varphi,p}$ if and only if $f\in BMO_{\varphi,p}(G)$ and there is $N<\infty$, only depending on f, φ, p and n, such that*

$$\Omega_p(f,K_r) \leqq N\cdot \int_r^k \varphi(t)/t\,dt,\quad r>0 \tag{2.4}$$

holds for every compact set $K \subset F$, *where* $k=c(r+d)$, $c>1$, *and* d *is the diameter of* K.

Let $N(f)$ *be the best constant* N *in* (2.4), *then*

$$c(B, n)\cdot\|f\|_{\varphi,p} \leqq \|f\|_{\varphi,p,G}+N(f) \leqq c(A, n)\cdot\|f\|_{\varphi,p}. \tag{2.5}$$

Corollary. *Let* G *and* F *be as in the theorem. Then* $f\in BMO_p$ *if and only if* $f\in BMO_p(G)$ *and* (2.2) *holds for every compact set* $K\subset F$. *If* $N(f)$ *is the best constant in* (2.2), *then*

$$c_1\cdot\|f\|_{*,p} \leqq \|f\|_{*,p,G}+N(f) \leqq c_2\cdot\|f\|_{*,p}.$$

To prove the corollary take $\varphi(r)\equiv 1$ in Theorem 2.4.

The necessity and sufficiency parts of Theorem 2.4 are proved using Theorems 2.1 and 2.2 respectively. Thus the sufficiency part is rather easy and therefore Theorem 2.4 is mainly a necessary condition for a function $f\in BMO_{\varphi,p}(G)$ to belong to $BMO_{\varphi,p}$. In section 7.2 we give an example where Theorem 2.4 is used for this purpose in the BMO-case.

2.3. For general open sets our methods only give necessary conditions on f. Assume that G is an open set and $f\in BMO_{\varphi,p}(G)$, $1\leqq p<\infty$. Then if f has an extension $\tilde{f}$ in $BMO_{\varphi,p}$ it is necessary that (2.1) holds for all compact sets K of measure zero, $K\subset G$ and $0<r<r_0$, where $r_0=\operatorname{dist}(K, \partial G)$. This follows immediately from Theorem 2.1. We do not go any further into this case.

Remark. P. W. Jones has studied the problem to extend *every* function $f\in BMO(G)$ to a function $\tilde{f}\in BMO$, when G is connected. He solved it in terms of the Whitney decomposition of G [7, Theorem 1]. It is for example well known that when G is the open unit ball in $\mathbf{R}^n$, then every $f\in BMO(G)$ has an extension $\tilde{f}$ in BMO.

2.4. The results in Theorem 2.4 can be considered as a method to generate equivalent norms on $BMO_{\varphi,p}$, $1\leqq p<\infty$. Let F be any non-empty closed set with measure zero Let $G=R^n\setminus F$ and define

$$\|f\|_{\varphi,p,F} = \|f\|_{\varphi,p,G}+N(f),$$

where $N(f)$ is as in Theorem 2.4. Then we have the following theorem

Theorem 2.5. *Let* $1\leqq p<\infty$ *and let* $\varphi(r)$ *satisfy* $(*)$ *and* $(**)$. *Then* $\|f\|_{\varphi,p,F}$ *is an equivalent norm on* $BMO_{\varphi,p}$ *and*

$$c(B, n)\cdot\|f\|_{\varphi,p} \leqq \|f\|_{\varphi,p,F} \leqq c(A, n)\cdot\|f\|_{\varphi,p}. \tag{2.6}$$

Note that the constants in (2.6) are independent of F. Theorem 2.5 follows easily from Theorem 2.4 and its proof.

2.5. Our next theorem is an extension theorem for continuous functions of a type studied by H. Wallin [16] and the author [13, 14]. Similar extension problems for Hölder continuous functions have been studied by A. Jonsson [8]. Let $C(K)$ denote the space of continuous functions on K.

