Continuous manifolds in R^n that are sets of interpolation for the Fourier algebra

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1. Introduction

We are concerned with the question of when a continuous k-dimensional manifold $E \subseteqq R^n$ is a Helson set. Therefore we are concerned also with how the transform $\hat{\mu}$ decays at infinity when μ is a bounded Borel measure with support contained in a manifold E. The object is to understand the extent to which an E of a given dimension, and perhaps a given smoothness, can "participate" in the arithmetic structure and the harmonic analysis of Euclidean space. J.-P. Kahane and N. Th. Varopoulos have used Baire category arguments to produce examples of interest; our productions are more nearly constructive but use similar ideas. Kahane has shown the existence of Helson curves in R^n for $n \geqq 2$; we include constructions of such a curve, and of a Helson surface in R^6 as well. The latter construction is sufficiently general to allow us to outline the construction of Helson k-manifolds in R^{nk} for $n \geqq k+1$ and to give reasonable upper and lower bounds for their Helson constants.

It is well-known that sufficiently smooth manifolds E support measures whose transforms decay at infinity at a rate which is related to the curvature of E. The lemmas of van der Corput offer one such result. Björk has shown that C^1 manifolds which have, in a general sense, no "flat spots" support measures whose transforms tend to zero at infinity. Helson manifolds cannot support such measures. However, T. Hedberg constructed a continuous Helson graph E_θ in R^2 which supports a measure μ_θ with transform tending to zero at infinity on the cone $C_\theta = \{(x, y) \colon |y/x| \geqq \tan \theta\}$, for each $0 < \theta < \pi/2$. Katznelson and Körner have a more restrictive construction which improves this result. We show that the decay phenomenon exhibited by Hed-

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berg is common to all continuous graphs in R^2. Various extensions of this result and that of Björk are also provided for k-dimensional manifolds in R^n. Those results yield a lower bound for the Helson constants of a k-manifold in R^n.

We begin by explaining our conventions and terminology. Let $M=M(R^n)$ be the Banach algebra under convolution whose elements are the bounded Borel measures μ on R^n, with the total-variation norm. The transform $\hat{\mu}$ is the function on R^n given by

$$\hat{\mu}(y) = \int_{R^n} e^{-iy\cdot x}\, d\mu(x).$$

For an element f of the subalgebra $L^1=L^1(R^n)$,

$$\hat{f}(y) = \int_{R^n} e^{-iy\cdot x} f(x)\, dx.$$

The transform maps L^1 one-to-one onto an algebra of continuous functions vanishing at infinity which we denote by $A=A(R^n)$ and endow with pointwise operations and the norm $\|\hat{f}\|_A=\|f\|_1$. The transform is then an isometric isomorphism. The Banach space duals of $C_0=C_0(R^n)$ and A are respectively M and $PM=PM(R^n)$. The elements ν of PM, called pseudomeasures, are the distributions whose transforms are in $L^\infty(R^n)$. For $\hat{f}\in A$ and $\nu\in PM$,

$$\langle \hat{f}, \nu\rangle = \int_{R^n} f(x)\overline{\hat{\nu}(x)}\, dx;$$

$$\|\nu\|_{PM} = \|\hat{\nu}\|_\infty .$$

For a closed set $E\subseteqq R^n$, let $M(E)$ be the subspace of M consisting of the measures with support contained in E. Let $M_c(E)$ and $M_d(E)$ be the subspaces of $M(E)$ consisting of the measures that are continuous and discrete, respectively. Let $PM(E)=\{\nu\in PM\colon E$ contains the support of $\nu\}$ and $PF(E)=\{\nu\in PM(E)\colon \hat{\nu}(y)\to 0$ as $y\to\infty\}$. The elements of $PF(R^n)$ are called pseudofunctions. Let $A(E)$ be the quotient algebra $A/I(E)$, where $I(E)$ is the ideal $\{f\in A\colon f^{-1}(0)\supseteqq E\}$, and define $C_0(E)$ analogously. The natural norm-decreasing inclusion map: $A(E)\subseteqq C_0(E)$ has adjoint: $M(E)\subseteqq A(E)^*$. A set E is a *Helson set* if it is a set of interpolation for the algebra A, that is if $A(E)=C_0(E)$, or equivalently if its Helson constant

$$\begin{aligned}\alpha(E) &= \sup\{\|f\|_{A(E)}\colon f\in A(E) \text{ and } \|f\|_{C_0(E)}\leqq 1\}\\ &= \sup\{\|\mu\|\colon \mu\in M(E) \text{ and } \|\hat{\mu}\|_\infty\leqq 1\}\end{aligned}$$

is finite. Let

$$\alpha_c(E) = \sup\{\|\mu\|\colon \mu\in M_c(E) \text{ and } \|\hat{\mu}\|_\infty\leqq 1\},$$

$$\alpha_d(E) = \sup\{\|\mu\|\colon \mu\in M_d(E) \text{ and } \|\hat{\mu}\|_\infty\leqq 1\}.$$

A set E is a *Sidon set* if $\alpha_d(E)$, called the *Sidon constant* of E, is finite.

Most negative results, as to when a manifold E cannot be a Helson set or when the constants of E must be large, are based on the connection between $\alpha(E)$ and the behavior, for measures $\mu \in M(E)$, of $\hat{\mu}$ near infinity. The term "Helson set" arose in honor of the paper [7], where it is shown that if $E \subseteqq R$ and $M(E) \cap PF(E)$ contains a nonzero element, then $\alpha(E)=\infty$. The same is true for R^n, and in fact the following stronger result is now well-known.

1.1 Proposition. *Let $E \subseteqq R^n$, and let τ be a unit vector in R^n. Suppose that there exists a nonzero measure $\mu \in M(E)$ such that for every $\varepsilon > 0$ there exists d such that $|\hat{\mu}(y)| < \varepsilon$ whenever $|\tau \cdot y| > d$. Then $\alpha(E)=\infty$.*

We give but an indication of a proof. If $r_1, \ldots, r_n \in R$ and

$$d\nu_n(x) = \sum_{k=1}^{n} \varepsilon_k e^{ir_k \tau \cdot x} d\mu(x),$$

where $\varepsilon_k = -1$ or $+1$, then $\nu_n \in M(E)$ and $\hat{\nu}_n(y) = \sum_{k=1}^{n} \varepsilon_k \hat{\mu}(y - r_k \tau)$. Evidently if $|r_k|$ increases sufficiently fast with k, then $\|\nu_n\|_{PM} < \left(1+\sum_{k=1}^{n} k^{-2}\right)\|\mu\|_{PM}$. But for many choices of the signs ε_k, $\|\nu_n\| > \sqrt{n/3}\,\|\mu\|$ — by an argument like that of [13, p. 143] or [15, p. 18]. The result follows. The method of [3] also works.

On smooth manifolds it is relatively easy to find cases of μ such that $\hat{\mu}$ decays quickly at infinity. In particular many authors, with interests in harmonic synthesis and number theory ([14], [15], [22], [8], [9], [2], to mention a few) have studied the decay at infinity of $\hat{\sigma}$ where σ is the surface area form of a smooth manifold. Littman [17] showed that if at each point of a smooth n-manifold $E \subseteqq R^{n+1}$ (smooth depends on n but is at least C^2), k of its n principal curvature vectors are nonzero, then $\hat{\sigma}(y) = 0(|y|^{-k/2})$ as $y \to \infty$. For many C^1-manifolds $E \subset R^{n+1}$ the result of Björk [1, Prop. 1.2] gives $\hat{\sigma} \in C_0(R^{n+1})$. Hence $\alpha(E)=\infty$ for all such E's. As we shall see, even for manifolds E that are not very smooth, the behavior of $\hat{\mu}$ at infinity for $\mu \in M(E)$ is an attractive condition from which to obtain lower bounds for $\alpha(E)$.

Positive results began with Kahane's 1968 study of the still-unanswered rearrangements problem of N. Lusin. He proved among other results that if $\beta < 1$, and H_β is the metric space of nondecreasing Lip (β) functions on $[0, 1]$, then every pair (φ_1, φ_2) in the metric space $H_\beta \times H_\beta$, except for a set of first category, parametrically defines a Helson curve in R^2; and similarly in R^3 (see [11] or [12, Section VII.9]). In 1970 Varopoulos (see [27], [24], [25]) proved that Sidon manifolds of dimension $n-1$ are abundant in R^n. In fact, except for a set of first category, all real-valued functions in $C^s(R^{n-1})$ have Sidon graphs in R^n (where C^s is the space of functions whose partials of order $[s]$ are continuous and in Lip $(s-[s])$). However, for $s \geqq 1$ none of those manifolds are Helson sets, as we shall show.

In Section 2 we treat negative results in R^2, as to when a curve cannot be a

Helson set or when $\alpha(E)$ must be large. In Section 3 we construct a Helson Lip (1) curve in R^2. In Section 4 we return to negative results, generalizing to R^n. In Section 5 we construct a Helson surface in R^6 and show how to construct a Helson k-manifold in R^{nk} for $n \geqq k+1$.

2. Negative results for plane curves

Let G_f be the graph in R^2 of a real-valued measurable function defined on some set Y of finite, positive Lebesgue measure. Let μ_f denote the measure on G_f obtained by lifting Lebesgue measure "dx" from Y to the graph; that is, let

$$\int \Phi \, d\mu_f = \int_Y \Phi(x, f(x))\, dx \quad \text{for} \quad \Phi \in C(G_f)$$

or equivalently

$$\hat{\mu}_f(u, v) = \int_Y e^{-i(ux+vf(x))}\, dx \quad \text{for} \quad (u, v) \in R^2.$$

For $X \subseteqq Y$, let $\mu_{f|X}$ denote the restriction of μ_f to the set $\{(x, f(x)) \colon x \in X\}$.

We are concerned with the behavior of $\hat{\mu}_f$ at infinity under various conditions on f. Our treatment will progress from simple results with easy proofs toward more general and technical ones. As the remarks make clear, we are in some cases just refining or simplifying earlier work.

2.1 Lemma. *Let $f \in C^1(I)$ and $g \in C^1(J)$, where I and J are compact intervals. If $f'(I) \cap g'(J) = \emptyset$, then $\mu_f * \mu_g$ is absolutely continuous.*

Proof. For $(x, y) \in I \times J$, let $(u, v) = T(x, y) = (x+y, f(x)+g(y))$. We claim that T is one-to-one. If it were not, then there would be points $x, x' \in I$ and $y, y' \in J$ such that $x+y = x'+y'$ and $f(x)+g(y) = f(x')+g(y')$. If $a = x+y$, then $f(x)+g(a-x) = f(x')+g(a-x')$, and it would follow that $f'(t)-g'(a-t) = 0$ for some $t \in J$, which cannot be. The claim is proved. The Jacobian of T is $F(x, y) = g'(y) - f'(x)$, which never vanishes on $I \times J$. Thus by a change of variables, $\hat{\mu}_f \hat{\mu}_g$ is the transform of an element of $L^1(R^2)$:

$$\hat{\mu}_f(s, t)\hat{\mu}_g(s, t) = \int_{I \times J} e^{-i(s,t)\cdot(x+y, f(x)+g(y))}\, dx\, dy$$

$$= \int_{T(I \times J)} e^{-i(s,t)\cdot(u,v)} F^{-1}(u, v)\, du\, dv.$$

The lemma is proved.

The following result is in Björk [1, Prop. 1.2]. Our 2-dimensional proof is elementary. Let $|S|$ denote the Lebesgue measure of a set S.

2.2 Theorem. *If $f \in C^1[a, b]$ and $|(f')^{-1}(y)| = 0$ for each y, then μ_f^2 is absolutely continuous. In particular, $\mu_f \in PF(G_f)$ and $\alpha(G_f) = \infty$.*

Proof. The hypothesis implies that the distribution function $y \to |(f')^{-1}[y, \infty)|$ is continuous. Thus for each $\varepsilon > 0$ there is a partition $y_1, \dots, y_n$ of the range of f' such that for each j from 1 to $n-1$, $|B_j| < \varepsilon$, where $B_j = (f')^{-1}[y_j, y_{j+1}]$. The set B_j is a union of disjoint closed intervals. Its interior contains a compact set A_j, a finite union of closed intervals, such that $|A_j| > |B_j| - (\varepsilon/n)$. Let μ_j denote $\mu_{f|A_j}$ and let $\nu = \sum_{j=1}^n \mu_j$. Then $\nu^2 = \omega + \sum \mu_j^2$, where ω is absolutely continuous by multiple applications of 2.1. Since $\|\sum \mu_j^2\| \leqq (\max \|\mu_j\|) \sum \|\mu_j\| < \varepsilon \|\mu_f\|$, it follows that $\|\mu_f^2 - \omega\| < 3\varepsilon \|\mu_f\|$. Since ε was arbitrary, it follows that μ_f^2 is absolutely continuous. The Theorem is proved.

Remarks. Slightly modified arguments show that 2.1 and 2.2 hold equally well for functions f that are strictly convex on $[a, b]$; but we shall deal with that class by another approach, which yields a rate of decay for $\hat{\mu}_f$.

One may study the behavior of $\hat{\mu}_f(r\tau)$ as $|r| \to \infty$, a distinct question for each unit vector $\tau \in R^2$. It is useful to write $\hat{\mu}_f(r\tau) = -\int e^{-irt}\, d\lambda(t)$, where $\lambda(t) = |\{x\colon g(x) \geqq t\}|$ is the distribution function for $g(x) = \tau \cdot (x, f(x))$. Hedberg (see [6] and [16, p. 48]) presented cases in which $\hat{\mu}_f$ vanishes at infinity in certain cones, sets of the form $\{r\tau\colon r \in R,\ \tau \in U\}$ where U is a closed arc, and cases in which the function $r \to \hat{\mu}_f(r\tau)$ is in $A(R)$ ($d\lambda$ is absolutely continuous) for certain τ. Under the assumptions of 2.2, the latter condition holds for every τ. Information about the rate of decay of $\hat{\mu}_f(r\tau)$ as $|r| \to \infty$ can be obtained from an integrable Lipschitz condition on the Radon—Nikodym derivative of $d\lambda$ (see [12, p. 14]). It is easy to obtain when f is strictly convex, as we are about to explain.

When f is continuous and convex on $[a, b]$, then f' is defined and continuous except at worst on a countable set. Wherever "$f'(x)$" is not defined in the usual sense, let it denote $f'(a^+)$ if $x = a$, $f'(x^-)$ if $x = b$, and either one if $a < x < b$. With that convention in force, f' is defined and nondecreasing on $[a, b]$.

2.3 Proposition. *Let f be continuous and convex on $[a, b]$. Let τ be a unit vector in R^2, and let $g(x) = \tau \cdot (x, f(x))$ and $g'(x) = \tau \cdot (1, f'(x))$, and choose c such that $|g'(c)| = \min\{|g'(x)|\colon a \leqq x \leqq b\}$.*

(i) *For each $t > 0$ and $r > 0$,*

$$|\hat{\mu}_f(r\tau)| \leqq 2(1 + 2\pi/t) \inf\left\{\varepsilon > 0\colon \frac{\varepsilon}{t} \min\left(|g'(c-\varepsilon)|, |g'(c+\varepsilon)|\right) \geqq \frac{1}{|r|}\right\}, \tag{1}$$

where we interpret $g'(x)$ as ∞ if $x \notin [a, b]$.

