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Linear resolvent growth test for similarity of a weak contraction to a normal operator

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Abstract. It is proved in Benamara–Nikolski [1] that if the spectrum $\sigma(T)$ of a contraction T with finite defects ($\operatorname{rank}(I-T^*T)=\operatorname{rank}(I-TT^*)<\infty$) does not coincide with $\bar{\mathbf{D}}$, then the contraction is similar to a normal operator if and only if

$$\mathcal{C}_1(T)=\sup_{\lambda\in\mathbf{C}\setminus\sigma(T)}\|(T-\lambda)^{-1}\|\operatorname{dist}(\lambda,\sigma(T))<\infty.$$

The examples of Kupin–Treil [9] show that the result is no longer true if we replace the condition $\operatorname{rank}(I-T^*T)<\infty$ by its weakened version $I-T^*T\in\mathfrak{S}_1$, where $\mathfrak{S}_1$ denotes the class of nuclear operators.

We prove in this paper that, however, the following theorem holds.

Theorem. *Let T be a contraction acting on a separable Hilbert space H, $\sigma(T)\neq\bar{\mathbf{D}}$. If*

(LRG) $$\mathcal{C}_1(T)=\sup_{\lambda\in\mathbf{C}\setminus\sigma(T)}\|(T-\lambda)^{-1}\|\operatorname{dist}(\lambda,\sigma(T))<\infty,$$

(UTB) $$\mathcal{C}_2(T)=\sup_{\mu\in\mathbf{D}}\operatorname{tr}(I-b_\mu(T)^*b_\mu(T))<\infty,$$

where $b_\mu(T)=(I-\bar{\mu}T)^{-1}(T-\mu)$, $\mu\in\mathbf{D}$, then the contraction T is similar to a normal operator.

This result answers a question put in [1] and gives a proof of a conjecture from [9].

1. Results and outline of the proof

Let T be a contraction acting on a separable Hilbert space H. *Throughout the paper we suppose that $\sigma(T)\neq\bar{\mathbf{D}}$, even if we do not emphasize this explicitly.* We say that a contraction has the (LRG) property if

(LRG) $$\mathcal{C}_1(T)=\sup_{\lambda\in\mathbf{C}\setminus\sigma(T)}\|R_\lambda(T)\|\operatorname{dist}(\lambda,\sigma(T))<\infty,$$

(LRG) stands for *the linear growth of the resolvent.* The results in [1] show, that the (LRG) condition itself is not at all sufficient for similarity of a given contraction to a normal operator. This suggests that we should require more than just the (LRG) condition for the operator T to get an efficient resolvent test for similarity to a normal operator. For instance, we can consider small perturbations of a unitary operator. To this end, it may be reasonable to look at the defect operators $I-T^*T$ and $I-TT^*$.

Let us turn to the (LRG) property. First, we observe that the condition is invariant with respect to the Möbius transformations of the unit disk $\mathbf{D}$. To be precise, let $b_\mu(T)=(I-\bar{\mu}T)^{-1}(T-\mu)$, $\mu\in\mathbf{D}$, be the Möbius transform of T.

Lemma 1.1. *Let T be a completely nonunitary contraction on a Hilbert space H. Then*

$$a_1\mathcal{C}_1(T)\le\mathcal{C}_1(b_\mu(T))\le A_1\mathcal{C}_1(T)$$

for every $\mu\in\mathbf{D}$. The numbers a_1 and A_1 are absolute constants.

We prove the lemma in Subsection 3.2. The reasoning is essentially based on the delicate "Y. Domar lemma" type result (see Subsection 3.1).

It is clear that both T and $b_\mu(T)$ are similar to a normal operator simultaneously. Hence, it seems natural to require that any addition to the (LRG) condition participating in a similarity test should also be Möbius invariant. On the other hand, the condition $\operatorname{tr}(I-T^*T)<\infty$ is not conformally invariant with respect to the linear-fractional transformations of $\mathbf{D}$, so we should modify it in an appropriate way (see (UTB) below).

The assumptions of the main theorem now look quite natural.

Theorem 1.1. *Let T be a contraction acting on a separable Hilbert space H, $\sigma(T)\neq\bar{\mathbf{D}}$. Then T is similar to a normal operator as long as*

(LRG) $$\mathcal{C}_1(T)=\sup_{\lambda\in\mathbf{C}\setminus\sigma(T)}\|R_\lambda(T)\|\operatorname{dist}(\lambda,\sigma(T))<\infty,$$

(UTB) $$\mathcal{C}_2(T)=\sup_{\mu\in\mathbf{D}}\operatorname{tr}(I-b_\mu(T)^*b_\mu(T))<\infty,$$

We call the second condition of the theorem the (UTB) property ((UTB) stands for *uniform trace boundedness*). Sometimes we write $T\in$(LRG) or $T\in$(UTB) to say that T possesses one or the other property.

Now, we explain some ideas underlying the proof of the theorem. Due to the canonical decomposition of a contraction on the orthogonal sum of a unitary operator and a completely nonunitary contraction (see [11, Chapter 1]), it suffices to consider the latter component. Next, given a completely nonunitary contraction satisfying the (LRG) and the (UTB) properties, we proceed in two steps. Assume first, that the contraction is complete, i.e. it has a complete family of eigenvectors.

In this case the (LRG) and the (UTB) properties imply that the set $\sigma(T)$ is sparse enough and a certain embedding theorem holds. By a result of [12] the embedding yields the unconditional basis property of the family of eigenvectors of the contraction. The latter is equivalent to the similarity we seek for.

Secondly, we join the "outer spectrum" to the point spectrum and we prove absence of the "singular" spectrum. This enables one to derive the theorem for general completely nonunitary contraction (and not only for complete ones). The tools used at this stage of the proof are adapted from [1].

We conclude the introduction with recalling some well-known definitions and the standard notation. Let E and E_* be separable Hilbert spaces. We denote by $L(E,E_*)$ the space of bounded linear operators mapping E into E_*. We put $L(E)=L(E,E)$. Further, we write $H^\infty(L(E,E_*))$ for the space of $L(E,E_*)$-valued bounded analytic functions on the unit disk $\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$. We put $BH^\infty(L(E,E_*))$ to be the unit ball of the space. Similarly, we denote by $L^\infty(L(E,E_*))$ the space of the $L(E,E_*)$-valued bounded measurable functions on the unit circle $\mathbf{T}=\{z\in\mathbf{C}:|z|=1\}$. As usual, we put $L^2(E)=L^2(\mathbf{T},E)$ to be the Hilbert space of measurable functions f on $\mathbf{T}$, taking values in E such that

$$\|f\|^2=\frac{1}{2\pi}\int_0^{2\pi}\|f(e^{i\phi})\|_E^2\,d\phi<\infty,$$

and $H^2(E)$ stands for the Hilbert space of E-valued analytic functions in $\mathbf{D}$ with

$$\sup_{0\le r<1}\frac{1}{2\pi}\int_0^{2\pi}\|f(re^{i\phi})\|_E^2\,d\phi<\infty.$$

We say that a function $\theta\in H^\infty(L(E,E_*))$ is *inner* (*$*$-inner*), if $\theta(t)^*\theta(t)=I$ a.e. on $\mathbf{T}$ ($\theta(t)\theta(t)^*=I$ a.e. on $\mathbf{T}$). The function θ is said to be *outer* (*$*$-outer*), if $\overline{\theta H^2(E)}=H^2(E_*)$ (the function $\theta(\bar z)^*$ is outer).

The operators $A_1\in L(H_1)$ and $A_2\in L(H_2)$ are called similar if there exists a boundedly invertible operator $W\in L(H_2,H_1)$ such that $A_2=W^{-1}A_1W$.

2. Preliminaries

The material presented in this section is of common knowledge and is cited here only for the reader's convenience.

2.1. Some facts on the Sz.-Nagy–Foiaş model

A wide panorama of the subject we discuss in this subsection can be found in the monographs [11], [10].

We introduce some notation to define the function model. Let us fix a function $\theta \in BH^\infty(L(E,E_*))$. We put $\Delta(t)=(I-\theta(t)^*\theta(t))^{1/2} \in L^\infty(L(E))$, $0 \le \Delta(t) \le I$ a.e. on $\mathbf{T}$.

Further, we consider the so-called model space

$$K_\theta = \begin{bmatrix} H^2(E_*) \\ \overline{\Delta L^2(E)} \end{bmatrix} \ominus \begin{bmatrix} \theta \\ \Delta \end{bmatrix} H^2(E). \tag{2.1}$$

We denote by P_θ the orthogonal projection onto K_θ,

$$P_\theta \colon \begin{bmatrix} H^2(E_*) \\ \overline{\Delta L^2(E)} \end{bmatrix} \longrightarrow K_\theta,$$

and by M_θ the operator acting on the space K_θ by means of the formula

$$M_\theta x = P_\theta z x, \quad x \in K_\theta.$$

The operator is a contraction, $\|M_\theta\| \le 1$, and it is called the model operator.

Now we recall some facts about contractions acting on a separable Hilbert space H. As was already mentioned, any contraction T can be represented in the form $T=U \oplus T_0$, where U is a unitary operator and T_0 is a completely nonunitary contraction, i.e. none of the restrictions of the latter to its reducing subspaces is unitary.

