Spectral analysis in weighted L^1 spaces on $\mathbf{R}$

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Introduction

Weighted L^1 algebras on $\mathbf{R}$ were introduced by Beurling in [1]. A Beurling algebra L^1_p is defined as the convolution algebra of (equivalence classes of) functions f, Lebesgue measurable on $\mathbf{R}$ and satisfying

$$\|f\|_p = \int_{\mathbf{R}} |f(x)|p(x)dx < \infty$$

where p is the weight-function associated with the algebra in question. In order that L^1_p be an algebra, a condition of the type

$$p(x + y) \leq p(x)p(y)$$

has to be fulfilled by p. According to the size of p, Beurling talks of different cases. If, for simplicity, we assume p to be even, we consider the limit

$$\alpha = \lim_{|x|\to\infty} \frac{\log p(x)}{|x|}$$

which can be shown to exist. If $\alpha > 0$, we have the *analytic* case. When $\alpha = 0$, the *quasi-analytic* and *non-quasianalytic* cases are distinguished according as

$$\int_{\mathbf{R}} \frac{\log p(x)}{1 + x^2} dx$$

diverges or converges, respectively.

A central problem in the study of any Banach algebra is that of its ideal structure, in particular the problem of *spectral analysis*. We say that spectral analysis holds in an algebra B if every closed (proper) ideal in B is contained in a regular maximal ideal of B. The General Tauberian Theorem of Wiener [15] tells us that spectral analysis does hold in ordinary $L^1(\mathbf{R})$, and this result has been extended to Beurling's non-quasianalytic algebras, which are, among other things, regular algebras; see [3] and [12]. In the quasianalytic cases (including the analytic ones), however, it is not evident that spectral analysis should hold. Indeed, Nyman [14] proved that in particular analytic cases it does not, and he also demonstrated that the same thing occurs in several non-analytic (though quasianalytic) cases. Korenbljum [11], apparently unaware of Nyman's work, gave a precise description of the "evasive" ideals in the analytic cases considered by Nyman.

In the present paper, imposing a few reasonable conditions on the weight-functions, we prove that spectral analysis fails in quite general quasi-analytic and analytic algebras, and we exhibit chains of ideals not contained in any regular maximal ideal. In particular, our results contain the description of Korenbljum's ideals. Whereas Korenbljum shows that these are *all* the "strange" ideals that exist in the case considered by him, we have so far not proved the corresponding facts in general cases.

Geĭsberg and Konjuhovskiĭ have dealt with the quasi-analytic case in a series of papers [5], [10], [6], [7], independently of the present author, who has only very recently become aware of their work. Their results overlap with ours; in certain respects they go further than this paper, in others they do not reach as far. Their methods are largely different from ours.

Following Domar [4], we shall not only consider *algebras* L_p^1, but also L^1-spaces where translations are bounded operators. The problem of spectral analysis is quite relevant here, if *closed translation invariant subspace* (CTIS) is substituted for closed ideal, and a CTIS is called *regular maximal* if its codimension is 1. We establish the failure of spectral analysis here, too, but the description of »strange» CTIS is not quite complete.

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1. Assumptions

Throughout the paper, we let p be a real-valued, continuous function on $\mathbf{R}$ satisfying the following set of conditions, (1.1—4).

$$p(\varrho x) \geq p(x) = p(-x) \geq p(0) = 1 \quad \text{for every} \quad x \in \mathbf{R} \quad \text{and} \quad \varrho > 1. \tag{1.1}$$

$$\int_{\mathbf{R}} \frac{\log p(x)}{1 + x^2}\, dx = \infty. \tag{1.2}$$

$$p(x) = \exp\left(\frac{\pi}{2}\, |x| q(x)\right), \quad x \in \mathbf{R}, \quad \text{where} \quad 0 < q(x) \leq q(0) = M \quad \text{and} \quad q(\varrho x) \leq q(x) \quad \text{if} \quad \varrho > 1. \tag{1.3}$$

$$\sup_{y\in\mathbf{R}} \frac{p(x+y)}{p(y)} \text{ is locally bounded as a function of } x \in \mathbf{R}. \tag{1.4}$$

Define L_p^1 to be the Banach space of all complex-valued f, measurable on **R** and with finite norm

$$\|f\| = \|f\|_p = \|f\|_{L_p^1} = \int_{\mathbf{R}} |f(x)|p(x)dx.$$

(As usual, we identify functions that are equal almost everywhere.) The dual space $(L_p^1)^*$ of linear functionals on L_p^1 can be represented as the space $L_{1/p}^\infty$ of measurable functions F with finite norm

$$\|F\| = \|F\|_p^* = \operatorname*{ess\,sup}_{x\in\mathbf{R}} \frac{|F(x)|}{p(x)}$$

via the duality

$$F(f) = F * f(0) = \int_{\mathbf{R}} f(x)F(-x)dx.$$

The translation operators T_x, $x \in \mathbf{R}$, are defined by

$$(T_xf)(y) = f_x(y) = f(y-x), \quad y \in \mathbf{R}. \tag{1.5}$$

As a consequence of our assumptions, T_x is bounded on L_p^1. Indeed, by (1.4),

$$\|T_x\| = \sup_{y\in\mathbf{R}} \frac{p(x+y)}{p(y)} < \infty.$$

Since p is an even function, $\|T_x\| = \|T_{-x}\|$, and by the monotonicity of p it is also clear that $\|T_x\| \leq \|T_n\|$ if $0 \leq x \leq n$. If n is a natural number, it is easy to see that $\|T_n\| \leq \|T_1\|^n$. From these facts we get the estimate

$$\|T_x\| \leq Ce^{A|x|} \tag{1.6}$$

for some constants C and A.

From (1.3) we see that the limit $q_0 = \lim_{x\to\infty} q(x)$ exists and is ≥ 0. In Beurling's classification, the analytic case corresponds to $q_0 > 0$ and the non-analytic to $q_0 = 0$. We shall have to treat these cases separately in a few proofs later on, and for the former case we shall also demand some extra hypothesis on p. For the sake of reference we state here

ADDITIONAL ASSUMPTION I.

$$q_0 > 0, \text{ and } x(q(x) - q_0) \text{ is non-decreasing for } x > 0. \tag{1.7}$$

ADDITIONAL ASSUMPTION II.

$$q_0 = 0. \tag{1.8}$$

We also need, occasionally, that L_p^1 is a Banach algebra, which is ensured by

ADDITIONAL ASSUMPTION III.

$$p(x+y) \leq Cp(x)p(y). \tag{1.9}$$

Here, as well as in the rest of the paper, the letter C is used to denote a constant, not necessarily the same constant on different occasions.

2. Fourier transforms

The Fourier transform $\hat{f}$ of an $f \in L_p^1$ is defined by

$$\hat{f}(\zeta) = \int_{\mathbf{R}} e^{-ix\zeta} f(x)dx,$$

where $\zeta = \xi + i\eta$ is a complex number; $\hat{f}(\zeta)$ is well-defined if $|\eta| \leq \frac{\pi}{2} q_0$. If L_p^1 is a Banach algebra, i.e. when the Additional Assumption III is fulfilled, the Gelfand space of regular maximal ideals of L_p^1 can be identified with the set $S_p = \left\{\zeta\colon |\eta| \leq \frac{\pi}{2} q_0\right\}$ in the complex plane under the usual topology; see Loomis [12], No. 23 D.

As noted in the introduction, the natural substitutes for regular maximal ideals in the general case are regular maximal CTIS. (We use the writing regular maximal to indicate that the codimension is 1.) We shall find that these correspond to the points of S_p in the same way as the regular maximal ideals in the algebra case. Indeed, a regular maximal CTIS can be described as the annihilator of a translation invariant one-dimensional subspace of $(L_p^1)^*$, which we can describe as

$$\{\lambda F\colon \lambda \in \mathbf{C}\}$$

where F is a fixed nonzero member of $(L_p^1)^*$. By translation invariance, for every $t \in \mathbf{R}$ there is a number $\lambda(t)$ so that

$$F(x+t) = \lambda(t)F(x) \tag{2.1}$$

for almost all x. We can assume F normalized so that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon F(x)dx \text{ exists and equals } 1. \tag{2.2}$$

Then

$$\frac{1}{\varepsilon}\int_0^\varepsilon F(t+x)dx = \frac{1}{\varepsilon}\int_0^\varepsilon \lambda(t)F(x)dx = \lambda(t)\frac{1}{\varepsilon}\int_0^\varepsilon F(x)dx. \tag{2.3}$$

The far left member of this tends to $F(t)$ a.e. as $\varepsilon \to 0$, and thus we have

$$F(t) = \lambda(t) \quad \text{a.e.} \tag{2.4}$$

If ε and C are suitably chosen we have from (2.3)

$$\lambda(t) = C\int_0^\varepsilon F(t+x)dx$$

from which one sees that λ is continuous. We can then as well assume F to be continuous, by (2.4) and the fact that F is only "defined a.e." The relation (2.1) is strengthened to

$$F(x+t) = F(t)F(x),$$

holding everywhere by continuity. This implies the existence of a constant C and a number $\zeta = \xi + i\eta \in \mathbf{C}$ such that

$$F(x) = Ce^{i\zeta x}, \quad x \in \mathbf{R}.$$

C must be 1, by (2.2), and to make $F \in (L_p^1)^*$ we must also have $|\eta| \leq \frac{\pi}{2} q_0$, so that $\zeta \in S_p$. We have thus shown that a regular maximal CTIS in L_p^1 corresponds to a number $\zeta \in S_p$ such that

$$\int_{\mathbf{R}} e^{-i\zeta x} f(x)dx = 0$$

for all f in the CTIS.

