On the adjoint of an elliptic linear differential operator and its potential theory

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1. Introduction

In her thesis [4], R.-M. Hervé develops Brelot's axiomatic potential theory. Within this theory she constructs an adjoint potential theory satisfying the same axioms. She applies this to the potential theory associated with an elliptic linear second-order differential operator L. When the adjoint operator L^* exists in the classical sense and has Hölder-continuous coefficients, the adjoint potential theory coincides with that of L^*. In Section 3 of this paper we generalize this fact to the case when the coefficients of L are assumed to be locally α-Hölder continuous and L^* is defined in the sense of distributions. This result easily implies some properties of supersolutions of the equation $L^*u = 0$ proved by Littman [5]. He shows that they satisfy a minimum principle and have some approximation properties.

Under the same assumptions, we prove in Section 4 that the distribution solutions of $L^*u = 0$ are locally α-Hölder continuous. In Section 5 we obtain a formula for Hervé's L^*-harmonic measure of a domain ω. This measure is shown to have an area density given simply by a conormal derivative of the Green's function of L in ω. Finally, we prove a Fredholm type theorem for the Dirichlet problems for L and L^* in a given domain.

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2. Preliminaries

Suppose we are given a domain $\Omega_0 \subset \mathbf{R}^n$, $n \geq 2$, and a differential operator

$$Lu = a^{ij}u_{ij} + b^i u_i + cu,$$

defined in Ω_0. We assume that $a^{ij} = a^{ji}$, that L is elliptic in Ω_0, and that the coefficients are locally α-Hölder continuous, for some α with $0 < \alpha < 1$. As Hervé shows, we can let the $C^{(2)}$ functions u satisfying $Lu = 0$ be the harmonic functions in Brelot's axiomatic potential theory presented in Brelot [2, 3]. In this way we obtain a potential theory satisfying Brelot's Axioms 1, 2, and 3′ (see Hervé [4]). We write »L-harmonic», »L-potential», etc. when we refer to concepts of this theory.

To make possible the construction of an adjoint theory, we must limit ourselves to a domain $\Omega \subset \Omega_0$ where a positive L-potential exists. Depending on the coefficient c of the operator, Ω may be chosen in the following way, as shown by Hervé [4, p. 562].

1. If $c \leq 0$ and $c \not\equiv 0$, we may take $\Omega \subset \Omega_0$ arbitrary.
2. If $c \equiv 0$, we may take any bounded Ω such that $\bar{\Omega} \subset \Omega_0$.
3. If c is arbitrary, any $x_0 \in \Omega_0$ has a neighbourhood which is an admissible Ω.

From now on we fix such an Ω. In the sequel $\omega, \omega_1, \ldots$ will always be subdomains of Ω or Ω_0.

We follow the notation of Brelot and Hervé and write $\hat{R}_v^E$ for the balayaged function of a nonnegative L-superharmonic function v and a set $E \subset \Omega$. If $\bar{\omega}$ is compact and contained in Ω, and f is defined and continuous on $\partial\omega$, then the solution of the Dirichlet problem for L in ω with boundary values f is denoted by H_f^ω. A point $x_0 \in \partial\omega$ is called L-regular for ω if $H_f^\omega(x) \to f(x_0)$ as $x \to x_0$, $x \in \omega$, for any continuous f. As shown by Hervé, x_0 is L-regular if and only if it is regular in classical potential theory.

Hervé [4, Prop. 35.1] constructs an L-potential P_y in Ω with support $\{y\}$ and such that the mapping $(x, y) \to P_y(x)$ is continuous for $x, y \in \Omega$, $x \neq y$. The support of a potential P is defined as the complement of the largest open set in which P is harmonic. The function $P_y(x)$ is a fundamental solution of L in Ω. Hervé [4, Theorem 18.2] shows that any L-potential P in Ω can be represented as

$$P(x) = \int P_y(x) d\mu(y) \tag{2.1}$$

for a unique positive measure μ in Ω. The support of μ coincides with the support of the L-potential P.

For any bounded ω of class $C^{(1,\lambda)}$ and such that $\bar{\omega} \subset \Omega$ the Green's function is given by

$$G^\omega(x, y) = P_y(x) - H_{P_y}^\omega(x). \tag{2.2}$$

The function G^ω can be used to solve a boundary value problem, as follows. If f is continuous in $\bar{\omega}$ and locally Hölder continuous in ω, the unique solution of the problem

$$Lu = f \text{ in } \omega, \quad u = 0 \text{ on } \partial\omega$$

is given by

$$u(x) = -\int\limits_{\omega} G^{\omega}(x, y)f(y)dy. \tag{2.3}$$

For this see Miranda [6].