The defect operators and defect subspaces of a contraction T are defined by

$$D_T = (I-T^*T)^{1/2} \colon H \longrightarrow H, \quad \mathfrak{D}_T = \overline{D_T H},$$
$$D_{T^*} = (I-TT^*)^{1/2} \colon H \longrightarrow H, \quad \mathfrak{D}_{T^*} = \overline{D_{T^*} H}.$$

We define an operator-valued function $\theta_T(\lambda)$ by the formula

$$\theta_T(\lambda) = -T + \lambda D_{T^*}(I-\lambda T^*)^{-1} D_T|_{\mathfrak{D}_T}$$

for $\lambda \in \mathbf{D}$, and it is called the characteristic function of T. It can be shown that $\theta_T \in BH^\infty(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ and that θ_T is pure, that is the only subspace $E \subset \mathfrak{D}_T$ where $\theta_T(t)|_E$ is a unitary constant a.e. on $\mathbf{T}$, is $E=\{0\}$.

The following theorem links the two series of definitions given above.

Theorem 2.1. ([11], Chapter 6, the model theorem)

(i) *Any completely nonunitary contraction T defined on a Hilbert space H is unitarily equivalent to T_{θ_T}, where $\theta_T \in BH^\infty(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ is the characteristic function of the contraction.*

(ii) *Let θ be a pure contractive function from $H^\infty(L(E,E_*))$. Then the contraction M_θ is completely nonunitary and its characteristic function coincides with the initial function θ.*

We say that two functions $\theta_1 \in H^\infty(L(E_1, E_{1*}))$ and $\theta_2 \in H^\infty(L(E_2, E_{2*}))$, E_1, E_{1*}, E_2 and E_{2*} being Hilbert spaces, coincide, if there exist unitary operators $U: E_1 \to E_2$ and $U_*: E_{1*} \to E_{2*}$ such that $\theta_2 = U_* \theta_1 U^*$.

Given a Möbius transformation $b_\mu(\lambda) = (\lambda - \mu)/(1 - \bar{\mu}\lambda)$, $\lambda, \mu \in \mathbf{D}$, of the unit disk $\mathbf{D}$, consider the operator $b_\mu(T) = (I - \bar{\mu}T)^{-1}(T - \mu)$. The functions $\theta_{b_\mu(T)}(b_\mu(\lambda))$ and $\theta_T(\lambda)$ coincide in the sense of the above definition (see [11, Section 6.1]):

$$U_* \theta_{b_\mu(T)}(b_\mu(\lambda)) U^* = \theta_T(\lambda), \tag{2.2}$$

where $\lambda \in \mathbf{D}$. The following two-sided inequality [11, Section 6.4] relates the resolvent $R_\lambda(T) = (T - \lambda)^{-1}$ and the inverse of $\theta_T(\lambda)$,

$$(1 - |\lambda|)\|R_\lambda(T)\| \le \|\theta_T(\lambda)^{-1}\| \le 1 + 2(1 - |\lambda|)\|R_\lambda(T)\| \tag{2.3}$$

for all $\lambda \in \mathbf{D} \setminus \sigma(T)$.

As follows from [6] and [11, Chapter 9], a contraction T, $\sigma(T) \subset \mathbf{T}$, is similar to a unitary operator if and only if

$$\|R_\lambda(T)\| \le \frac{C}{|1 - |\lambda||} \tag{2.4}$$

for all $\lambda \in \mathbf{C} \setminus \mathbf{T}$. Equivalently, $\|\theta_T(\lambda)^{-1}\| \le C < \infty$, $\lambda \in \mathbf{D}$.

2.2. Angles between invariant subspaces and Bezout equations

In this subsection we mainly follow [1, Section 1.6]. Information on regular factorizations can be found in [11].

It is well known that every invariant subspace L of the model operator M_θ ($M_\theta L \subset L$) defines a certain regular factorization $\theta = \theta_2 \theta_1$ [11, Chapter 7]. The converse is also true, i.e. every regular factorization of the characteristic function $\theta_T = \theta_2 \theta_1$ of a contraction T makes it possible to construct a T-invariant subspace L_{θ_1, θ_2}. We refer to [11, Chapter 7] for the definition of regular factorizations, as well as for their basic properties. In particular, it is shown there that factorizations $\theta = \theta_2 \theta_1$, where θ_2 is inner or θ_1 is $*$-inner, are regular.

Let L and L' be two invariant subspaces of a completely nonunitary contraction T defined on a Hilbert space H and let $\theta = \theta_2 \theta_1$ and $\theta = \theta_2' \theta_1'$ be the corresponding regular factorizations. Assume that $L + L'$ is dense in H. We are interested in conditions for the angle between these subspaces to be positive (or, in other words, when $L \cap L' = \{0\}$ and the sum $L + L'$ is closed). This means that

$$\cos(L, L') = \sup_{\substack{l \in L \\ l' \in L'}} \frac{|(l, l')|}{\|l\| \, \|l'\|} < 1.$$

This is equivalent to saying that the skew projection $\mathcal{P}_{L\|L'}: L+L' \to L$ defined by the relation $\mathcal{P}(l+l')=l$, $l\in L$, $l'\in L'$, is bounded.

It is proved in [14] that $L+L'$ is a direct decomposition of H if and only if the Bezout equation

$$\Gamma_1(\lambda)\theta_1(\lambda)+\Gamma_1'(\lambda)\theta_1'(\lambda)=I \tag{2.5}$$

is solvable with $\Gamma_1\in H^\infty(L(F,E))$, $\Gamma_1'\in H^\infty(L(F',E))$ and $\lambda\in\mathbf{D}$, F and F' being some intermediate Hilbert spaces, and an additional equation of the same type is solvable in certain L^∞ spaces as well. It is known (see references in [14]) that if the space E is of finite dimension, the sole equation (2.5) is sufficient to have $H=L\dot{+}L'$. There are some other special cases, where the solvability of the equation (2.5) implies the conclusion. The following theorem, for instance, is a corollary of the general considerations from [1].

Theorem 2.2. ([1], Section 1.6) *Let L and L' be invariant subspaces, defined by regular factorizations $\theta=\theta_2\theta_1$ and $\theta=\theta_2'\theta_1'$, and let the sum $L+L'$ be dense in H. The sum $L+L'$ is a direct sum (and hence $H=L\dot{+}L'$) whenever θ_1' is a $*$-inner function.*

We will need to apply the theorem to a quite particular situation. Namely, we put the first factorization to be the canonical factorization of θ, $\theta=\theta_{\text{in}}\theta_{\text{out}}$, and we put the second factorization to be the $*$-canonical one, $\theta=\theta_{\text{out}*}\theta_{\text{in}*}$. This means that the function θ_{in} ($\theta_{\text{in}*}$) is inner ($*$-inner), and the function θ_{out} ($\theta_{\text{out}*}$) is outer ($*$-outer), respectively. Note that these factorizations always exist [11, Chapter 5] and are regular. We denote the corresponding invariant subspaces by L_{out} and $L_{\text{in}*}$, respectively. Theorem 2.2 shows that the sum $L_{\text{out}}+L_{\text{in}*}$ is direct whenever equation (2.5) is solvable.

2.3. Sparse and Carleson subsets in the unit disk D

Detailed information about these subjects can be found in [10], [4] and [8].

Let us set $\varrho(\lambda,\mu)=|b_\mu(\lambda)|$, $\lambda,\mu\in\mathbf{D}$, and, further, $B_\delta(\mu)=\{\lambda\in\mathbf{D}:|b_\mu(\lambda)|\le\delta\}$, $0<\delta<1$, $\mu\in\mathbf{D}$. The disk $B_\delta(\mu)$ is called a *pseudo-hyperbolic neighborhood* of the point μ with radius δ. We say that the set $\sigma=\{\lambda_k\}_{k=1}^\infty$ is *sparse*, if there exists a number $\delta>0$ such that

$$B_\delta(\lambda_1)\cap B_\delta(\lambda_2)=\emptyset, \tag{2.6}$$

where $\lambda_1,\lambda_2\in\sigma$ and $\lambda_1\ne\lambda_2$. The set σ is called *Carleson* if

$$\inf_{\mu\in\sigma}\prod_{\lambda\in\sigma\setminus\{\mu\}}|b_\mu(\lambda)|\ge\delta_0>0.$$

There is another characterization of Carleson sets, and it is sometimes more convenient than the original one.

Theorem 2.3. ([10], Chapter 6, the embedding theorem) *Let* $\sigma\subset\mathbf{D}$. *The following assertions are equivalent*:

(i) σ *is a Carleson set*;

(ii) σ *is sparse and*

$$\sup_{\mu\in\mathbf{D}}\sum_{\lambda\in\sigma}(1-|b_\mu(\lambda)|^2)<\infty.$$

We say that a set σ is N-*Carleson* (N-*sparse*) if it is a union of N Carleson sets (N sparse sequences).

The following lemma is proved in [1] under somewhat weaker assumptions.

Lemma 2.1. ([1], Section 1.4) *Let* σ *be an* N-*Carleson set. Then there exists a number* $\varepsilon>0$ *such that*

$$[(1-\delta)\xi,\xi)\cap\Omega_\varepsilon\neq\emptyset$$

for all $\xi\in\mathbf{T}$ *and* $\delta\in(0,1]$. *Here* $\Omega_\varepsilon=\{z\in\mathbf{D}:\operatorname{dist}(z,\sigma)\geq\varepsilon(1-|z|)\}$. *In an equivalent way, for any* $\xi\in\mathbf{T}$ *there always exists a sequence* $\{z_n\}_{n=1}^\infty\subset[0,\xi)$ *such that* $|z_n|\to1$ *and*

$$\operatorname{dist}(z_n,\sigma)\geq\varepsilon(1-|z_n|). \tag{2.7}$$

Here and below, $\operatorname{dist}(z,\sigma)=\inf_{\lambda\in\sigma}|z-\lambda|$.

