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Algebraic approach to the interval linear static identification, tolerance, and control problems, or One more application of Kaucher arithmetic

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In this paper, the *identification problem*, the *tolerance problem*, and the *control problem* are treated for the interval linear equation $\mathbf{A}x = \mathbf{b}$. These problems require computing an inner approximation of the *united solution set* $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(Ax \in \mathbf{b})\}$, of the *tolerable solution set* $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}$, and of the *controllable solution set* $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\mathbf{A}x \ni b)\}$ respectively. An *algebraic approach* to their solution is developed in which the initial problem is replaced by that of finding an *algebraic solution* of some auxiliary interval linear system in Kaucher extended interval arithmetic. The algebraic approach is proved almost always to give inclusion-maximal inner interval estimates of the solution sets considered. We investigate basic properties of the algebraic solutions to the interval linear systems and propose a number of numerical methods to compute them. In particular, we present the simple and fast *subdifferential Newton method*, prove its convergence and discuss numerical experiments.

Алгебраический подход к интервальным линейным статическим задачам идентификации, о допусках и об управлении, или Еще одно применение арифметики Каухера

С. П. Шарый

В этой работе рассматриваются *задача идентификации*, *задача о допусках* и *задача об управлении* для интервальной линейной системы $\mathbf{A}x = \mathbf{b}$, требующие нахождения внутренней оценки для *объединенного множества решений* $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(Ax \in \mathbf{b})\}$, *допустимого множества решений* $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(Ax \in \mathbf{b})\}$ и *управляемого множества решений* $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\mathbf{A}x \ni b)\}$ соответственно. Развивается *алгебраический подход* к их решению, при котором исходная задача заменяется задачей отыскания *алгебраического решения* для некоторой вспомогательной интервальной линейной системы в расширенной интервальной арифметике Каухера. Показано, что алгебраический подход почти всегда дает максимальные по включению внутренние оценки для рассматриваемых множеств решений. Исследуются основные свойства алгебраических решений интервальных линейных систем, обсуждаются численные методы для их нахождения. В частности, мы предалагаем простой и быстрый *субдифференциальный метод Ньютона*, доказываем его сходимость и приводим результаты численных экспериментов с ним.

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1. Introduction

1.1. Notation

In this paper, intervals and other interval objects are denoted by boldface letters, for example, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$, while non-interval (real) objects are not distinguished in any way. Also, we need the following notation:

$\mathbb{IR}$—the set of all real intervals $[\underline{x}, \overline{x}]$ on $\mathbb{R}$, $\underline{x} \le \overline{x}$,

$\mathbb{IR}^n$—the set of n-dimensional interval vectors,

$\overline{\mathbf{x}}$, $\underline{\mathbf{x}}$—upper and lower bounds of $\mathbf{x}$ respectively,

$\operatorname{mid} \mathbf{x} = (\overline{\mathbf{x}} + \underline{\mathbf{x}})/2$—mean value (midpoint) of $\mathbf{x}$,

$\operatorname{rad} \mathbf{x} = (\overline{\mathbf{x}} - \underline{\mathbf{x}})/2$—radius of $\mathbf{x}$,

$|\mathbf{x}| = \max\{|\overline{\mathbf{x}}|, |\underline{\mathbf{x}}|\}$—absolute value (magnitude) of $\mathbf{x}$,

$\langle \mathbf{x} \rangle = \begin{cases} \min\{|\overline{\mathbf{x}}|, |\underline{\mathbf{x}}|\}, & \text{if } 0 \notin \mathbf{x}, \\ 0, & \text{otherwise} \end{cases}$ —mignitude of $\mathbf{x}$ or the least distance between points of $\mathbf{x}$ and zero, in some sense the opposite of the absolute value,

$(\cdot, \cdot)$—standard scalar multiplication in $\mathbb{R}^n$, that is, the sum of component products.

If $\mathbf{x} = (\mathbf{x}_i)_{i=1}^n$ is an interval vector, then all of the operations defined above are to be understood componentwise, so that rad $\mathbf{x}$, for example, is the real vector $(\operatorname{rad} \mathbf{x}_i)_{i=1}^n$. We shall assume the topology on the interval space $\mathbb{IR}^n$ to be defined in the standard way, that is, by the Hausdorff metric

$$\varrho(\mathbf{x}, \mathbf{y}) = \max\{\|\underline{\mathbf{x}} - \underline{\mathbf{y}}\|, \|\overline{\mathbf{x}} - \overline{\mathbf{y}}\|\} \tag{1}$$

and the norm of the interval vector $\mathbf{x} \in \mathbb{IR}^n$ is

$$\|\mathbf{x}\| \stackrel{\text{def}}{=} \big\||\mathbf{x}|\big\|, \tag{2}$$

where $\|\cdot\|$ is a monotonic vector norm on $\mathbb{R}^n$.