(ii) *If $\tau = (\cos\beta, \sin\beta)$ and $|f'(x+h) - f'(x)| \geqq \varrho |h|$ on $[a, b]$, then*

$$|\hat{\mu}_f(r\tau)| \leqq 4\sqrt{\frac{2\pi}{\varrho |r \sin\beta|}} \quad \text{for all} \quad r \neq 0. \tag{2}$$

(iii) *If* $|g'(c)| \geqq m > 0$, *then*

$$|\hat{\mu}_f(r\tau)| \leqq 2\pi/m\,|r| \quad \text{for all} \quad r \neq 0.$$

(iv) *If* f *is strictly convex, then* $\hat{\mu}_f(r\tau) \to 0$ *as* $|r| \to \infty$.

Proof. (i) Since the statement (1) is the same for $-\tau$ as for τ, we may suppose that the second coordinate of τ is nonnegative and hence that g is convex and g' does not vanish identically on any open subinterval of $[a, b]$. We leave to the reader the cases $c=a$ and $c=b$, and suppose that $a<c<b$. Using an estimate of van der Corput [28, V.4.3(i)], we find that

$$\left|\int_{c+\varepsilon}^{b} e^{-ir\tau\cdot(x, f(x))}\,dx\right| = \left|\int_{c+\varepsilon}^{b} e^{-irg(x)}\,dx\right| \leqq \frac{2\pi}{|rg'(c+\varepsilon)|}.$$

Therefore

$$\left|\int_{c}^{b} e^{-ir\tau\cdot(x, f(x))}\,dx\right| \leqq \varepsilon + \frac{2\pi}{|rg'(c+\varepsilon)|},$$

which is bounded by $\varepsilon\left(1+\dfrac{2\pi}{t}\right)$ provided $|rg'(c+\varepsilon)| \geqq \dfrac{t}{\varepsilon}$. A similar estimate applies to the integral from a to c. The statement (1) follows.

(ii) The choice of c and the hypothesis of (ii) applied to $g'(x)=\cos\beta+f'(x)\sin\beta$ yield that $|g'(c\pm\varepsilon)| \geqq \varrho\varepsilon\,|\sin\beta|$. By (1), with $t=2\pi$,

$$|\hat{\mu}_f(r\tau)| \leqq 4 \inf\{\varepsilon > 0\colon\ |\sin\beta|\,\varrho\varepsilon^2/2\pi \geqq 1/|r|\}$$

for $r \neq 0$, and (2) follows.

(iii) Apply (1), taking the limit as $t \to 0$.

(iv) In (1) set: for $\varepsilon > 0$ set $t=\sqrt{\varepsilon}$ and let $|r| \to \infty$.

2.4 Theorem. *If* f *is strictly convex on* $[a, b]$, *then* $\mu_f \in PF$ *and hence* $\alpha(G_f)=\infty$.

Proof. Let n be an arbitrary positive integer. For $1 \leqq j \leqq n$, let $[a_j, b_j] \subset (x_{j-1}, x_j)$, where $x_j=a+(j/n)(b-a)$, such that $\sum (b_j-a_j)\hat{\mu} > (b-a)(1-(1/n))$. Let $\mu_j=\mu_{f\,|[a_j, b_j]}$. Then $\|\mu_j\| < (b-a)/n$ for each j, and $\|\mu_f - \sum \mu_j\| < (b-a)/n$. Let

$$U_j = \{\tau\colon\ \tau\cdot(1, f'(x)) = 0 \text{ for some } x \in (x_{j-1}, x_j)\}.$$

Since f' is strictly increasing on $[a, b]$, the sets U_j are disjoint and the quantities $m_j=\inf\{|\tau\cdot(1, f'(x))|\colon a_j \leqq x \leqq b_j,\ \tau \notin U_j\}$ are positive. Let m be the smallest m_j. Then $|\hat{\mu}_j(r\tau)| \leqq 2\pi/m|r|$ for every j and every unit vector $\tau \notin U_j$, by 2.3 (iii). For each τ the inequality fails for at most one j; hence $|\hat{\mu}_f(r\tau)| \leqq 2\pi(n-1)/m|r| + 2(b-a)/n$. The theorem follows.

Remarks. Every convex curve γ has a well-defined arc length parametrization: $\gamma=\{p(s)\colon 0\leqq s\leqq L\}$. Let μ_γ be the measure in $M(\gamma)$ defined by:

$$\int \varphi\, d\mu_\gamma = \int_0^L \varphi(p(s))\, ds \quad \text{for} \quad \varphi\in C(\gamma).$$

If γ is convex, the estimate (1) holds for $\hat\mu_\gamma(r\tau)$ when $g(s)=\tau\cdot p(s)$. Thus the decay of $\hat\mu_\gamma(r\tau)$ as $|r|\to\infty$ depends on the "curvature" of γ at the points where τ is the normal. For example, suppose that s_0 is the only zero of $g'(s)=\cos\theta(s)$, where $\theta(s)=\arg p'(s)-\arg\tau$ is increasing with s. Then, using the usual convention for θ',

$$|g'(s_0\pm\varepsilon)| \geqq \frac{1}{2}(\sin\theta(s_0))|\theta'(s_0)|\varepsilon = \frac{1}{2}|\theta'(s_0)|\varepsilon$$

for all ε sufficiently small. If $\theta'(s_0)\neq 0$, then by (1) (putting $t=2\pi$)

$$|\hat\mu_\gamma(r\tau)| \leqq 4\sqrt{|4\pi/r\theta'(s_0)|} = 4\sqrt{|4\pi s'(\theta_0)/r|},$$

for all sufficiently large $|r|$ (depending on $\theta_0=\theta(s_0)$). Assume now that γ is strictly convex. Then s is a function of $\omega=\arg p'(s)$. If we also assume that $s(\omega)$ is absolutely continuous, then we can use the maximal function for s',

$$S(\omega) = \sup_{x>0}\frac{1}{x}\int_{|\omega-u|<x} s'(u)\, du,$$

as follows: If $\omega_0=\theta_0+\arg\tau$, then

$$|s(\omega_0)-s(\omega)|/|\omega_0-\omega| \leqq S(\omega_0) \quad \text{for all } \omega.$$

Thus $|\hat\mu_\gamma(r\tau)|\leqq 4\sqrt{|4\pi S(\omega_0)/r|}$ for all $r\neq 0$. This is the basic ingredient in a result of Svensson [26, Lemma 2.1 and Theorem 5.1].

If f is convex but not strictly convex, then G_f contains a linear segment L and $|\hat\mu_f|$ will converge to $\mu_f(L)$ along the line normal to L. One might hope that if f is C^1 and not affine on the whole interval $[a, b]$, there would still be *some* nonzero measure $\mu\in PF(G_f)$. According to 2.2, that is the case when the distribution $\sigma(y)=|(f')^{-1}[y,\infty)|$ is continuous. The result in fact depends on the behavior of σ near its discontinuities and on the sets $(f')^{-1}(y)$ which have positive measure and give rise to those discontinuities. The next result suggests a sense in which 2.2 and 2.4 are best possible.

2.5 Proposition. *Let Y be a perfect compact subset of* $[0, 1]$ *such that* $PF(Y)=\{0\}$. *(For example, Y could be the Cantor set* $\{2\sum_{j=1}^\infty \varepsilon_j 3^{-j}\colon \varepsilon_j=0$ *or* $1\}$.*) Let g be a continuous increasing function on* $[0, 1]$ *with range* $[0, 1]$ *that is constant on each interval contiguous to Y. Let* $f(x)=\int_0^x g(t)\,dt$. *Then* $PF(G_f)=\{0\}$.

Proof. Let $\nu\in PF(G_f)$. Let $H=\{(x, f(x))\colon x\in Y\}$, and set $H'=G_f\setminus H$. If $z\in H'$, then z lies on the interior of a linear segment $L\subseteq H'$; if z belonged also to the support of ν, then there would be a function $g\in A(R^2)$ such that $0\neq g\nu\in PF(L)$, which cannot be. Therefore $\operatorname{supp}\nu\subseteqq H$. For every $t\in R$, the element ν_t defined by: $\hat{\nu}_t(s)=\hat{\nu}(s,t)$ for $s\in R$, belongs to $PF(Y)$ and hence is zero. Therefore $\nu=0$ and the proposition is proved.

So far we have discussed only graphs that are fairly smooth, but now we turn to cases when f is at best continuous. We shall show that whenever f is continuous and $\theta\in[0,\pi)$, there is a measure $\mu\in M(G_f)$ and an arc U of length θ on the unit circle such that $\hat{\mu}$ vanishes at infinity in the cone $\{r\tau\colon r\in R,\ \tau\in U\}$. It will follow that $\alpha(G_f)\geqq 2$ whenever f is a continuous function of bounded variation. Those results depend on a differentiability condition that is set forth in the next lemma.

For a set $Q\subseteqq R$, considered as a subset of the range of the tangent function, let $P(Q)$ denote the cone $\left\{(r\cos\theta,\ r\sin\theta)\colon r\in R \text{ and } \tan\left(\theta-\frac{\pi}{2}\right)\notin Q\right\}$. Thus for example,

$$P(\{0\})=\left\{(r\cos\theta,\ r\sin\theta)\colon r\in R \text{ and } -\frac{\pi}{2}<\theta<\frac{\pi}{2}\right\},$$

$$P([0,1])=\left\{(r\cos\theta,\ r\sin\theta)\colon r\in R \text{ and } -\frac{\pi}{4}<\theta<\frac{\pi}{2}\right\}.$$

2.6 Lemma. *Let f be a real-valued measurable function defined on a set Y of finite, positive Lebesgue measure. Suppose that for some $m\in R$ and $\varrho>0$,*

$$\limsup_{x\in Y,\,x\to x_0}\left|\frac{f(x)-f(x_0)}{x-x_0}-m\right|<\varrho \quad \text{for all} \quad x_0\in Y. \tag{3}$$

Choose θ_1 and θ_2 such that $P([m-\varrho,m+\varrho])=\{(r\cos\theta,\ r\sin\theta)\colon r\in R \text{ and } \theta_1<\theta<\theta_2\}$. For $\theta_1<\theta<\theta_2$, let $\hat{h}_\theta(r)=\hat{\mu}_f(r\cos\theta,\ r\sin\theta)$. Then the mapping $\theta\to h_\theta$ is a continuous mapping from (θ_1,θ_2) into $L^1(R)$. In particular, $\hat{\mu}_f\in C_0\big(P([m-\varrho-\varepsilon,m+\varrho+\varepsilon])\big)$ for each $\varepsilon\to 0$.

Proof. Consider first the case $m=0$. We may suppose that θ_1 lies between $-\pi/2$ and 0, and of course $\theta_1=-\theta_2$. Since μ_f is approached in norm by measures $\mu_{f|X}$ for compact sets $X\subseteqq Y$ that approach Y in Lebesgue measure, it suffices to prove the result for the case of compact Y. When Y is compact (3) implies first that f is uniformly continuous and bounded on Y and then that f is $\mathrm{Lip}\,(1)$ on Y. In particular f extends to a $\mathrm{Lip}\,(1)$ function on R which is thus differentiable a.e. on Y in the usual sense. In any event, the sets

$$D_n=\left\{x\in Y\colon \left|\frac{f(x)-f(y)}{x-y}\right|>\varrho \text{ for some } y\in Y,\ 0<|x-y|<n^{-1}\right\}$$

are open relative to Y and form a decreasing sequence that, in light of (3), satisfies $\cap D_n=\emptyset$. Let $0<\delta<1$. Then there is some D_m such that $|Y\setminus D_m|>(1-\delta)|Y|$. Set $Y_\delta=Y\setminus D_m$. Decompose Y_δ into subsets $X_1, \dots, X_n$, each with diameter $<m^{-1}$. Let X be any one of the X_j's with positive measure.

Fix $\theta\in(\theta_1, \theta_2)$, let $\tau=(\cos\theta, \sin\theta)$, and let $g_\theta(x)=\tau\cdot(x, f(x))$ for $x\in Y$. Since $|f'|<\varrho$, the restriction on θ implies that $g'_\theta(x)\equiv\cos\theta+f'(x)\sin\theta$ is positive and bounded away from 0 on Y. Moreover $g_\theta|X$ is strictly increasing. Denote its inverse by u_θ. Then u_θ is Lip (1), and differentiable a.e., on its domain, and the distribution function $\lambda_\theta(t)=\{x\in X\colon g_\theta(x)\geqq t\}$ is absolutely continuous, with λ'_θ equal to $-u'_\theta=-1/g'_\theta(u_\theta(t))$ on $g_\theta(X)$ and zero elsewhere. Let $h_\theta=-\lambda'_\theta$. Then $h_\theta\in L^1(R)$ and

$$\hat{h}_\theta(r)=\int_{-\infty}^{\infty}e^{-irt}h_\theta(t)\,dt=-\int_{-\infty}^{\infty}e^{-irt}\,d\lambda_\theta(t)=\int_X e^{-irg_\theta(x)}\,dx=\hat{\mu}_f(r\tau).$$

Let

$$v_\theta(t)=\begin{cases}g'_\theta(u_\theta(t))=\cos\theta+f'(u_\theta(t))\sin\theta & \text{for} \quad t\in g_\theta(X)\\ 1 \quad \text{for} \quad t\notin g_\theta(X).\end{cases}$$

Let χ_θ be the indicator of $g_\theta(X)$. Then

$$\|h_\theta-h_\eta\|_1=\int_{-\infty}^{\infty}\left|\frac{\chi_\theta(t)}{v_\theta(t)}-\frac{\chi_\eta(t)}{v_\eta(t)}\right|dt=\int_{-\infty}^{\infty}\left|\frac{\chi_\theta(t)v_\eta(t)-\chi_\eta(t)v_\theta(t)}{v_\theta(t)v_\eta(t)}\right|dt,$$

which an exercise will show tends to zero as $\theta\to\eta$. The lemma is proved in the case $m=0$. A suitable rotation of the coordinate system allows the reduction of the general case to that one.

Lemma 2.6 gives no information on the rate of decay of $\hat{\mu}_f$, that depends on the set Y. Rather it is an extension of the earlier results concerning C^1 and convex functions and the result of Hedberg [6]. We present some of its implications in the next theorem. Since (3) is equivalent to "$f=h|Y$ (a.e.) for some h that is differentiable a.e. on Y," as noted, we will use the simpler terminology.

2.7 Theorem. *Let f be a real-valued function defined and differentiable a.e. on a set Y of finite, positive Lebesgue measure. Let σ be the distribution function,*

$$\sigma(t)=|\{x\in Y\colon f'(x)\geqq t\}|.$$

(i) *If σ is continuous at each point of a closed set Q, then $\hat{\mu}_f\in C_0(P(R\setminus Q))$.*

(ii) *If σ is discontinuous at t_0 and $Y_0=(f')^{-1}(t_0)$, then*

$$0\neq\hat{\mu}_{f|Y_0}\in C_0(P([t_0-\varepsilon, t_0+\varepsilon]))$$

for every $\varepsilon>0$.