2.4. Some properties of the trace class operators

A good reference on the subject of this subsection is [5].

We start with standard definitions. Let H be a Hilbert space and $\mathfrak{S}_\infty$ denote the ideal of compact operators. The Schatten–von Neumann ideals $\mathfrak{S}_p$, $0<p\leq\infty$, are defined in the usual way,

$$\mathfrak{S}_p=\left\{A\in\mathfrak{S}_\infty:\sum_{k=1}^\infty s_k(A)^p<\infty\right\},$$

$s_k(A)=\lambda_k(A^*A)^{1/2}$, where $\lambda_k(A)$ are the eigenvalues of the operator A and $s_k(A)$ is called the k-*th singular number* of A.

Let $A\in\mathfrak{S}_1$ and $\{e_k\}_{k=1}^\infty$ be an arbitrary orthonormal basis of H. It is known that the sum $\operatorname{tr}A=\sum_{k=1}^\infty(Ae_k,e_k)$ converges and does not depend on the choice of the orthonormal basis.

It is clear that if $A=A^*\geq 0$ and $A\in\mathfrak{S}_1$, then

$$\operatorname{tr} A=\sum_{j=1}^{\infty}\lambda_j(A)=\sum_{j=1}^{\infty}s_j(A).$$

This relation implies that $\operatorname{tr} PAP\leq\operatorname{tr} A$ for any orthogonal projection P and any operator A with the properties stated above. In particular, if $k=\operatorname{rank} P<\infty$, then

$$\operatorname{tr} A\geq\operatorname{tr} PAP=\sum_{j=1}^{k}s_j(PAP)\geq k\min_{1\leq j\leq k}\lambda_j(PAP). \tag{2.8}$$

The determinant of the operator $I-A$, $A\in\mathfrak{S}_1$, can be defined as $\det(I-A)=\prod_{k=1}^{\infty}(1-\lambda_k(A))$. We conclude the subsection with the following criterion.

Theorem 2.4. ([5], Chapter 5) *Let T be a complete nonunitary contraction on a Hilbert space H, $I-T^*T\in\mathfrak{S}_1$ and $\sigma(T)\neq\overline{\mathbf{D}}$. If the system of the root subspaces of T is complete in H, then*

$$\det T^*T=\prod_{\lambda\in\sigma_p(T)}|\lambda|^2, \tag{2.9}$$

where the product is computed counting the Riesz multiplicities of the eigenvalues $\lambda\in\sigma_p(T)$.

3. Proof of Theorem 1.1 for complete contractions

3.1. Lemma on subharmonic functions

In what follows, we rely on a very delicate result on majorization of subharmonic functions. Apparently, the first result of this type was obtained in [2]. We cite here its refinement proved in [7].

Lemma 3.1. ([7], Section 23) *Let σ be a closed set of the disk $\overline{\mathbf{D}}$ and let u be a subharmonic function on $\mathbf{C}\backslash\sigma$ satisfying the inequality*

$$u(\lambda)\leq\max\left\{\frac{1}{\operatorname{dist}(\lambda,\sigma)},\ \frac{1}{\big|1-|\lambda|\big|}\right\}.$$

Then

$$u(\lambda)\leq\frac{A_0}{\operatorname{dist}(\lambda,\sigma)}$$

for all $|\lambda|\geq\frac{1}{2}$, $\lambda\in\mathbf{C}\backslash\sigma$, and where $A_0=447$.

3.2. (LRG) property is invariant with respect to Möbius transformations

Let T be a complete completely nonunitary contraction on a Hilbert space H. We assume that $\sigma(T)\neq\overline{\mathbf{D}}$ and, by virtue of the model theorem, we may suppose that the contraction T coincides with a model operator M_θ^* defined by a certain function $\theta\in BH^\infty(L(E))$, E being a Hilbert space.

The purpose of the subsection is to prove Lemma 1.1. The main idea of the proof is to express the (LRG) property in terms of the characteristic function. It is convenient to put

$$\mathcal{C}_3(T)=\sup_{\lambda\in\mathbf{D}\setminus\sigma}\|\theta(\lambda)^{-1}\|\inf_{\mu\in\sigma}|b_\mu(\lambda)|. \tag{3.1}$$

We note that $\mathcal{C}_1(T)\geq 1$ and $\mathcal{C}_3(T)\geq 1$ (see the (LRG) property for the definition of $\mathcal{C}_1(T)$). The first inequality follows from the fact that $\|R_\lambda(T)\|\operatorname{dist}(\lambda,\sigma)\geq 1$, $\lambda\in\mathbf{C}\setminus\sigma$, and the proof of the second one is presented in Subsection 3.3. The proof of Lemma 1.1 is based on the mutual estimates between these constants.

Lemma 3.2. *Let T be a completely nonunitary contraction on a Hilbert space H and θ be its characteristic function. Then there exist two absolute constants a_2 and A_2 such that*

$$a_2\mathcal{C}_3(T)\leq\mathcal{C}_1(T)\leq A_2\mathcal{C}_3(T). \tag{3.2}$$

Proof. The proof is essentially based on inequality (2.3). We proceed with the left part of inequality (3.2). We have

$$\begin{aligned}\|\theta(\lambda)^{-1}\|&\leq 1+2\mathcal{C}_1(T)\sup_{\mu\in\sigma}\frac{1-|\lambda|}{|\lambda-\mu|}\leq 1+2\mathcal{C}_1(T)\sup_{\mu\in\sigma}\frac{|1-\bar{\mu}\lambda|}{|\lambda-\mu|}\\&\leq(1+2\mathcal{C}_1(T))\sup_{\mu\in\sigma}\frac{1}{|b_\mu(\lambda)|},\end{aligned}$$

because $|b_\mu(\lambda)|\leq 1$, $\lambda\in\mathbf{D}$. The bound, together with $\mathcal{C}_1(T)\geq 1$, implies

$$\mathcal{C}_3(T)=\sup_{\lambda\in\mathbf{D}\setminus\sigma}\|\theta(\lambda)^{-1}\|\inf_{\mu\in\sigma}|b_\mu(\lambda)|\leq 3\mathcal{C}_1(T),$$

and we can take $a_2=\frac{1}{3}$.

We continue with the right part of inequality (3.2). The reasoning is more complicated; it uses the nontrivial Lemma 3.1. We have for $|\lambda|\leq\frac{1}{2}$,

$$\|R_\lambda(T)\|\leq\frac{\|\theta(\lambda)^{-1}\|}{1-|\lambda|}\leq\frac{\mathcal{C}_3(T)}{1-|\lambda|}\sup_{\mu\in\sigma}\frac{|1-\bar{\mu}\lambda|}{|\lambda-\mu|}\leq 4\mathcal{C}_3(T)\sup_{\mu\in\sigma}\frac{1}{|\lambda-\mu|}=\frac{4\mathcal{C}_3(T)}{\operatorname{dist}(\lambda,\sigma)}, \tag{3.3}$$

since $1-|\lambda|\geq\frac{1}{2}$.

Let $|\lambda|>\frac{1}{2}$ now. We consider the cases $\{\lambda:\frac{1}{2}<|\lambda|<1\}$ and $\{\lambda:|\lambda|>1\}$ separately. We get for $\frac{1}{2}<|\lambda|<1$,

$$\|R_\lambda(T)\|\leq\frac{\|\theta(\lambda)^{-1}\|}{1-|\lambda|}\leq\mathcal{C}_3(T)\sup_{\mu\in\sigma}\frac{|1-\bar{\mu}\lambda|}{(1-|\lambda|)|\lambda-\mu|}.$$

Using the obvious inequalities $|1-\bar{\mu}\lambda|\leq1-|\lambda|^2+|\lambda|\,|\lambda-\mu|\leq2(1-|\lambda|)+|\lambda-\mu|$, we continue

$$\mathcal{C}_3(T)\sup_{\mu\in\sigma}\left\{\frac{2}{|\lambda-\mu|}+\frac{1}{1-|\lambda|}\right\}\leq2\mathcal{C}_3(T)\max\left\{\frac{1}{\operatorname{dist}(\lambda,\sigma)},\ \frac{1}{1-|\lambda|}\right\}. \tag{3.4}$$

The computation for $|\lambda|>1$ is much more simple

$$\|R_\lambda(T)\|\leq\frac{1}{|\lambda|-1}\leq\max\left\{\frac{1}{\operatorname{dist}(\lambda,\sigma)},\ \frac{1}{|\lambda|-1}\right\}. \tag{3.5}$$

Summing up (3.5), (3.4) and the inequality $\mathcal{C}_3(T)\geq1$, we obtain

$$\|R_\lambda(T)\|\leq2\mathcal{C}_3(T)\max\left\{\frac{1}{\operatorname{dist}(\lambda,\sigma)},\ \frac{1}{|1-|\lambda||}\right\}$$

for all $|\lambda|>\frac{1}{2}$, $\lambda\in\mathbf{C}\setminus\sigma$. Since the norm $\|R_\lambda(T)\|$ is a subharmonic function of λ, Lemma 3.1 yields

$$\|R_\lambda(T)\|\leq\frac{2A_0\mathcal{C}_3(T)}{\operatorname{dist}(\lambda,\sigma)}, \tag{3.6}$$

for all $\lambda\in\mathbf{C}\setminus\sigma$ and where $A_0=447$. It gives us the conclusion of the lemma with $A_2=894$. □