To construct the adjoint potentials, Hervé uses the concept of completely-determining open set in Ω, which is defined in Hervé [4, p. 451]. If ω is L-completely determining, Hervé defines the L^*-harmonic measure σ_y^{ω} for ω at $y \in \omega$ by the equation

$$\hat{R}_{P_y}^{\Omega \setminus \omega}(x) = \int P_z(x) d\sigma_y^{\omega}(z). \tag{2.4}$$

The left side of (2.4) is an L-potential in Ω, so because of (2.1), the measure σ_y^{ω} is uniquely determined by (2.4). Hervé now calls a function L^*-harmonic in ω_1 if it is continuous there and satisfies

$$u(y) = \int u(x) d\sigma_y^{\omega}(x), \quad y \in \omega,$$

for any L-completely determining ω such that $\bar{\omega} \subset \omega_1$.

Hervé shows that the L^*-harmonic functions satisfy the axioms of Brelot's potential theory. In the adjoint theory the function $P_y^*(x) = P_x(y)$ is a potential with support $\{y\}$ and plays the role of $P_y(x)$. Following Hervé's notations, we shall write »L^*-superharmonic», »L^*-potential», etc., for concepts pertaining to this adjoint theory. From the definition it can be proved that the property of L^*-harmonicity in ω is independent of the domain Ω considered, $\Omega \supset \omega$. If the adjoint operator

$$L^*u = \frac{\partial^2}{\partial x_i \partial x_j}(a^{ij}u) - \frac{\partial}{\partial x_i}(b^i u) + cu \tag{2.5}$$

exists in the classical sense and has Hölder-continuous coefficients, then the L^*-harmonic functions are simply the solutions of $L^*u = 0$. In the general case, we interpret (2.5) in the sense of distributions, for any locally integrable u.

For each $\varepsilon > 0$ we fix a nonnegative $C^{(\infty)}$ function w_ε in $\mathbf{R}^n$, with support contained in $\{|x| \leq \varepsilon\}$, and such that

$$\int w_\varepsilon(x) dx = 1.$$

We let $H(x, y)$ be the fundamental solution of the operator $a^{ij}(y)\partial^2/\partial x_i \partial x_j$ defined by

$$H(x, y) = \frac{1}{2\pi\sqrt{A(y)}} \log \left(\sum a_{ij}(y)(x_i - y_i)(x_j - y_j)\right)^{-1/2} \text{ if } n = 2,$$
$$= \frac{1}{(n-2)\omega_n\sqrt{A(y)}} \left(\sum a_{ij}(y)(x_i - y_i)(x_j - y_j)\right)^{(2-n)/2} \text{ if } n > 2.$$

Here ω_n is the area of the unit sphere in $\mathbf{R}^n$, and $A(y)$ and $(a_{ij}(y))$ are the determinant and the inverse, resp., of the matrix $(a^{ij}(y))$.

3. The fundamental equivalence

We start with a preliminary regularity property of the distribution solutions of $L^*u = 0$ in a domain $\omega \subset \Omega$.

LEMMA 1. *If* $u \in L^1_{\mathrm{loc}}(\omega)$ *satisfies* $L^*u = 0$ *in the sense of* $\mathcal{D}'(\omega)$, *then* u *coincides a.e. in* ω *with a continuous function.*

Proof. Assume $n > 2$, and take a fundamental solution $F(x, y)$ of L in ω. Let U be a relatively compact open subset of ω, and pick $y \in U$ so that

$$\int_{B_\varrho} |u(x) - u(y)|dx = o(\varrho^n) \tag{3.1}$$

as $\varrho \to 0$, where B_ϱ is the ball $\{x\colon |x - y| \leq \varrho\}$. Choose $\varphi \in \mathcal{D}(\omega \setminus \bar{B}_{\varrho/2})$ equal to 1 in $U \setminus B_\varrho$. In $B_\varrho \setminus B_{\varrho/2}$ we let the derivatives of φ satisfy $\varphi_i = O(\varrho^{-1})$ and $\varphi_{ij} = O(\varrho^{-2})$ as $\varrho \to 0$, and outside U we take φ independent of ϱ and y. Since $x \to \varphi(x)F(x, y)$ is a $C^{(2)}$ function, we conclude that