Theorem 2.6. *Let K be a compact set in $\mathbf{R}^n$. Then every $f_0 \in C(K)$ has an extension f which is continuous in $\mathbf{R}^n$ and has arbitrary small norm in BMO, if and only if K has measure zero.*

Theorem 2.6 is proved by a method to build up continuous Riesz potentials due to H. Wallin [16] and used by the author for non-linear potentials [13]. The building blocks are formed from a lemma by J. B. Garnett and P. W. Jones [4, Lemma 2.2]. Since both these results are known we only indicate how the pieces are put together (Section 6).

3. Lemmas

Let $\{E_i\}_1^N$ be a class of pairwise disjoint, bounded sets with positive measure, $E=\bigcup_1^N E_i$ and let f be a function. Let $c_i=C_{E_i}$ be such that

$$\left(|E_i|^{-1}\int_{E_i}|f(x)-c_i|^p\,dx\right)^{1/p} = \Omega_p(f, E_i), \quad 1 \leqq i \leqq N.$$

Lemma 3.1.

(a) *Let* $0<p\leqq 1$, *then*

$$\Omega_p(f, E)^p \leqq \max_i \Omega_p(f, E_i)^p + \max_{i,j} |c_i-c_j|^p$$

(b) *Let* $1\leqq p<\infty$, *then*

$$\Omega_p(f, E) \leqq \max_i \Omega_p(f, E_i) + \max_{i,j} |c_i-c_j|.$$

Proof of Lemma 3.1. We give the proof for the case $0<p\leqq 1$. Let $\alpha_i=|E_i|/|E|$ and $c=\sum_1^N \alpha_i\cdot c_i$, then using Minkowski's Inequality we get

$$\begin{aligned}\Omega_p(f, E)^p &\leqq |E|^{-1}\int_E |f(x)-c|^p\,dx \\ &= \sum_{i=1}^N \alpha_i\cdot|E_i|^{-1}\int_{E_i}|f(x)-c|^p\,dx \\ &\leqq \sum_{i=1}^N \alpha_i\cdot\left(|E_i|^{-1}\int_{E_i}|f(x)-c_i|^p\,dx+|c_i-c|^p\right) \\ &\leqq \max_i \Omega_p(f, E_i)^p + \max_{i,j}|c_i-c_j|^p.\end{aligned}$$

This proves Lemma 3.1 when $0<p\leqq 1$. The case $1<p<\infty$ is treated analogously and the proof is therefore omitted.

Lemma 3.2. *Let E be a bounded measurable set in $\mathbf{R}^n$, $r>0$ and $k\geqq 1$. Then*

$$|E_{k\cdot r}| \leqq k^n \cdot |E_r|.$$

Lemma 3.2 is a measure-theoretical consequence of the following purely geometrical lemma.

Lemma 3.3. *Let $k\geqq 1$ and $r>0$, then*

$$\Big|\bigcup_{i=1}^{N} B(x_i, k\cdot r)\Big| \leqq k^n \cdot \Big|\bigcup_{i=1}^{N} B(x_i, r)\Big|. \tag{3.1}$$

where B stands for balls in $\mathbf{R}^n$.

Proof. If we use the formula $|D|=\int_D 1\,dx$ with $D=\bigcup_1^N B(x_i, k\cdot r)$, then a simple change of variables shows that (3.1) is equivalent to

$$\left|\bigcup_1^N B\left(\frac{x_i}{k}, r\right)\right| \leqq \left|\bigcup_1^N B(x_i, r)\right|. \tag{3.2}$$

We prove (3 2) by induction over N. For $N=1$ there is nothing to prove. Assume that (3.2) holds for any collection of N balls and that we are given $(N+1)$ balls.