(iii) *If σ is discontinuous at each of n distinct points $t_1, \dots, t_n$, then $\alpha(G_f)\geqq n$.*

Proof. Let $\varepsilon>0$. Choose an integer $N>0$ such that

$$\sigma(N-1)<\varepsilon \quad\text{and}\quad |Y|-\varepsilon<\sigma(-N+1).$$

Choose an integer $K>0$ such that

$$|\sigma(s)-\sigma(t)|<\varepsilon \quad\text{whenever}\quad s\in Q\cap[-N,N] \quad\text{and}\quad |s-t|<1/K.$$

Let $Q_1, \ldots, Q_J$ be a listing, from left to right, of the intersections

$$Q\cap[-N+k/K, -N+(k+1)/K] \quad (0\leqq k\leqq 2NK-1)$$

that are non-empty. Choose η such that $0<\eta<1$ and

$$|\sigma(s)-\sigma(t)|<\varepsilon/(J+1) \quad\text{whenever}\quad s\in\bigcup_{j=1}^{J} Q_j \quad\text{and}\quad |s-t|<\eta.$$

Let $a_j=\min\{s\colon s\in Q_j\}$, $b_j=\max\{s\colon s\in Q_j\}$, and $I_j=[a_j+\eta, b_j-\eta]$. Then if $A_j=\{x\in Y\colon f'(x)\in I_j\}$, $\hat{\mu}_{f|A_j}\in C_0(P([a_j,b_j]))$ by Lemma 2.6. Let

$$B=\left\{x\in Y\colon f'(x)\in[-N+\eta, a_1-\eta]\cup\bigcup_{j=1}^{J-1}[b_j+\eta, a_{j+1}-\eta]\cup[b_J+\eta, N-\eta]\right\},$$

noting that some or all of the intervals in that union may be empty. Since f' is bounded away from Q on B, $\hat{\mu}_{f|B}\in C_0(P(R\setminus Q))$ by Lemma 2.6. If C is the complement in Y of $B\cup\,\cup A_j$, then $|C|<6\varepsilon$ by the choices of N and η.

Choose M so that

$$|\hat{\mu}_{f|B}(y)|<\varepsilon \quad\text{whenever}\quad \|y\|>M \quad\text{and}\quad y\in P(R\setminus Q) \quad\text{and}$$

$$|\hat{\mu}_{f|A_j}(y)|<\varepsilon/J \quad\text{whenever}\quad \|y\|>M,\quad y\in P([a_j,b_j]),$$

$$\text{and}\quad 1\leqq j\leqq J.$$

Then for every $y\in P(R\setminus Q)$ such that $\|y\|>M$, since $y\notin P([a_j,b_j])$ for at most one value of j, and since $|\{x\in Y; f'(x)\in Q_j\}|<\varepsilon$,

$$|\hat{\mu}_f(y)|=\left|\sum_{j=1}^{J}\hat{\mu}_{f|A_j}(y)+\hat{\mu}_{f|B}(y)+\hat{\mu}_{f|C}(y)\right|\leqq((J-1)\varepsilon/J)+2\varepsilon+|C|<9\varepsilon.$$

Part (i) is proved.

(ii) Apply Lemma 2.6 with $f|Y_0$, Y_0, and t_0 in the roles of f, Y, and m respectively. Part (ii) is proved.

(iii) Let $Y_k=(f')^{-1}(t_k)$, and let ν_k be the probability measure $|Y_k|^{-1}\mu_{f|Y_k}$. By part (ii) we see that we may select symmetric closed cones ζ_k $(1\leqq k\leqq n)$ such that $\hat{\nu}_k\in C_0(\zeta_k)$ and such that the complement of ζ_k is contained in $\{0\}\cup\zeta_j$ for each $j\neq k$. It follows that if $\nu=n^{-1}\sum_{k=1}^{n}\nu_k$, then for $\|y\|\to\infty$ $\limsup|\hat{\nu}(y)|\leqq 1/n$. (In fact, equality holds, since for each k $|\hat{\nu}_k(rt_k,-r)|\to 1$ as $|r|\to\infty$.) For each $\varepsilon>0$ we may choose vectors $y_1, \ldots, y_n$ such that if μ is defined by: $\hat{\mu}(y)=$

$n^{-1}\sum_{k=1}^{n}\hat{\nu}_k(y-y_k)$, then $|\hat{\mu}(y)|\leqq\varepsilon+n^{-1}$ for all y, while of course $\|\mu\|=1$. The Theorem is proved.

Remark. Let f be as in the theorem and let $Y_0=(f')^{-1}(t_0)$. Let τ be the unit vector orthogonal to $(1, t_0)$. Then the distribution function for the derivative of $f|(Y\setminus Y_0)$ is of course continuous at t_0, so that the behavior of $\hat{\mu}_f(r\tau)$ as $|r|\to\infty$ depends entirely on $f|Y_0$.

The next theorem organizes the results about continuous graphs. First, a lemma.

2.8 Lemma. *Let f be a continuous function with domain $[a, b]$ and range $[c, d]$. Then there exists a strictly monotone function g defined on $[c, d]$ such that $(g(y), y)\in G_f$ for $c\leqq y\leqq d$.*

Proof. Choose x_1 and x_2 in $[a, b]$ such that $f(x_1)=c$ and $f(x_2)=d$, and let x_0 be the smaller of the two. Let

$$g(y) = \min\{x\colon\ x \geqq x_0 \text{ and } f(x) = y\} \quad \text{for} \quad y\in[c, d].$$

It is easy to see that if $x_0=x_1$, then g is strictly decreasing and if $x_0=x_2$, then g is strictly increasing. In either case, the lemma is proved.

2.9 Theorem. *Let f be a continuous real-valued function defined on $[a, b]$, where $a<b$.*

(i) *There is a unit vector τ and a positive measure $\nu\in M(G_f)$ such that $\hat{\nu}\in C_0(\zeta)$ for every closed cone ζ not containing the directions $\pm\tau$.*

(ii) *If f is of bounded variation, then $\alpha_c(G_f)\geqq 2$.*

(iii) *If f is not of bounded variation, then $\alpha(G_f)\geqq\pi/2$.*

Proof. (i) If f is constant, then $\tau=(1, 0)$ satisfies the statement. If not, then G_f contains the graph of the function g provided by Lemma 2.8 and supports the measure μ_g (for g, the roles of the x and y axes are the reverse of the usual). Applying Theorem 2.7 to g, we find that either the function $t\to|(g')^{-1}[t,\infty)|$ is continuous and $\hat{\mu}_g\in C_0(R^2)$, or else it has a discontinuity at some t_0 and $\tau=(-1, g'(t_0))$ satisfies the statement.

(ii) If G_f contains a straight line segment, then $\alpha_c(G_f)=\infty$. If there are disjoint intervals I_j such that both $Y_1=(f')^{-1}(I_1)$ and $Y_2=(f')^{-1}(I_2)$ have positive measure, then 2.7, (i) and (ii), applied to $f|Y_j$, yields two probability measures μ_j in $M(G_f)$ such that $\hat{\mu}_j\in C_0(\zeta_j)$, where ζ_1 and ζ_2 are closed cones and $\zeta_1\cup\zeta_2=R^2$, and it follows (as in the proof of 2.7 (iii)) that $\alpha_c(G_f)\geqq 2$. In all the remaining cases, $f'=r$ a.e. for some r, but the complement D of $f((f')^{-1}(r))$ in the range of f has nonzero measure. Then $\hat{\mu}_f\in C_0(\zeta)$ for every closed cone not containing the direction $(r, -1)$. Obtain g as in Lemma 2.8 and let h be the restriction of g to D. Apply

2.7 (i) or (ii) to h, obtaining a probability measure $\nu\in M(G_f)$ that is mutually singular with μ_f and such that (at worst) $\hat{\nu}\in C_0(\zeta')$ where ζ' is a closed cone with $(r, -1)$ in its interior. It follows that $\alpha_c(G_f)\geqq 2$.

(iii) For each positive integer n, choose a sufficiently small $h>0$ so that $f(x+h)=f(x)$ for at least $2n$ values of x in $[a, b-h]$. Choose n such points $x_1, \dots, x_n$ such that the $2n$ points x_j, x_k+h (for $1\leqq j, k\leqq n$) are all distinct. Let μ_n be the measure that places mass n^{-1} at $(x_j, f(x_j))$ and mass $n^{-1}e^{2\pi ij/n}$ at $(x_j+h, f(x_j+h))$ for each j. Then $\|\mu_n\|=2$ and

$$|\hat{\mu}_n(u, v)| \leqq n^{-1}\sum_{j=1}^{n}|1+e^{2\pi ij/n}e^{ih\mu}| \to \int_0^1 \sqrt{2+2\cos\theta}\,d\theta = \frac{4}{\pi},$$

where the convergence is uniform for $(u, v)\in R^2$. Note that the sum equals a Riemann sum for the function $g(x)=|1+e^{2\pi ix}|$ on [0, 1]. Therefore $\|\mu_n\|/\|\mu_n\|_{PM}\geqq\frac{\pi}{2}-\varepsilon_n$ where $\varepsilon_n\to 0$, and hence $\alpha_d(G_f)\geqq\frac{\pi}{2}$.

Remark. We suspect, but do not know, that every continuous curve γ in R^2 possesses (i) of 2.9. Indeed this is the case when one of the coordinate functions is of bounded variation for then 2.8 applied to the other coordinate function yields a $G_f\subset\gamma$ where f is of bounded variation (not necessarily continuous), hence differentiable a.e. (see the argument in 4.4). There are continuous Helson graphs that support measures that decay more than is indicated in (i) of 2.9. In particular in [15, Section 5, Theorem 2] Körner presents a construction, due to Katznelson, of a Kronecker set γ_h in R^2 which is a (discontinuous) graph and which supports a measure μ satisfying $\mu\in C_0(\{(x, y)\colon |y|\leqq h(|x|)\})$, where h is any prescribed increasing function on R. One can connect the "pieces" of their γ_h using the methods we summarize in Section 3 to obtain a (continuous) Lip (1) Helson graph which contains γ_h.

3. A Lipschitz Helson curve in the plane

The technique used to prove the following lemma is standard. For $\varrho>0$, let $U(\varrho)=\{z\in R^n\colon \|z\|\leqq\varrho\}$. Let $T=\{z\in C\colon |z|=1\}$.

3.1 Lemma. *Let F be a finite independent subset of R^n. Let $0<\varepsilon<1$ and $0<\sigma$. Then there exist $\delta=\delta(F, \varepsilon, \sigma)>0$ and $\varrho=\varrho(F, \varepsilon, \sigma)>0$ such that for every function $f\colon F\to T$ there exists $g\in A(R^n)$ such that:*

(i) $|f(x)-g(x+z)|<\sigma$ *for* $x\in F$ *and* $z\in U(\delta)$*;*
(ii) $|\arg f(x)-\arg g(x+z)|<\sigma$ *for* $x\in F$ *and* $z\in U(\varrho)$*;*
(iii) $|g(y)|\leqq\varepsilon^2$ *for* $y\notin F+U(\varrho)$*; and*
(iv) $\|g\|_A\leqq\varepsilon^{-1}$.

Proof. Let $F=\{x_1, \dots, x_m\}$. For $k=(k_1, \dots, k_m)\in Z^m$, let

$$p(k)=\exp((\log\varepsilon)\textstyle\sum_{j=1}^m k_j^2).$$

Then p is positive-definite, $p\in A(Z^m)$, and $p(0)=1=\|p\|_A$. Let h be the characteristic function of the coset $\sum k_j=1$ in Z^m; it is the transform of an idempotent measure on T^m with norm one. Then $hp\in A(Z^m)$ and $\|hp\|_A\leqq 1$. Define φ on R_d^n by letting $\varphi(\sum_{j=1}^m k_j x_j)=(hp)(k_1, \dots, k_m)$ and letting $\varphi(x)$ vanish for x not in the span of F. Then $\varphi\in A(R_d^n)$, $\|\varphi\|_A\leqq 1$, and $\varphi=\varepsilon$ on F. For $f\colon F\to T$, let γ be an element of $(R_d^n)^\wedge=\bar{R}^n$ (the Bohr compactification of R^n) such that $\gamma=f$ on F. Let $\psi=\varepsilon^{-1}\gamma\varphi$. Then $\psi\in A(R_d^n)$, $\|\psi\|_A\leqq\varepsilon^{-1}$, ψ vanishes off the span of F, and

$$\psi\left(\textstyle\sum_1^m k_j x_j\right) = \varepsilon^{-1+\Sigma k_j^2}\prod_1^m f(x_j)^{k_j}$$

whenever $\sum k_j=1$. Thus $\psi=f$ on F, and $|\psi|\leqq\varepsilon^2$ elsewhere.

Let u be a nonnegative function in $A(R_d^n)$ with finite support and norm one such that $1-(\sigma/2)\leqq u\leqq 1$ on F (e.g. a Bochner—Fejér type kernel). If $q=u\psi$, then q may be regarded as a discrete measure on R^n and $\hat{q}$ as a trigonometric polynomial on $\bar{R}^n$. We now extend q as a function on $\operatorname{supp} u$ to an element of $A(R^n)$, as follows. For $\eta>0$, let k_η be a multiple of $\chi*\chi$, where χ is the indicator function of $U(\eta)$, chosen so that $\|k_\eta\|_{A(R^n)}=1=k_\eta(0)$. Then $\hat{k}_\eta\geqq 0$, and as $\eta\to 0$, $\sup\hat{k}_\eta\to 0$ and $\operatorname{supp} k_\eta\to\{0\}$. Thus $k_\eta(y)dy$ converges weak $*$ to Haar measure on $\bar{R}^n$. It follows that

$$\|q*k_\eta\|_A = \|\hat{q}\hat{k}_\eta\|_{L^1(R^n)} \to \|\hat{q}\|_{L^1(\bar{R}^n)} = \|q\|_{A(R_d^n)} \leqq \varepsilon^{-1}$$

as $\eta\to 0$. Therefore for η sufficiently small, the norm $\|q*k_\eta\|_{A(R^n)}$ is at most slightly greater than ε^{-1} and for $x\in F$ and z near zero, $(q*k_\eta)(x+z)=q(x)k_\eta(z)$. Therefore if δ and ϱ are sufficiently small (depending only on F, ε, and σ), we may take g to be a multiple (at most slightly different from one) of $q*k_\eta$. The lemma is proved.

3.2 Theorem. *There exists a curve $\Gamma\subset R^2$ which is the graph of a* Lip (1) *function and such that $\alpha(\Gamma)\leqq 3^{3/2}\cong 5.196$.*

Proof. Let $n\geqq 2$, and let $D=\{d_1<d_2<\dots<d_n\}$ and $E=\{e_1<e_2<\dots<e_n\}$ be two independent subsets of $[0, 1]$. Let τ and η be independent unit vectors in R^2, and let τ' and η' be unit vectors perpendicular to τ and η respectively. We define $P=P(D, E, \tau, \eta)$ to be the polygonal path whose $2n-1$ vertices, in order, are $d_1\tau+e_1\eta$, $d_1\tau+e_2\eta$, $d_2\tau+e_2\eta$, $\dots$, $d_n\tau+e_n\eta$. We shall call such a path an I-*polygonal path.* Let $s(P)$ denote the largest distance between two consecutive vertices of P.