This is precisely the same description as for the regular maximal ideals in case L_p^1 is a Banach algebra.

3. A chain of functionals with empty "Carleman Spectrum"

In this section we shall construct non-zero functions $G_\alpha \in (L_p^1)^*$ ($\alpha \in \mathbf{R}$), that will later be seen to annihilate certain CTIS in L_p^1, although these CTIS are not annihilated by any exponential $e^{i\zeta x}$, $\zeta \in S_p$. A very similar construction is found in [8], and similar ideas occur in [6] and [7].

LEMMA 3.1. *Let* $z = x + iy$, $y > 0$, *and define*

$$u(z) = u(x, y) = \frac{y}{\pi} \int_{\mathbf{R}} \left(\frac{1}{(t-x)^2 + y^2} - \frac{1}{t^2+1} \right) \log p(t) dt. \tag{3.1}$$

Then u *is harmonic in* $y > 0$ *and has boundary values equal to* $\log p(x)$ *on* $\mathbf{R}$.

Proof. For fixed $R > 0$, let $p_R(t) = p(t)$ if $|t| \leq R$ and $p_R(t) = 1$ otherwise. Define $u_R(z)$ by (3.1) using p_R instead of p; u_R is clearly harmonic in $y > 0$ and has boundary values $\log p(x)$ for $|x| < R$. Staying inside a (semi-)circle $x^2 + y^2 \leq r^2$ we estimate

$$|u(z) - u_R(z)| \leq 2y \int_{|t|>R} \frac{|1 + 2xt - x^2 - y^2|}{((t-x)^2 + y^2)(1 + t^2)} |t| q(t) dt \leq$$

$$\leq Cy \int_{|t|>R} \frac{1 + 2r|t| + r^2}{|t|((t-x)^2 + y^2)} dt \leq Cy \left(2r + \frac{r^2+1}{R} \right) \int_{|t|>R} \frac{dt}{(t-x)^2 + y^2} \to 0 \quad \text{as } R \to \infty,$$

uniformly for $x^2 + y^2 \leq r^2$ (C is independent of R). Thus u is harmonic in $y > 0$. To check the boundary values, fix $x \in \mathbf{R}$ and choose an arbitrary $\varepsilon > 0$. Take $R (> |x|)$ large enough to make $|u(x, y) - u_R(x, y)| < \varepsilon$ for, say, $y < 1$. Then

$$|u(x, y) - \log p(x)| \leq \varepsilon + |u_R(x, y) - \log p(x)|, \quad 0 < y < 1,$$

so that

$$\limsup_{y \to 0+} |u(x, y) - \log p(x)| \leq \varepsilon,$$

and, since ε is arbitrary, this completes the proof of the lemma.

Let v be the conjugate harmonic function of u in the upper half-plane, chosen such that $v(i) = 0$, and define

$$G(z) = G(x + iy) = G_0(z) = e^{u(z) + iv(z)}, \quad y > 0.$$

$$G_\alpha(z) = G(z) e^{i\alpha z}, \quad \alpha \in \mathbf{R}.$$

The boundary values $G_\alpha(x) = \lim_{y \to 0+} G_\alpha(x + iy)$ exist a.e., since G_α is locally bounded. They satisfy $|G_\alpha(x)| = p(x)$, and thus $G_\alpha \in (L_p^1)^*$. (Here and in the sequel we use somewhat abusively the same notation for an analytic function and its restriction to, or boundary values on, the line.)

By inspection it is seen that

$$u(z) + iv(z) = \frac{z}{2i} \int_{\mathbf{R}} \frac{1 + zt}{t(t-z)(t^2+1)} |t| q(t) dt + iC, \quad y > 0,$$

and taking the imaginary part of this for $z = iy$ we have

$$v(iy) = \frac{y^2}{2} \int_{\mathbf{R}} \frac{|t|q(t)}{t(t^2 + y^2)} \, dt + C = C$$

the integrand being an odd function of t. To make $v(i) = 0$, we must take $C = 0$. Thus

$$G_\alpha(iy) = e^{u(iy)-\alpha y}$$

is real-valued and positive. We introduce

$$H_0(y) = -\frac{1}{y} u(iy), \quad y > 0. \tag{3.2}$$

An explicit formula is

$$H_0(y) = (y^2 - 1) \int_0^\infty \frac{tq(t)}{(t^2 + y^2)(t^2 + 1)} \, dt. \tag{3.3}$$

This can be differentiated to yield

$$H_0'(y) = 2y \int_0^\infty \frac{tq(t)}{(t^2 + y^2)^2} \, dt > 0, \quad y > 0.$$

Thus H_0 increases monotonically, and we can define its inverse function $L_0 = H_0^{-1}$. The domain of L_0 will include $[0, \infty[$; indeed, $H_0(y) \to +\infty$ as $y \to +\infty$ as a consequence of (1.2).

We need a few elementary properties of the functions introduced in this section before we can proceed.

LEMMA 3.2.
(a) $|u(z + h) - u(z)| \leq \log \|T_h\|, \; y > 0, \; h \in \mathbf{R}$.
(b) $|G_\alpha(z + t)| \leq \|T_t\| |G_\alpha(z)|, \; t \in \mathbf{R}$.
Here, T_h is the translation operator defined in (1.5).

Proof. (a) If $h > 0$, we can write

$$u(z + h) - u(z) = \frac{y}{\pi} \lim_{R\to\infty} \int_{-R}^{R} \left\{ \frac{1}{(x + h - t)^2 + y^2} - \frac{1}{(x - t)^2 + y^2} \right\} \log p(t) dt =$$

$$= \frac{y}{\pi} \int_{\mathbf{R}} \frac{\log \dfrac{p(u + h)}{p(u)}}{(u - x)^2 + y^2} \, du + \frac{y}{\pi} \lim_{R\to\infty} \left(\int_{R}^{R+h} \frac{\log p(u + h)}{(u - x)^2 + y^2} - \int_{-R}^{-R+h} \frac{\log p(u)}{(u - x)^2 + y^2} \right) \leq$$

$$\leq \log \|T_h\| + 0.$$

If $h < 0$, the same proof applies except for the fact that the residual terms that tend to zero have a slightly different appearance.

(b) We obtain directly

$$|G_\alpha(z + t)| = e^{u(z+t)-\alpha y} \leq e^{u(z)+\log\|T_t\|-\alpha y}$$

by the preceding estimate. The lemma is proved.

Now consider the Fourier-Carleman integrals

$$\varphi_1(\zeta) = \int_0^\infty e^{-i\zeta x} G(x) dx \tag{3.4}$$

$$\varphi_2(\zeta) = -\int_{-\infty}^0 e^{-i\zeta x} G(x) dx. \tag{3.5}$$

They represent functions analytic in $\eta < -\frac{\pi}{2} q_0$ and $\eta > \frac{\pi}{2} q_0$, respectively ($\zeta = \xi + i\eta$). For any fixed ζ in the respective regions, the function $z \mapsto G(z)e^{-i\zeta z}$ is analytic in $y = \operatorname{Im} z > 0$, and by Lemma 3.2(b),

$$|G(z)e^{-i\zeta z}| \leq G(iy)\|T_x\|e^{\eta x + \xi y}.$$

Since $H_0(y) \to \infty$ as $y \to \infty$, $G(iy)$ decreases faster than exponentially, and we can change the paths of integration in (3.4) and (3.5) to obtain, in both cases,

$$\varphi(\zeta) = i \int_0^\infty e^{\zeta y} G(iy) dy. \tag{3.6}$$

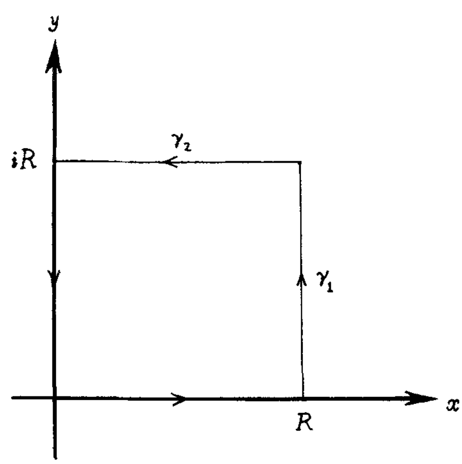

Fig. 1.

In fact, we can integrate $e^{-i\zeta z}G(z)$ around, say, the contour Γ_R in Figure 1. The integrals over parts of axes tend to (3.4) and (3.6), and the other parts tend to zero; using (1.6),

$$\left|\int_{\gamma_1}\right| \leq \int_0^R G(iy)\|T_R\|e^{R\eta+\xi y}dy \leq Ce^{R(\eta+A)}\int_0^\infty e^{y(\xi-H_0(y))}dy \to 0, \quad \text{if} \quad \eta < -A,$$

$$\left|\int_{\gamma_2}\right| \leq e^{-RH_0(R)+\xi R}\int_0^R Ce^{(\eta+A)x}dx \to 0, \qquad \text{if} \quad \eta < -A.$$

(Details concerning $G(x)$ as *boundary* values of G are left to the reader.)