It follows from the equivalence between (3.1) and (3.2) that we may assume $x_{N+1}=0$. Then we get

$$\left|\bigcup_1^{N+1} B\left(\frac{x_i}{k}, r\right)\right| = \left|\bigcup_1^N B\left(\frac{x_i}{k}, r\right)\right| + |B(0, r)| - \left|B(0, r) \cap \left(\bigcup_1^N B\left(\frac{x_i}{k}, r\right)\right)\right|.$$

By the induction hypothesis we have

$$\left|\bigcup_1^N B\left(\frac{x_i}{k}, r\right)\right| \leqq \left|\bigcup_1^N B(x_i, r)\right|.$$

We complete the proof by showing that

$$B(0, r)\cap\Big(\bigcup_1^N B(x_i, r)\Big)\subset B(0, r)\cap\left(\bigcup_1^N B\left(\frac{x_i}{k}, r\right)\right). \tag{3.3}$$

It suffices to prove (3.3) when $N=1$, i.e.

$$B(0, r)\cap B(x_1, r)\subset B(0, r)\cap B\left(\frac{x_1}{k}, r\right). \tag{3.4}$$

Now (3.4) follows from the inequality

$$\left|z-\frac{a}{k}\right| = \left|\left(1-\frac{1}{k}\right)\cdot z+\frac{1}{k}\cdot(z-a)\right|$$

$$\leqq \left(1-\frac{1}{k}\right)\cdot |z|+\frac{1}{k}\cdot |z-a|, \quad z\in\mathbf{R}^n \text{ and } k \geqq 1, \text{ with } a = x_1.$$

Lemma 3.4. *Let I and J be two cubes with side lengths r and s respectively, $I\subset J$. Then if $\varphi(r)$ satisfies $(*)$, $0<p<\infty$ and $f\in BMO_{\varphi,p}$ we have*

$$|C_I-C_J|\leqq c^{1+\frac{1}{p}}\cdot A\cdot\|f\|_{\varphi,p}\cdot\int_r^{2s}\varphi(t)/t\,dt.$$

The proof is by repeated use of Lemma 1.1 (c).
See S. Spanne [14] for the case $p=1$. The general case is proved analogously. Compare also [7, Lemma 1.1].

Lemma 3.5. *Let E and F be bounded sets with positive measure, $E\subset F$. Then for $0<p<\infty$,*

$$\Omega_p(f,E)\leqq(|F|/|E|)^{1/p}\cdot\Omega_p(f,F).$$

Proof of Lemma 3.5. By definition we have

$$\Omega_p(f,E)=\left(|E|^{-1}\int_E|f(x)-C_E|^p\,dx\right)^{1/p}$$

$$\leqq\left(|E|^{-1}\int_E|f(x)-C_F|^p\,dx\right)^{1/p}\leqq(|F|/|E|)^{1/p}\cdot\Omega_p(f,F),$$

and the lemma is proved.

4. Proof of Theorem 2.1

Let K be a compact set with measure zero and let $f\in BMO_{\varphi,p}$, $0<p<\infty$. We are going to prove that

$$\Omega_p(f,K_r)\leqq c^{1+\frac{1}{p}}\cdot A\cdot\|f\|_{\varphi,p}\cdot\int_r^k\varphi(t)/t\,dt,\quad r>0,$$

where $k=4\cdot(d+2r(1+\sqrt{n}))$.

Let M be a net of congruent cubes of side length r and let $M_0=\{I_k\}_1^N$ be the collection of those cubes in M which intersect K_r, $E=\bigcup_1^N I_k$, $K_r\subset E$.

Let c_k be such that

$$\left(|I_k|^{-1}\int_{I_k}|f(x)-c_k|^p dx\right)^{1/p}=\Omega_p(f,I_k),\quad 1\leqq k\leqq N.$$

Then we have by Lemma 3.2 and Lemma 3.5

$$\Omega_p(f,K_r)\leqq\left(\frac{|E|}{|K_r|}\right)^{1/p}\cdot\Omega_p(f,E)\leqq c^{1/p}\cdot\Omega_p(f,E),$$

since $E\subset K_u$ and $|K_u|\leqq(1+\sqrt{n})^n\cdot|K_r|$, for $u=r(1+\sqrt{n})$.