Proof of Lemma 1.1. By virtue of relation (2.2) we have

$$\|\theta_{b_\mu(T)}(b_\mu(\lambda))^{-1}\|=\|\theta(\lambda)^{-1}\|$$

for all $\lambda\in\mathbf{D}\setminus\sigma$ and, consequently,

$$\begin{aligned}\mathcal{C}_3(b_\mu(T))&=\sup_{\lambda\in\mathbf{D}\setminus\sigma_\mu}\|\theta^{-1}_{b_\mu(T)}(b_\mu(\lambda))\|\inf_{\zeta\in\sigma_\mu}|b_\zeta(b_\mu(\lambda))|\\&=\sup_{\lambda\in\mathbf{D}\setminus\sigma}\|\theta_T^{-1}(\lambda)\|\inf_{\zeta\in\sigma}|b_\zeta(\lambda)|=\mathcal{C}_3(T),\end{aligned}$$

where $\sigma_\mu=b_\mu(\sigma)$ stands for the spectrum of $b_\mu(T)$. The calculation shows that the constant $\mathcal{C}_3(T)$ is invariant with respect to Möbius transformations, and the conclusion of the lemma easily follows from the two-sided estimate (3.2). □

3.3. Some more notation and corollaries of the model theorem

Let T be a complete completely nonunitary contraction on a Hilbert space H. We denote by σ_p its point spectrum $\sigma_p(T)$. We assume that $\sigma(T)\neq\overline{\mathbf{D}}$, $I-T^*T\in\mathfrak{S}_1$ and the eigenvalues $\lambda\in\sigma_p$ of the operator are algebraically simple, $\operatorname{Ker}(T-\lambda I)=\operatorname{Ker}(T-\lambda I)^2$. We put $\mathcal{X}$ to be the family of the eigenspaces $\{X_\lambda\}_{\lambda\in\sigma_p}$, where $X_\lambda=\operatorname{Ker}(T-\lambda I)$, of the operator. Note that the completeness of the contraction means, under these restrictions, that

$$H=\bigvee_{\lambda\in\sigma_p} X_\lambda,$$

where $\bigvee$ stands for the closed linear span.

For an arbitrary subset ω of σ_p we define a subspace X_ω and an operator T_ω by the formulas

$$X_\omega=\bigvee_{\lambda\in\omega} X_\lambda \quad\text{and}\quad T_\omega=T|_{X_\omega}.$$

The just defined operator is a contraction, and $\sigma_p(T_\omega)=\omega$. We put, for brevity, $b_\mu(\omega)=\omega_\mu$. The equality $\sigma_p(b_\mu(T))=\sigma_{p\mu}$ is just a consequence of the spectral mapping theorem [3]. Similarly, we have $b_\mu(T_\omega)=b_\mu(T)_{\omega_\mu}$, and thus $\sigma_p(b_\mu(T_\omega))=\omega_\mu$. We mention also that

$$R_\lambda(T_\omega)=R_\lambda(T)|_{X_\omega}, \tag{3.7}$$

and this fact will be often used in the sequel.

Suppose now that $T=M_\theta^*$ for some $\theta\in BH^\infty(L(E))$. The kernel spaces $X_\lambda=\operatorname{Ker}(M_\theta^*-\lambda I)$, $\lambda\in\sigma_p$, have the form $X_\lambda=\varphi_\lambda E_\lambda\subset H^2(E)$, where $E_\lambda=\operatorname{Ker}\theta(\lambda)^*\neq\{0\}$, $E_\lambda\subset E$, and $\varphi_\lambda(z)=(1-|\lambda|^2)^{1/2}/(1-\bar\lambda z)$ is the reproducing kernel of the Hardy space H^2. The left factor in the corresponding regular factorization $\theta=\theta_\lambda\tilde\theta^\lambda$ has the form $\theta_\lambda(z)=b_\lambda(z)P_{E_\lambda}+(I-P_{E_\lambda})$, P_{E_λ} being the orthogonal projection from E to E_λ. In particular, we have

$$\frac{1}{|b_\lambda(z)|}=\|\theta_\lambda(z)^{-1}\|\le\|\theta(z)^{-1}\|$$

for any $\lambda\in\sigma_p$ and $z\in\mathbf{D}$. Hence, $\mathcal{C}_3(T)\ge 1$ (see (3.1) for the definition of the constant).

We set $X_\lambda^\mu=\operatorname{Ker}(b_\mu(T)-\lambda I)$, $\lambda\in\sigma_{p\mu}$, and $\mathcal{X}_\mu=\{X_\lambda^\mu\}_{\lambda\in\sigma_{p\mu}}$. The above mentioned description of X_λ and relation (2.2) yield, for instance, that $\dim X^\mu_{b_\mu(\lambda)}=\dim X_\lambda$, $\lambda\in\sigma_p$.

We conclude this subsection with a lemma from [1].

Lemma 3.3. ([1], Section 1.3) *Let T be a contraction with the* (LRG) *property and $\mathcal{X}$ be its family of eigenspaces. Then*

(i) *every $\lambda\in\sigma_p$ is algebraically simple*;

(ii) *the system $\mathcal{X}$ is uniformly minimal.*

3.4. Local properties of the spectrum of a contraction $T\in$(LRG)$\cap$(UTB)

We need the following Möbius invariant constant

$$\mathcal{C}_2(T)=\sup_{\mu\in\mathbf{D}}\operatorname{tr}(I-b_\mu(T)^*b_\mu(T)). \tag{3.8}$$

Further, given a number $\delta>0$ and a point $\mu\in\mathbf{D}$, we set $\sigma_{\mu,\delta}=\sigma_p\cap B_\delta(\mu)$. For brevity, we put $\omega=\sigma_{\mu,\delta}$, $T_\omega=T_{\sigma_{\mu,\delta}}$ and $X_\omega=X_{\sigma_{\mu,\delta}}$.

It turns out that the contractions $T\in$(LRG)$\cap$(UTB) have sufficiently sparse spectra (even if one counts the Riesz multiplicities of the eigenvalues).

Lemma 3.4. *Let T be a complete completely nonunitary contraction on a Hilbert space H and $0<\delta<1$. If $T\in$*(LRG)$\cap$(UTB), *then*

$$\dim X_\mu\le\mathcal{C}_2(T),\qquad \mu\in\sigma_p; \tag{3.9}$$

$$\min\left\{\frac{2}{3}\dim X_{\sigma_{\mu,\delta}},\ \frac{2}{3\sqrt{3}}\frac{1}{B\delta\mathcal{C}_1(T)}-1\right\}\le\mathcal{C}_2(T), \tag{3.10}$$

for every $\mu\in\mathbf{D}$, where $B=2A_1/a_1$, and a_1 and A_1 are the absolute constants from Lemma 1.1.

Proof. Let $\mu\in\sigma_p$. Due to a remark from Subsection 3.3, we have $\dim X^\mu_{b_\mu(\mu)}=\dim\operatorname{Ker}b_\mu(T)=\dim X_\mu$. It follows from relation (2.8) that

$$\mathcal{C}_2(T)\ge(1-\|b_\mu(T_{\{\mu\}})\|)\dim X_\mu=\dim X_\mu,$$

and relation (3.9) is proved.

To prove (3.10), consider the operator $b_\mu(T_\omega)$. It is convenient to put $X_{\mu,\omega}=\bigvee_{\lambda\in\omega}X^\mu_\lambda$. We see that $\sigma_p(b_\mu(T_\omega))=\omega_\mu=b_\mu(\sigma_{\mu,\delta})=\sigma_{p\mu}\cap B_\delta(0)\subset\{z:|z|\le\delta\}$, by definition.

We are going to estimate the norm of the operator $b_\mu(T_\omega)$ with the help of the Riesz–Dunford calculus. To this end, we surround every point $\lambda\in\omega_\mu$ by a circle $\gamma_\lambda=\{z\in\mathbf{C}:|z-\lambda|=\varepsilon\}$, $0<\varepsilon\le\delta$, with ε small enough. Then we set $\gamma=\bigcup_{\lambda\in\omega_\mu}\gamma_\lambda$ and apply the calculus formula to estimate the norm of the operator $b_\mu(T_\omega)$. We get

$$\|b_\mu(T_\omega)\|\le\left\|\frac{1}{2\pi i}\int_\gamma zR_z(b_\mu(T_\omega))\,dz\right\|\le\frac{|\gamma|}{2\pi}\sup_{z\in\gamma}|z|\sup_{z\in\gamma}\|R_z(b_\mu(T_\omega))\|.$$

Now, we use relation (3.7) and the equality $\operatorname{dist}(z,\omega_\mu)=\varepsilon$,

$$\|b_\mu(T_\omega)\| \le 2\delta\mathcal{C}_1(b_\mu(T))\#\omega_\mu \le B\delta\mathcal{C}_1(T)\#\omega, \tag{3.11}$$

where the sign # means the cardinality of a set. To get the last inequality we apply Lemma 1.1, and the absolute constant B equals $2A_1/a_1$.