1.2. Problem statement

The subject matter of this paper is certain problems relating to interval linear algebraic systems, but we shall not consider them in the context of the so-called self-validating computations, validated numerics etc. as is fashionable among modern numerical analysts. To our mind, that artificially narrows the scope of interval analysis, of the very interval idea. Personally, I prefer to take interval analysis primarily as a convenient and computationally efficient tool to deal with a specific kind of uncertainty, a special case of the currently popular *bounded uncertainty*, that is, as a tool of data analysis that is alternative to probabilistic and fuzzy models. It is this viewpoint that underlies our work, and I am sure the outlined area will be the major realm of applications of interval analysis in the years to come.

So our main object under consideration is the interval system of linear algebraic equations

$$\mathbf{A}x = \mathbf{b} \tag{3}$$

with an interval $n \times n$-matrix $\mathbf{A}$ and an interval right-hand side n-vector $\mathbf{b}$. It is common knowledge that (3) is only a formal symbol, which in itself can mean, for instance, a collection of all point linear algebraic systems $Ax = b$ with elements belonging to $\mathbf{A}$ and $\mathbf{b}$ respectively. To pose the problem correctly, let us define what is meant by the solution or the solution set to (3). In interval analysis, the following four solution sets have been the subject of more or less vigorous enquiry so far:

- the *united solution set* formed by solutions of all point systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$, i.e., the set

$$\Sigma_{\exists\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\} \tag{4}$$

  historically first and undoubtedly the most popular of the solution sets; it is called by Western authors simply as *solution set* and usually is denoted by $\Sigma(\mathbf{A}, \mathbf{b})$ (see [1, 17, 19] and the extensive references there);

- the *tolerable solution set*, formed by all point vectors x such that the product $Ax \in \mathbf{b}$ for any $A \in \mathbf{A}$, i.e., the set

$$\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\} \tag{5}$$

  (see [8, 13, 18, 19, 25, 28] et al). Neumaier [18, 19] and some other authors use the term "restricted solution set" for (5), denoting it $\Sigma_0(\mathbf{A}, \mathbf{b})$, but in our work we keep to the more adequate term "tolerable". The history of the set $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$ and of some related problems was described comprehensively in the papers by Neumaier [18] and by Kelling and Oelschlägel [13];

- the *controllable solution set*

$$\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b)\} \tag{6}$$

formed by all point vectors $x \in \mathbb{R}^n$, such that for any desired $b \in \mathbf{b}$ we can find a corresponding $A \in \mathbf{A}$ satisfying $Ax = b$ (see [26]);

- the *algebraic solution* [22], i.e., such an interval vector $\mathbf{x}_a$ that, substituting into (3) and executing all interval arithmetic operations, results in the valid equality $\mathbf{A}\mathbf{x}_a = \mathbf{b}$.

In particular, one can readily see from these definitions that

$$\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) \subseteq \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b}) \quad \text{and} \quad \Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) \subseteq \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b}).$$

As a visual example to illustrate the above concepts, we have chosen the popular interval linear system

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix} \tag{7}$$

from [3] repeatedly considered by various authors. Its solution sets are depicted in Figure 1.

Aside from the formal definitions (4)–(6), there also exist more convenient characterizations for the solution sets under study. It is well known that for any interval matrix $\mathbf{A}$ and a point vector x

$$\mathbf{A} \cdot x = \{Ax \mid A \in \mathbf{A}\}$$

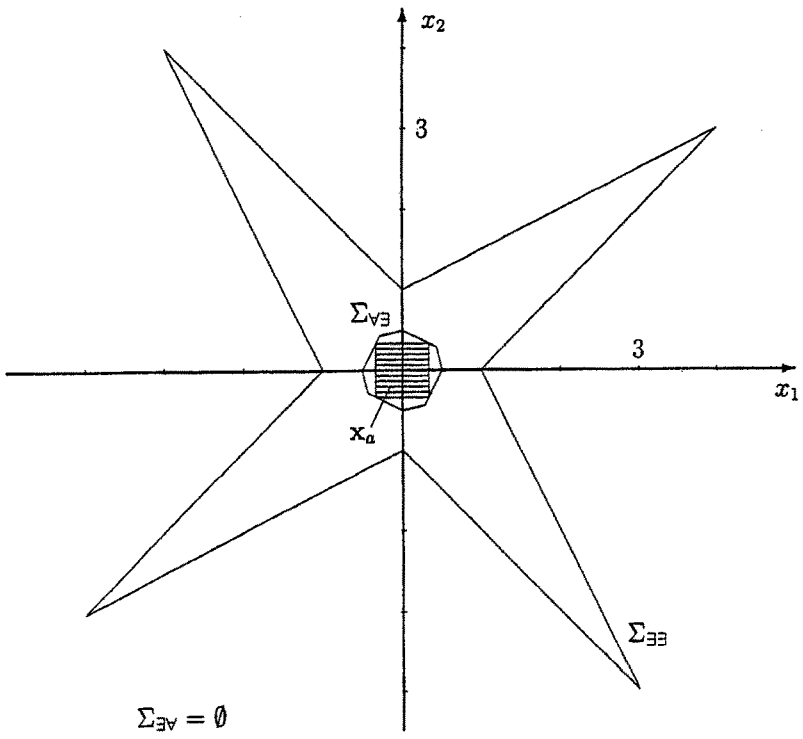

Figure 1. The solution sets to (4)–(7)