The function φ, defined by (3.6), is an entire function, furnishing the analytic continuation of the Fourier-Carleman transform. Since it is entire, the spectrum of G, in the sense of Beurling in [2], is empty.

We note that for real $\zeta = \xi$, $\varphi(\xi)$ is purely imaginary, and $|\varphi(\xi)| = -i\varphi(\xi)$.

If the preceding reasoning is carried through with G_α instead of G, we see that the Carleman transform of G_α is the entire function $\zeta \mapsto \varphi(\zeta - \alpha)$.

4. Statement of results

We begin by defining a number Q by

$$Q = q_0 - q_0 \log q_0, \tag{4.1}$$

which is interpreted as 0 if $q_0 = 0$. For an $f \in L^1_p$ we define the following number, that actually measures "how small" $f(\xi)$ is at $+\infty$:

$$\gamma_+(f) = \limsup_{\xi\to+\infty} (\xi - H_0(-\log|\hat{f}(\xi)|)) - Q. \tag{4.2}$$

THEOREM 1. *For every real number* α *there exists* $f \in L^1_p$ *with* $\gamma_+(f) = \alpha$.

Now consider the following properties of an $f \in L^1_p$.

(A) $\gamma_+(f) \leq \alpha$.

(B) For every $\varepsilon > 0$, $\displaystyle\int_0^\infty |\hat{f}(\xi)||\varphi(\xi - \alpha - \varepsilon)|d\xi < \infty$.

(C) For every $\varepsilon > 0$, $G_\alpha(iy) \cdot \displaystyle\int_{\mathbf{R}} |\hat{f}(\xi)|e^{y\xi}d\xi = O(e^{\varepsilon y})$ as $y \to +\infty$.

(D) f has an analytic extension to the lower half-plane, and the function $z \mapsto f(-z)G_\alpha(z)$ is in H^1, where H^1 denotes the Hardy space of the upper half-plane.

(E) For every $\beta \geq \alpha$, $f * G_\beta(x) = \int_{\mathbf{R}} f(x-t)G_\beta(t)dt = 0.$

(F) $$f * G_\alpha(x) = \int_{\mathbf{R}} f(x-t)G_\alpha(t)dt = 0.$$

THEOREM II. *For every real number α, the following implications hold:*

$$A \Rightarrow B \Leftrightarrow C \Rightarrow D \Leftrightarrow E \Rightarrow F.$$

THEOREM III. *If f satisfies any one of the conditions A through F for all real α, then $f = 0$. In particular, if $\gamma_+(f) = -\infty$, then $f = 0$.*

To round off, we can "close the circuit" as follows.

THEOREM IV A. *Under Additional Assumptions I and III, $F \Rightarrow A$, so that, in this case, the statements A, B, C, D, E, and F are all equivalent.*

THEOREM IV NA. *Under Additional Assumptions II and III, $F \Rightarrow B$, so that, in this case, the statements B, C, D, E, and F are all equivalent.*

The algebraic implications of Theorems I and II are the following. The space L_p^1 contains a continuous chain of closed translation invariant subspaces I_α, $\alpha \in \mathbf{R}$ (ideals, under Additional Assumption III), which are not contained in any regular maximal CTIS, where the word "regular" is used to indicate that the codimension is 1. Thus spectral analysis fails in L_p^1. Here I_α is defined as the set of all $f \in L_p^1$ satisfying, e.g. (E). Under Additional assumptions such that the conclusions of Theorem IV A (A for Analytic) or IV NA (NA for Non-Analytic) hold, the picture is clearer than otherwise: we have then several alternative descriptions of the I_α.

In fact, I_α is nonvoid by Theorem I; it is closed, since it is defined by (E) as an annihilator; its translation invariance is obvious from (E). The fact that it is not contained in any regular maximal CTIS can be demonstrated in the following manner. It is easily seen from the representation (E) that I_α along with any f also contains the functions $x \mapsto e^{i\beta x}f(x)$ with $\beta \leq 0$ which have for Fourier transforms the left translates of $\hat{f}$. Since the L_p^1 classes considered by us have quasi-analytic transform classes (see e.g. [13]), $\hat{f}$ can only have isolated zeros (of finite multiplicity), provided f is not equivalent to zero. It follows that no point $\zeta \in S_p$ can be a common zero to the transforms of *all* $f \in I_\alpha$. (In the analytic case, the zeros on the boundary of S_p may not be isolated. However, it is well known that they form a set of linear measure zero. For our purposes it is sufficient that they cannot fill an interval; if they did we could continue $\hat{f}(\zeta)$ across that interval by the Schwarz principle of reflection, and we would find $\hat{f}(\zeta) \equiv 0$. The argument above using translates will hold equally well in this context.) It is also clear that the I_α form a chain of nested sets; indeed

$$I_\alpha \underset{\neq}{\subset} I_\beta \quad \text{if} \quad \alpha < \beta,$$

and also, by Theorem III,

$$I_{-\infty} = \bigcap_{\alpha \in \mathbf{R}} I_\alpha = \{0\}.$$

An interesting feature about I_α is the connexion with H^1. The annihilator of H^1, considered as an ideal in $L^1(\mathbf{R})$, is the set $\overline{H^\infty}$ of bounded functions on $\mathbf{R}$ with spectrum in $-\infty < \xi \leq 0$. Such functions have an analytic extension to the lower half-plane or, equivalently, a conjugate-analytic extension to the upper half-plane. The description of I_α given in (D) can be interpreted to say that the set $G_\alpha \cdot \overline{H^\infty} = \{G_\alpha F: F \in \overline{H^\infty}\}$ is the annihilator of I_α. The inclusion $I_\alpha \subset I_\beta$ for $\alpha < \beta$ corresponds to the dual inclusion $G_\alpha \overline{H^\infty} \supset G_\beta \overline{H^\infty}$, which also follows from the fact that $e^{i\delta x} \in \overline{H^\infty}$ if and only if $\delta < 0$.

Instead of considering the behaviour of $\hat{f}(\xi)$ as $\xi \to +\infty$, we can equally well define

$$\gamma_-(f) = \limsup_{\xi \to -\infty} (|\xi| - H_0(-\log |\hat{f}(\xi)|)) - Q$$

and obtain a corresponding chain of CTIS $\{I_\alpha^-\}$, "localized at $-\infty$". Thus we have a doubly indexed chain

$$\{I_{\alpha,\beta} = I_\alpha \cap I_\beta^- : -\infty < \alpha, \beta \leq +\infty\},$$

$I_{+\infty}$ and $I_{+\infty}^-$ denoting L_p^1.

In the particular case when $p(x) = \exp\left(\frac{\pi}{2} a|x|\right)$, $a > 0$, the ideals identified by us coincide with those of Korenbljum in [11]. He defines for $f \in L_p^1$ the number

$$\delta_+(f) = \limsup_{\xi \to +\infty} e^{-\xi/a} \log |\hat{f}(\xi)|,$$

and considers the ideals J_β of all f with $\delta_+(f) \leq \beta$, where $-\infty < \beta \leq 0$. Our $\gamma_+(f)$ is equal to

$$-a(\log |\delta_+(f)| - \log a + 1) = -a \log |\delta_+(f)| - Q. \tag{4.3}$$

In fact, for this case we get $G(z) = \exp(iaz \cdot \log(-iz))$, where the log is the principal branch (imaginary part in $(-\pi, \pi)$). Thus $G(iy) = \exp(-ay \log y)$ and $H_0(y) = a \log y$. The expression in the definition (4.2) turns out to be

$$\xi - a \log(-\log |\hat{f}(\xi)|) - Q = -a \log(-e^{-\xi/a} \log |\hat{f}(\xi)|) - Q,$$

from which (4.3) follows. Korenbljum does not use our G_α for annihilators. Instead he introduces the function

$$g_\mu(x) = \frac{e^{-i\mu x}}{\Gamma(1 + iax)}$$

which he shows will annihilate f and its translates if (our) $\gamma_+(f) < \infty$ and $\mu \geq \gamma_+(f) + Q$.

Nyman's thesis [14] also essentially contains these results, although they are not all explicitly stated. For these algebras, Nyman and Korenbljum also prove the "generalized Tauberian theorem" that any f with $\hat{f}(\zeta) \neq 0$ for $\zeta \in S_p$ and $\gamma_\pm(f) = +\infty$ generates the whole algebra L^1_p. Korenbljum takes a further step to show that any proper closed ideal in L^1_p which is not included in a regular maximal ideal must indeed be of the type $I_\alpha \cap I^-_\beta$. The corresponding questions for general L^1_p, where $p(x)$ may not be $\exp\left(\frac{\pi}{2}\,a|x|\right)$, $a > 0$, have not been attacked in the present work, but the author hopes to have the opportunity to return to them.