$$\int u(x)L_x(\varphi(x)F(x, y))dx = 0.$$

But $L_xF(x, y) = 0$ for $x \neq y$, so

$$\begin{aligned}\int_{B_\varrho} u(x)a^{ij}(x)\varphi_{ij}(x)F(x, y)dx + 2\int_{B_\varrho} u(x)a^{ij}(x)\varphi_i(x)F_{x_j}(x, y)dx + \\ + \int_{B_\varrho} u(x)b^i(x)\varphi_i(x)F(x, y)dx + \int_{\omega\setminus U} u(x)L_x(\varphi(x)F(x, y))dx = 0.\end{aligned} \tag{3.2}$$

We know that $F(x, y) = O(|x - y|^{2-n})$ and that

$$F(x, y) - H(x, y) = O(|x - y|^{\alpha-2-n}).$$

(Cf. Miranda [6, pp. 18—20]). Hence, (3.1) implies that the third term in (3.2) is $o(1)$ as $\varrho \to 0$, and the first term equals

$$\int_{B_\varrho} u(y)a^{ij}(y)\varphi_{ij}(x)H(x, y)dx + o(1). \tag{3.3}$$

By means of an integration by parts, we find that the second term in (3.2) equals the same expression, except for a factor -2. But H is a fundamental solution of $a^{ij}(y)\partial^2/\partial x_i\partial x_j$, and $\varphi - 1$ can be considered as a function with compact support in $\bar{B}_\varrho$, so the integral in (3.3) equals $u(y)$. Letting $\varrho \to 0$, we get

$$u(y) = \int_{\omega \setminus U} u(x)L_x(\varphi(x)F(x, y))dx \tag{3.4}$$

for a.a. y in U. Since $F(x, y)$ is continuous in (x, y) for $x \neq y$, the integral in (3.4) is a continuous function of y in U, and the lemma is proved for $n > 2$.

If $n = 2$, we introduce a new variable x_3 and put $M = L + \partial^2/\partial x_3^2$ and $v(x, x_3) = u(x)$ in $\omega \times R$. Then M is elliptic, and v satisfies $M^*v = 0$ in the sense of $\mathcal{D}'(\omega \times R)$, since for $\psi \in \mathcal{D}(\omega \times R)$ we have

$$\int vM\psi dx dx_3 = \int dx_3 \int_\omega u(x)L\psi(x, x_3)dx + \int_\omega u(x)dx \int \psi_{x_3 x_3}(x, x_3)dx_3 = 0.$$

Thus we can make v and hence also u continuous by changing them on null sets, and the proof is complete.

Remark. As we shall see later, u is in fact Hölder continuous, and therefore the proof of (3.4) holds also in the two-dimensional case.

THEOREM 1. *Let $\omega \subset \Omega$. A locally integrable function u in ω satisfies $L^*u = 0$ in the sense of $\mathcal{D}'(\omega)$ if and only if u coincides a.e. in ω with a function which is L^*-harmonic in ω. Similarly, u is locally integrable in ω and satisfies $L^*u \leq 0$ in the sense of $\mathcal{D}'(\omega)$ if and only if u coincides a.e. in ω with a function which is L^*-superharmonic in ω.*

Proof. Suppose u is L^*-harmonic in ω, and let $\psi \in \mathcal{D}(\omega)$. Take ω_1 and ω_2 such that

$$\operatorname{supp} \psi \subset \omega_1 \subset \bar{\omega}_1 \subset \omega_2 \subset \bar{\omega}_2 \subset \omega,$$

and let ω_2 be bounded and of class $C^{(1,\lambda)}$. If $u \geq 0$, define

$$v = (\hat{R}_u^{*\omega_1})_{\omega_2}$$

which is the balayaged function of u in the L^*-potential theory in ω_2. Then v is an L^*-potential in ω_2 with support contained in $\partial\omega_1$, and v coincides with u in ω_1. By Theorems 33.1 and 18.2 in Hervé [4], the L^*-potentials can be represented as in (2.1). In this case we obtain

$$v(y) = \int G^{\omega_2}(x, y) d\mu(x), \tag{3.5}$$

for some positive measure μ with support contained in $\partial\omega_1$.

There exists a positive L^*-potential P_y^* in Ω and thus also a positive L^*-harmonic function in a neighbourhood of $\bar{\omega}_2$. An L^*-harmonic u of arbitrary sign is therefore in ω_2 a difference between two positive L^*-harmonic functions. Hence, we obtain a representation similar to (3.5) for any u, but where μ need not be positive.