We pick an arbitrary invariant subspace $X\subset X_{\mu,\omega}$ of the contraction $b_\mu(T_\omega)$. We set $T_X=b_\mu(T_\omega)|_X$, $\sigma_X=\sigma(T_X)$ and $k=\dim X$, $0\le k\le \dim X_{\mu,\omega}$. It is evident that

$$\|T_X\| \le B\delta\mathcal{C}_1(T)\#\sigma_X.$$

Then, applying inequality (2.8), we get

$$\mathcal{C}_2(T) \ge (1-\|T_X\|^2)k \ge (1-(B\delta\mathcal{C}_1(T)\#\sigma_X)^2)k \ge (1-(B\delta\mathcal{C}_1(T)k)^2)k =: f(k),$$

since $\#\sigma_X\le k$, and this is true for all $0\le k\le\dim X_{\mu,\omega}$, $k\in\mathbf{N}$.

It is easy to see that the maximum value of the function $f(k)$, $k\ge 0$, equals $f_{\max}=2/3\sqrt{3}\,B\delta\mathcal{C}_1(T)$ and it is attained at the point $k_{\max}=1/\sqrt{3}\,B\delta\mathcal{C}_1(T)$.

We keep in mind two possibilities, $\dim X_{\mu,\omega}\le k_{\max}$ or $\dim X_{\mu,\omega}>k_{\max}$. If the first case occurs, we put $k=\dim X_{\mu,\omega}$, and, consequently,

$$\mathcal{C}_2(T) \ge (1-(B\delta\mathcal{C}_1(T)\dim X_{\mu,\omega})^2)\dim X_{\mu,\omega} \ge \tfrac{2}{3}\dim X_{\mu,\omega}.$$

Further, we see that $f([k_{\max}])\ge f(k_{\max})-1$ by the Lagrange formula and the inequality $f'(k)\le 1$; the square brackets $[\,\cdot\,]$ stand for the entire part of a number. So, we get in the second case

$$\mathcal{C}_2(T) \ge \max_{k\in\mathbf{N}} f(k) \ge \frac{2}{3\sqrt{3}}\frac{1}{B\delta\mathcal{C}_1(T)}-1,$$

which completes the proof. □

Corollary 3.1. *The inequality*

$$\dim X_{\sigma_{\mu,\delta}} \le \tfrac{3}{2}\mathcal{C}_2(T) \tag{3.12}$$

holds true whenever

$$\delta < \frac{2}{3\sqrt{3}}\frac{1}{B\mathcal{C}_1(T)(\mathcal{C}_2(T)+1)}. \tag{3.13}$$

Proof. Indeed, inequality (3.13) means that $2/3\sqrt{3}\,B\delta\mathcal{C}_1(T)-1>\mathcal{C}_2(T)$, and, by virtue of (3.10) we get

$$\min\left\{\frac{2}{3}\dim X_{\sigma_{\mu,\delta}},\ \frac{2}{3\sqrt{3}}\frac{1}{B\delta\mathcal{C}_1(T)}-1\right\} = \frac{2}{3}\dim X_{\sigma_{\mu,\delta}} \le \mathcal{C}_2(T). \quad \square$$

It is convenient to put

$$\mathcal{N}(T) = \left[\tfrac{3}{2}\mathcal{C}_2(T)+1\right]. \tag{3.14}$$

Corollary 3.2. *We have*

$$\#(\sigma_p \cap B_\delta(\mu)) = \#\sigma_{\mu,\delta} \le \mathcal{N}(T) \tag{3.15}$$

for every $\mu \in \mathbf{D}$ *and* δ *from* (3.13).

This corollary means that the point spectrum $\sigma_p(T)$ of a contraction $T \in$ (LRG)$\cap$(UTB) is a union of at most $\mathcal{N}(T)$ sparse sequences (see (2.6)).

Remark 3.1. Note that the proof of (3.9) uses the (UTB) property only.

It is worth mentioning that in order to prove estimate (3.10), we were forced to drop some subspaces from the family of subspaces $\mathcal{X}_\mu$. This is a manifestation of the so-called "round-off-error" phenomenon, pointed out in [13, Lemma 12.2].

For the next lemma, we need some more notation. Let T be as in Lemma 3.4. We fix $\varrho_0 = \varrho_0(\delta) = \delta/4(\mathcal{N}(T)+1)$ and surround every point $\lambda \in \sigma_p$ by a disk $B_{\varrho_0}(\lambda)$, here δ and $\mathcal{N}(T)$ are values from (3.13) and (3.14). Then we consider an open set $G = \bigcup_{\lambda\in\sigma_p} B_{\varrho_0}(\lambda)$ and denote by $\{G_n\}_{n=1}^\infty$ a collection of its connected components. Further, we set $\sigma_n = \sigma_p \cap G_n$ and $\sigma^n = \sigma_p \setminus \sigma_n$.

The following lemma states that the subsets just defined satisfy, modulo Möbius transformations, the conditions of Lemma 3.6.

Lemma 3.5. ([10], [14]) *The following relations hold true*

$$\#\sigma_n \le \mathcal{N}(T); \tag{3.16}$$

$$\operatorname{diam}_{\mathrm{ph}} G_n \le \delta; \tag{3.17}$$

$$\varrho(\sigma_n, \sigma^n) \ge 2\varrho_0 > 0. \tag{3.18}$$

The expression $\operatorname{diam}_{\mathrm{ph}}$ *stands for the pseudo-hyperbolic diameter of a set.*

Proof. Observe that (3.17) follows immediately from the definition of the sets G_n and relation (3.16).

We start with the proof of (3.16). Suppose that (3.16) is false, and hence that it is possible to find a set σ_n such that $\#\sigma_n \ge \mathcal{N}(T)+1$. We renumber the set $\{\lambda_k\}_{k=1}^{\mathcal{N}(T)+1} \subset \sigma_n$ in such a way that $B_{\varrho_0}(\lambda_k) \cap B_{\varrho_0}(\lambda_{k+1}) \ne \emptyset$. Since $|b_{\zeta_1}(\zeta_3)| \le |b_{\zeta_1}(\zeta_2)| + |b_{\zeta_2}(\zeta_3)|$ for arbitrary points ζ_1, ζ_2 and ζ_3 in $\mathbf{D}$, we have $|b_{\lambda_k}(\lambda_{k+1})| \le 2\varrho_0$, and $|b_{\lambda_1}(\lambda_k)| \le 2k\varrho_0$, $1 \le k \le \mathcal{N}(T)+1$. Consequently, we get $\#B_\delta(\lambda_1) \cap \sigma_p \ge \mathcal{N}(T)+1$, which contradicts (3.15). Therefore, (3.16) is proved.

The proof of (3.18) is also simple. We suppose that $\varrho(\sigma_n, \sigma^n) < 2\varrho_0$. This yields that there exist points $\lambda \in \sigma_n$ and $\mu \in \sigma^n$ such that $\varrho(\lambda, \mu) < 2\varrho_0$ and hence, by the construction of G_n, the point μ has to lie in σ_n. The contradiction completes the proof. $\square$

3.5. Local properties of the resolvent of a contraction $T\in(\mathrm{LRG})\cap(\mathrm{UTB})$

Here we obtain some additional information on the behavior of the norm $\|R_\lambda(T)\|$ for a contraction possessing the properties (LRG) and (UTB). The technique we use here is similar to that from [10, Section 9.5]. In fact, it relies on the subharmonicity of the function $\|R_\lambda(T)\|$ only.

Firstly, we prove the following "spreading of estimates" lemma for subharmonic functions of a very special type.

Lemma 3.6. *Let $\sigma=\sigma_1\cup\sigma_2$ be disjoint subsets of the unit disk $\mathbf{D}$. Let u and u_1 be subharmonic functions defined on $\mathbf{C}\backslash\sigma$ and $\mathbf{C}\backslash\sigma_1$, respectively. Assume that*

$$u_1(\lambda)\le u(\lambda)\le\frac{c_3}{\operatorname{dist}(\lambda,\sigma)};\quad \operatorname{dist}(\sigma_2,\sigma_1)=\delta>0, \tag{3.19}$$

for all $\lambda\in\mathbf{C}\backslash\sigma$. Then

$$u_1(\lambda)\le\frac{18c_3}{\delta}\frac{1}{\operatorname{dist}(\lambda,\sigma_1)} \tag{3.20}$$

for all $\lambda\in\mathbf{C}\backslash\sigma_1$.

Proof. Let $|\lambda|\ge2$. As $\sigma_j\subset\mathbf{D}$, we have $|\lambda|-1\le\operatorname{dist}(\lambda,\mathbf{T})\le\operatorname{dist}(\lambda,\sigma_j)\le|\lambda|+1$, $j=1,\ 2$. So, we get

$$u_1(\lambda)\le\frac{c_3}{\operatorname{dist}(\lambda,\sigma)}\le\frac{c_3}{|\lambda|-1}\le\frac{3c_3}{\operatorname{dist}(\lambda,\sigma_1)}$$

for all $|\lambda|\ge2$ since $(|\lambda|+1)/(|\lambda|-1)\le3$ for these λ.

Now, let $|\lambda|<2$. Let $G=\{\lambda\in\mathbf{D}_2:\operatorname{dist}(\lambda,\sigma_2)<\operatorname{dist}(\lambda,\sigma_1)\}$, where $\mathbf{D}_2=\{\lambda:|\lambda|<2\}$. Evidently, G is an open set containing σ_2. We consider the cases $\lambda\in\mathbf{D}_2\backslash G$ and $\lambda\in G$ separately.