Some further comment should be afforded the non-analytic case $q_0 = 0$. The first treatment of this situation seems to be by Nyman [14]. His reasoning starts from a function, which is essentially our φ, and a contour Γ on which $\varphi(\zeta)$ is bounded and integrable; the contour has a general appearance as in Figure 2, which has given rise in closer circles to the name "Nyman's bottle" for the whole idea. The functional corresponding to our G is defined as

$$G(x) = \int_\Gamma e^{i\zeta x}\varphi(\zeta)\,d\zeta,$$

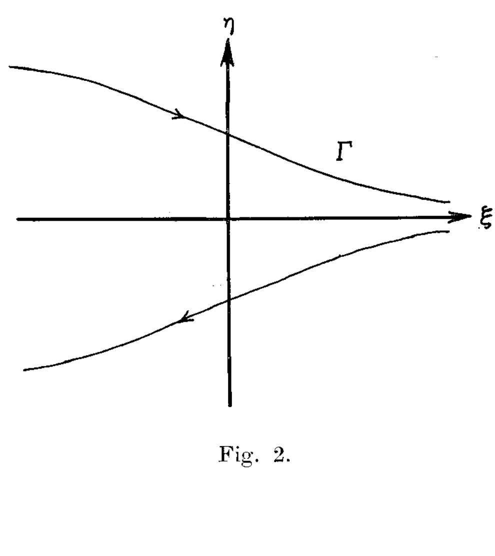

Fig. 2.

and the weight-function p is introduced now, to make $G \in (L_p^1)^*$. One explicit case is demonstrated, where $\log p(x)$ turns out to be asymptotic to $\frac{\pi}{2} \frac{x}{\log |x|}$. Nyman shows that there exists a non-zero $f \in L_p^1$ with $\hat{f}(\xi) = \varphi(\xi)^{-a}$, $a > 1$, or at least something similar to this, such that

$$\int_{\mathbf{R}} f(x + y)G(x)dx \equiv 0.$$

The ideas of the proof lean heavily on contour integration and are essentially inspired from the treatment of the case when $\log p(x) = \frac{\pi}{2} a|x|$, $a > 0$.

As mentioned in the introduction, Geĭsberg and Konjuhovskiĭ have also treated the case $q_0 = 0$. They consider algebras, i.e., they assume (1.9), and require explicitly certain additional smoothness and monotonicity conditions on p. Under such assumptions they can actually prove what essentially amounts to the equivalence of conditions A and D. Their technique involves the function $g(x) = 1/G(x)$, which is in $L^1(\mathbf{R})$ and has a Fourier transform $\hat{g}$. Their version of condition D is that $\hat{f}$ be representable as the convolution of $\hat{g}$ and some L^1-transform that vanishes on a certain half-line.

Theorem IV is inspired by Hirschman's paper [8]. Actually it states a sharper version of his Theorem 2, which is formulated so as to deal with the relations between the size of a function and its Fourier transform at infinity. It says that if for some $\varepsilon > 0$, f satisfies

$$\xi - H_0(-\log |\hat{f}(\xi)|) \leq -\varepsilon\xi + C, \quad \xi > \xi_0,$$

then $f = 0$. Our theorem requires only that the left member tend to $-\infty$, no matter how slowly. (Hirschman has, admittedly, fewer "technical" assumptions on p than we do, and his assumption is that $f \cdot p \in L^2$, not L^1.) In the same connexion, our Theorem I is nothing but Hirschman's Theorem 3, where we merely take somewhat more out of the proof. It could be mentioned that many of the ideas employed in the present investigations, notably the use of the functions H_0 and L_0, more or less come from reading [8].

5. Technicalities

In the proofs, we use a number of properties of our constructs H_0 and φ. For convenience in reading, we collect them here; the reader is advised not to indulge in this section at a first reading, but consult it when it is referred to.

The function H_0 was defined in (3.2) and (3.3). We introduce the computationally simpler function

$$H(y) = \int_0^y \frac{tq(t)}{1+t^2}\,dt \tag{5.1}$$

and its inverse $L = H^{-1}$ (cf. [7]). The connexion between H_0 and H is expressed in the first Lemma.

LEMMA 5.1. *For* $y \geq 1$, $H_0(y) \leq H(y)$, *and*

$$H_0(y) - H(y) \to 0 \quad as \quad y \to +\infty.$$

Proof. Writing $H_0 - H$ explicitly, we have

$$H_0(y) - H(y) = -\int_0^y \frac{tq(t)}{t^2+y^2}\,dt + (y^2-1)\int_y^\infty \frac{tq(t)}{(t^2+y^2)(t^2+1)}\,dt = A(y) + B(y).$$

Here,

$$A(y) = -\int_0^1 \frac{sq(ys)}{s^2+1}\,ds \to -\frac{1}{2}q_0 \log 2 \quad \text{as} \quad y \to +\infty,$$

and for finite y, $A(y) \leq -\frac{1}{2}q(y)\log 2$. Furthermore, if $y \geq 1$,

$$B(y) = (y^2-1)\int_1^\infty \frac{sq(ys)}{(s^2+1)(s^2y^2+1)}\,ds.$$

For $1 \leq s < \infty$, $q(ys)$ lies between $q(y)$ and q_0, and thus

$$B(y) \leq q(y)(y^2-1)\int_1^\infty \frac{sds}{(s^2+1)(s^2y^2+1)} \leq \frac{1}{2}q(y)\log 2,$$

and similarly,

$$B(y) \geq \frac{1}{2}q_0 \log \frac{2y^2}{y^2+1}.$$

As $y \to +\infty$, $B(y) \to \frac{1}{2}q_0 \log 2$, being $\leq \frac{1}{2}q(y)\log 2$ all the way. This proves the lemma.

Some facts about H and L are listed in the following lemma.

LEMMA 5.2.

(a) $H(y) \leq \frac{M}{2}\log(y^2+1)$, *where* $M = q(0)$.

(b) *There exist positive constants* C, C_0 *such that*

$$L(\xi) > C_0 e^{C\xi}, \quad \xi > 1.$$

(c) *If* $a > 0$, *then*

$$H(ay) - H(y) = q(\theta) \log a + O(y^{-2}), \quad y \to +\infty,$$

where θ *lies between* y *and* ay. *Also,*

$$H\left(\frac{y}{q(y)}\right) - H(y) = q(\theta) \log q(y) + O(y^{-2}), \quad y \to +\infty,$$

with θ *between* $y/q(y)$ *and* y.

(d) *For every* $t > -H(y)$,

$$L(H(y) + t) \leq (y^2 + 1)^{1/2} e^{t/q(\theta)},$$

where θ *is between* y *and* $L(H(y) + t)$.

(e) *If* $\alpha > 0$,

$$\int_0^\infty e^{L(\xi) - L(\xi + \alpha)} d\xi < \infty.$$

(f) *For every* $\varepsilon > 0$, *there exists a number* $Y = Y(\varepsilon)$ *such that for all* $y \geq Y$ *and all* $\lambda \geq 0$,

$$yH(\lambda) - \lambda \leq yH(y) - Qy + \varepsilon y,$$

where Q *is defined by* (4.1).

Proofs. (a) is immediate from the definition (5.1), and (b) follows directly from (a).

(c) If $a < 1$, the Mean Value Theorem yields

$$H(y) - H(ay) = \int_{ay}^{y} \frac{tq(t)}{1 + t^2} dt = q(\theta) \int_{ay}^{y} \frac{tdt}{1 + t^2}$$

where the integral is

$$\frac{1}{2} \log \frac{y^2 + 1}{a^2 y^2 + 1} = \log \frac{1}{a} + O(y^{-2}).$$

When $a > 1$, as well as when a is not constant but equal to $1/q(y)$, the proof is practically identical.

(d) First assume $t \geq 0$. By the definition of L as the inverse of H, we have

$$t = \int_{y}^{L(H(y)+t)} \frac{xq(x)}{1 + x^2} dx = q(\theta) \cdot \frac{1}{2} \log \frac{1 + L^2}{1 + y^2},$$

where L stands for $L(H(y)+t)$ and $y < \theta < L$. Solving for L we obtain

$$L = ((y^2+1)e^{2t/q(\theta)} - 1)^{1/2},$$

from which the expected formula follows. An analogous proof holds if $t < 0$.

(e) By the Mean Value Theorem of Lagrange, $L(\xi + \alpha) - L(\xi) = \alpha L'(\xi + \theta\alpha)$, $0 < \theta < 1$, and since

$$L'(y) = \frac{1}{H'(L(y))} = \frac{1 + L(y)^2}{L(y)q(L(y))} \geq \frac{L(y)}{M},$$

this is greater than $\dfrac{\alpha}{M} L(\xi + \theta\alpha) \geq \dfrac{\alpha}{M} L(\xi)$; the statement follows, using (b).

(f) We claim that $H(\lambda) - H(y) + Q - \dfrac{\lambda}{y} \leq \varepsilon$ for all $\lambda \geq 0$ if y is large enough. Differentiating the left member of this with respect to λ we find that it has a maximum (for fixed y) when λ and y are connected by the relation

$$y = \frac{1+\lambda^2}{\lambda q(\lambda)}.$$

The maximum value is found to be

$$Q + q(\theta)\log q(\lambda) - q(\lambda) + O(\lambda^{-2}),$$

where θ is between λ and y; we have used (c). As λ and y tend to $+\infty$, appropriately connected, this expression tends to zero, and so it is surely $< \varepsilon$ if y is large enough, ε being any prescribed positive number. The lemma is proved.