Because of (2.3), the function ψ satisfies

$$\psi(x) = -\int_{\omega_2} G^{\omega_2}(x, y) L\psi(y) dy. \tag{3.6}$$

Now (3.5—6) and Fubini's theorem imply that

$$\int u L\psi dx = -\int \psi d\mu = 0,$$

and thus $L^*u = 0$.

Conversely, suppose u is locally integrable in ω and satisfies $L^*u = 0$. We take a completely determining ω_1 with $\bar{\omega}_1 \subset \omega$ and a point $y \in \omega_1$. Put

$$f(x) = \int P_z(x) d(\varepsilon_y(z) - \sigma_y^{\omega_1}(z)),$$

where ε_y is the measure consisting of a unit mass at y. From (2.4) it follows that $f(x) = 0$ for $x \notin \bar{\omega}_1$, since $\hat{R}_{P_y}^{\Omega \setminus \omega_1} = P_y$ in $\Omega \setminus \bar{\omega}_1$. Now define

$$g_\varepsilon = \varepsilon_y * w_\varepsilon, \quad h_\varepsilon = \sigma_y^{\omega_1} * w_\varepsilon,$$

and

$$f_\varepsilon(x) = \int P_z(x)(g_\varepsilon(z) - h_\varepsilon(z)) dz.$$

Since $P_z(x)$ is a fundamental solution, we see that f_ε is L-harmonic outside the supports of g_ε and h_ε and that $f_\varepsilon \to f = 0$ in $\Omega \setminus \bar{\omega}_1$ as $\varepsilon \to 0$, uniformly on compact subsets of $\Omega \setminus \bar{\omega}_1$. From Theorem 35, IV in Miranda [6], it follows that the first- and second-order derivatives of f_ε tend to 0 in $\Omega \setminus \bar{\omega}_1$ as $\varepsilon \to 0$, uniformly on compact subsets. Take $\varphi \in \mathcal{D}(\omega)$ equal to 1 in a neighbourhood U of $\bar{\omega}_1$. Then

$$L(\varphi f_\varepsilon) = -g_\varepsilon + h_\varepsilon$$

in U, and $L(\varphi f_\varepsilon) \to 0$ uniformly in $\omega \setminus U$ as $\varepsilon \to 0$.

Since φf_ε is of class $C^{(2)}$, it is clear that

$$\int u L(\varphi f_\varepsilon) dx = 0$$

or equivalently,

$$\int_{\omega\setminus U} uL(\varphi f_\varepsilon)dx - \int ug_\varepsilon dx + \int uh_\varepsilon dx = 0.$$

By Lemma 1 we can assume that u is continuous. Letting $\varepsilon \to 0$, we get

$$u(y) = \int u d\sigma_y^{\omega_1},$$

and the first part of Theorem 1 is proved.

The proof that $L^*u \leq 0$ for an L^*-superharmonic function u is quite similar to the corresponding proof for L^*-harmonic functions and is omitted.

Conversely, suppose that L^*u is a negative measure $-\mu$. Take a bounded ω_1 of class $C^{(1,\lambda)}$ and such that $\bar{\omega}_1 \subset \omega$. Because of (2.3), any $\psi \in \mathcal{D}(\omega_1)$ satisfies

$$\int uL\psi dx = \int\int G^{\omega_1}(x, y)d\mu(x)L\psi(y)dy,$$

where the double integral is absolutely convergent. The first part of Theorem 1 now shows that the function v defined by

$$v(y) = u(y) - \int G^{\omega_1}(x, y)d\mu(x) \tag{3.7}$$

is equal to an L^*-harmonic function a.e. in ω_1. The integral in (3.7) represents an L^*-potential, so u coincides with an L^*-superharmonic function a.e. in ω_1 and thus also in ω. The proof of Theorem 1 is complete.