Fix $\varepsilon>0$. Using repeatedly the case $n=1$ of Lemma 3.1, we find that for each $\sigma>0$ there exist $\alpha=\alpha(P, \sigma)$ and $\varrho=\varrho(P, \sigma)$ such that:

(I) for each function $f\colon D\to T$ there exists $g\in A(R)$ such that

(1) $|g(d\tau\cdot\eta')-f(d)|<\sigma$ for each $d\in D$,

(2) $\big|g(t\tau\cdot\eta')-f(d)|g(t\tau\cdot\eta')|\big|<\sigma$ whenever $d\in D$ and $|t-d|<p$,

(3) $|g(t\tau\cdot\eta')|\leqq\varepsilon^2$ whenever $\operatorname{dist}(t,D)\geqq\varrho$,

(4) $\|g\|_A\leqq\varepsilon^{-1}$, and

(5) $|g(x)-g(y)|<\sigma$ whenever $|x-y|<\alpha$;

and such that

(II) for every function $f\colon E\to T$ there exists $g\in A(R)$ such that conditions (1)—(5) hold with e, η, τ, and E taking the roles of d, τ, η, and D respectively.

To construct a Helson curve, devise a sequence of I-polygonal paths $P_m=P(D_m, E_m, \tau_m, \eta_m)$ such that $s(P_m)\downarrow 0$, that every point of P_{m+1} lies within distance $\alpha(P_m, m^{-1})$ away from P_m, and such that $\Gamma=\lim P_m$ is a curve and not a singleton. One may make Γ the graph of an increasing Lip (1) function by choosing τ_m and η_m so that (say) $\arg\tau_m$ decreases to 0 and $\arg\eta_m$ increases to $\dfrac{\pi}{4}$.

Let $\mu\in M(\Gamma)$ be a measure of norm one. To prove that Γ is a Helson set it suffices to find a positive lower bound for $\|\mu\|_{PM}$. Let Q be a compact rectangle whose interior contains Γ. Let $\beta>0$ and choose a continuous function $h\colon Q\to T$ such that $\|h\mu-|\mu|\|<\beta$. Pick $\alpha>0$ such that $|h(u)-h(v)|<\beta$ whenever $u, v\in Q$ and $\operatorname{dist}(u,v)<\alpha$. Pick m large enough so that $m^{-1}<\beta/2$, $\alpha_m=\alpha(P_m, m^{-1})<\alpha$, and $s(P_m)<\alpha$. Let $\{d_1,\ldots,d_n\}=D_m$, $\{e_1,\ldots,e_n\}=E_m$. Let

$$S_1=\bigcup_{j=1}^{n}\{a\eta_m'+b\eta_m\colon\ b\in R,\ |a-d_j\tau_m\cdot\eta_m'|<\alpha_m\},$$

$$S_2=\bigcup_{j=1}^{n}\{a\tau_m+b\tau_m'\colon\ a\in R,\ |b-e_j\eta_m\cdot\tau_m'|<\alpha_m\}.$$

Since Γ lies within distance α_m from P_m, $\Gamma\subseteqq S_1\cup S_2$. Therefore either $|\mu|(S_1)\geqq\dfrac{1}{2}$ or $|\mu|(S_2)\geqq\dfrac{1}{2}$; we shall assume the former, the other case being equivalent to deal with. Let g be a function in $A(R)$ satisfying (1)—(5), with $\sigma=m^{-1}$ and with f given by: $f(d_j)=h(d_j\tau_m+e_j\eta_m)$. Define g_1 on R^2 by: $g_1(t\eta_m'+s\eta_m)=g(t)$. Then g_1 is the Fourier—Stieltjes transform of a measure with norm bounded by ε^{-1}. The η_m' coordinate of each point of $S_1\cap\Gamma$ lies within distance α_m from that of a vertex of P_m; hence $|g(z)-h(z)|<3\beta$ for $z\in S_1\cap\Gamma$. The set $\Gamma\setminus S_1$ is the union of the disjoint sets

$$R_1=\big\{z\in\Gamma\setminus S_1\colon\ \big|g(z)-h(z)|g(z)|\big|<\beta\big\},$$

$$R_2=\{z\in\Gamma\setminus(S_1\cup R_1)\colon\ |g(z)|<\varepsilon^2\}.$$

Then $\Gamma=(S_1\cap\Gamma)\cup R_1\cup R_2$ and

$$\|\mu\|_{PM}\varepsilon^{-1}\geqq\left|\int_\Gamma g\,d\mu\right|\geqq\left|\int_{S_1} g\,d\mu+\int_{R_1} g\,d\mu\right|-\frac{1}{2}\varepsilon^2\|\mu\|.$$

Since

$$\int_{S_1} g\,d\mu+\int_{R_1} g\,d\mu=\int_{S_1} h\,d\mu+\int_{R_1}|g|h\,d\mu+\int_{S_1} g-h\,d\mu+\int_{R_1} g-|g|h\,d\mu$$
$$=|\mu|(S_1)+\beta_1+|g\mu|(R_1)+\beta_2+3\beta_3|\mu|(S_1)+\beta_4|\mu|(R_1),$$

where $|\beta_j|\leqq\beta$ for $1\leqq j\leqq 4$, it follows that

$$\|\mu\|_{PM}\varepsilon^{-1}\geqq\frac{1}{2}\|\mu\|-2\beta-4\beta\|\mu\|-\frac{1}{2}\varepsilon^2\|\mu\|.$$

Since $\beta>0$ was arbitrary and since $\|\mu\|=1$,

$$\|\mu\|_{PM}\geqq\frac{1}{2}(\varepsilon-\varepsilon^3),$$

which is at its maximum $3^{-3/2}$ when $\varepsilon^2=1/3$. This proves the theorem.

Remarks. The proof above is a variant on that of [18]. The Lemma 3.1 is related to the powerful separation results that emerged with the solution to the union problem for Helson sets (see [10], [24], [21]). For those results we recommend [5, Chapter 2].

Varopoulos [27] showed the existence of convex Sidon curves in R^2. He showed that in fact there exist continuous and strictly convex functions f such that G_f is the union of two independent sets E_1 and E_2. One may show that for such graphs $\alpha_d(G_f)\leqq\frac{1}{2}5^{5/4}$ as follows. If $\gamma_i=(x_i, f(x_i))$ for $1\leqq i\leqq 4$ are distinct points on G_f, then $\gamma_1-\gamma_2\neq\gamma_3-\gamma_4$, by the strict convexity of f. In particular, if $\gamma_1, \gamma_2, \gamma_3\in E_1$, then $\gamma_1-\gamma_2+\gamma_3\notin E_2$. Let μ be a discrete measure in $M(G_f)$ with norm one, and with finite support. Then either $|\mu|(E_1)\geqq\frac{1}{2}$ or $|\mu|(E_2)\geqq\frac{1}{2}$; let us assume the former. The first paragraph in the proof of 3.1 (see also, Theorem 2.1.3 of [6]) implies that if $\varepsilon>0$ and $\beta>1$, there exists $g\in A(R_d^2)$ such that $\int_{E_1} g\,d\mu=|\mu|(E_1)$, $|g|\leqq\beta\varepsilon$ on E_2, and $\|g\|_A\leqq\beta\varepsilon^{-1/4}$. Then $\beta\varepsilon^{-1/4}\|\mu\|_{PM}\geqq\left|\int g\,d\mu\right|\geqq|\mu|(E_1)-\frac{1}{2}\beta\varepsilon$. Therefore $\|\mu\|_{PM}\geqq\frac{1}{2}(\varepsilon^{1/4}-\varepsilon^{5/4})$, which has its maximum $2(5^{-5/4})$ when $\varepsilon=1/5$.

4. Higher dimensions

Lemma 2.8 does not generalize, and we do not know whether 2.9(i) does. But most parts of Section 2 have analogues in dimensions $n>2$.

Let E be a set of finite measure in R^l and let f be a measurable function from E into R^n, where $1\leqq l<n$. Let $\mu_f\in M(f(E))$ be the nonnegative measure of norm $|E|$ given by: $\int_{R^n} g\,d\mu_f=\int_E g(f(x))\,dx$. Let u be a unit vector in R^l. For a measurable set $F\subseteqq E$, we say that f is *differentiable relative to F in the direction u* if for each $x\in F$, the limit

$$D_{u,F}f(x) = \lim_{\substack{t\to 0\\ x+tu\in F}} \frac{f(x+tu)-f(x)}{t}$$

exists. When u and F are understood, we denote that limit by $f'(x)$.

4.1 Lemma. *Let f: $E\to R^n$ be a measurable function, where E is a set of finite measure in R^l. Let u be a unit vector in R^l. Suppose that $f'(x)\equiv D_{u,E}f(x)$ exists for each $x\in E$, and suppose that $M=\sup_{x\in E}\|f'(x)\|$ is finite. Let $\varrho>0$, and let*

$$U = \{\tau\in R^n\colon \|\tau\| = 1 \text{ and } |\tau\cdot f'(x)| \geqq \varrho \text{ for all } x\in E\}.$$

Let $\hat{h}_\tau(r)=\hat{\mu}_f(r\tau)$. Then

(i) *$\hat{h}_\tau\in A(R)$ for each $\tau\in U$, and*

(ii) *the mapping $\tau\to h_\tau$ is continuous from U into $L^1(R)$.*

Proof. We may suppose that $|E|>0$. Since $f'(x)$ is the (a.e.) pointwise limit of

$$f_n(x) = \Big[\int_{|t|\leqq n^{-1}} t\chi_E(x+tu)\,dt\Big]^{-1}\int_{|t|\leqq n^{-1}} (f(x+tu)-f(x))\chi_E(x+tu)\,dt,$$

it is a measurable function. Therefore E can be partitioned into disjoint measurable sets E_i $(1\leqq i\leqq m+1)$ such that $\left|\bigcup_{i=1}^m E_i\right|>\left(1-\dfrac{\delta}{2}\right)|E|$ and $\|f'(x)-f'(y)\|<\varrho/3$ whenever $x, y\in E_i$ for some $i\leqq m$. For each $i\leqq m$, choose a closed set $E_i'\subseteqq E_i$ and then $F_i\subseteqq E_i'$ such that $\left|\bigcup_{i=1}^m F_i\right|>(1-\delta)|E|$, and such that for some $\varepsilon\in(0, 1)$,

$$\|f(x+\alpha u)-f(x)-\alpha f'(x)\| \leqq |\alpha|\varrho/3$$

whenever

$$x, x+\alpha u\in F_i,\quad |\alpha| \leqq \varepsilon,\quad \text{and}\quad 1 \leqq i \leqq m.$$

Let F be any of the sets F_i that has positive measure. Decompose F into subsets F_j' $(1\leqq j\leqq k)$, each with diameter less than ε. Let P be any of the sets F_j' that has positive measure. Fix z in the orthogonal complement H of Ru, fix $\tau\in U$, and let $g_z(\alpha)=\tau\cdot f(z+\alpha u)$. Now if $z+\alpha_0u$ and $z+\alpha u$ belong to P, then

$$|g_z(\alpha)-g_z(\alpha_0)-(\alpha-\alpha_0)\tau\cdot f'(z+\alpha_0 u)| < |\alpha-\alpha_0|\varrho/3 < \varepsilon\varrho/3.$$

Hence, assuming $\alpha > \alpha_0$ and $\tau \cdot f' > 0$ (recall $\tau \cdot f'$ is one of $\leqq -\varrho$ or $\geqq \varrho$ on P),

$$(\alpha-\alpha_0)(M+\varrho) > g_z(\alpha)-g_z(\alpha_0) > (\alpha-\alpha_0)2\varrho/3.$$

Thus g_z is strictly monotone in α on the set $\{\alpha:\ z+\alpha u \in P\}$, and the measure $dg_z(\alpha)$ is absolutely continuous. If $\tau \cdot f'$ is suitably extended, then

$$dg_z(\alpha) = |\tau \cdot f'(z+\alpha u)|\, \chi_P(z+\alpha u)\, d\alpha.$$

Define $T\colon P \to R^l$ by: $T(z+\alpha u) = z+g_z(\alpha)u = z+\beta u$. Then T is injective and both T and T^{-1} are continuous in measure when restricted to P and $T(P)$ respectively and, for almost all $z \in H$,

$$dg_z^{-1}(\beta) = \tau \cdot f'\big(T^{-1}(z+\beta u)\big)^{-1} \chi_P\big(T^{-1}(z+\beta u)\big)\, d\beta.$$

If λ is the distribution,

$$\lambda(\alpha) = |\{x \in P:\ \tau \cdot f(x) \geqq \alpha\}|$$

then

$$\begin{aligned}\lambda(\alpha_1)-\lambda(\alpha_2) &= \int_H \int_{T^{-1}\{(\{z\}\times[\alpha_1,\alpha_2))\cap T(P)\}} \chi_P(z+\alpha u)\, d\alpha\, dz \\ &= \int_H \int_{(\{z\}\times[\alpha_1,\alpha_2))\cap T(P)} \chi_P\big(T^{-1}(z+\beta u)\big) \big|\tau \cdot f'\big(T^{-1}(z+\beta u)\big)\big|^{-1} d\beta\, dz\end{aligned}$$

where T^{-1} and $\tau \cdot f'(T^{-1})$ are suitably extended to all of $[\alpha_1, \alpha_2]$. It follows that $d\lambda$ is absolutely continuous and that in fact

$$h_\tau(\beta)\, d\beta \equiv d\lambda(\beta) = \Big[\int_H \chi_P\big(T^{-1}(z+\beta u)\big) \big|\tau \cdot f'\big(T^{-1}(z+\beta u)\big)\big|^{-1} dz\Big]\, d\beta.$$

We shall show now that the mapping $\tau \to h_\tau$ is continuous from U into $L^1(R)$ and leave it to the reader to replace P by E in the conclusions. Denote the dependence of T and H on $\tau \in U$ by writing T_τ, H_τ. Suppose that $\sigma \in U$ and let τ be an element of U that is near σ. Let χ_σ be the characteristic function of $T_\sigma(P)$. The maps T_τ, T_σ are "projections" of f on P; hence T_τ, T_σ are related by a map which is bicontinuous in measure, providing τ is near σ. Since $T_\tau \to T_\sigma$ pointwise as $\tau \to \sigma$, this implies $\chi_\sigma(T_\tau) \to \chi_\sigma(T_\sigma)$ in $L^1(R^l)$. Thus $\big|(T_\sigma(P) \setminus T_\tau(P)) \cup (T_\sigma(P) \setminus T_\tau(P))\big| \to 0$ and hence $T_\tau^{-1} \to T_\sigma^{-1}$ pointwise a.e. It follows by the convergence theorems of Lebesgue that $\|h_\tau - h_\sigma\|_1 \to 0$ as $\tau \to \sigma$.