Now we connect the function φ of (3.6) with the function L.

LEMMA 5.3. *If* $\varepsilon > 0$, *there exists a number* $\xi_0 = \xi_0(\varepsilon)$ *such that*

$$e^{L(\xi - Q - \varepsilon)} \leq |\varphi(\xi)| \leq e^{L(\xi - Q + \varepsilon)} \quad \textit{for} \quad \xi > \xi_0. \tag{5.2}$$

Proof. Observing that

$$|\varphi(\xi)| = \int_0^\infty e^{y\xi - yH_0(y)}dy,$$

we begin with the right-hand inequality. Splitting the integration at

$$y = y_0 = L(\xi - Q + \varepsilon)/\xi,$$

we have safely

$$\int_0^{y_0} \leq \sup_{y\geq 0} e^{-yH_0(y)} \cdot \int_0^{y_0} e^{y\xi}dy \leq \frac{1}{2}\, e^{L(\xi - Q + \varepsilon)},$$

if ξ is large enough. In the other integral we put $\lambda = L(\xi - Q + \varepsilon)$ so that $\xi = H(\lambda) + Q - \varepsilon$. We are required to prove that

$$\int_{\lambda/(H(\lambda)+Q-\varepsilon)}^{\infty} e^{y(H(\lambda)-H_0(y)+Q-\varepsilon)} dy \leq \frac{1}{2} e^{\lambda},$$

if λ is large. The left member is majorized by

$$\sup \exp\left(y\left(H(\lambda) - H_0(y) + Q - \frac{\varepsilon}{2}\right)\right) \cdot \int e^{-\varepsilon y/2} dy.$$

The integral tends to zero as $\lambda \to \infty$, so we are left with proving that the supremum indicated is $\leq Ce^{\lambda}$. In fact, when $y > \lambda/(H(\lambda) + Q + \varepsilon)$,

$$y\left(H(\lambda) - H_0(y) + Q - \frac{\varepsilon}{2}\right) \leq y\left(H(\lambda) - H(y) + Q - \frac{\varepsilon}{4}\right) = A,$$

if λ is large enough, by Lemma 5.1. Differentiating A with respect to y we get

$$\frac{\partial A}{\partial y} = H(\lambda) - H(y) - q_0 \log q_0 - \frac{\varepsilon}{4} + q_0 - \frac{y^2}{1 + y^2} q(y). \tag{5.3}$$

When $\partial A/\partial y = 0$, for $y = y_1$, A can be written

$$y_1 \cdot \frac{y_1^2 q(y_1)}{1 + y_1^2} \leq y_1 q(y_1).$$

Since the last two terms in (5.3), grouped together, tend to zero as $y \to +\infty$, and since y_1 is large when λ is large, we see that $\partial A/\partial y$ is negative for all y such that $H(y) > H(\lambda) - q_0 \log q_0 - \frac{\varepsilon}{8}$. Thus the maximum of A is assumed when y satisfies the opposite inequality, i.e.,

$$y_1 \leq L\left(H(\lambda) - q_0 \log q_0 - \frac{\varepsilon}{8}\right). \tag{5.4}$$

If $q_0 = 0$ we have directly $y_1 \leq \lambda$ and

$$A \leq \lambda q(y_1) < \lambda$$

as soon as λ is large enough to force y_1 past the point where $q(y_1) = 1$. Otherwise, apply Lemma 5.2 (d) to (5.4). With the appropriate θ we get

$$y_1 \leq (\lambda^2 + 1)^{1/2} \exp\left\{\frac{-q_0 \log q_0 - \varepsilon/8}{q(\theta)}\right\} \leq$$

$$\leq (\lambda^2 + 1)^{1/2} e^{-\delta} q_0^{-q_0/q(\theta)}, \ \delta = \frac{\varepsilon}{8q(0)} > 0,$$

and

$$A = (\lambda^2 + 1)^{1/2} e^{-\delta} q(y_1) q_0^{-q_0/q(\theta)} \leq \sqrt{\lambda^2 + 1} \leq \lambda + 1$$

for λ large enough, since both y_1 and θ tend to infinity with λ. This proves half of the Lemma.

The left-hand inequality in (5.2) remains to prove. Since $H_0(y) \leq H(y)$, we have to show that

$$\int_0^\infty e^{-yH(y)+y(H(\lambda)+Q+\varepsilon)} dy \geq e^\lambda$$

for sufficiently large λ, where $\lambda = L(\xi - Q - \varepsilon)$. First consider the case $q_0 > 0$. Then

$$\int_0^\infty \geq \int_0^{\lambda/q_0} \geq \exp\left(-\frac{\lambda}{q_0} H(\lambda/q_0)\right) \int_0^{\lambda/q_0} e^{y(H(\lambda)+Q+\varepsilon)} dy \geq$$

$$\geq C \exp\left\{\frac{\lambda}{q_0}\left(H(\lambda) - H\left(\frac{\lambda}{q_0}\right) - q_0 \log q_0\right) + \lambda + \frac{\varepsilon\lambda}{2q_0}\right\}$$

where C is of the general size $e^{\varepsilon\lambda/2q_0}/H(\lambda) \geq 1$ if λ is large. By Lemma 5.2(c),

$$H(\lambda) - H(\lambda/q_0) = q(\theta) \log q_0 + O(\lambda^{-2}).$$

If $q_0 \geq 1$, we have here $\theta > \lambda/q_0$. Introducing this and observing that $\log q_0 \geq 0$, we have the desired estimate. If $0 < q_0 < 1$, we have instead $\theta > \lambda$, and if λ is large

$$H(\lambda) - H(\lambda/q_0) \geq q_0 \log q_0 - \frac{\varepsilon}{2}$$

which takes care of that case. If $q_0 = 0$ we have defined $Q = 0$ and are required to prove

$$\int_0^\infty e^{-yH(y)+y(H(\lambda)+\varepsilon)} dy \geq e^\lambda.$$

We get

$$\int_0^\infty \geq \int_0^{\lambda/q(\lambda)} \geq \frac{C}{H(\lambda)} \exp\left\{\frac{\lambda}{q(\lambda)}\left(H(\lambda) - H\left(\frac{\lambda}{q(\lambda)}\right) + \varepsilon\right)\right\}.$$

Here, by Lemma 5.2(c)

$$H\left(\frac{\lambda}{q(\lambda)}\right) - H(\lambda) \leq q(\lambda) \log \frac{1}{q(\lambda)} + O(\lambda^{-2}),$$

and so

$$\int_0^\infty \geq \frac{C}{H(\lambda)} \exp\left\{\lambda\left(\log q(\lambda) + \frac{\varepsilon}{q(\lambda)}\right)\right\}.$$

Since $q(\lambda) \to 0$ as $\lambda \to \infty$, the "coefficient" $\log q(\lambda) + \varepsilon/q(\lambda)$ is ultimately > 1 so that it both takes care of $H(\lambda)$ in the denominator and gives the desired estimate. The proof is complete.

Concerning the growth of $|\varphi(\xi)|$ we give another lemma.

LEMMA 5.4. *Assume that the Additional Assumption II is fulfilled, i.e. that* $q_0 = 0$. *Then for every* $\varepsilon > 0$,

$$\int_0^\infty \frac{|\varphi(\xi - \varepsilon)|^2}{|\varphi(\xi)|} d\xi < \infty.$$

Proof. Let $\varepsilon = 4\eta$, and note that $Q = 0$. Using the preceding lemma we find ξ_0 such that $L(\xi - \eta) \leq \log |\varphi(\xi)| \leq L(\xi + \eta)$ for $\xi > \xi_0$. Then the integrand is majorized by $\exp(2L(\xi - 3\eta) - L(\xi - \eta))$. In the proof of Lemma 5.2(e) we saw that

$$L(\xi - \eta) - L(\xi - 3\eta) = 2\eta \frac{1 + L_\theta^2}{L_\theta q(L_\theta)},$$

where $L_\theta = L(\xi - (1 + 2\theta)\eta)$, $0 < \theta < 1$. Thus we get

$$A = 2L(\xi - 3\eta) - L(\xi - \eta) = \frac{L(\xi - 3\eta)q(L_\theta) - 2\eta L_\theta}{q(L_\theta)} - \frac{2\eta}{L_\theta \cdot q(L_\theta)}.$$

Since we suppose that $q(x) \to 0$, we can take ξ_0 so large that $q(L_\theta) < \eta$ for all $\xi > \xi_0$; since L is an increasing function we get

$$A \leq \frac{L_\theta \eta - 2\eta L_\theta}{q(L_\theta)} - \frac{2\eta}{L_\theta q(L_\theta)} = -\frac{\eta}{q(L_\theta)}\left(L_\theta + \frac{2}{L_\theta}\right) \leq -L_\theta$$

Thus

$$\frac{|\varphi(\xi - \varepsilon)|^2}{|\varphi(\xi)|} \leq e^{-L(\xi - 3\eta)},$$

which proves the lemma.

6. Proofs of Theorems II and III

In the chain $A \Rightarrow B \Leftrightarrow C \Rightarrow D \Leftrightarrow E \Rightarrow F$ we proceed from left to right. In the proofs we can assume $\alpha = 0$. The general case can be reduced to this upon replacing $f(x)$ by $f(x)e^{-i\alpha x}$, which corresponds to replacing $\hat{f}(\xi)$ by $\hat{f}(\xi+\alpha)$. The details are left to the reader.