Let $\lambda\in\mathbf{D}_2\backslash G$. We have, by definition of G, $\operatorname{dist}(\lambda,\sigma_2)\ge1/3\operatorname{dist}(\lambda,\sigma_1)$, and, consequently,

$$\operatorname{dist}(\lambda,\sigma)=\min\{\operatorname{dist}(\lambda,\sigma_2),\ \operatorname{dist}(\lambda,\sigma_1)\}\ge1/3\operatorname{dist}(\lambda,\sigma_1).$$

This implies

$$u_1(\lambda)\le\frac{3c_3}{\operatorname{dist}(\lambda,\sigma_1)}$$

for all $\lambda\in\mathbf{D}_2\backslash G$.

Let now $\lambda\in\partial G$. It follows that $\operatorname{dist}(\lambda,\sigma_j)\le3\operatorname{dist}(\lambda,\sigma)$, and we have

$$\delta\le\operatorname{dist}(\sigma_2,\sigma_1)\le\operatorname{dist}(\lambda,\sigma_2)+\operatorname{dist}(\lambda,\sigma_1)\le6\operatorname{dist}(\lambda,\sigma).$$

Hence

$$u_1(\lambda) \le u(\lambda) \le \frac{6c_3}{\delta}$$

for all $\lambda \in \partial G$, and the maximum principle provides

$$u_1(\lambda) \le \frac{18c_3}{\delta} \frac{1}{\operatorname{dist}(\lambda, \sigma_1)}$$

for all $\lambda \in G$. The proof is finished. □

We note that, due to Corollary 3.1, we may assume that δ is as small as we want, $\delta < \frac{1}{2}$, for example. Now, we use Lemmas 3.5 and 3.6 to prove the following fact.

Lemma 3.7. *Let T be a contraction possessing the* (UTB) *and the* (LRG) *properties. Then*

$$\|R_\lambda(b_\mu(T_{\sigma^n}))\| \le \frac{c_4}{\operatorname{dist}(\lambda, \sigma_\mu^n)} \tag{3.21}$$

for all $\lambda \in \mathbf{C} \setminus \sigma_\mu^n$ and for all n and $\mu \in \mathbf{D}$ such that $\varrho(\mu, G_n) \le \varrho_0$. Here c_4 is a constant that depends only on δ, $C_1(T)$, and $\mathcal{N}(T)$.

Proof. Since $\sigma_n \subset G_n \subset B_\delta(\mu)$, we can easily see that $\operatorname{diam}_{\text{ph}} \sigma_n \le \operatorname{diam}_{\text{ph}} G_n \le \delta$. Further, we obtain from (3.16)–(3.18) that

$$\sigma_{n,\mu} = b_\mu(\sigma_n) \subset G_{n,\mu}, \qquad G_{n,\mu} = b_\mu(G_n),$$

$$G_{n,\mu} \subset B_\delta(0) = \{z : |z| \le \delta\}, \tag{3.22}$$

$$\varrho(\sigma_{n,\mu}, \sigma_\mu^n) \ge \varrho_0. \tag{3.23}$$

Since $c_5|\lambda - \mu| \le |b_\lambda(\mu)| \le c_6|\lambda - \mu|$, whenever $\lambda, \mu \in B_{1/2}(0)$, relation (3.23) implies that $\operatorname{dist}(\sigma_{n,\mu}, \sigma_\mu^n) \ge c_7 \varrho_0$. Moreover, the functions $u(\lambda) = \|R_\lambda(b_\mu(T))\|$ and $u_1(\lambda) = \|R_\lambda(b_\mu(T_{\sigma^n}))\|$ are defined on $\mathbf{C} \setminus \sigma_{p\mu}$ and $\mathbf{C} \setminus \sigma_\mu^n$, respectively, and, obviously, $\|R_\lambda(b_\mu(T_{\sigma^n}))\| \le \|R_\lambda(b_\mu(T))\|$. Condition (3.19) is true since the (LRG) property is invariant with respect to Möbius transformations (see Lemma 1.1). The application of Lemma 3.6 leads to the claimed result. □

3.6. Global properties of the spectrum of a contraction with (LRG) and (UTB)

We start this subsection with a lemma characterizing the "individual" properties of a complete contraction T.

Lemma 3.8. *Let T be a complete nonunitary contraction, $\sigma(T)\neq\bar{\mathbf{D}}$ and $\mathcal{X}=\{X_\lambda\}_{\lambda\in\sigma_p}$ be the system of its eigenspaces. If*

$$I-T^*T\in\mathfrak{S}_1;$$

$$T\in(\mathrm{LRG});$$

(3.24) *there exists a number $0<\delta<\frac{1}{2}$ such that $B_\delta(0)\cap\sigma_p=\emptyset$,*

then we have

$$\sum_{\lambda\in\sigma_p}(1-|\lambda|^2)\dim X_\lambda\leq c_8(\delta)\operatorname{tr}(I-T^*T), \tag{3.25}$$

where $c_8(\delta)=\mathcal{C}_1(T)^2/\delta^3\log(1/\delta-1)$.

Proof. Condition (3.24) yields that the operator T is invertible, and, by Theorem 2.4,

$$\det T^*T=\prod_{\lambda\in\sigma_p}|\lambda|^2.$$

We denote by β_j the eigenvalues of the operator T^*T, so the eigenvalues of the operator $I-T^*T$ are equal to $\alpha_j=1-\beta_j$, $0\leq\alpha_j,\beta_j\leq 1$. We rewrite the previous relation as

$$\log\frac{1}{\prod_{\lambda\in\sigma_p}|\lambda|^2}=\log\frac{1}{\prod_j\beta_j}.$$

We estimate the left-hand side of the equality from below

$$\log\frac{1}{\prod_{\lambda\in\sigma_p}|\lambda|^2}=2\sum_{\lambda\in\sigma_p}\log\left(1+\left(\frac{1}{|\lambda|}-1\right)\right)\geq\frac{\delta\log(1/\delta-1)}{1-\delta}\sum_{\lambda\in\sigma_p}(1-|\lambda|^2),$$

we have used here that $\log(1+x)\geq\delta\log(1/\delta-1)x/(1-\delta)$ whenever $1\leq x\leq 1/\delta$. Similarly, we have for the other side of the equality

$$\log\frac{1}{\prod_j\beta_j}=\sum_j\log\left(\left(\frac{1}{\beta_j}-1\right)+1\right)\leq\sum_j\frac{1-\beta_j}{\beta_j}.$$

We observe that, if $\lambda=0$, the condition (LRG) means that

$$\|T^{-1}\|\leq\frac{\mathcal{C}_1(T)}{\operatorname{dist}(0,\sigma)},$$

or, which is the same, $\inf_j|\lambda_j|^2\leq\mathcal{C}_1(T)^2\inf_j|\beta_j|$. Taking into account (3.24), we obtain $|\beta_j|^2\geq\delta^2/\mathcal{C}_1(T)^2$. Hence,

$$\sum_j\frac{1-\beta_j}{\beta_j}\leq\frac{\mathcal{C}_1(T)^2}{\delta^2}\sum_j(1-\beta_j)\leq\frac{\mathcal{C}_1(T)^2}{\delta^2}\operatorname{tr}(I-T^*T),$$

and the proof is completed. □

The results of Subsection 3.5 allow us to remove the supplementary restriction (3.24).

Lemma 3.9. *Let T be a completely nonunitary contraction and $T\in(\mathrm{LRG})\cap(\mathrm{UTB})$. Then*

$$\sum_{\lambda\in\sigma_{p\mu}}(1-|\lambda|^2)\le c_9 \tag{3.26}$$

for all $\mu\in\mathbf{D}$, where c_9 depends on δ, $\mathcal{C}_1(T)$, $\mathcal{C}_2(T)$, and $\mathcal{N}(T)$ only.

Proof. Take $\mu\in\mathbf{D}$. At first, we suppose that $\sigma_{p\mu}\cap B_{\varrho_0}(0)=\emptyset$. We may apply Lemma 3.8, which gives us immediately

$$\sum_{\lambda\in\sigma_{p\mu}}(1-|\lambda|^2)\le c_8(\varrho_0)\,\mathrm{tr}(I-b_\mu(T)^*b_\mu(T))\le c_8(\varrho_0)\mathcal{C}_2(T).$$

Now, assume that $\omega_1=\sigma_{p\mu}\cap B_{\varrho_0}(0)\ne\emptyset$. We observe that there exists some $\sigma_{n_0,\mu}$ such that $\omega_1\subset\sigma_{n_0,\mu}$, and, in particular, we have $\#\omega_1\le\mathcal{N}(T)$. We apply Lemma 3.7 to the contraction $b_\mu(T_{\sigma^{n_0}})$,

$$\|R_\lambda(b_\mu(T_{\sigma^{n_0}}))\|\le\frac{c_4}{\mathrm{dist}(\lambda,\sigma_\mu^{n_0})}\quad\text{for all }\lambda\in\mathbf{C}\backslash\sigma_\mu^{n_0}.$$