In the proof of Theorem 2.9, the part of a curve that at least behaves like a linear segment (i.e., constant slope) is separated from the rest. The situation for manifolds is analogous. Let $E_0 \subseteqq R^l$, and suppose that $f\colon E_0 \to R^n$ is differentiable a.e. relative to the set E_0 of finite measure in the directions $u_1, \ldots, u_m$. Let $A(x)$ be the $n \times m$ matrix whose j^{th} column is $D_{u_j, E_0} f(x)$. Let H_s be the collection of all subspaces of R^n with dimension s. For each $S \in H_s$, the set

$$E_0(S) \equiv \{x \in E_0:\ \text{range}\, A(x) \subseteqq S\}$$

is measurable. Thus there is a countable collection $\{E_0(S_j)\}$ such that $S_j \in H_0$ and $|E_0(S_j)|>0$ (at this zero-th stage the only S_j is $\{0\}$). Let $E_1=E_0 \setminus \cup_j E_0(S_j)$. Repeating, let $E_2=E_1 \setminus \cup \{E_1(S_j)\colon S_j \in H_1$ and $|E_1(S_j)|>0\}$. Continuing, we obtain $E_0 \supseteqq E_1 \supseteqq \dots \supseteqq E_k \supseteqq E_{k+1}$ where $k \leqq n$, $|E_{k+1}|=0$, and for each s from 1 to k, $|E_s(S)|=0$ for all $S \in H_{s-1}$ and either $E_{s+1}=E_s$ or $E_s \setminus E_{s+1} = \cup \{E_s(S)\colon S \in H_s$ and $|E_s(S)|>0\}$. Thus there is a countable collection $\{H_j\}$ of distinct subspaces of R^n and a measurable decomposition $\{F_j\}$ of E_0 such that

(1) range $A(x) \subseteqq H_j$ for a.a. $x \in F_j$,

(2) $|F_j|>0$, and

(3) if $H \subsetneqq H_j$ is a subspace, then

$$|\{x \in F_j\colon \text{range } A(x) \subseteqq H\}| = 0.$$

The one bad extreme case, when $A(x)=0$ a.e. (i.e., $|E_1|=0$) does not occur with graphs; we will deal with it in our extension of Theorem 2.9.

4.2 Theorem. *Let $E \subset R^l$, and suppose that $f\colon E \to R^n$ is differentiable a.e. relative to the set E of finite measure in the directions $u_1, \dots, u_m \in R^l$. Let $A(x)$ be the $n \times m$ matrix whose j^{th} column, $A_j(x)$, is $D_{u_j, E} f(x)$. Then there is a measurable decomposition $\{F_j\}$ of E and corresponding subspaces H_j of R^n satisfying* (1), (2), (3) *above. The pairs (F_j, H_j) are unique up to sets of measure zero, the directions $u_1, \dots, u_m$, and a reordering of subscripts. Moreover, let $\pi_j\colon R^n \to H_j$ be the projection map and let $C(H_j, \varrho)=\{\omega \in R^n\colon \|\pi_j(\omega)\| \geqq \varrho \|\omega\|\}$ for any subspace H_j and $\varrho>0$. Then:*

(i) *For each $\varrho>0$, $\hat{\mu}_{f|F_j} \in C_0(C(H_j, \varrho))$.*

(ii) *Suppose that there are at least k distinct pairs $(F_1', H_1'), \dots, (F_k', H_k')$ among the $\{(F_j, H_j)\}$. For each nonzero vector ω in the span H of $H_1', \dots, H_k'$ let $r(\omega)$ be the number of nonzero projections among $\pi_j(\omega)$, $1 \leqq j \leqq k$. Set $s=\min\{r(\omega)\colon \omega \in H\}$. Then $\nu=\sum^k |F_j'|^{-1} \mu_{f|F_j'}$ satisfies*

$$\limsup_{\substack{\omega \in C(H,\varrho) \\ \|\omega\| \to \infty}} |\hat{\nu}(\omega)| \leqq \|\nu\| (1-s/k), \quad \text{for} \quad \varrho > 0.$$

(iii) *Suppose that there exist k distinct pairs $(F_1', H_1'), \dots, (F_k', H_k')$ among the $\{(F_j, H_j)\}$ and $l \geqq 1$ such that every choice of l distinct elements from $\{H_1', \dots, H_k'\}$ spans R^n. Then $\nu=\sum^k |F_j'|^{-1} \mu_{f|F_j'}$ has norm k and* $\limsup |\hat{\nu}| \leqq$ *$l-1$. Thus $\alpha_c(E) \geqq k/(l-1)$.*

Proof. The existence of the pairs (F_j, H_j) was established in the preceding paragraph, and the uniqueness follows from that discussion.

To prove (i) set $F=F_j$, $H=H_j$, and $\pi=\pi_j$ for some fixed j. We may assume that $H \neq \{0\}$ and that $A(x)$ exists everywhere on F. The subsets

$$D_M = \{x \in F\colon \|A(x)\| \leqq M\}$$

tend to F in measure as $M \to \infty$. Thus we may assume $\|A(x)\|$ is bounded on F. Let $U=\{u \in R^n: \|u\|=1, \|\pi(u)\| \geqq \varrho\}$, and fix $\varepsilon > 0$. Condition (3) implies that for each $u \in U$ there exists $\delta(u) > 0$ satisfying $|\{x \in F: \|uA(x)\| \leqq \delta(u)\}| < \varepsilon$. Since U is compact, $\|A(x)\|$ is bounded, and $u \to uA(x)$ is continuous, we may choose $\delta(u) \geqq \delta' > 0$ for some δ' and all $u \in U$. In particular there exist $\delta > 0$, a finite open cover $U_1, \ldots, U_k$ of U, and measurable partitions $\{P_{ir}\}_r$ of F where $0 \leqq r \leqq m$ for each $1 \leqq i \leqq k$ such that, for each i, $|P_{i0}| < \varepsilon$ and $\|uA_r(x)\| > \delta$ for all $u \in U_i$ and all $x \in P_{ir}$ for each $1 \leqq r \leqq m$. Lemma 4.1 now yields the existence of an N_{ir} satisfying $|\hat{\mu}_{f|P_{ir}}(su)| < \varepsilon/m$ whenever $u \in U_i$, $|s| \geqq N_{ir}$, $r \neq 0$. Thus, setting $N = \max N_{ir}$, we have

$$|\hat{\mu}_{f|F}(su)| \leqq \sum_{r=0}^{m} |\hat{\mu}_{f|P_{ir}}(su)| < \varepsilon + (\varepsilon/m)m = 2\varepsilon,$$

whenever $u \in U_i$ and $|s| \geqq N$ for each $1 \leqq i \leqq k$. Since $\varepsilon > 0$ is arbitrary, (i) is valid.

To obtain (ii) we observe that for each $\varrho > 0$, there exists $\varrho' > 0$ such that each $\omega \in C(H, \varrho)$ must lie in at least s of the cones $C(H_j, \varrho')$, $1 \leqq j \leqq k$; hence (ii) is an immediate corollary to (i). The hypothesis in (iii) implies $s = k-(l-1)$ in (ii) and this yields the desired property of ν. Proposition 1.1 now implies $\alpha_c(E) \geqq k/(l-1)$.

The multiplicity properties of a given manifold in R^n can now be estimated quite easily providing a parametrization f: $E \subset R^l \to R^n$ can be given which is sufficiently nice. That is: $\mu_f \in C_0(H)$ for a subspace $H \subseteqq R^n$ if, for each nonzero $u \in H$, $|\{x \in E: \|uA(x)\| = 0\}| = 0$, in the notation of 4.2; just note that $H \cap H_j^\perp = \{0\}$ for each j, and apply part (i) of 4.2. Thus any manifold in R^n without "flat spots" of positive measure (in particular, a strictly convex $(n-1)$-manifold) supports a probability measure in $PF(R^n)$. As we have seen (2.5), C^1 manifolds need not support such a measure. Never-the-less they are never Helson sets. We need only verify this for C^1 curves in R^n, since subsets of Helson sets are Helson sets. Our proof is basically a verification of the conditions in 4.2 part (iii).

4.3 Theorem. *Suppose that $\gamma \subset R^n$, $n \geqq 1$, is a C^1 curve. Then $\alpha_c(\gamma) = \infty$.*

Proof. The set of all linear manifolds in R^n that contain a segment of γ contains a manifold of minimal dimension k, for some $k \leqq n$. Hence a translate of γ has a segment in R^k. Let σ be this segment. Since k is minimal, no segment of σ lies in a proper linear submanifold of R^k. Of course, $\alpha_c(\gamma) \geqq \alpha_c(\sigma)$. If $k=1$, σ is a linear segment, and we are done. Thus we may assume $k = n \geqq 2$ and $\sigma = \gamma$. Let $U = \{x \in R^n: \|x\| = 1\}$ and let f: $[a, b] \to R^n$ be a C^1 parametrization of γ.

Suppose that we have chosen l points $t_1, \ldots, t_l \in (a, b)$ such that the $f'(t_j)$ are distinct and such that every subset of the vectors $\{f'(t_1), \ldots, f'(t_l)\}$ of cardinality $\min(n, l)$ is independent in R^n. Let S be the collection of all r-dimensional subspaces spanned by every choice of $r=\min(n-1, l)$ of those vectors. Each distinct choice of r subscripts gives a distinct element of the finite set S. Let $J\subset[a, b]$ be any set with nonempty interior. If $J\subseteqq\cup(f')^{-1}(H)$, $H\in S$, then one of these closed sets, say $(f')^{-1}(H_0)$, must contain an interval I; hence $f(t)\cdot v$ must be constant on I for each $v\in H_0^{\perp}$; hence the segment $f(I)$ lies in a translate of H_0, contrary to our assumption. It follows that there exists a $t_{l+1}\in J$ such that $f'(t_{l+1})\notin\cup H$, $H\in S$. Therefore every subset of $\{f'(t_1), \dots, f'(t_{l+1})\}$ of cardinality $\min(n, l+1)$ is independent. Thus for any positive integer l and any choice of open sets $J_1, \dots, J_l\subset[a, b]$ there exist $t_j\in J_j$ such that $\{f'(t_1), \dots, f'(t_l)\}$ satisfies our independence property. Fix such an l and choice of points. Note that every choice of n vectors $x_1, \dots, x_n$ from $f'(t_1), \dots, f'(t_l)$ with distinct indices forms an independent set in R^n. Thus $U\cap(\cap^n\{x_j\}^{\perp})=\emptyset$; hence there exists $\delta>0$ such that the set

$$B_\delta = \{x\in R^n\colon \|x\| < \delta\}$$

satisfies $U\cup(\{x_1\}^{\perp}+B\cap\dots\cap\{x_n\}^{\perp}+B_\delta)=\emptyset$ for all choices of the $x_1, \dots, x_n$. Let I_j, $1\leqq j\leqq l$ be disjoint intervals (need only sets of positive measure) contained in $(f')^{-1}[f'(t_j)+B_{\delta/2}]$, respectively, and choose $\varrho>0$ such that $U\setminus C(f'(t)R, \varrho)\subset(\{f'(t_j)\}^{\perp}+B_\delta)\cap U$ for all $t\in I_j$ and each $1\leqq j\leqq l$, where $C(f'(t)R, \varrho)$ is the cone defined in 4.2. Note that $\hat{\mu}_{f|I}(rz)\to 0$ as $|r|\to\infty$ for each $z\in U\setminus(\{f'(t_j)\}^{\perp}+B_\delta)$ for $1\leqq j\leqq l$ by 4.1. Our choice of the t_j's forces each $z\in U$ to lie in all but at most $n-1$ of these sets. Hence the measure $v=\sum|I_j|^{-1}\mu_{f|I_j}$ has norm l and $\limsup|\hat{v}|\leqq n-1$. Hence $\alpha_c(\gamma)\geqq l/(n-1)$ by a variant of Proposition 1.1. Since l was arbitrary, the theorem is proved.

The next two results are concerned with lower bounds for the Helson constants of certain classes of k-manifolds in R^n that satisfy a property related to 4.2 (iii). These classes include all the continuous Helson manifolds known to us.

4.4 Theorem. *Suppose the curve $\gamma\subset R^n$, $n\geqq 2$, can be parametrized by continuous coordinate functions $f=(f_1, \dots, f_n)$ which are of bounded variation on $[a, b]$. Then $\alpha(\gamma)\geqq\dfrac{n}{n-1}$.*

Proof. Since $n/n-1$ decreases, we may assume as before that the smallest linear manifold $\mathfrak{M}$ that contains γ is R^n. (If $n=1$, then $\alpha(\gamma)=\infty$.)

Using Theorem 4.2, choose from the pairs (F_j, H_j) a minimal collection (E_j, K_j), $1\leqq j\leqq k$, so that $K\equiv K_1+\dots+K_k=\sum H_j$. Then $k\leqq\dim(K)\leqq n$. If $k=n$, then $K=R$ and part (ii) of that theorem implies this result. If $k<n$, then choose any orthonormal basis $\beta=\{u_1, \dots, u_n\}$ so that u_1 is orthogonal to K. The component functions $f\cdot u_1, \dots, f\cdot u_n$ are of bounded variation and nonconstant since $\mathfrak{M}=R^n$. Lemma 2.8 applied to $f\cdot u_1$ therefore yields a monotone function $g(y)$

defined on an interval in the range of $f\cdot u_1$ so that

$$h(y) = \big(y, f(g(y))\cdot u_2, \ldots, f(g(y))\cdot u_n\big),$$

relative to the basis β, parametrizes a piece of γ. Since g is monotone and the f_j's are of bounded variation, the components $f(g)\cdot u_j$ are also of bounded variation. Again let $(L_1, S_1), \ldots, (L_l, S_l)$ be a minimal collection from the pairs (F_i, H_i) associated with h so that $K+S_1+\ldots+S_l=K+\sum H_j$. Set $S=S_1+\ldots+S_l$ and observe that $l\geqq 1$ since $h'\notin K$ a.e. Part (ii) of Theorem 4.2 yields probability measures $\mu, \nu\in M(\gamma)$ so that $\limsup|\hat{\mu}|\leqq 1-1/k$ and $\limsup|\hat{\nu}|\leqq 1-1/l$ on $C(K, \varrho)$ and $C(S, \varrho)$, respectively, for $\varrho>0$. In particular $\omega=\mu k/(k+l)+\nu l/(k+l)$ is a probability measure on γ with $\limsup|\hat{\omega}|\leqq 1-1/l+k\leqq 1-1/r$, where $r=\dim(K+S)$, on $C(K+S, \varrho)$. Continue in this manner until R^n is exhausted. Call the resulting probability measure ω. It follows that $\limsup|\hat{\omega}|\leqq 1-1/n$ on R^n. This is sufficient to imply $\alpha(\gamma)\geqq(1-1/n)^{-1}$.