$A \Rightarrow B$. Using Lemma 5.1 we see that we could have defined $\gamma_+(f)$ equivalently by the formula

$$\gamma_+(f) = \limsup_{\xi\to+\infty} (\xi - H(-\log|\hat{f}(\xi)|)) - Q. \tag{6.1}$$

The assumption $\gamma_+(f) \leq 0$ is then easily rewritten in the following form: For every $\varepsilon > 0$, there exists $\xi_0(\varepsilon)$ such that

$$|\hat{f}(\xi)| \leq e^{-L(\xi-Q-\varepsilon)} \quad \text{if} \quad \xi > \xi_0(\varepsilon). \tag{6.2}$$

From Lemma 5.3 follows that if we take $\xi_0(\varepsilon)$ large enough we will also have $|\varphi(\xi-3\varepsilon)| \leq e^{L(\xi-Q-2\varepsilon)}$, so that

$$\int_{\xi_0(\varepsilon)}^{\infty} |\hat{f}(\xi)||\varphi(\xi-3\varepsilon)|d\xi \leq \int_{\xi_0(\varepsilon)}^{\infty} e^{L(\xi-Q-2\varepsilon)-L(\xi-Q-\varepsilon)}d\xi < \infty,$$

by Lemma 5.2(e). The integral over $(0, \xi_0(\varepsilon))$ is of course finite. Since ε is arbitrary, the proof of $A \Rightarrow B$ is complete.

$B \Rightarrow C$. We assume that for every $\varepsilon > 0$,

$$\int_0^{\infty} |\hat{f}(\xi)||\varphi(\xi-\varepsilon)|d\xi < \infty,$$

and we consider

$$J(y) = G(iy)\int_{\mathbf{R}} |\hat{f}(\xi)|e^{y\xi}d\xi.$$

The part of this that corresponds to the integral over $(-\infty, 0)$ clearly tends to zero, and the other half is

$$\leq e^{-yH_0(y)}\int_0^{\infty} |\hat{f}(\xi)||\varphi(\xi-\varepsilon)|d\xi \cdot \sup_{\xi>0} \frac{e^{y\xi}}{|\varphi(\xi-\varepsilon)|} \leq$$

$$\leq Ce^{\varepsilon y-yH(y)} \sup_{\xi} e^{y\xi-L(\xi-Q-2\varepsilon)}, \quad C = C(\varepsilon).$$

Here we have used Lemmas 5.1 and 5.3. Let $\lambda = L(\xi-Q-2\varepsilon)$ to get

$$J(y) \leq Ce^{3\varepsilon y}e^{y(Q-H(y))} \cdot \sup_{\lambda>0} e^{yH(\lambda)-\lambda}.$$

Now invoke Lemma 5.2 (f) to see that if y is large enough we have

$$J(y) \leq Ce^{3\varepsilon y} \cdot e^{\varepsilon y} = Ce^{4\varepsilon y}.$$

Since $\varepsilon > 0$ was arbitrary, we have proved (C).

$C \Rightarrow B$. Assuming (C), we can write, with any $\varepsilon > 0$,

$$\int_0^\infty |\hat{f}(\xi)||\varphi(\xi - 2\varepsilon)|d\xi = \int_0^\infty |\hat{f}(\xi)|d\xi \int_0^\infty G(iy)e^{(\xi-2\varepsilon)y}dy =$$

$$= \int_0^\infty e^{-2\varepsilon y}G_\alpha(iy)dy \int_0^\infty |\hat{f}(\xi)|e^{y\xi}d\xi \leq \int_0^\infty e^{-2\varepsilon y} \cdot Ce^{\varepsilon y}dy < \infty,$$

where the order of integration can be changed due to positivity.

$C \Rightarrow D$. Fix $\varepsilon > 0$. Under the assumption (C) it is clear that f has an analytic continuation to $y < 0$. Let us denote $\check{f}(z) = f(-z)$. In order to show that $\check{f}G \in H^1$ we begin by considering $F = \check{f}G_\varepsilon$, i.e. $F(z) = f(-z)G(z)e^{i\varepsilon z}$. Introduce the smoothed function F_h by the relation

$$F_h(z) = \frac{1}{2h}\int_{-h}^{h} F(x + t + iy)dt, \quad h > 0, \quad z = x + iy.$$

F_h is holomorphic in $y > 0$. Since $F \in L^1$ on the real axis, the boundary values of F_h will be bounded and in L^1. We claim that $F_h \in H^1$. Since H^1 is a closed subspace of L^1 and $\|F_h - F\|_{L^1} \to 0$ as $h \to 0+$, this will imply that $F \in H^1$. We estimate

$$|F_h(iy)| \leq \frac{1}{2h}\int_{-h}^{h} |f(-t-iy)G_\varepsilon(t+iy)|dt \leq \int_{\mathbf{R}} |\hat{f}(\xi)|e^{\xi y}d\xi \cdot G(iy)e^{-\varepsilon y} \cdot \max_{|t|\leq h} \|T_t\| \leq C,$$

where we have used Lemma 3.2 (b). Thus F_h is bounded on the positive imaginary axis, as well as on $\mathbf{R}$. Returning for a moment to F, we have, by virtue of the relation (1.5),

$$|F(z)| \leq G(iy)\|T_x\| \int_{\mathbf{R}} |\hat{f}(\xi)|e^{y\xi}d\xi \cdot e^{-\varepsilon y} \leq C\|T_x\| \leq Ce^{A|z|}.$$

F_h, being a local mean-value of F, is of the same order, and thus Phragmén-Lindelöf's principle tells us that F_h is bounded in $y \geq 0$. But a bounded function with boundary values in L^1 must, in fact, be in H^1. This is seen by representing it with Poisson's formula and estimating the L^1 norms on parallels to $\mathbf{R}$.

So we have proved that $\check{f}G_\varepsilon \in H^1$ and thus for $\xi \leq 0$,

$$0 = \int_{\mathbf{R}} e^{-ix\xi}f(-x)G_\varepsilon(x)dx = (\check{f}G)^\wedge(\xi - \varepsilon).$$

But here ε was an arbitrary positive number, and $(\check{f}G)^\wedge$ is continuous, which implies that $(\check{f}G)^\wedge(\xi) = 0$ for all $\xi \leq 0$; this concludes the proof of (D).

$D \Rightarrow E$. By assumption, f is analytic in $y < 0$ and the integrals

$$\int_{\mathbf{R}} |f(-x-iy)G(x+iy)|dx$$

are bounded for $y > 0$. Then, using Lemma 3.1 (b),

$$\int_{\mathbf{R}} |f(t-x-iy)G(x+iy)|dx = \int_{\mathbf{R}} |f(x-iy)G(t-x+iy)|dx \leq$$
$$\leq \|T_t\| \cdot \|\check{f}G\|_{H^1},$$

which is independent of $y > 0$. This means that all the functions $z \mapsto f(t-z)G(z)$ are in H^1. By the H^p theory (see e.g. Hoffman [9]), we have then

$$\int_{\mathbf{R}} f(t-x)G(x)e^{-i\xi x}dx = 0, \quad \xi \leq 0, \quad t \in \mathbf{R}.$$

Thus if $\beta \geq 0$ we get

$$\int_{\mathbf{R}} f(t-x)G_\beta(x)dx = \int_{\mathbf{R}} f(t-x)G(x)e^{-i(-\beta)x}dx = 0, \quad t \in \mathbf{R},$$

which proves (E).

$E \Rightarrow D$. The assumption (E) has the following consequence. Define $g(x) = f(-x)G(x)$, $x \in \mathbf{R}$. Then $g \in L^1(\mathbf{R})$ and $\hat{g}(\xi) = 0$ for $\xi \leq 0$. Thus $g \in H^1$, so that $g(z)$ can be defined for $y = \operatorname{Im} z > 0$, and f has an analytic continuation to $y < 0$ defined by $f(z) = g(-z)/G(-z)$, which makes sense since $G(z)$ is never zero, G being defined as $\exp(u + iv)$. The statement (D) follows.

$E \Rightarrow F$. Trivial.

Theorem II is proved. Theorem III is now a simple consequence of Theorem II. By this, any one of the assumptions A, B, C, D, and E implies F. Now consider an $f \in L^1_p$ that satisfies F for all real α. Then, in particular,

$$0 = f * G_\alpha(0) = \int_{\mathbf{R}} f(-t)G(t)e^{i\alpha t}dt = (\check{f}G)\hat{}\,(-\alpha), \quad \alpha \in \mathbf{R}.$$

Since $\check{f}G \in L^1(\mathbf{R})$, Fourier's inversion formula implies $f(-x)G(x) = 0$, a.e., and since $|G(x)| = p(x) > 0$ we obtain $f(x) = 0$ a.e., as required.

7. Proof of Theorem I

First we observe that it is sufficient to find an $f \neq 0$ with $\gamma_+(f) < +\infty$. By Theorem III, this f must also have $\gamma_+(f) > -\infty$. Since

$$\gamma_+(x \mapsto e^{i\beta x}f(x)) = \gamma_+(f) + \beta,$$

we can then obtain any real value for γ_+.