Further, we note that $\sigma(b_\mu(T_{\sigma^{n_0}}))\cap B_{\varrho_0}(0)=\emptyset$. Hence, by Lemma 3.8

$$\begin{aligned}\sum_{\lambda\in\sigma_\mu^{n_0}}(1-|\lambda|^2)&\le c_8(\varrho_0)\,\mathrm{tr}(I-b_\mu(T_{\sigma^{n_0}})^*b_\mu(T_{\sigma^{n_0}}))\\&\le c_8(\varrho_0)\,\mathrm{tr}(I-b_\mu(T)^*b_\mu(T))\le c_8(\varrho_0)\mathcal{C}_2(T).\end{aligned}$$

This gives us

$$\sum_{\lambda\in\sigma_{p\mu}}(1-|\lambda|^2)\le c_8(\varrho_0)\mathcal{C}_2(T)+\mathcal{N}(T),$$

and the lemma is proved. □

Corollary 3.3. *If T is a contraction from Lemma 3.9, then $\sigma_p(T)$ is $\mathcal{N}(T)$-Carleson.*

Proof. We have, by Lemma 3.9,

$$\sum_{\lambda\in\sigma_p}(1-|b_\mu(\lambda)|^2)\le c_9$$

for every $\mu\in\mathbf{D}$. Now the statement follows from the embedding theorem, Theorem 2.3, and Corollary 3.2. □

3.7. Basis properties of the family of eigenspaces

We introduce some notation to formulate a theorem. Let $\{\theta_n\}_{n=1}^{\infty}$ be a family of $L(E)$-valued two-sided inner functions, E be a Hilbert space. We consider subspaces K_{θ_n} (see (2.1)) and we define $L(E)$-valued inner functions θ and θ^n via the model spaces

$$K_\theta = \bigvee_{n=1}^{\infty} K_{\theta_n} \quad \text{and} \quad K_{\theta^n} = \bigvee_{k\neq n} K_{\theta_k}.$$

The spaces K_{θ_n} and K_{θ^n} are invariant with respect to the model operator M_θ^*, and, as was mentioned in Subsection 2.2, they define regular factorizations $\theta=\theta_n\tilde{\theta}^n$, $\theta=\theta^n\tilde{\theta}_n$ of the function θ.

The following theorem states that the property of being an unconditional basis is equivalent to the uniform minimality and two embedding theorems.

Theorem 3.1. ([12]) *Let $\{\theta_n\}_{n=1}^{\infty}$ be a family of $L(E)$-valued two-sided inner functions, and* $\dim K_{\theta_n}\leq M$ *for some $M>0$.*

The family is a Riesz basis in its linear span if and only if it is uniformly minimal and the two following embedding theorems hold true

$$\sum_{n=1}^{\infty}(1-\|\theta_n(\lambda)^*e\|^2)\leq C<\infty; \tag{3.27}$$

$$\sum_{n=1}^{\infty}(1-\|\tilde{\theta}_n(\lambda)^*e\|^2)\leq C<\infty, \tag{3.28}$$

for every $e\in E$, $\|e\|=1$, and every $\lambda\in\mathbf{D}$.

We put our situation in the frame of the theorem by setting

$$K_\theta = \bigvee_{\lambda\in\sigma_p} X_\lambda$$

and $\mathcal{X}=\{X_\lambda\}_{\lambda\in\sigma_p}$. The regular factorizations of θ

$$\theta=\theta_\lambda\tilde{\theta}^\lambda \quad \text{and} \quad \theta=\theta^\lambda\tilde{\theta}_\lambda$$

for $\lambda\in\sigma_p$ are defined by the corresponding invariant subspaces

$$K_{\theta_\lambda}=X_\lambda \quad \text{and} \quad K_{\theta^\lambda} = \bigvee_{\mu\in\sigma_p\setminus\{\lambda\}} X_\mu.$$

Proof of Theorem 1.1 *for complete contractions.* The family $\mathcal{X}$ of eigenspaces of T is uniformly minimal by virtue of Lemma 3.3. The same lemma implies that all the eigenvalues of T are algebraically simple. The inequality $\dim K_{\theta_\lambda} = \dim X_\lambda \le \mathcal{N}(T)$ is guaranteed by (3.9). So, we should verify only embeddings (3.27) and (3.28). We have

$$\|\theta_\lambda(\mu)^* e\| \ge |\det \theta_\lambda(\mu)| \quad \text{and} \quad \|\tilde{\theta}_\lambda(\mu)^* e\| \ge |\det \theta_\lambda(\mu)|,$$

where $\|e\|=1$, and $|\det \theta_\lambda(\mu)| \ge |b_\lambda(\mu)|^{\mathcal{N}(T)}$, $\lambda \in \sigma_p$. Hence, (3.27) and (3.28) follow from already proved relation (3.26),

$$\sum_{\lambda\in\sigma_p} (1-|b_\lambda(\mu)|^{2\mathcal{N}(T)}) \le 2\mathcal{N}(T) \sum_{\lambda\in\sigma_p} (1-|b_\lambda(\mu)|) \le 2\mathcal{N}(T) \sum_{\lambda\in\sigma_{p\mu}} (1-|\lambda|) \le 2\mathcal{N}(T) c_9.$$

Theorem 1.1 for complete contractions is proved. □

4. Proof of Theorem 1.1 for general contractions

The main lines of the reasoning of this section follow [1].

4.1. Some inequalities for trace-class operators

The first important step is separating the point spectrum of a contraction from the unitary one. We obtain some infinite dimensional counterparts of the inequalities

$$|\det A|^{-1} \le \|A^{-1}\|^n \quad \text{and} \quad \|A^{-1}\| \le \|A\|^{n-1} |\det A|^{-1}$$

valid in finite dimensional spaces (A is an $n\times n$ matrix here).

Lemma 4.1. *Let θ be an operator on a separable Hilbert space E, $\|\theta\|\le 1$, and let $I-\theta^*\theta$ be a trace-class operator. Then*

$$\|(\theta^*\theta)^{-1}\| \le \frac{1}{\det \theta^*\theta} = \frac{1}{|\det \theta|^2}, \tag{4.1}$$

$$\frac{1}{|\det \theta|^2} \le \exp(\|\theta^{-1}\|^2 \operatorname{tr}(I-\theta^*\theta)). \tag{4.2}$$

Proof. The assumptions of the lemma imply that the operator $\theta^*\theta$ admits a spectral decomposition of the form

$$\theta^*\theta x = \sum_{k=1}^{\infty} \mu_k (x, e_k) e_k,$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal system, and $\{\mu_k\}_{k=1}^{\infty}$ is the sequence of eigenvalues of $\theta^*\theta$, $0\le\mu_k\le 1$. Inequality (4.1) is almost obvious

$$\|(\theta^*\theta)^{-1}\| \le \frac{1}{\inf_{k\ge1}|\mu_k|} \le \frac{1}{\prod_{k=1}^{\infty}|\mu_k|} = \frac{1}{\det\theta^*\theta}.$$

The proof of (4.2) is also relatively simple

$$\begin{aligned}\log\frac{1}{\det\theta^*\theta} &\le \log\frac{1}{\prod_{k=1}^{\infty}\mu_k} = \sum_{k=1}^{\infty}\log\left(1+\left(\frac{1}{|\mu_k|}-1\right)\right)\\ &\le \sum_{k=1}^{\infty}\left(\frac{1}{|\mu_k|}-1\right) \le \frac{1}{\inf_{k\ge1}|\mu_k|}\sum_{k=1}^{\infty}(1-\mu_k) = \|(\theta^*\theta)^{-1}\|\operatorname{tr}(I-\theta^*\theta).\end{aligned}$$

Taking into account the inequality $\|(\theta^*\theta)^{-1}\|\le\|\theta^{-1}\|^2$, we get the lemma. □

4.2. The (LRG) and (UTB) properties yield the triviality of the unitary spectrum of a contraction

Let T be a completely nonunitary contraction (we do not require its completeness now), possessing the (LRG) and the (UTB) properties. We denote by θ its characteristic function. Further, we define its regular factorization $\theta=\theta_2\theta_1$ (see Subsection 2.2) and the corresponding invariant subspace L_1 by the relation $L_1=\bigvee_{\lambda\in\sigma_p}X_\lambda$. We put $T_1=T|_{L_1}$. Evidently, T_1 is a complete contraction.

Lemma 4.2. *Let T be a completely nonunitary contraction. If* $T\in(\text{LRG})\cap(\text{UTB})$, *then* $T_1\in(\text{LRG})\cap(\text{UTB})$.

Proof. It is clear that

$$\mathcal{C}_2(T_1) = \sup_{\mu\in\mathbf{D}}\operatorname{tr}(I-b_\mu(T_1)^*b_\mu(T_1)) \le \sup_{\mu\in\mathbf{D}}\operatorname{tr}(I-b_\mu(T)^*b_\mu(T)) = \mathcal{C}_2(T),$$

and $T_1\in(\text{UTB})$.

It remains to get the second property. Note that $\sigma_p(T_1)=\sigma_p$. We consider two cases, $|\lambda|<\frac{1}{2}$ and $|\lambda|\ge\frac{1}{2}$. If $|\lambda|<\frac{1}{2}$, we get immediately

$$\|R_\lambda(T_1)\| \le \|R_\lambda(T)\| \le \frac{4\mathcal{C}_1(T)}{\operatorname{dist}(\lambda,\sigma_p)},$$

where $\lambda\in\mathbf{C}\setminus\sigma_p$. If $|\lambda|\ge\frac{1}{2}$, we obtain

$$\|R_\lambda(T_1)\| \le \|R_\lambda(T)\| \le \mathcal{C}_1(T)\max\left\{\frac{1}{\operatorname{dist}(\lambda,\sigma_p)},\ \frac{1}{\big|1-|\lambda|\big|}\right\} \le \frac{A_0\mathcal{C}_1(T)}{\operatorname{dist}(\lambda,\sigma_p)},$$

by virtue of Lemma 3.1, and so $\mathcal{C}_1(T_1)\le A_0\mathcal{C}_1(T)$. The proof is finished. □

Hence, the results of Section 3 can be applied to the operator T_1, and $\sigma_p(T_1)$ is $\mathcal{N}(T_1)$-Carleson (see Corollary 3.3).