A subset $\mathfrak{M}\subseteqq R^n$ *has projection property* $P(m, k)$ if there is an $m\geqq k$-dimensional subspace $S\subseteqq R^n$ which contains a translate $\mathfrak{M}'$ of $\mathfrak{M}$ such that for each k-dimensional subspace K of S there is a set $E\subset K$ of positive k-dimensional measure and a function $F\colon E\to K^{\perp}$ such that (1) $(z, F(z))\in\mathfrak{M}'$ for all $z\in E$ and (2) the directional derivatives of F relative to E exist a.e. in E for a fixed set of k independent directions in K. The curve of the previous theorem is a 1-manifold satisfying $P(m, 1)$, for some m. Also, the Helson k-manifolds in R^n which we construct later will satisfy $P(n, k)$.

4.5 Theorem. *Suppose* $\mathfrak{M}\subseteqq R^n$ *satisfies* $P(m, k)$. *Then* $\alpha(\mathfrak{M})\geqq m/m-k\geqq n/n-k$.

Proof. It is sufficient to assume $m=n$ since $\mathfrak{M}'\subseteqq R^m$ and $\alpha(\mathfrak{M}')=\alpha(\mathfrak{M})$. We can also assume $k<n$; otherwise $\alpha(\mathfrak{M})=\infty$. The partition $P_j=\{jk+l\colon 1\leqq l\leqq k\}$, $0\leqq j<n$, of the first nk positive integers will be used as an indexing aid in the finite induction process to follow. The partitions P_j which contain a multiple of n will be called boundary partitions. Denote their subscripts by $0<j_1<\ldots<j_k=n-1$ and set $j_0=-1$. The process to follow would be considerably simplified if k divides n, but as given includes, for example, the possibility that $(k, n)=1$.

To begin, set $B_{1j_0}=B_{2j_0}=H_{j_0}=\{0\}$, set $j=j_0+1$, let K_j be any k-dimensional subspace of $B_{2j_0}^{\perp}$, and let E, F be the set and function corresponding to K_j which is guaranteed by (1) of $P(n, k)$. With $g(z)=(z, F(z))$ for $z\in E$, Theorem 4.2 together with (2) of $P(n, k)$ implies there is a subspace $H_j\subseteqq R^n$, with $\dim(H_j)\geqq k$, such that $\hat{\nu}_j\equiv\hat{\mu}_{g/E'}\in C_0\big(C(H_j, \varrho)\big)$ for any $\varrho>0$, where $E'\subseteqq E$. Note that H_j projects onto K_j, so we can (by picking a subspace if necessary) assume that this projection is an isomorphism. Suppose K_j, H_j, ν_j have been defined for $j_0<j<l<j_1$. Choose for K_l any k-dimensional subspace of $(B_{2j_0}+H_{j_0+1}+\ldots+H_{l-1})^{\perp}$ and

obtain H_l, v_l as above with l replacing j, again choosing H_l so its projection onto K_l is an isomorphism. Notice that $\dim(B_{2j_0}+H_{j_0+1}+\ldots+H_l)=(l-j_0)k+\dim B_{2j_0}$. For the boundary index one of two possible constructions is required. If P_{j_1} ends with a multiple of n (i.e. n divides $(j_1+1)k$), obtain H_{j_1} as above with $l=j_1$, then set $B_{1j_1}=B_{2j_1}=\{0\}$ and proceed as above with j_l, j_{l+1} replacing j_{l-1}, j_l. Otherwise pick any $K_{j_1}\supset(B_{2j_0}+H_{j_0+1}+\ldots+H_{j_1-1})^{\perp}\equiv S$ of dimension k and obtain the corresponding H_{j_1}, v_{j_1} as before. Let $B_{1j_1}\subset H_{j_1}$ be the subspace which projects isomorphically onto S, and let $B_{2j_1}=B_{1j_1}^{\perp}\cap H_{j_1}$. Now continue the construction by replacing j_{l-1}, j_l with j_l, j_{l+1} and repeating with $j=j_l+1$ and $K_j\subset B_{2j_l}^{\perp}$, for $1\leqq l\leqq k-1$. One obtains n k-dimensional subspaces H_j and corresponding positive measures $v_j\in M(\mathfrak{M})$ for $0\leqq j<n$. Notice also that $B_{2j_0}=B_{2j_k}=\{0\}$. We claim that any nonzero $z\in R^n$ satisfies $\pi_j(z)\neq 0$ for at least k of the projections $\pi_j\colon R^n\to H_j$, $j_0<j\leqq j_k$. To see this fix any string $j_l<j\leqq j_{l+1}$ for some fixed $0\leqq l\leqq k-1$. If $\pi_j(z)=0$ for each such j, then $l>0$ and $z\in B_{2j_l}$ which is orthogonal to B_{1j_l}; hence $z\notin(B_{2j_{l-1}}+H_{j_{l-1}+1}+\ldots+H_{j_l-1})^{\perp}$; hence $\pi_{j_l}(z)\neq 0$ and $\pi_r(z)\neq 0$ for some $j_{l-1}\leqq r<j_l$. The claim follows by applying this to each string in order as $0\leqq l\leqq k-1$. In particular Theorem 4.2 (i) implies that for each nonzero $z\in R^n$ the probability measure $v=n^{-1}\sum\|v_j\|^{-1}v_j$, $0\leqq j\leqq n-1$, satisfies $\limsup|\hat{v}(rz)|\leqq(n-k)/n$ as $|r|\to\infty$.

To complete the argument note that the collection

$$\{z\in R^n\colon\|z\|=1,\quad z\in C(H_j,\ \varrho),\quad j_{l-1}\leqq j\leqq j_l\}$$

for fixed nonzero $l\leqq k$ and any $\varrho>0$ is an open cover of the unit sphere. Hence a compactness argument yields $\limsup|\hat{v}(x)|\leqq(n-k)/n$ as $\|x\|\to\infty$ on R^n. Since $v\in M(\mathfrak{M})$ with norm 1, we can conclude $\alpha(\mathfrak{M})\geqq n/n-k$.

5. A Helson surface in $\mathbf{R}^6$

Using Lemma 3.1, one may construct a Helson k-manifold in R^n whenever $n=lk$ where l is an integer no less than $k+1$. We shall explain the procedure for the case $k=2$, $n=6$, carefully, and outline the rest.

5.1 Theorem. *There exists a surface $\mathfrak{M}\subseteq R^6$ such that $\alpha(\mathfrak{M})\leqq 9\sqrt{\frac{3}{2}}\approx 11.02$.*

Proof. Let us describe R^6 as the product of three planes: the $X=(x_1,x_2)$-plane, the $Z=(z_1,z_2)$-plane, and the $W=(w_1,w_2)$-plane, thus:

$$R^6=\{(x_1,x_2,z_1,z_2,w_1,w_2)\}.$$

Let π_X, π_Z, and π_W be the canonical projections. We proceed by inductive steps; at the i^{th} step we obtain a surface S_i made up of planar faces, on each of which at least one of the three projections is constant-valued. As $i\to\infty$, S_i approaches a 2-manifold $\mathfrak{M}$, which will be a graph over the X-plane, and a Helson set.

For the entire construction, fix an $\varepsilon>0$, to be prescribed later.

Let V be a partition of $[-2, 2]$ of mesh less than $\frac{1}{2}$. Let $\{R_{l,1}\}_l$ be the collection of closed rectangles, with nonoverlapping interiors $R^0_{l,1}$, with the set of vertices $V_1=V\times V$. Let $c_{l,1}$ be the center of $R_{l,1}$, and let $\{C_{k,1}\}$ be the like collection of closed rectangles with vertex set $\{c_{l,1}\}$. The union of each collection certainly covers $\left[-\frac{3}{2}, \frac{3}{2}\right]^2$.

Given such a collection $\mathfrak{L}=\{L_k\}$ of rectangles, denote by $\mathfrak{J}(\mathfrak{L})$ the set of continuous one-to-one mappings $F\colon \cup_k L_k\to R^2$ that are affine on each L_k and such that the image of the set of vertices is independent. Thus each $F\in\mathfrak{J}(\mathfrak{L})$ preserves vertices, edges, and line segments contained within one L_k. Let $s(E)$ denote the minimum distance between distinct points in a finite set E.

Choose $F_{X,1}\in\mathfrak{J}(\{R_{l,1}\})$ such that

$$\operatorname{mesh} F_{X,1} \equiv \max_l \operatorname{diam} F_{X,1}(R_{l,1}) \leqq \operatorname{mesh}(\{R_{l,1}\})$$

and such that

$$[\bigcup_l F_{X,1}(R_{l,1})]^0 \supset \left[-\frac{3}{2}, \frac{3}{2}\right]^2.$$

Let $0<\sigma<\frac{1}{2}\, s(F_{X,1}(V_1))$. Let

$$\delta_{X,1} = \delta(F_{X,1}(V_1), \varepsilon, \sigma),$$
$$\varrho_{X,1} = \varrho(F_{X,1}(V_1), \varepsilon, \sigma)$$

as provided by Lemma 3.1. Choose $t>0$ sufficiently small so that if τ_1 is the vector (t, t), then $\{R_{l,t}+\tau_1\}_l$ covers $\left[-\frac{3}{2}, \frac{3}{2}\right]^2$ and

$$\|F_{X,1}(x-\tau_1)-F_{X,1}(x)\| < \delta_{X,1}/2$$

for all $x\in(\cup_l(R_{l,1}+\tau_1))\cap(\cup_l R_{l,1})$. Then for each pair k, l, the two or fewer points of the set $\partial F_{X,1}(R_{l,1}-\tau_1)\cap\partial F_{X,1}(R_{k,1})$ are contained in $F_{X,1}(V_1)+U(\delta_{X,1}/2)$.

Let $F_{Z,1}$ and $F_{W,1}$ belong to $\mathfrak{J}(\{C_{k,1}\})$.

The surface S_1 is defined as follows. For $x\in F_{X,1}(\bigcup_l R_{l,1})$, define $E_1(x)$ to be the set of all points (z, w) such that for some pair k, l,

$$z = F_{Z,1}(c_{l,1}),\quad w = F_{W,1}(c_{k,1}),$$

and

$$x\in F_{X,1}(R_{l,1})\cap F_{X,1}(R_{k,1}-\tau_1).$$

Let

$$S_1 = \{(x, z, w)\in R^6\colon\ x\in F_{X,1}(\bigcup_l R_{l,1}) \text{ and } (z, w)\in \mathrm{co}\, E_1(x)\},$$

where $\mathrm{co}\,E$ denotes the convex hull of the set E.

Now we shall describe S_1 another way. Let Q_1, Q_2, Q_3, and Q_4 denote the images under $F_{X,1}$ of a rectangle $R_{l,1}$ and its neighbors to the east, southeast, and south respectively. Denote the images of the translates by $-\tau_1$ of those four rectangles by $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, and $\tilde{Q}_4$, respectively. Let c_1, c_2, c_3, and c_4 be the centers of $R_{l,1}$ and those neighbors, respectively, and let C be the rectangle that they determine. Let

$$(z_1', \zeta_2) = F_{Z,1}(c_1), \quad (\zeta_1', \zeta_2') = F_{Z,1}(c_2),$$
$$(z_1, z_2) = F_{Z,1}(c_4), \quad (\zeta_1, z_2') = F_{Z,1}(c_3).$$

Make the corresponding definitions with w and W in the roles of z and Z. At this point, the reader should draw a picture of the eight rectangles Q. The part of S_1 whose X projection is the union of Q_4^0 with its northeast vertex and with the interior of its north and east edges consists of the following 16 planar faces, where we describe each according to the maximum number of intersections among the Q's and $\tilde{Q}$'s. Note that each of the two coordinate pairs Z and W is constant on 9 of the 16 faces; X is constant on 4 of the 16 and on 5 more faces of S_1 which are above the remainder of Q_4.

1. $\{(x, z, w)\colon x \in Q_4^0 \cap \tilde{Q}_4^0,\ z = (z_1, z_2),\ w = (w_1, w_2)\}$
2. $\{(x, z, w)\colon x \in \tilde{Q}_3 \cap \tilde{Q}_4 \cap Q_4^0,\ z = (z_1, z_2),\ w \in co[(w_1, w_2), (\omega_1, w_2')]\}$
3. $\{(x, z, w)\colon x \in Q_4^0 \cap \tilde{Q}_3^0,\ z = (z_1, z_2),\ w = (\omega_1, w_2')\}$
4. $\{(x, z, w)\colon x \in Q_4 \cap Q_3 \cap \tilde{Q}_3^0,\ z \in co[(z_1, z_2), (\zeta_1, z_2')],\ w = (\omega_1, w_2')\}$
5. $\{(x, z, w)\colon \{x\} = Q_4 \cap Q_3 \cap \tilde{Q}_3 \cap \tilde{Q}_2,\ z \in co[(z_1, z_2), (\zeta_1, z_2')],$
$w \in co[(\omega_1, w_2'), (\omega_1', \omega_2')]\}$
6. $\{(x, z, w)\colon x \in \tilde{Q}_3 \cap \tilde{Q}_2 \cap Q_4^0,\ z = (z_1, z_2),\ w \in co[(\omega_1, w_2'), (\omega_1', \omega_2')]\}$
7. $\{(x, z, w)\colon \{x\} = \tilde{Q}_4 \cap \tilde{Q}_3 \cap \tilde{Q}_2 \cap \tilde{Q}_1 \cap Q_4^0,\ z = (z_1, z_2),\ w \in F_{W,1}(C)\}$
8. $\{(x, z, w)\colon x \in \tilde{Q}_4 \cap \tilde{Q}_1 \cap Q_4^0,\ z = (z_1, z_2),\ w \in co[(w_1, w_2), (w_1', \omega_2)]\}$
9. $\{(x, z, w)\colon x \in Q_4^0 \cap \tilde{Q}_1^0,\ z = (z_1, z_2),\ w = (w_1', \omega_2)\}$
10. $\{(x, z, w)\colon x \in \tilde{Q}_1 \cap \tilde{Q}_2 \cap Q_4^0,\ z = (z_1, z_2),\ w \in co[(w_1', \omega_2), (\omega_1', \omega_2')]\}$
11. $\{(x, z, w)\colon x \in Q_4^0 \cap \tilde{Q}_2^0,\ z = (z_1, z_2),\ w = (\omega_1', \omega_2')\}$
12. $\{(x, z, w)\colon x \in Q_4 \cap Q_3 \cap \tilde{Q}_4^0,\ z \in co[(z_1, z_2), (\zeta_1, z_2')],\ w = (\omega_1', \omega_2')\}$
13. $\{(x, z, w)\colon \{x\} = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \cap \tilde{Q}_2^0,\ z \in F_{Z,1}(C),\ w = (\omega_1', \omega_2')\}$
14. $\{(x, z, w)\colon x \in Q_1 \cap Q_4 \cap \tilde{Q}_2^0,\ z \in co[(z_1, z_2), (z_1', \zeta_2)],\ w = (\omega_1', \omega_2')\}$
15. $\{(x, z, w)\colon \{x\} = Q_4 \cap Q_1 \cap \tilde{Q}_1 \cap \tilde{Q}_2,\ z \in co[(z_1, z_2), (z_1', \zeta_2)],$
$w \in co[(w_1', \omega_2), (\omega_1', \omega_2')]\}$
16. $\{(x, z, w)\colon x \in Q_1 \cap Q_4 \cap \tilde{Q}_1^0,\ z \in co[(z_1, z_2), (z_1', \zeta_2)],\ w = (w_1', \omega_2)\}$.