The construction is identical to that of Hirschman [8]. Let

$$f(x) = \frac{1}{\pi(i - x)^2 G(-x)}.$$

Obviously $f \in L_p^1$ and $f \neq 0$. Consider the formula defining $u(x + iy)$:

$$u(x + iy) = 2y \int_{\mathbf{R}} \left\{ \frac{1}{(t - x)^2 + y^2} - \frac{1}{t^2 + 1} \right\} |t| q(t) dt. \tag{7.1}$$

From this it is seen that u is bounded from below in every strip $0 < y < r$; in particular

$$u(x + iy) \geq 0, \quad 0 < y < 1. \tag{7.2}$$

It is permitted to differentiate (7.1) with respect to x under the sign of integration. The result shows that

$$u(x + iy) \geq u(iy), \quad x \in \mathbf{R}, \quad y > 0. \tag{7.3}$$

The formulas (7.2) and (7.3) imply that the analytic continuation of f to the lower half-plane satisfies

$$|f(x - iy)| \leq \frac{1}{\pi(1 + x^2)}, \quad 0 < y < 1,$$

and that it follows easily by contour integration and Lebesgue's dominated convergence theorem that

$$\hat{f}(\xi) = e^{-y\xi} \int_{\mathbf{R}} e^{-ix\xi} f(x - iy) dx$$

independently of $y > 0$. Estimate the size of $|\hat{f}(\xi)|$:

$$|\hat{f}(\xi)| \leq e^{-y\xi} \int_{\mathbf{R}} |f(x - iy)| dx \leq e^{-y\xi - u(iy)} \cdot \frac{1}{\pi} \int_{\mathbf{R}} \frac{dx}{x^2 + (y + 1)^2} \leq e^{-y\xi + yH_0(y)}.$$

Thus $\log |\hat{f}(\xi)| \leq yH_0(y) - y\xi$. We choose y to suit our purposes: $y = L_0(\xi - 1)$ gives $H_0(y) = \xi - 1$ and $\log |\hat{f}(\xi)| \leq - L_0(\xi - 1)$, whence immediately $\gamma_+(f) \leq 1 - Q < + \infty$, and the proof is complete.

8. Proof of Theorem IV

In the proof we use the following lemma, to which the author has found no convenient reference.

LEMMA 8.1. *If f is in H^1 of the upper half-plane, then*

$$\int_{\Gamma_\theta} f(z)dz = e^{i\theta} \int_0^{\to\infty} f(re^{i\theta})dr$$

is independent of θ in $0 \leq \theta \leq \pi$.

Here Γ_θ, of course, denotes the oriented half-line $\{re^{i\theta}: r \geq 0\}$.

Remark. It is actually true that the integral is absolutely convergent. This is not needed in our application of the lemma, and since there does not seem to exist a very simple proof of it, we abstain from including one.

Proof. Consider the contour integral

$$\int_\Gamma f(z)dz = 0,$$

where $\Gamma = \Gamma(R, S, \delta)$ is the closed quadrilateral with vertices at $\delta(\cot\theta + i)$, $S + i\delta$, $S + iR$, $R(\cot\theta + i)$; see Figure 3 for further notation. As for the integral over γ_2 it is well known (see [9]) that $f(z) \to 0$ uniformly as $|z| \to \infty$ inside any half-plane $y = \operatorname{Im} z \geq \delta > 0$. Thus $\int_{\gamma_2} \to 0$ as $S \to \infty$ for fixed R and δ. We are left with

$$\int_{\Gamma'} f(z)dz = 0,$$

where $\Gamma' = \Gamma'(R, \delta)$ is shown in Figure 4. Let first $\delta \to 0+$; with $f \in H^1$ the passage to the boundary values has no hazards. Turning to γ_3 we represent f by the Poisson integral

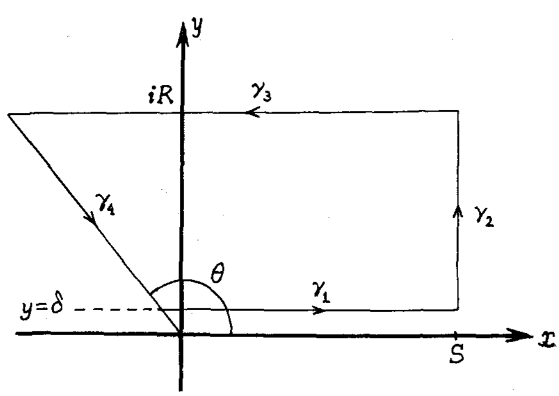

Fig. 3.

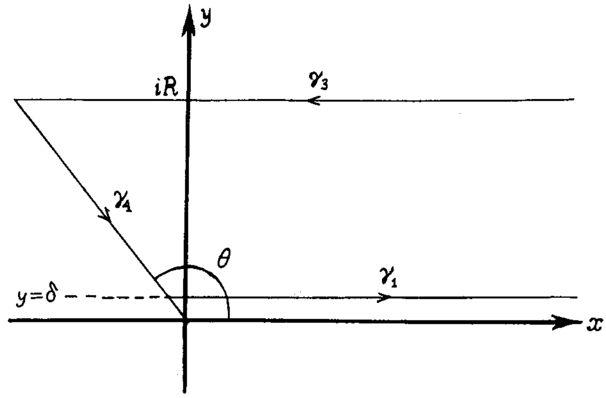

Fig. 4.

$$f(x+iR)=\frac{R}{\pi}\int_{\mathbf{R}}\frac{f(t)}{(t-x)^2+R^2}\,dt,$$

and, invoking absolute convergence, interchange the order of integrations to get

$$-\int_{\gamma_3}=\int_{R\cot\theta}^{\infty}f(x+iR)dx=\frac{1}{\pi}\int_{\mathbf{R}}f(t)\left(\frac{\pi}{2}+\arctan\left(\frac{t}{R}-\cot\theta\right)\right)dt.$$

Since

$$\int_{\mathbf{R}}f(t)dt=0 \tag{8.1}$$

for $f\in H^1$, this reduces to

$$\frac{1}{\pi}\int_{\mathbf{R}}f(t)\arctan\left(\frac{t}{R}-\cot\theta\right)dt\to-\frac{\arctan\cot\theta}{\pi}\int_{\mathbf{R}}f(t)dt=0$$

as $R\to\infty$, where we use Lebesgue's theorem on dominated convergence. Thus we have, with γ_4 extended to infinity,

$$\int_0^{\infty}f(x)dx+\int_{\gamma_4}f(z)dz=0$$

and since $\gamma_4=-\Gamma_\theta$ the Lemma is proved for $0\le\theta<\pi$; the case $\theta=\pi$ is immediate from (8.1).

When proving the Theorems IV A and NA we assume, as we did previously, that $\alpha=0$. Thus our assumption is

$$\int_{\mathbf{R}} f(x-t)G(t)dt = 0, \quad x \in \mathbf{R}. \tag{8.2}$$

Introduce the function h by

$$h(x) = \int_0^\infty f(x-t)G(t)dt = -\int_{-\infty}^0 f(x-t)G(t)dt. \tag{8.3}$$

For $x < 0$ we can estimate $|h(x)|$ using the first representation:

$$|h(x)| \leq \int_0^\infty |f(x-t)|p(x-t)\frac{p(t)}{p(x-t)}dt \leq \|f\|_p \sup_{t\geq 0}\frac{p(t)}{p(t+|x|)}. \tag{8.4}$$

and an analogous estimate holds for $x > 0$, obtained from the second integral in (8.3). By our assumptions on p we conclude that $h \in L^\infty(\mathbf{R})$. We can consider h as a tempered distribution; as such it has a Fourier transform $\hat{h}$, a pseudo-measure. We might try to compute $\hat{h}$ quite formally, and find

$$\hat{h}(\xi) = \hat{f}(\xi)\varphi(\xi). \tag{8.5}$$

However, the integrals involved in the formal argument are not convergent. Nevertheless we shall prove the following lemma.

LEMMA 8.2. *Under the Additional Assumption III, ensuring that L_p^1 is a Banach algebra,* (8.5) *holds true in the sense that $\hat{h}$ is actually a function and as such equal to $\hat{f}\cdot\varphi$.*

Proof of the lemma. Use Theorem I to find a function $k \in L_p^1$, $k \neq 0$, with small $\gamma_+(k) = \alpha$. This can be done so that the function $\hat{k}_1 = \varphi\hat{k}$ is in $L^1(\mathbf{R})$ and we can form

$$k_1(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{k}(\xi)\varphi(\xi)e^{i\xi x}d\xi. \tag{8.6}$$

In fact, by Lemma 5.3, $|\varphi(\xi)| \leq Ce^{L(\xi+\beta)}$ for some β, and so if we take $|\alpha| = -\alpha$ large enough,

$$|\hat{k}_1(\xi)| \leq Ce^{L(\xi+\beta)-L(\xi-\alpha)}, \quad \xi > 0,$$

which is in $L^1(0, \infty)$ by Lemma 5.2 (e); for negative $\xi, \varphi(\xi) = O(|\xi|^{-1})$ as $\xi \to -\infty$, and it is sufficient to require $\hat{k}$ to be in $L^1(-\infty, 0)$ or even slightly less; this is clearly feasible. (Convolve k by anything in L_p^1 with $\gamma_- < +\infty$.) It follows from (8.6) that $k_1 \in L^\infty(\mathbf{R})$.