For every I_k we have $\Omega_p(f,I_k)\leqq\|f\|_{\varphi,p}\cdot\varphi(r)$ and Lemma 3.1 (a) gives for $1\leqq p<\infty$,

$$\Omega_p(f,K_r)\leqq c^{1+\frac{1}{p}}\cdot\left(\|f\|_{\varphi,p}\cdot\varphi(r)+\max_{i,j}|c_i-c_j|\right).$$

Assume that K has diameter d, then $\operatorname{diam} E \leqq d+2r(1+\sqrt{n})$. There is a cube I_0 with side length $r_0=2(d+2r(1+\sqrt{n}))$ containing E. Lemma 3.4 gives

$$\Omega_p(f, K_r) \leqq c^{1+\frac{1}{p}} \cdot (\|f\|_{\varphi,p} \cdot \varphi(r) + \max_i |c_i - c_0|)$$

$$\leqq c^{1+\frac{1}{p}} \cdot A \cdot \|f\|_{\varphi,p} \cdot \int_r^{2r_0} \varphi(t)/t \, dt, \tag{4.1}$$

where c_0 is such that

$$\Omega_p(f, I_0) = \left(|I_0|^{-1} \int_{I_0} |f(x)-c_0|^p \, dx\right)^{1/p}.$$

Using Lemma 3.1 (b) we can analogously prove (4.1) for $0<p<1$. Now putting $k=2 \cdot r_0$ we get

$$\Omega_p(f, K_r) \leqq c^{1+\frac{1}{p}} \cdot A \cdot \|f\|_{\varphi,p} \cdot \int_r^k \varphi(t)/t \, dt$$

and thereby Theorem 2.1 is proved.

5. Proof of Theorems 2.2 and 2.4

5.1. Proof of Theorem 2.2. Let $I=I(a, r)$ be a cube and put $K=\{a\}$, $s=r\sqrt{n}$. Then by Lemma 3.5 and (2.3) we get for $0<p<\infty$

$$\Omega_p(f, I) \leqq \left(\frac{|K_s|}{|I|}\right)^{1/p} \cdot \Omega_p(f, K_s) \leqq c^{1/p} \cdot \Omega_p(f, K_s)$$

$$\leqq c^{1/p} \cdot N \cdot \int_s^{c \cdot s} \frac{\varphi(t)}{t} \leqq c^{1/p} \cdot c(B, n) \cdot N \cdot \varphi(r).$$

It follows that $\|f\|_{\varphi,p} \leqq c^{1/p} \cdot c(B, n) \cdot N$, which completes the proof of Theorem 2.2.

5.2. Proof of Theorem 2.4. Let $1 \leqq p<\infty$, G is an open set and $F=\mathbf{R}^n \setminus G$ is a non-empty set with measure zero. Let f be a function in $\mathbf{R}^n$. If $f \in BMO_{\varphi,p}$, then $f \in BMO_{\varphi,p}(G)$ and $\|f\|_{\varphi,p,G} \leqq \|f\|_{\varphi,p}$. Theorem 2.1 gives that (2.4) holds and

$$\|f\|_{\varphi,p,G} + N(f) \leqq c^{1+\frac{1}{p}} \cdot A \cdot \|f\|_{\varphi,p}. \tag{5.2}$$

Conversely, assume that $f \in BMO_{\varphi,p}(G)$ and (2.4) holds for every compact set $K \subset F$. Let I be a cube If $I \subset G$, then $\Omega_p(f, I) \leqq \varphi(r) \cdot \|f\|_{\varphi,p,G}$. Suppose that I intersects F and define $K=I \cap F$. Then if $s=r\sqrt{n}$ we get

$$I \subset K_s \subset (1+2\sqrt{n}) \cdot I. \tag{5.3}$$

Then by Lemma 3.5, (2.4) and (5.3) we have

$$\Omega_p(f, I) \leqq c^{1/p} \cdot \Omega_p(f, K_s) \leqq c(B, n) \cdot N \cdot \varphi(r),$$

since $\operatorname{diam} K \leqq r \cdot \sqrt{n}$.

It follows that

$$\|f\|_{\varphi, p} \leqq c(B, n) \cdot (\|f\|_{\varphi, p, G} + N(f)), \tag{5.4}$$

where $N(f)$ is the best constant N in (2.4). Combining (5.2) and (5.4) we have proved Theorem 2.4.