The following lemma characterizes the (UTB) property in terms of the characteristic function of a contraction.

Lemma 4.3. *Let T be a completely nonunitary contraction satisfying* (UTB). *Then*

$$\operatorname{tr}(I-\theta(\mu)^*\theta(\mu))=\operatorname{tr}(I-b_\mu(T)^*b_\mu(T))\le\mathcal{C}_2(T).$$

Proof. Recall that (see (2.2))

$$U_{*a}\theta_{b_a(T)}(b_a(\mu))U_a^*=\theta_T(\mu)$$

for $a\in\mathbf{D}$. Here $U_{*a}:\mathcal{D}_{b_a(T)^*}\to\mathcal{D}_T$ and $U_a:\mathcal{D}_{b_a(T)}\to\mathcal{D}_T$ are some unitary mappings. This implies the equality

$$I-\theta_T(\mu)^*\theta_T(\mu)=U_a(I-\theta_{b_a(T)}(b_a(\mu))^*\theta_{b_a(T)}(b_a(\mu)))U_a^*.$$

We take $a=\mu$ and, since $\theta_T(0)=-T|_{\mathcal{D}_T}$, it gives us

$$I-\theta_T(\mu)^*\theta_T(\mu)=U_\mu(I-b_\mu(T)^*b_\mu(T))U_\mu^*.$$

We conclude the computation with

$$\operatorname{tr}(I-\theta(\mu)^*\theta(\mu))=\operatorname{tr}U_\mu(I-b_\mu(T)^*b_\mu(T))U_\mu^*=\operatorname{tr}(I-b_\mu(T)^*b_\mu(T))\le\mathcal{C}_2(T).$$

The proof is finished. □

Lemma 4.4. *Let T be a completely nonunitary contraction, $\theta(\lambda)$ be its characteristic function, and $T\in$*(LRG)$\cap$(UTB). *Then*

(i) *the function* $\det\theta$ *contains no singular inner factor in its canonical factorization;*

(ii) *the outer part of the function $\theta(\lambda)$ is boundedly invertible:*

$$\sup_{\lambda\in\mathbf{D}}\|\theta_{\text{out}}(\lambda)^{-1}\|\le C<\infty. \tag{4.3}$$

Proof. By virtue of Lemmas 4.1 and 4.3,

$$\frac{1}{|\det\theta(\lambda)|^2}\le\exp(\|\theta(\lambda)^{-1}\|^2\operatorname{tr}(I-\theta(\lambda)^*\theta(\lambda)))\le\exp(\mathcal{C}_2(T)\|\theta(\lambda)^{-1}\|^2). \tag{4.4}$$

To get (i) we just repeat a part of the reasoning from [1, Theorem 1.5.2]. Indeed, if $\det\theta(\lambda)$ contains a nontrivial singular factor, we have

$$\lim_{r\to1-0}|\det\theta(r\xi)^{-1}|=+\infty,$$

where $\xi\in\mathbf{T}$. The relation (4.4) gives us $\lim_{r\to1-0}\|\theta(r\xi)^{-1}\|=+\infty$. Inequality (2.3) implies that $\lim_{r\to1-0}(1-r)\|R_{r\xi}(T)\|=+\infty$. Lemma 2.1 provides us with a sequence $\{z_n\}_{n=1}^{\infty}$, $z_n=r_n\xi$, $r_n\to1$, such that $\operatorname{dist}(z_n,\sigma_p)\ge\varepsilon(1-|z_n|)$. We conclude that

$$\lim_{r\to1-0}\|R_{z_n}(T)\|\operatorname{dist}(z_n,\sigma_p)=+\infty,$$

which contradicts the (LRG) property.

We pass to a sketch of the proof of (4.3) now. It is not difficult to see that $\sup_{\lambda\in\mathbf{D}}\|\theta_{\mathrm{out}}(\lambda)^{-1}\|<\infty$ if and only if the function $\det\theta_{\mathrm{out}}(\lambda)$ is boundedly invertible. This fact follows from the relations

$$\|(\theta_{\mathrm{out}}(t)^*\theta_{\mathrm{out}}(t))^{-1}\|=\|(\theta(t)^*\theta(t))^{-1}\|$$

a.e. on $\mathbf{T}$ and

$$\begin{aligned}\frac{1}{|\det\theta_{\mathrm{out}}|^2}&\le\exp(\|\theta_{\mathrm{out}}^{-1}\|^2\operatorname{tr}(I-\theta_{\mathrm{out}}^*\theta_{\mathrm{out}}))\le\exp(\|\theta_{\mathrm{out}}^{-1}\|^2\operatorname{tr}(I-\theta^*\theta))\\&\le\exp(\|\theta_{\mathrm{out}}^{-1}\|^2\operatorname{tr}(I-b_\mu(T)^*b_\mu(T)))\le\exp(\mathcal{C}_2(T)\|\theta_{\mathrm{out}}^{-1}\|),\end{aligned}$$

because $\theta_{\mathrm{out}}^*\theta_{\mathrm{out}}\ge\theta^*\theta$ by the definition of an outer function [11, Chapter 5], and, consequently, $\operatorname{tr}(I-\theta_{\mathrm{out}}^*\theta_{\mathrm{out}})\ge\operatorname{tr}(I-\theta^*\theta)$.

To get the desirable conclusion for the outer part of $\theta(\lambda)$, we apply arguments similar to those used while considering the inner part of the function, see [1, Theorem 1.5.2] for the details.

The proof is finished. □

4.3. Separating the spectra and completing the proof of Theorem 1.1

We write down the canonical and $*$-canonical factorizations of the characteristic function θ of the contraction T. The factorizations are regular, and we consider the corresponding invariant subspaces $L=L_{\mathrm{out}}$ and $L'=L_{\mathrm{in}*}$. It is obvious that the sum $L+L'$ is dense in $H=K_\theta$, and Theorem 2.2 states that the sum is direct if and only if the corresponding Bezout equation is solvable.

The following lemma yields the solvability of the equation.

Lemma 4.5. *Let $\theta_1, \theta_1' \in H^\infty(L(E))$ be analytic functions possessing the scalar multiples ω_1 and ω_1', respectively. Let, further, $|\omega_1(z)| + |\omega_1'(z)| \geq \varepsilon$, $z \in \mathbf{D}$. Then there exist functions $\Gamma_1, \Gamma_1' \in H^\infty(L(E))$ that solve the Bezout equation* (2.5).

Proof. The definition of a scalar multiple implies that there exist operator-valued functions $\Omega_1, \Omega_1' \in H^\infty(L(E))$ with the properties

$$\Omega_1 \theta_1 = \theta_1 \Omega_1 = \omega_1 I \quad \text{and} \quad \Omega_1' \theta_1' = \theta_1' \Omega_1' = \omega_1' I,$$

and $\omega_1, \omega_1' \in H^\infty$. Since $|\omega_1(z)| + |\omega_1'(z)| \geq \varepsilon$, $z \in \mathbf{D}$, we find γ_1 and γ_1' in H^∞ solving the scalar Bezout equation $\gamma_1 \omega_1 + \gamma_1' \omega_1' = 1$. We note that the functions $\Gamma_1 = \gamma_1 \Omega_1$ and $\Gamma_1' = \gamma_1' \Omega_1'$ satisfy the equations

$$\Gamma_1 \theta_1 + \Gamma_1' \theta_1' = \gamma_1 \Omega_1 \theta_1 + \gamma_1' \Omega_1' \theta_1' = (\gamma_1 \omega_1 + \gamma_1' \omega_1') I = I,$$

and the proof is finished. □

Remark 4.1. We have $\theta_1 = \theta_{\text{out}}$ and $\theta_1' = \theta_{\text{in}*}$. Lemma 4.4 states that, under the assumptions mentioned there, $|\det \theta_{\text{out}}(\lambda)| \geq \varepsilon > 0$, so the lemma is valid, and the conclusion of Theorem 2.2 follows.

Proof of Theorem 1.1. Lemma 4.4 affirms that $\theta_{\text{out}}^{-1} \in H^\infty(L(E))$ and $\det \theta_{\text{out}}^{-1} \in H^\infty$. Lemma 4.5 and the remark after it imply that the sum $H = L + L'$ is direct. The first part of Lemma 4.4 gives that $\det \theta_{\text{in}*}$ is a Blaschke product and, hence, $T_{\text{in}*} = T_1$ is a complete contraction (see Lemma 4.2). Since $T_1 \in (\text{LRG}) \cap (\text{UTB})$, it is similar to a normal operator by the particular case of Theorem 1.1, proved in Section 3. The similarity of the operator $T|_{L_{\text{out}}}$ to a unitary operator follows at once from relation (2.4). To complete the proof, we use the positivity of the angle between $L_{\text{in}*}$ and L_{out}.

The proof is finished. □

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