Note that the only points x above which both z and w are moving lie within $\delta_{X,1}/2$ of the northeast vertex.

By describing how S_2 is defined after S_1, we shall make clear how the sequence of surfaces $\{S_i\}$ is defined.

Let $\{s_i\}$ and $\{t_j\}$ be increasing finite sequences such that $\{(s_i, t_j)\}$ is the set of vertices of a collection $\mathfrak{L}$ of rectangles. Let $\bar{s}_i=(s_i+s_{i+1})/2$, $\bar{t}_j=(t_j+t_{j+1})/2$. Let $\mathfrak{B}$ be a similarly described collection of rectangles, with vertex set $\{(p_m, q_n)\}$, such that $\{p_m\}\cap\{s_i\}=\emptyset=\{q_n\}\cap\{t_j\}$, such that each rectangle in $\mathfrak{L}$ contains several in $\mathfrak{B}$, and such that the union of $\mathfrak{B}$ contains the union of $\mathfrak{L}$. When those properties hold, we call $\mathfrak{B}$ an *overlay of* $\mathfrak{L}$.

Fix i. Let $0<\beta<\frac{1}{2}$. Identify m such that $\bar{s}_i\in(p_m, p_{m+1})$. For each l such that $p_l\in(s_i, s_{i+1})$, except the smallest and the largest such l, define

$$p_l' = \bar{s}_i+\beta(p_l-\bar{p}_m).$$

For the value of l such that $s_i\in(p_{l-1}, p_l)$, define

$$p_l' = s_i+\beta(p_l-s_i).$$

For the value of l such that $s_{i+1}\in(p_l, p_{l+1})$, define

$$p_l' = s_{i+1}-\beta(s_{i+1}-p_l).$$

Note that for all l except those two,

$$p_{l+1}'-p_l' = \beta(p_{l+1}-p_l).$$

Obtain $\{q_k'\}$ similarly. The collection of rectangles $\mathfrak{P}=\mathfrak{P}(\mathfrak{B})$ with vertex set $\{(p_l', q_k')\}$ is the *β-distortion of $\mathfrak{B}$ relative to $\mathfrak{L}$*. The rectangles of $\mathfrak{B}$ are in one-to-one correspondence with those in $\mathfrak{P}(\mathfrak{B})$, by means of the mapping

$$\varphi\colon [p_l, p_{l+1}]\times[q_k, q_{k+1}] \to [p_l', p_{l+1}']\times[q_k', q_{k+1}'].$$

Let $\mathfrak{C}$ be the collection of rectangles whose vertices are the centers of rectangles in $\mathfrak{B}$. The mapping F defined on the centers of $\mathfrak{B}$ by:

$$F(\text{center } B) = \text{center } \varphi(B) \quad \text{for} \quad B\in\mathfrak{B},$$

extends to a continuous map F from the union of $\mathfrak{B}$ onto the union of $\mathfrak{P}$ which is affine on each rectangle in $\mathfrak{C}$. If $B\in\mathfrak{B}$, $L\in\mathfrak{L}$, and $B\subseteq L^0$, then

$$F(\text{center } B)\in\beta(L-\text{center } L)+\text{center } L.$$

If on the other hand $B\cap\partial L\neq\emptyset$, then $F(\text{center } B)$ lies on the same side of L as center B.

We obtain S_2 as follows. Let

$$\sigma_1 = \min\{s(F_{X,1}(V_1)),\ s(F_{T,1}(\{c_{l,1}\})\colon T = Z \text{ or } W\}$$
$$\delta_1 = \min\{\delta_{X,1}/4,\ \delta(F_{T,1}(\{c_{l,1}\}), \varepsilon, \sigma_1)\colon T = Z \text{ or } W\},$$
$$\varrho_1 = \min\{\varrho_{X,1},\ \varrho(F_{T,1}(\{c_{l,1}\}), \varepsilon, \sigma_1)\colon T = Z \text{ or } W\}.$$

Choose $\beta_1 > 0$ and $0 < \delta_1' < \|\tau_1\|/10$ so that on $\bigcup_k R_{k,1}$:

$$\|x-y\| < \delta_1' \Rightarrow \|F_{X,1}(x) - F_{X,1}(y)\| < \delta_1/12,$$

and

$$\|x-y\| < \beta_1 \operatorname{mesh}(\{R_{k,1}\}) \Rightarrow \|F_{T,1}(x) - F_{T,1}(y)\| < \delta_1/12$$
$$\text{for} \quad T = Z \quad \text{and } W.$$

Let $\mathfrak{B}$ be a collection of rectangles with mesh $\mathfrak{B} < \delta_1'$ such that both $\mathfrak{B}$ and $\mathfrak{B}+\tau_1$ are overlays of $\{R_{k,1}\}$. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be respectively the β_1-distortions of $\mathfrak{B}$ and $\mathfrak{B}+\tau_1$ relative to $\{R_{k,1}\}$, and let F, φ; F', φ' be the corresponding maps. Let $\{R_{k,2}\} = \mathfrak{B}$, $c_{k,2} = \text{center } R_{k,2}$; and let $\mathfrak{C} = \{C_{k,2}\}$ be the collection of rectangles with vertex set $\{c_{k,2}\}$. Denote the vertices of $\{R_{k,2}\}$ by V_2. Choose $F_{Z,2}, F_{W,2} \in \mathfrak{J}(\{C_{k,2}\})$ so that on $\bigcup_k R_{k,2}$:

$$\|F_{Z,2}(x) - F_{Z,1}(F(x))\| < \frac{1}{4} s(F_{Z,1}(F(V_2)))$$

and

$$\|F_{W,2}(x) - F_{W,1}(F'(x+\tau_1))\| < \frac{1}{4} s(F_{W,1}(F'(V_2+\tau_1))).$$

Were it not for the requirement that $\{F_{Z,2}(C_{k,2})\}$ be an independent set, $F_{Z,1}\circ F$ would serve as $F_{Z,2}$; as it is, we must choose a slight perturbation. Then we choose $F_{X,2} \in \mathfrak{J}(\{R_{k,2}\})$ such that on V_2

$$\|F_{X,2}(x) - F_{X,1}(x)\| < \frac{1}{4} s(F_{X,1}(V_2)).$$

Finally, we carry out the selection of τ_2, $\delta_{x,2}$, $\varrho_{x,2}$, E_2, and the definition of the surface S_2 following the procedures described above for when the subscript has value 1.

We claim that

$$S_2 \subseteq S_1 + U(\delta_1/2)$$

and

$$\operatorname{mesh} S_2 \equiv \max\{\operatorname{diam} P\colon P \text{ is a planar face of } S_2\} < \frac{3}{4} \operatorname{mesh} S_1.$$

To verify the latter note that if $R_{l2} \cap C_{k1} \neq \emptyset$, then $F(c_{l2}) \in C_{k1}$. Moreover, if c_{l2}, c_{j2} are adjacent centers, then $\|F(c_{l2}) - F(c_{j2})\| < \frac{1}{2} \operatorname{mesh}(\pi_1(c_{l1}))$ or $\frac{1}{2} \operatorname{mesh}(\pi_2(\{c_{l1}\}))$

depending on whether c_{l2}, c_{j2} lie on a horizontal or vertical segment (π_j denotes projection onto j^{th} coordinate). It follows that diam $F(C_{k2})<\frac{1}{2}$ mesh $\{C_{k1}\}$; hence mesh $F(\{C_{k2}\})<\frac{1}{2}$ mesh $\{C_{k1}\}$. But each $F(C_{k2})\subset C_{j1}$ for some j and $F_{Z,1}$ is affine on C_{j1}. Hence mesh $F_{Z,1}(F(\{C_{k2}\}))<\frac{1}{2}$ mesh $F_{Z,1}(\{C_{k1}\})$. Since $F_{Z,2}$ is only a slight perturbation of $F_{Z,1}(F)$ (they differ by less than $\delta_1/16$), mesh $F_{Z,2}(\{C_{k2}\})<\frac{5}{8}$ mesh $F_{Z,1}(\{C_{k1}\})$. A similar argument applied to F' and $R_{l2}+\tau_1$ yields the same result for $F_{W,2}$. That, together with mesh $F_{X,2}(\{R_{l2}\})\ll$mesh $F_{X,1}(\{R_{l1}\})$ and the manner in which S_2 is constructed yields mesh $(S_2)<\frac{3}{4}$ mesh (S_1).

To show that $S_2\subset S_1+U(\delta_1/2)$, we must be more careful. Let $x\in\pi_X S_2$ and $x_2=F_{X,2}^{-1}(x)$. Recall that for any $R_{k,2}$, the image $F_{X,1}(R_{k,2})$ is contained in a disc of radius $\alpha_1\equiv\delta_1/12$ (by our choice of mesh $(\{R_{k,2}\})$), $F_{X,1}$ and $F_{X,2}$ differ by at most $\alpha_1/4$ on the vertices of $R_{k,2}$, and $F_{X,2}$ is affine on $R_{k,2}$. Hence $\|F_{X,2}(u)-F_{X,1}(v)\|<3\alpha_1/2$ for any $u, v\in R_{k,1}$. Let φ_j be the maps given by $\varphi_j(R_{k,j})=$center $R_{k,j}$ for $j=1, 2$. We will say that u, v are *linked* if there are three or less distinct rectangles $R_1, R_2, R_3\in\{R_{k,2}\}$ satisfying $v\in R_1$, $R_1\cap R_2\neq\emptyset\neq R_2\cap R_3$, and $u\in R_3$. To establish our claim we consider whether or not x_2 is "close" to the boundary of some $R_{k,1}$, $R_{k,1}-\tau_1$, or not, where close is defined as linked. In particular, each v that is linked to x_2 satisfies either (1) $v\in V_1-\tau_1\cup V_1$, or (2) $v\in\cup[\{\partial R_{k,1}\}\cup\{\partial R_{k,1}-\tau_1\}]$, or (3) all other cases. Pick a v whose case number is minimum. Suppose it is (1) and $v\in V_1$. Then x_2 is not linked to any $v'\in\{\partial R_{k,1}-\tau_1\}$; otherwise $\|v'-v\|\leqq 6$ mesh $(\{R_{k,2}\})<6\|\tau_1\|/10$, but $\|u-v-\tau_1\|\geqq\|\tau_1\|/\sqrt{2}$ for any $u\in\bigcup_k\partial R_{k,1}$ since $\|\tau_1\|\ll s(V_1)$ and $\tau_1=(t, t)$, some t. Let $x_2+\tau_1\in R_{l,1}^0$ and $x_2+\tau_2\in R_4\in\{R_{k,2}\}$. It follows that $F'(\varphi_2(R_4)+\tau_1)\in\varphi_1(R_{l,1})+U(\alpha_2)$, where $\alpha_2<\beta_1$ mesh $(\{R_{k,1}\})$; hence $F_{W,1}[F'(\varphi_2(R_4)+\tau_1)]\in F_{W,1}(\varphi_1(R_{l,1}))+U(\alpha_1)$; hence $F_{W,2}(\varphi_2(R_4))\in F_{W,1}(\varphi_1(R_{l,1}))+U(3\alpha_1/2)$. Since $W_2\equiv\pi_W[\pi_X^{-1}(x)\cap S_2]=$ co $\{F_{W,2}(\varphi_2(R_4))\colon R_4\in\{R_{k,2}\},\ x_2+\tau_2\in R_4\}$ and $W_1\equiv\pi_W[\pi_X^{-1}(v)\cap S_1]=\{F_{W,1}(\varphi_1(R_{l,1}))\}$, we conclude that $W_2\subset W_1+U(3\alpha_1/2)$. For the z-projections observe that $v=$center $C_{k,1}$, some k; hence $R_3\subset C_{k,1}$ (mesh $(\{R_{k,2}\})\ll s(V_1)$); hence $F(R_3)\subset C_{k,1}$; hence $F_{Z,1}[F(\varphi_2(R_3))]\in F_{Z,1}(C_{k,1})$; hence $F_{Z,2}(\varphi_2(R_3))\in F_{Z,1}(C_{k,1})+U(3\alpha_1/2)$. Thus, as before for $Z_2=\pi_Z[\pi_X^{-1}(x)\cap S_2]$ and $Z_1=\pi_Z[\pi_X^{-1}(v)\cap S_1]$ we have $Z_2\subset Z_1+U(3\alpha_1/2)$. Since

$$\|F_{X,1}(v)-F_{X,2}(x_2)\|\leqq\|F_{X,1}(v)-F_{X,1}(u)\|+\|F_{X,1}(u)-x\|<5\alpha_1$$

for $u\in R_3$, we conclude $\Pi_X^{-1}(x)\cap S_2\subset[\Pi_X^{-1}F_{X,1}^{-1}(v)\cap S_1]+U(6\alpha_1)$. If $v\in V_1-\tau_1$ a similar argument gives the same conclusion.

Next suppose (2) is the minimum case number. Observe that if x_2 is linked to $v\in\partial R_{k,1}$ and to $v'\in\partial R_{l,1}-\tau_1$, then x_2 is linked (possibly thru corners) to a $v\in\partial R_{l,1}-\tau_1\cap\partial R_{k,1}$; when that happens, pick such a common v. Using the notation and estimates of the preceding paragraph, we conclude that at least one of Z_1 or W_1 is a segment with the other possibly a singleton, that $F_{Z,1}[F(\varphi_2(R_3))]\in Z_1+U(\alpha_1)$, and that $F_{W,1}[F'(\varphi_2(R_4)+\tau_1)]\in W_1+U(\alpha_1)$; hence

$$F_{Z,1}(\varphi_2(R_3))\in Z_1+V(3\alpha_1/2), \quad F_{X,2}(\varphi_2(R_4))\in W_1+U(3\alpha_1/2),$$

and $\|F_{X,1}(v)-x\|<5\alpha_1$. Thus $\pi_X^{-1}(x)\cap S_2\subset[\pi_X^{-1}F_{X,1}(v)\cap S_1]+U(6\alpha_1)$.

Finally, if (3) is the minimum case number, then $x_2\in R^0_{l,1}-\tau_1\cap R^0_{k,1}$ for some k, l, $F(\varphi_2(R_3))\in\varphi_1(R_{k,1})+U(\alpha_2)$, and $F'(\varphi_2(R_4)+\tau_1)\in\varphi(R_{l,1})_1+U(\alpha_2)$; hence $F_{Z,2}(\varphi_2(R_3))\in F_{Z,1}(\varphi_1(R_{k,1}))+U(3\alpha_1/2)$, $F_{W,2}(\varphi_2(R_4))\in F_{W,1}(\varphi_1(R_{l,1}))+U(3\alpha_1/2)$, and $\|F_{X,1}(x_2)-x\|<3\alpha_1/2$. Thus $\pi_X^{-1}(x)\cap S_2\subset[\pi_X^{-1}(F_{X,1}(x))\cap S_1]+U(3\alpha_1)$.