4. Regularity of the L*-harmonic functions

THEOREM 2. *If u is L^*-harmonic in $\omega \subset \Omega$, then $u \in C_{\mathrm{loc}}^{(0,\alpha)}(\omega)$.*

Proof. Suppose $n > 2$, take a compact set $K \subset \omega$, and let $\varphi \in \mathcal{D}(\omega)$ be 1 in a neighbourhood U of K. If $y \in K$, we have

$$\int u(x)a^{ij}(y)\psi_{ij}(x)dx = \int u(x)((a^{ij}(y) - a^{ij}(x))\psi_{ij}(x) - b^i(x)\psi_i(x) - c(x)\psi(x))dx$$

for any $\psi \in \mathcal{D}(\omega)$. As in the proof of Lemma 1 we find that

$$\begin{aligned} u(y) = &\int u(x)H(x, y)a^{ij}(y)\varphi_{ij}(x)dx + 2\int u(x)a^{ij}(y)H_{x_i}(x, y)\varphi_j(x)dx + \\ &+ \int u(x)(a^{ij}(x) - a^{ij}(y))\frac{\partial^2}{\partial x_i \partial x_j}(\varphi(x)H(x, y))dx + \\ &+ \int u(x)(b^i(x)\frac{\partial}{\partial x_i}(\varphi(x)H(x, y)) + c(x)\varphi(x)H(x, y))dx. \end{aligned} \tag{4.1}$$

Now take y and $z \in K$ with $\varrho = |z - y|$ so small that

$$B = \{x: |x - y| \leq 2\varrho\} \subset U,$$

and consider (4.1) and the corresponding formula for $u(z)$. From the regularity of the a^{ij} it follows that

$$H(x, y) - H(x, z) = O(\varrho^{\alpha}|x - y|^{2-n} + \varrho|x - y|^{1-n}),$$
$$H_{x_i}(x, y) - H_{x_i}(x, z) = O(\varrho^{\alpha}|x - y|^{1-n} + \varrho|x - y|^{-n}),$$

and

$$H_{x_ix_j}(x, y) - H_{x_ix_j}(x, z) = O(\varrho^{\alpha}|x - y|^{-n} + \varrho|x- y|^{-1-n}),$$

if $x \notin B$. Since u is continuous, these estimates easily imply that the difference between the first terms in the formulas for $u(y)$ and $u(z)$ is $O(\varrho^{\alpha})$ as $\varrho \to 0$, and the same is true for the second and fourth terms.

The third term in (4.1) we split as $\int_B + \int_{U\setminus B} + \int_{\omega\setminus U}$, and the integrals over B in this expression and in the corresponding expression for $u(z)$ are $O(\varrho^{\alpha})$. The difference between the integrals over $\omega\setminus U$ is also $O(\varrho^{\alpha})$. The remaining difference can be written as

$$\begin{aligned}&\int_{U\setminus B} u(x)(a^{ij}(z) - a^{ij}(y))H_{x_ix_j}(x, z)dx + \\ &+ \int_{U\setminus B} u(x)(a^{ij}(x) - a^{ij}(y))(H_{x_ix_j}(x, y) - H_{x_ix_j}(x, z))dx.\end{aligned} \tag{4.2}$$

Here the second term is $O(\varrho^{\alpha})$, and the first term is $O(\varrho^{\alpha} \log 1/\varrho)$. Hence, $u \in C^{(0,\beta)}_{\mathrm{loc}}$ for some $\beta > 0$. But then we can improve the last estimate. Since

$$\int_{|x-z|=r} H_{x_ix_j}(x, z)dS_x = 0$$

for all r, the first term in (4.2) equals

$$\int_{U\setminus B} (u(x) - u(z))(a^{ij}(z) - a^{ij}(y))H_{x_ix_j}(x, z)dx + O(\varrho^{\alpha}),$$

which is bounded by

$$O(\varrho^{\alpha}) \cdot \int_{U\setminus B} |x - z|^{\beta-n}dx + O(\varrho^{\alpha}) = O(\varrho^{\alpha}).$$

The case $n = 2$ now follows as in the proof of Lemma 1, and Theorem 2 is proved.

Remark. It is clear that the exponent α is best possible. In a similar way, one can prove regularity properties of solutions of nonhomogeneous equations $L^*u = f$. For example, if $f \in L^p_{\text{loc}}$, then $u \in C^{(0,\alpha)}_{\text{loc}}$ if $p = n/(2-\alpha)$, and u is continuous if $p > n/2$.

5. A formula for the L*-harmonic measure

We shall approximate the coefficients of L with more regular functions, and start by examining how the Green's function varies with the coefficients. For $\varepsilon \to 0$, assume that

$$L_\varepsilon u = a^{ij}_\varepsilon u_{ij} + b^i_\varepsilon u_i + c_\varepsilon u$$

is an operator with coefficients in $C^{(0,\alpha)}(\omega)$, for some $\omega \subset \Omega$. Let G^ω_ε be the corresponding Green's function, whenever it exists.