6. Proof of Theorem 2.6

Assume that K has the extension property and $|K|>0$. Then for every $f \in C(K)$ there is a sequence $\{f_n\}_1^\infty$ of continuous extensions converging to zero in BMO. Let I be a large cube containing K in its interior. Then there is a constant $\alpha(I)$ and a subsequence converging to $\alpha(I)$ pointwise a.e. in I [10, p. 67]. It follows that f is constant a.e. on K. This can hold for every $f \in C(K)$ only if K has measure zero and thus we have proved the necessity part of the theorem.

As we stated in the introduction we only sketch the proof of the sufficiency part of the theorem. We start with the following construction in BMO due to J. B. Garnett and P. W. Jones [4, Lemma 2.2]. For any cube I in $\mathbf{R}^n$ we let $\tilde{I} = 3 \cdot I$.

Lemma 6.1. *Let m be a positive integer, $\varepsilon > 0$ and $\lambda = m \cdot \varepsilon$. Let Q be a dyadic cube and let $\{Q_i\}_1^\infty$ be a class of pairwise disjoint dyadic subcubes of Q. If*

$$\sum_1^\infty |Q_i| \leqq 4^{-m \cdot n} \cdot |Q|,$$

then there is $g \in BMO$, g continuous if $\{Q_i\}$ is finite, such that

(a) $\operatorname{supp} g \subset \tilde{Q}$, *(b)* $0 \leqq g(x) \leqq \lambda$,

(c) $g(x) \equiv \lambda$ *on* $\bigcup_1^\infty Q_i$, *(d)* $\|g\|_* \leqq C \cdot \varepsilon$,

(e) $|\operatorname{supp} g| \leqq 3^n \cdot 4^{m \cdot n} \cdot \sum_1^\infty |Q_i|$.

The constant C only depends on n.

Let K be a compact set in $\mathbf{R}^n$ with measure zero and V an open set containing K. We cover K with congruent dyadic cubes I_ν, such that $\tilde{I}_\nu \subset V$, $1 \leqq \nu \leqq N$, i.e.

$$K \subset \bigcup_1^N I_\nu.$$

Let Q in Lemma 6.1 be one of the cubes I_ν and cover $I_\nu \cap K$ with a finite set of dyadic subcubes of $Q = I_\nu$, whose total measure is so small that the function $g = g_\nu$

constructed in Lemma 6.1 with $\lambda=1$ satisfies

$$\|g_\nu\|_* < \frac{\varepsilon}{2N}, \quad 1 \leqq \nu \leqq N. \tag{6.1}$$

Now define

$$g(x) = \sum_1^N g_\nu(x), \; f(x) = 2\cdot\min\left(g(x), \frac{1}{2}\right).$$

Lemma 6.2. *Let K be a compact set with measure zero contained in an open set V. Then for every $\varepsilon>0$ there is a continuous function $f\in BMO$ such that*

(a) $0\leqq f(x)\leqq 1$, (b) $f(x)\equiv 1$ *in a neighbourhood of K,* (c) $\operatorname{supp} f\subset V$, *and* (d) $\|f\|_*<\varepsilon$.

Proof of Lemma 6.2. The properties (a), (b) and (c) are obvious, since the set where $g(x)>\frac{1}{2}$ is open. Part (d) now follows from (6.1) and the text following Lemma 1.1. This proves the lemma.

The proof of Theorem 2.6 is now by means of Lemma 6.2 and the method used by H. Wallin in [16]. We omit the details since they are easily found there. We note that our Lemma 6.2 corresponds to [16, Lemma 1].

7. Some Examples

We start with some examples to show that the estimate (2.1) in Theorem 2.1 is in general best possible when $p=1$.