At the i^{th} step we obtain $\{R_{ki}\}_k$, $\{C_{li}\}_l$, V_i, $\{c_{ki}\}_k$, $\delta_{X,i}$, $\varrho_{X,i}$, τ_i, E_i, S_i, δ_i, ϱ_i with $S_{i+1}\subset S_i+U(\delta_i/2)$. Since $\delta_{i+1}<\delta_i/2$, it follows that $\{S_i\}$ must approach some set $\mathfrak{M}'$, contained in $S_i+U(\delta_i)$ for each i. Set $\mathfrak{M}=\mathfrak{M}'\cap\pi_X^{-1}([0,1]^2)$. We claim that $\mathfrak{M}$ is the graph of some continuous H: $[0,1]^2>\mathbf{R}^4$ and that $\mathfrak{M}$ is Helson. At least $\mathfrak{M}$ is the graph of some H, since $\operatorname{mesh}(S_i)\to 0$. If H is not continuous, then there is an $x\in[0,1]^2$, i_0, and a $\delta>0$ so that for all $\varrho>0$ and all $i\geqq i_0$ we have $(\pi_X^{-1}(x)\cap S_i)+U(\delta)\not\supseteq\pi_X^{-1}(x+U(\varrho))$. But fix $i\geqq i_0$ so that $\operatorname{mesh}(S_i)<\delta/2$, set $\varrho=\tau_i/2$, and let $y\in x+U(\varrho)$. Then $E_i(y)\cup E_i(x)\subset F_{Z,i}(C_{li})\times F_{W,i}(C_{li})$ for some l, a set whose diameter $\leqq\sqrt{2}\operatorname{mesh}(S_i)<\delta$. It follows that H must be continouus.

Finally, we verify that $\mathfrak{M}$ is Helson and give an upper bound for its Helson constant. Let μ be a regular Borel measure supported on $\mathfrak{M}$ with $\|\mu\|=1$. Let $\varrho>0$ and choose $f\in C(\mathfrak{M})$ so that $|f|\leqq 1$ and $|1-\int f\,d\mu|<\varrho$. Extend f to a uniformly continuous function on $\mathbf{R}^6$ of norm 1. Denote that extension by f also. Choose $0<\delta<\varrho$ so that $\|f-f_y\|_\infty<\varrho/4$ whenever $\|y\|<\delta$. Fix i so that $\operatorname{mesh}(S_i)<\delta$. Then $\delta_i\ll\varrho_i\leqq\varrho_{X,i}\operatorname{mesh}(S_i)$ and $\mathfrak{M}\subset S_i+U(\delta_i)$. Set $H_X(\alpha)=\pi_X^{-1}(F_{X,i}(V_i)+U(\alpha))$ for $\alpha=\delta_{X,i}$ and $\varrho_{X,i}$ and $H_T(\alpha)=\pi_T^{-1}(F_{T,i}(\{c_{ki}\})+U(\alpha))$ for $\alpha=\delta_i$ and ϱ_i and $T=Z$ and W. Let $(x_k,z_k,\omega_k)\in\pi_Z^{-1}(F_{Z,i}(c_{ki}))\cap S_i$ where (of course) $z_k=F_{Z,i}(c_{ki})$ and let $(x,z,\omega)\in T_{Z,k}(\alpha)\equiv\pi_Z^{-1}(F_{Z,i}(c_{ki})+U(\alpha))\cap(S_i+U(\delta_i))$ where $\alpha=\delta_i$ and ϱ_i. For now, consider only $\alpha=\varrho_i$. Then $z\in z_k+U(\varrho_i)$ and there must be an $(x'z',\omega')\in S_i$ such that $z'\in z_k+U(\varrho_i+\delta_i)$ and such that $(x,z,\omega)\in(x',z',\omega')+U(\delta_i)$. Since $[F_{Z,i}(c_{ki})+U(\varrho_i+\delta_i)]\cap F_{Z,i}(\{c_{ji}\}_j)=\{F_{Z,i}(c_{ki})\}$, it follows that $x_k,x'\in F_{X,i}(R_{ki})$ where center $R_{k,i}=c_{ki}$; hence $\omega_k,\omega'\in F_{W,i}(C_{li})$ where C_{li} has c_{ki} for a vertex. That is,

$\|(x_k,z_k,\omega_k)-(x,z,\omega)\|\leqq\|(x_k,z_k,\omega_k)-(x',z',\omega')\|+\delta_i\leqq\|(x_k,\omega_k)-(x',\omega')\|+$
$+|z_k-z'|+\delta_i\leqq\sqrt{2}\operatorname{mesh}(S_i)+\varrho_i+2\delta_t<4\delta.$

It follows that $|f(y)-f(x_k,z_k,\omega_k)|<\varrho$ for $y\in T_{Z,k}(\varrho_i)$, that the $\{T_{Z,k}(\varrho_i)\}_k$ are disjoint, and that $H_Z(\varrho_i)\cap[S_i+U(\delta_i)]=\bigcup_k T_{Z,k}(\varrho_i)$. Since $\delta_i<\varrho_i$, the same is true when δ_i replaces ϱ_i; since Z, W are symmetric cases, the same is true for $T_{W,k}(\alpha)$

where $\alpha=\delta_i$ or ϱ_i. The argument for X is slightly different. Let $x_{li}\in V_i$, let $(x_l, z_l, \omega_l)\in\pi_X^{-1}(F_{X,i}(x_{li}))\cap S_i$ where $x_l=F_{X,1}(x_{li})$, and let $(x, z, \omega)\in T_{X,l}(\alpha)\equiv\pi_X^{-1}(F_{X,i}(x_{li})+U(\alpha))\cap(S_i+U(\delta_i))$ where $\alpha=\delta_{X,i}$ and $\varrho_{X,i}$. Again set $\alpha=\varrho_{X,i}$. Then there must be some $(x', z', \omega')\in S_i$ such that $x'\in x_l+U(\varrho_{X,i}+\delta_i)$ and such that $(x, z, \omega)\in(x', z', \omega')+U(\delta_i)$. Thus $z_l, z'\in F_{Z,i}(C_{ki})$ and $\omega_l, \omega'\in F_{W,i}(C_{ki})$ where $x_{li}\in C_{ki}$. As before, it follows that $\|(x_l, z_l, \omega_l)-(x, z, \omega)\|<4\delta$ and hence that $|f(y)-f(x_l, z_l, \omega_l)|<\varrho$ for $y\in T_{X,l}(\alpha)$, where $\alpha=\delta_{X,i}$ and $\varrho_{X,i}$. Now recall that the only planar sections in S_i on which both z and ω vary lie in $H_X(\delta_{X,i}/2)$. Thus if $(x, z, \omega)\in S_i+U(\delta_i)$ and if neither $|z-F_{Z,i}(c_{ki})|<\delta_i$ nor $|\omega-F_{W,i}(c_{ki})|<\delta_i$ for some k, then $|x-x_l|<\delta_{X,i}/2+\delta_i<3\delta_{X,i}/4$. Thus $S_i+U(\delta_i)\subset H_X(\delta_{X,i})\cup H_Z(\delta_i)\cup H_W(\delta_i)$. Indeed each of those three sets contains at least 9/16 of $S_i+U(\delta_i)$ in the sense that each contains a δ_i-band about 9 of the 16 planar sections of $\pi_X^{-1}(F_{X,i}(R'_{k,i}))\cap S_i$, where $R'_{k,i}$ is the union of $R^0_{k,i}$ with its north vertex and the interior of its north and east edges. If μ_X, μ_Z, and μ_W denote the restrictions of μ to the sets $H_X(\delta_{X,i})$, $H_Z(\delta_i)$, and $H_W(\delta_i)$, respectively, then at least one of them, say μ_T, must satisfy $\|\mu_T\|\geqq 1/3$. Of course $\left|\int f d\mu_T-\|\mu_T\|\right|<\varrho$. If $T=X$, let $\delta_T=\delta_{X,i}$, $\varrho_T=\varrho_{X,i}$, and let F_T be the finite set $F_{X,i}(V_i)$; otherwise, let $\delta_T=\delta_i$, $\varrho_T=\varrho_i$, $F_T=F_{T,i}(\{c_{k,i}\})$. Let $\sigma=s(F_{X,i}(V_i))$. Then $\sigma_i\leqq\sigma<\delta<\varrho$. Applying Lemma 3.1 with δ_T, ϱ_T, F_T in the roles of δ, ϱ, F, we obtain a certain $g\in A(R^2)$. (Note that the present f restricted to the finite F_T is a convex combination of functions of modulus 1 on F_T.) Hence there exists $h\in A(R^6)$ such that: $|f(y)-h(y)|<2\varrho$ on $H_T(\delta_T)\cap[S_i+U(\delta_i)]$; $|\arg f(y)-\arg h(y)|<2\varrho$ on $H_T(\varrho_T)\cap[S_i+U(\delta_i)]$; $|h(y)|\leqq\varepsilon^2$ elsewhere; and $\|h\|_A\leqq\varepsilon^{-1}$. By denoting the characteristic functions of $H_T(\delta_T)$ and $H_T(\varrho_T)\setminus H_T(\delta_T)$ by χ_1 and χ_2, we can therefore write $\left|\int h d\mu\right|=\left|\int h\chi_1 d\mu+\int h\chi_2 d\mu+\int h(1-\chi_1-\chi_2)d\mu\right|=\left|\int f\chi_1 d\mu+\int |h| f\chi_2 d\mu\right|+\Delta\geqq\int\chi_1 d|\mu|+\int\chi_2 d|\mu|-\varrho+\Delta\geqq 1/3-\varrho+\Delta$, where $|\Delta|\leqq 2\varrho+\varepsilon^2 2/3$. It follows that $\|\hat\mu\|_\infty\geqq\varepsilon(1/3-3\varrho-\varepsilon^2 2/3)$. Letting $\varrho\to 0$ yields $\|\hat\mu\|_\infty\geqq\varepsilon(1-2\varepsilon^2)/3$. That expression is maximized when $\varepsilon^2=1/6$. With that value of ε in use throughout the construction we obtain $\alpha(\mathfrak{M})<9\sqrt{3/2}$. The theorem is proved.

Evidently a similar construction using $n+1$ copies of R^2 instead of 3 and $n-1$ distinct translation vectors $\tau_{ij}=(t_{ij}, t_{ij})$ for $1\leqq j\leqq n-1$ at the i^{th} stage of the construction yields the following situation. The surface S_i above $F_{X,i}(R_{k,i})$ has $n^2+2n^2+n^2$ planar faces (not counting those on the south or west edges of $F_{X,i}(R_{k,i})$; they are counted for a different k). The first n^2 are X-planar faces; the $2n^2$ occur over edges but not the intersection of edges; the last n^2 occur over the intersections of edges. Form the $n+1$ corresponding $H_X(\alpha)$, $H_Z(\alpha)$, $H_W(\alpha)$, ... Each point in S_i lies in at least $n-1$ of these sets since at most two variable pairs are non-constant on each planar section of S_i. In particular if $\mu\in M(S_i)$ and if $\mu_1, \dots, \mu_{n+1}$ denotes the restrictions of μ to $H_X, H_Z, \dots$, then $\sum\|\mu_j\|\geqq(n-1)\|\mu\|$; hence $\|\mu_j\|\geqq[(n-1)/(n+1)]\|\mu\|$ for some j. It follows that $\|\hat\mu\|_\infty\geqq\varepsilon(n-1-2\varepsilon^2)/(n+1)$ for all $0<\varepsilon<1$. Thus $\alpha(\mathfrak{M})\leqq 2(n+1)[3/2(n-1)]^{3/2}$ for $n\leqq 7$ and $\alpha(\mathfrak{M})\leqq(n+1)/(n-3)$

for $n \geqq 7$. In terms of the even dimension d of the space containing the 2-manifold $\mathfrak{M}$, that becomes $\alpha(\mathfrak{M}) \leqq d[3/(d-4)]^{3/2}$ for $d \leqq 16$ and $\alpha(\mathfrak{M}) \leqq d/d-8$ for $d \geqq 16$.

The construction of a Helson k-manifold using $n+1$ copies of R^k for $n \geqq k$ can be carried out in a similar manner that we briefly describe using the obvious extensions in notation. Denote the $n+1$ copies of R^k as $X_0, X_1, \ldots, X_n$ (previously, $X=X_0$, $Z=X_1$, $W=X_2$) and the $n-1$ translation vectors at the i^{th} stage by $\tau_{i,j}=(t_{ij}, \ldots, t_{ij})$ for $1 \leqq j \leqq n-1$. Define the k-dimensional faces of S_i in a manner analogous to $k=2$ and observe that each of those faces occurs either above the interior of at least one of $F_{X_0,i}(R_{l,i})$, $F_{X_0,i}(R_{l,i}-\tau_{i,1})$, $\ldots$, $F_{X_0,i}(R_{l,i}-\tau_{i,n-1})$ or otherwise lies above the intersection of the boundaries of k-distinct choices of these k-rectangles. This latter case can only occur close to the vertices of $F_{X_0,i}(R_{l,i})$, where "close" is determined completely by the selection of the translation vectors. Thus on each face of S_i either at least one of the X_j coordinates is a vertex of $F_{X_{j,i}}(\{C_{l,i}\})$ for $1 \leqq j \leqq n$ or the X_0 coordinate is essentially in $F_{X_0,i}(V_i)$. The remainder of the construction offers no surprises. For $n=k$, this yields the optional $\varepsilon^2=1/3k$ and the corresponding $\alpha(\mathfrak{M}) < \sqrt{3k}\, 3(k+1)/2$. For $n>k$, one obtains $\sum \|\mu_j\| \geqq (n+1-k)\|\mu\|$; hence $\|\mu_j\| > (1-k/n+1)\|\mu\|$; hence $\|\hat{\mu}\|_\infty > \varepsilon(n+1-k-\varepsilon^2 k)/n+1$; hence the optimal $\varepsilon^2=\min\left((n+1-k)/3k, 1\right)$; hence $\alpha(\mathfrak{M}) < \frac{1}{2}(n+1)\sqrt{27k/(n+1-k)^3}$ if $n+1 \leqq 4k$ and $\alpha(\mathfrak{M}) \leqq (n+1)/(n+1-2k)$ if $n+1 \geqq 4k$.

In terms of the dimension d, our efforts yield a Helson k-manifold $\mathfrak{M}$ in R^d where $d=lk$ and $l \geqq 4k$ with $d/(d-k) \leqq d/(d-2k^2)$. For $k+1 \leqq l < 4k$, the upper bound is somewhat worse. The spread between the lower and upper bounds obtained may well reflect our lack of attention to the wealth of arithmetical relationships which exist on these highly nonconvex manifolds. The right analysis there would presumably raise the lower bound, but it eluded us. Those same relationships may prevent the occurrence of Helson k-manifolds in R^d for d too close to k. For example, it seems quite likely that a Helson 2-manifold cannot exist in R^3 but can in R^4. The best we can do is R^6.

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