LEMMA 2. *Assume that ω is a bounded $C^{(2,\lambda)}$ domain with $\bar{\omega} \subset \Omega$. Let the $C^{(0,\alpha)}(\omega)$ norms of the a^{ij}_ε be bounded for small ε, and let $a^{ij}_\varepsilon \to a^{ij}$ uniformly in ω as $\varepsilon \to 0$, and analogously for b^i_ε and c_ε. Then $G^\omega_\varepsilon(x, y) \to G^\omega(x, y)$ uniformly on any compact subset of $\omega \times \omega$ which is disjoint with the diagonal.*

Proof. Since there is a positive L-potential in Ω, there exists a $C^{(2,\alpha)}$ function v in $\bar{\omega}$ which is positive and satisfies $Lv < 0$ in $\bar{\omega}$. Since then also $L_\varepsilon v < 0$ if ε is small enough, G^ω_ε exists for such ε.

In Boboc and Mustată [1, Chap. 4], there is a construction of the Green's function for L in the form of an L-potential whose support is the point y. From this construction it can be seen that $G^\omega_\varepsilon(x, y)$ is uniformly continuous in y when x and y stay within disjoint compact subsets of ω, and this continuity is uniform in ε for small ε. Boboc and Mustată assume that the coefficient c is nonpositive, but f this is not the case, we can apply their proof to the operator $M_\varepsilon\colon u \to v^{-1}L_\varepsilon(vu)$, n which the coefficient of u is negative. Then the Green's function of L_ε is given by

$$G^\omega_{L_\varepsilon}(x, y) = G^\omega_{M_\varepsilon}(x, y)\frac{v(x)}{v(y)},$$

and the same equicontinuity follows.

For $f \in C^{(0,\lambda)}(\omega)$ and ε small, let u_ε be the solution of the problem

$$L_\varepsilon u_\varepsilon = f \ \text{ in } \omega, \quad u_\varepsilon = 0 \ \text{ on } \partial\omega.$$

If we define u similarly by means of L, then the $C^{(2)}(\omega)$ norms of u and u_ε are uniformly bounded for small ε, as follows from Miranda [6, Theorem 35, IV]. Now $L(u - u_\varepsilon) = (L_\varepsilon - L)u_\varepsilon$, so $\sup_\omega |L(u - u_\varepsilon)| \to 0$ as $\varepsilon \to 0$. From Theorem 35, IX in [6], we then conclude that $u_\varepsilon \to u$ as $\varepsilon \to 0$, uniformly in ω. Since

$$u(x) - u_\varepsilon(x) = -\int (G^\omega(x, y) - G_\varepsilon^\omega(x, y))f(y)dy,$$

the lemma then follows if we choose f suitably and use the equicontinuity of G_ε^ω.

Suppose that ω is a bounded $C^{(1,\alpha)}$ domain with $\bar{\omega} \subset \Omega$. Then $\partial\omega$ is a compact $(n-1)$-manifold, imbedded in $\mathbf{R}^n$, and for topological reasons each of its components separates $\mathbf{R}^n$. It follows that $\partial\bar{\omega} = \partial\omega$, which means that at each point of this manifold, ω lies on one side and $\mathbf{R}^n \setminus \bar{\omega}$ on the other. It is easily shown that ω is L-completely determining (see the proof of this fact for open balls in Hervé [4, p. 565]).

If $x \in \partial\omega$, we let $n_x = (\cos\alpha_1, \ldots, \cos\alpha_n)$ be the exterior unit normal of $\partial\omega$ at x. The conormal derivative at x is defined by $\partial/\partial\nu = a^{ij}(x)\cos\alpha_j\partial/\partial x_i$. The area measure of $\partial\omega$ is denoted dS.

THEOREM 3. *If y is a point in the $C^{(1,\alpha)}$ domain ω described above, then the L^*-harmonic measure σ_y^ω is absolutely continuous with respect to dS, and*

$$\frac{d\sigma_y^\omega}{dS} = -\frac{\partial G^\omega(x, y)}{\partial\nu_x} \tag{5.1}$$

for $x \in \partial\omega$. This density of σ_y^ω is α-Hölder continuous and positive on $\partial\omega$.