7.1. Example 1. Assume that $\varphi(r)$ is non-decreasing and concave and that $\varphi(r)/r$ is almost decreasing. Let $K=\{-l, l\}$ be a two-point set in $\mathbf{R}^1$ with diameter $d=2l>0$. We put

$$g_1(x) = \max\left(\int_{|x-l|}^{2l} \varphi(t)/t\,dt, \; 0\right)$$

$$g_2(x) = \begin{cases}\varphi(x-3l), & x\geqq 3l\\ 0, & x<3l\end{cases}$$

and define $g(x)=g_1(x)+g_2(x)$, $f(x)=g(x)-g(-x)$, $x\in\mathbf{R}^1$. We can now prove that $f\in BMO_\varphi$ and

$$\Omega(f, K_r) \geqq c(\varphi)\cdot\|f\|_\varphi\cdot\int_r^{\frac{4}{3}(r+d)}\varphi(t)/t\,dt, \; r>0. \tag{7.1}$$

We only give a short sketch of the proof. Clearly $\|g_1\|_\varphi\leqq c(\varphi)$ by Lemma 1.1 (d) and (e), and $\|g_2\|_\varphi\leqq 2$ by Lemma 1.1 (b) and the concavity of $\varphi(r)$. It is easily seen that $f(K_r)=0$, and $|K_r|\leqq c\cdot r$, $r>0$. A direct calculation now gives (7.1) and completes the example.

Our next example is of the same type, but with an infinite set K.

Example 2. Let $K=\{\pm 2^{-\nu},\ \nu=0, 1, \ldots\}\cup\{0\}$ be a set in $\mathbf{R}^1$. Define for $k=0, 1, \ldots, l_k=2^{-k-2}$, $g_k(x)=\ln^+(l_k/|x|)$ and $f_k(x)=g_k(x-2^{-k})$.

Then let $f=\sum_1^\infty f_k(x)$, $0\leqq x\leqq 2$, $f(x)=1$, for $x\geqq 2$ and $f(-x)=-f(x)$, $x\in\mathbf{R}^1$. We are going to prove that $f\in BMO$ and

$$\Omega(f, K_r) \geqq c\cdot\ln(2+2/r), \quad r>0. \tag{7.2}$$

We prove that $f\in BMO$ in a lemma which is of interest in itself. It is a variant of [4, Lemma 2.1] with the same method of proof.

Lemma 7.1. *Let $\{I_j\}_1^\infty$ be a class of pairwise disjoint cubes in R^n and let $\{f_j\}_1^\infty$ be functions such that*

$$\operatorname{supp} f_j \subset I_j, \quad \|f_j\|_1 \leqq a\cdot|I_j|, \quad \|f_j\|_* \leqq b, \quad \textit{for} \quad j=1, 2, \ldots .$$

Then $f=\sum_1^\infty f_j$ is in BMO and $\|f\|_*\leqq c(a+b)$.

Proof of Lemma 7.1. Let I be a cube in $\mathbf{R}^n$ and define

$$A=\{j;\ I_j\cap I\neq\emptyset \text{ and } l(I_j)\leqq l(I)\},$$

$$g(x)=\sum\nolimits_A f_j(x),\ f(x)=g(x)+h(x).$$

Then

$$\Omega(g, I)\leqq 2\cdot|I|^{-1}\cdot a\cdot\sum\nolimits_A|I_j|\leqq 2\cdot 3^n\cdot a$$

since all cubes I_j, $j\in A$, are contained in the cube $3\cdot I$. Let

$$B=\{j;\ I_j\cap I\neq\emptyset \text{ and } l(I_j)>l(I)\},$$

then $\operatorname{Card} B\leqq 3^n$ and

$$\Omega(h, I)\leqq\sum\nolimits_B\Omega(f_j, I)\leqq 3^n\cdot b.$$

Thus $\|f\|_*\leqq c\cdot(a+b)$ and Lemma 7.1 is proved.