Now study

$$K(x) = \int_0^\infty k(x-t)G(t)dt = -\int_{-\infty}^0 k(x-t)G(t)dt, \tag{8.7}$$

where the double representation is a consequence of the choice of k and the fact that A $\Rightarrow$ E (Theorem II). We have also $k(x-\cdot)G(\cdot) \in H^1$ for every x (translating k does not affect $\gamma_+(k)$, and A $\Rightarrow$ D). By Lemma 8.1 with $\theta = \pi/2$,

$$K(x) = i\int_0^{\to\infty} k(x-iy)G(iy)dy.$$

But

$$k(x-iy) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{k}(\xi)e^{i\xi x+\xi y}d\xi,$$

and we get for any $A > 0$, by absolute convergence,

$$\frac{i}{2\pi}\int_0^A G(iy)dy\int_{\mathbf{R}} \hat{k}(\xi)e^{i\xi x+\xi y}d\xi = \frac{i}{2\pi}\int_{\mathbf{R}} \hat{k}(\xi)e^{i\xi x}\int_0^A G(iy)e^{\xi y}dy.$$

Since $\hat{k}\varphi$ is integrable, and the inner integral in the right member grows monotonically to $|\varphi(\xi)| = -i\varphi(\xi)$ as $A \to \infty$, the passage to the limit is legitimate and yields

$$K(x) = k_1(x), \quad x \in \mathbf{R}.$$

Thus k_1 is also given by the integrals in (8.7). Now since both f and k are in L^1_p and we assume the latter to be a Banach algebra, the following calculations make sense:

$$f*k_1(x) = \int_{\mathbf{R}} f(x-t)k_1(t)dt = \int_{\mathbf{R}} f(x-t)dt\int_0^\infty k(t-u)G(u)du =$$

$$= \int_0^\infty G(u)du\int_{\mathbf{R}} f(x-t)k(t-u)dt = \int_0^\infty G(u)\cdot f*k(x-u)du,$$

and similarly

$$k*h(x) = \int_0^\infty G(u)\cdot k*f(x-u)du,$$

so that $f*k_1 = k*h$. Taking Fourier transforms (in the distribution sense), we have

$$\hat{f}(\xi)\hat{k}_1 = \hat{k}(\xi)\hat{h}$$

or, since we know that $\hat{k}_1 = \hat{k}\varphi$,

$$\hat{f}(\xi)\hat{k}(\xi)\varphi(\xi) = \hat{k}(\xi)\hat{h}.$$

We see that $\hat{h}$ can be represented by the function $\hat{f}(\xi)\varphi(\xi)$ near every point where $\hat{k}(\xi) \neq 0$. By small translations of $\hat{k}$ (as in the discussion following the statement of Theorem IV NA) we conclude that $\hat{h}(\xi) = \hat{f}(\xi)\varphi(\xi)$ everywhere (h does not really depend on k), which proves the lemma.

Proof of Theorem IV A. Under the Additional Assumption I, the formula (8.4) will actually imply that $h \in L^1(\mathbf{R})$. Indeed, if

$$p(t) = \exp\left(\frac{\pi}{2} t q_0\right) \cdot \exp\left(\frac{\pi}{2} t(q(t) - q_0)\right),$$

where the second factor is non-decreasing for $t \geq 0$, we obtain

$$|h(x)| \leq Ce^{-\frac{\pi}{2} q_0 |x|}.$$

But then $\hat{h}$ is a bounded function, and we have from (8.5) that

$$|\hat{f}(\xi)| \leq \frac{C}{|\varphi(\xi)|}.$$

Lemma 5.3 now contains all the hard work needed to establish the relation

$$|\hat{f}(\xi)| \leq Ce^{-L(\xi - Q - \varepsilon)} \quad \text{if} \quad \xi > \xi_0(\varepsilon),$$

which is equivalent to $\gamma_+(f) \leq 0$. The proof is complete.

Remark. The only place in the proof where the Additional Assumption I is applied is to establish that $h \in L^1(\mathbf{R})$. Any other assumption that accomplishes the same feat can be substituted for it.

Proof of Theorem IV NA. First we observe that the Schwarz inequality gives

$$\left(\int_0^\infty |\hat{f}(\xi)\varphi(\xi - \varepsilon)| d\xi\right)^2 \leq \int_0^\infty |\hat{f}(\xi)|^2 |\varphi(\xi)| d\xi \cdot \int_0^\infty \frac{|\varphi(\xi - \varepsilon)|^2}{|\varphi(\xi)|} d\xi.$$

By Lemma 5.4 it is then sufficient to prove that

$$\int_0^\infty |\hat{f}(\xi)|^2 |\varphi(\xi)| d\xi < \infty.$$

But $|\hat{f}|^2 = (f * \tilde{f})\hat{}$, where $\tilde{f}(x) = \overline{f(-x)}$. Our assumption $f * G = 0$ implies that also $f * \tilde{f} * G = 0$. We conclude that it is sufficient to prove

$$\int_0^\infty \hat{f}(\xi)|\varphi(\xi)|d\xi < \infty$$

for $f \in L_p^1$ satisfying $\hat{f}(\xi) \geq 0$ and $f * G = 0$. We also remember that $\varphi(\xi) = i|\varphi(\xi)|$.

Let g be an even function in $L^1(\mathbf{R})$ such that the Fourier transform $\hat{g}$ has compact support and satisfies $\hat{g}(\xi) \geq 0$, $\hat{g}(0) = 1$, $\hat{g}(\varrho\xi) \leq \hat{g}(\xi)$ if $\varrho \geq 1$. Remembering the function h obtained from f and G in Lemma 8.2, we get for all $\varepsilon > 0$, by Parseval's theorem,

$$0 \leq -i \int_{\mathbf{R}} \hat{f}(\xi)\varphi(\xi)\hat{g}(\varepsilon\xi)d\xi = -i \cdot \frac{2\pi}{\varepsilon} \int_{\mathbf{R}} h(x)g\left(\frac{x}{\varepsilon}\right)dx,$$

where the right hand member can be estimated:

$$\leq \frac{2\pi}{\varepsilon}\|h\|_{L^\infty} \int_{\mathbf{R}} \left|g\left(\frac{x}{\varepsilon}\right)\right| dx = 2\pi\|h\|_{L^\infty}\|g\|_{L^1},$$

independently of $\varepsilon > 0$. Thus the integral of $|\hat{f}(\xi)\varphi(\xi)\hat{g}(\varepsilon\xi)|$ is uniformly bounded. As $\varepsilon \to 0+$, $\hat{g}(\varepsilon\xi)$ tends monotonically to 1 from below, and Beppo Levi's convergence theorem gives (8.8). The proof is complete.

References

1. Beurling, A., Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionelle. *9ème congrès des mathématiciens scandinaves, août* 1938, Helsinki (1939), 345—366.
2. —»— Sur les spectres des fonctions. *Colloques internationaux du C.N.R.S., Analyse harmonique,* Nancy (1947), 9—29.
3. Domar, Y., Harmonic analysis based on certain commutative Banach algebras. *Acta Math.* 96 (1956), 1—66.
4. —»— Spectral analysis in spaces of sequences summable with weights. *J. Functional Analysis* 5 (1970), 1—13.
5. Geĭsberg, S. P., Completeness of function shifts in certain spaces. *Dokl. Akad. Nauk* SSSR 173 (1967), 741—744. English translation in *Soviet Math. Dokl.* 8 (1967), 451—454.
6. Geĭsberg, S. P., & Konjuhovskiĭ, V. S., On primary ideals in rings of functions integrable with weight. (Russian.) *Tartu Riikl. Ül. Toimetised,* 253 (1970), 134—144. (RŽMat. 9 (1971), p. 78.)
7. —»— Description of certain systems of primary ideals in rings of functions integrable with weight. (Russian.) *Leningrad. Gos. Ped. Inst. Učen. Zap.*, 404 (1971), 349—361. (RŽMat. 12 (1971), p. 106.)

8. Hirschman, Jr., I. I., On the behavior of Fourier transforms at infinity and on quasi-analytic classes of functions. *Amer. J. Math.* 72 (1950), 200–213.
9. Hoffman, K., *Banach spaces of analytic functions.* Prentice-Hall, Englewood Cliffs, N. J. (1962).
10. Konjuhovskiĭ, V. S., On the closedness of certain ideals in the ring $L^1_\alpha(R)$. (Russian.) *Doklady XXVIII konferencii LISI,* (1969), 5–8.
11. Korenbljum, B. I., A generalization of Wiener's Tauberian theorem and spectrum of fast-growing functions. (Russian.) *Trudy Moskov. Mat. Obšč.* 7 (1958), 121–148. (MR 21, 266.)
12. Loomis, L., *An introduction to abstract harmonic analysis.* Van Nostrand, New York (1953).
13. Mandelbrojt, S., *Séries adhérentes, régularisation des suites, applications.* Gauthier-Villars, Paris (1952).
14. Nyman, B., *On the one-dimensional translation group and semigroup in certain function spaces.* Thesis, Uppsala (1950).
15. Wiener, N., *The Fourier integral and certain of its applications.* Cambridge (1933).

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