Proof. Since σ_y^ω is independent of $\Omega \supset \bar{\omega}$, we can assume that Ω is bounded and of class $C^{(2,\lambda)}$, and that the coefficients of L are defined in a slightly larger domain, so that Lemma 2 applies to Ω. Then we can write $P_y(x) = G(x, y)$. Put

$$u = \hat{R}_{P_y}^{\Omega\setminus\omega}, \tag{5.2}$$

which is an L-harmonic function in $\Omega\setminus\partial\omega$. Since all the points of $\partial\omega$ are regular, u equals P_y in $\Omega\setminus\omega$ and $H_{P_y}^\omega$ in ω, and u is continuous in $\bar{\Omega}$ and zero on $\partial\Omega$. Due to Theorem 3.1 in Widman [7], grad u is α-Hölder continuous in $\bar{\omega}$ and in $\bar{\Omega}\setminus\omega$, but its boundary values on $\partial\omega$ need not coincide. We write $\partial u'/\partial\nu$ and $\partial u''/\partial\nu$ for the conormal derivatives obtained from the values of grad u in ω and $\Omega\setminus\bar{\omega}$, resp. Further, we put $\Delta\partial u/\partial\nu = \partial u'/\partial\nu - \partial u''/\partial\nu$.

Now define

$$a_\varepsilon^{ij} = a^{ij} * w_\varepsilon$$

and analogously for b_ε^i and c_ε, for $\varepsilon > 0$, and write L_ε and $G_\varepsilon = G_\varepsilon^\Omega$ as before. By $\partial/\partial\nu_\varepsilon$ we shall mean the conormal derivative on $\partial\omega$ associated with L_ε, and $\Delta\partial u/\partial\nu_\varepsilon$ is defined analogously. We also need the auxiliary function

$$b_\varepsilon(x) = \sum_j \cos\alpha_j\left(b_\varepsilon^j(x) - \sum_i \frac{\partial a_\varepsilon^{ij}(x)}{\partial x_i}\right),$$

defined for $x \in \partial\omega$.

From Lemma 3.3 in Widman [7], it follows that each second derivative u_{ij} is integrable in Ω. Therefore, we can apply Green's and Stokes's formulas, and for $x \in \omega$ we obtain

$$0 = -\int_{\Omega\setminus\omega} G_\varepsilon(x, z)L_\varepsilon u(z)dz - \int_{\partial\omega} G_\varepsilon(x, z)\frac{\partial u''(z)}{\partial\nu_\varepsilon}dS_z + \\ + \int_{\partial\omega} \frac{\partial G_\varepsilon(x, z)}{\partial\nu_{\varepsilon,z}}u(z)dS_z - \int_{\partial\omega} b_\varepsilon(z)G_\varepsilon(x, z)u(z)dS_z$$

and

$$u(x) = -\int_{\omega} G_\varepsilon(x, z)L_\varepsilon u(z)dz + \int_{\partial\omega} G_\varepsilon(x, z)\frac{\partial u'(z)}{\partial\nu_\varepsilon}dS_z - \\ - \int_{\partial\omega} \frac{\partial G_\varepsilon(x, z)}{\partial\nu_{\varepsilon,z}}u(z)dS_z + \int_{\partial\omega} b_\varepsilon(z)G_\varepsilon(x, z)u(z)dS_z.$$

(Cf. Miranda [6, pp. 12—20]). Adding, we get

$$u(x) = -\int_{\Omega} G_\varepsilon(x, z)L_\varepsilon u(z)dz + \int_{\partial\omega} G_\varepsilon(x, z)\Delta\frac{\partial u(z)}{\partial\nu_\varepsilon}dS_z, \tag{5.3}$$

and in a similar way, this formula can be proved for $x \in \Omega\setminus\bar{\omega}$. Now let $\varepsilon \to 0$. From the construction of the Green's function in Boboc and Mustată [1], we conclude that $G_\varepsilon(x, z) = O(H(x, z))$ in $\Omega\times\Omega$, uniformly in ε. Hence, it follows from Lemma 2 that the first integral in (5.3) tends to $\int G(x, z)Lu(z)dz = 0$, and so

$$u(x) = \int_{\partial\omega} G(x, z)\Delta\frac{\partial u(z)}{\partial\nu}dS_z. \tag{5.4}$$

From (2.2) and (5.2) we see that

$$\Delta\frac{\partial u(z)}{\partial\nu} = -\frac{\partial G^\omega(z, y)}{\partial\nu_z} \geq 0$$

for $z \in \partial\omega$, so (2.4) and (5.4) imply (5.1). It follows from Theorem 3.1 in Widman [7] that $d\sigma_y^\omega/dS \in C^{(0,\alpha)}(\partial\omega)$.