It is easy to check that Lemma 7.1 applies to $f=\sum_1^\infty f_k$ and hence $f\in BMO$. In view of the fact that $|K_r|\leqq c\cdot r\cdot\ln 1/r$, $0<r<\dfrac{1}{4}$, we must prove that

$$\int_{K_r}|f(x)|\,dx\geqq c\cdot r\cdot(\ln 1/r)^2,\quad 0<r<\frac{1}{4}.$$

Let $2^{-2-m}\leqq r<2^{-1-m}$, $m\geqq 1$, then

$$\int_{K_r}|f|dx\geqq 2\cdot\sum\nolimits_0^{m-1}\int_0^r g_k(x)\,dx$$

$$\geqq 2\cdot r\cdot\sum\nolimits_0^{m-1}\ln\left(\frac{r}{l_k}\right)\geqq c\cdot r\cdot m^2$$

$$\geqq c\cdot r\cdot\left(\ln\frac{1}{r}\right)^2,\quad 0<r<\frac{1}{4}.$$

It is easily seen that $\Omega(f, K_r) \geqq c$, when $r \geqq \frac{1}{4}$. This completes the proof of (7.2). We can construct similar examples in higher dimensions in the following way. Consider

$$\mathbf{R}^{m+n} \text{ with points } (x, y),\ x \in \mathbf{R}^m,\ y \in \mathbf{R}^n.$$

Let $g \in BMO_{\varphi, p}(\mathbf{R}^m)$, $0 < p < \infty$, and define

$$f(x, y) = g(x),\ (x, y) \in \mathbf{R}^{m+n}.$$

Then $\|f\|_{\varphi,p} = \|g\|_{\varphi,p}$, where the norms are taken in the respective spaces. Let E and F be compact sets in $\mathbf{R}^m$ and $\mathbf{R}^n$ and define $K = E \times F$. Assume that E has measure zero in $\mathbf{R}^m$ and let E_r, F_r and K_r be defined with respect to the respective distance functions. Then K has measure zero in $\mathbf{R}^{m+n}$ and

$$\Omega_p(f, K_r) \geqq 2^{-\frac{m}{2p} - \frac{n}{2p}} \cdot \Omega_p\left(g, E_{\frac{r}{\sqrt{2}}}\right),\ r > 0.$$

Our Examples 1 and 2 above can be generalized to higher dimensions in this way.

7.2. Our next example illustrates the situation in the corollary of Theorem 2.4.

Example 3. We let (x, y) denote points in $\mathbf{R}^2$ and define

$$F = \{(x, y);\ |y| = e^{-1/|x|},\ x \neq 0\} \cup \{(0, 0)\},$$

$$G = \mathbf{R}^2 \setminus F,\ g(r) = \int_r^1 e^{1/t}\, dt,\ r > 0.$$

We let

$$f(x, y) = \begin{cases} g(|x|), & |y| < e^{-\frac{1}{|x|}},\ x > 0 \\ -g(|x|), & |y| < e^{-\frac{1}{|x|}},\ x < 0 \\ 0, & \text{elsewhere.} \end{cases}$$

We are going to prove that $f \in BMO(G)$ but that $\Omega(f, K_r)$ grows much faster to infinity than $\ln 1/r$, as $r \to 0.$, when $K = F \cap \{|x| \leqq 1\}$. Hence by the corollary of Theorem 2.4 it follows that $f \notin BMO$.

Let $I = [a, a+r] \times [b, b+r]$, $a > 0$ be a cube in G such that $f \not\equiv 0$ on I. Then $r \leqq 2 \cdot e^{-1/a}$ and

$$\Omega(f, I) \leqq \frac{1}{r^2} \cdot \int_a^{a+r} \int_a^{a+r} |g(x_1) - g(x_2)|\, dx_1\, dx_2 \leqq r \cdot e^{1/a} \leqq 2,$$

Hence $f \in BMO(G)$ and $\|f\|_{*,G} \leqq 2$. Let $K = F \cap \{|x| \leqq 1\}$, then $f(K_r) = 0$ and $|K_r| \leqq c \cdot r$, $r > 0$.

Thus we get with $\alpha=\left(\ln\frac{1}{r}\right)^{-1}$, $0<r<e^{-2}$,

$$\Omega(f, K_r) \geqq c\cdot\int_\alpha^1 (t-\alpha)e^{1/t}\,dt \geqq \frac{c}{\sqrt{r}}\cdot\left(\ln\frac{1}{r}\right)^{-2},$$

and hence $\Omega(f, K_r)/\ln 1/r\to\infty$, $r\to 0$. This completes the example.

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