To prove that $\partial G^\omega(x, y)/\partial\nu_x$ is negative on $\partial\omega$, we note that $G^\omega(x, y)$, considered as a function of x, is L-harmonic and positive in $\omega\setminus\{y\}$ and zero on $\partial\omega$. Let ω_1 be a domain obtained by taking away from ω a small ball centred at y. If the coefficient c is nonpositive, the result follows directly from Theorem 3,IV in Miranda [6], applied in ω_1. For arbitrary c, we take v as in the proof of Lemma 2 and apply the same theorem to G/v and the operator $u \to v^{-1}L(vu)$.

Theorem 3 is proved.

6. The Dirichlet problems for L and L*

For subdomains of Ω_0 the Dirichlet problems for L and L^* need not be uniquely solvable, but we have the following Fredholm type theorem.

THEOREM 4. *Let ω be a bounded $C^{(1,\lambda)}$ domain with $\bar{\omega} \subset \Omega_0$. Consider the problems*

$$Lu = f \quad in \quad \omega, \quad u = 0 \quad on \quad \partial\omega, \tag{6.1}$$

and

$$L^*v = g \quad in\ the\ sense\ of \quad \mathscr{D}'(\omega), \quad v = \varphi \quad on \quad \partial\omega, \tag{6.2}$$

where $f \in C_{\mathrm{loc}}^{(0,\lambda)}(\omega)$ and f is continuous in $\bar{\omega}$, $g \in L^p(\omega)$ for some $p > n/2$, and φ is continuous on $\partial\omega$. Then either (6.1) *and* (6.2) *are both uniquely solvable, or else the corresponding homogeneous problems have the same finite number of linearly independent solutions u_i and v_i, $i = 1, \ldots, m$. In the second case* (6.1) *is solvable if and only if*

$$\int_\omega f v_i dx = 0, \quad i = 1, \ldots, m,$$

and (6.2) *if and only if*

$$\int_\omega g u_i dx + \int_{\partial\omega} \varphi \frac{\partial u_i}{\partial \nu} dS = 0, \quad i = 1, \ldots, m.$$

Proof. If we let M be the operator $L - \gamma$ and choose the constant γ large enough, there exists a Green's function G of M in ω. Then u is a solution of (6.1) if and only if $Mu = f - \gamma u$ in ω and $u = 0$ on $\partial\omega$, which is equivalent to

$$u(x) = \gamma \int_\omega G(x, y) u(y) dy - \int_\omega G(x, y) f(y) dy. \tag{6.3}$$

Similarly, v solves (6.2) if and only if

$$v(y) = \gamma \int_\omega G(x, y) v(x) dx - \int_\omega G(x, y) g(x) dx - \int_{\partial\omega} \frac{\partial G(x, y)}{\partial \nu_x} \varphi(x) dS_x. \tag{6.4}$$

To this pair of integral equations, Fredholm's theory is applicable (see Miranda [6]). Hence, the equations are either both uniquely solvable, or else the corresponding homogeneous equations have the same finite number of linearly independent solutions u_i and v_i, $i = 1, \ldots, m$. These functions are then also solutions of the homogeneous problems (6.1) and (6.2). Moreover, in the second case (6.3) is solvable if and only if

$$\int_\omega v_i(x)dx \int_\omega G(x, y)f(y)dy = 0, \quad i = 1, \ldots, m,$$

which is equivalent to

$$\int_\omega f v_i dx = 0, \quad i = 1, \ldots, m.$$

For (6.4) the condition of solvability is

$$\begin{aligned} &\int_\omega u_i(y)dy \int_\omega G(x, y)g(x)dx + \\ &+ \int_\omega u_i(y)dy \int_{\partial\omega} \frac{\partial G(x, y)}{\partial \nu_x} \varphi(x)dS_x = 0, \quad i = 1, \ldots, m. \end{aligned} \tag{6.5}$$

Using the methods of Widman [7 or 8], one shows that $\partial G(x, y)/\partial \nu_x = O(|x - y|^{1-n})$ for $x \in \partial\omega$, $y \in \omega$. It follows that (6.5) is equivalent to

$$\int_\omega g u_i dx + \int_{\partial\omega} \varphi \frac{\partial u_i}{\partial \nu} dS = 0, \quad i = 1, \ldots, m,$$

and the proof of Theorem 4 is complete.

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