where "$\cdot$" denotes the common interval matrix-vector multiplication [1, 17, 19]. Because of this,

$$\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid \mathbf{A}\cdot x \subseteq \mathbf{b}\},$$
$$\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid \mathbf{A}\cdot x \supseteq \mathbf{b}\}$$

and for the united solution set, as Beeck first pointed out in [4],

$$\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b}) = \{x \in \mathbb{R}^n \mid \mathbf{A}\cdot x \cap \mathbf{b} \neq \emptyset\}.$$

The direct descriptions of the solution sets (4)–(6), since their computational complexity grows exponentially with n, become laborious and practically useless, even for systems with relatively small dimension. Besides, Lakeyev and Noskov [16] managed to prove that the problems of recognition whether $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$ or $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$ is empty or not are NP-complete for the systems with rectangular $\mathbf{A}$ (see also Rohn [24]). For this reason, one usually confines oneself to finding simple subsets of $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$ and $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$, since for all their points the properties $(\mathbf{A}\cdot x \cap \mathbf{b} \neq \emptyset)$, $(\mathbf{A}\cdot x \subseteq \mathbf{b})$ or $(\mathbf{A}\cdot x \supseteq \mathbf{b})$ remain valid, respectively. Put it differently, we replace $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$, $\Sigma_{\forall\exists}(\mathbf{A},\mathbf{b})$, and $\Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$ by their inner approximations, formulating the problems to be solved in the following form:

Find an interval vector that is included in the tolerable solution set (if nonempty) of the interval linear system (8)

and

Find an interval vector that is included in the controllable solution set (if nonempty) of the interval linear system. (9)

We are going to seek the inner approximation to the united solution set too, but such a problem statement, though not completely new (see, e.g., [4]), requires justification. Traditionally, outer component-wise estimates for the united solution set are computed, and the standard form of this problem—the so-called "outer problem" for the interval linear algebraic systems—is as follows:

Find an interval vector that includes the united solution set of the interval linear system. (10)

The problem (10), being in fact a generalized sensitivity problem in interval form, is a classical interval analysis problem, and a large number of papers has been devoted to various aspects of its solution from the early 1960's to now. However, the outer approximation of $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$ contains points that have nothing to do with solutions of the system $Ax = b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$, and due to this, such a problem statement may turn out unacceptable in many practical situations. The latter is especially typical for the observation and identification problems. So the third problem we shall deal with is

Find an interval vector that is included in the united solution set of the interval linear system. (11)

The problem (8) is the classical *linear tolerance problem* [8, 13, 18, 19, 28] (sometimes referred to as the *inner problem* for the interval linear system), with numerous and fruitful practical applications, and the problem (9) is a new promising interval algebraic problem that is believed to have extensive potential use in the automatic control. We shall call the problems (9) and (11) the *control problem* and the *identification problem* for the interval linear algebraic system, respectively.[1]

[1]The author realizes that the terms "identification problem" and "control problem" may seem a poor choice, somewhat vague and pretentious. The words "identification" and "control" have very wide meaning, which is in no

1.3. Algebraic approach

The aim of this work is to present a new efficient algorithmic approach to the solution of the problems (8), (9), and (11), but the principles that underlie our construction are to a great extent unusual. Our construction is based upon the concept of the *algebraic solution* to the system (3) mentioned above.

Definition 1. *An interval vector is said to be algebraic solution to the interval system of equations if substitution it into the system and execution of all interval arithmetic operations results in a valid equality.*

This concept was first considered by Ratschek and Sauer in [22], but only a few papers on this subject have appeared for the last decade. Apparently, until now Western researchers have taken the algebraic solution solely as a theoretical curiosity. Nonetheless, if $\mathbf{x}_a$ is an algebraic solution of the interval system $\mathbf{A}x = \mathbf{b}$, then, due to inclusion monotonicity, there holds

$$\mathbf{A}x \subseteq \mathbf{A}\mathbf{x}_a = \mathbf{b}$$

for any $x \in \mathbf{x}_a$, so $\mathbf{x}_a \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. We have proved

Proposition 1. *If the interval vector $\mathbf{x}_a$ is an algebraic solution to the system $\mathbf{A}x = \mathbf{b}$, then $\mathbf{x}_a \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, that is, $\mathbf{x}_a$ is a solution to the interval linear tolerance problem (8).*

The linear tolerance problem thus reduces to a purely algebraic one: find an algebraic solution of the system (3). That is a very attractive feature, notwithstanding that the algebraic solution to the interval linear system does not need to exist even if the corresponding linear tolerance problem is compatible. It is demonstrated, for instance, by the one-dimensional linear tolerance problem with $\mathbf{A} = [-1, 1]$, $\mathbf{b} = [-2, 3]$. The interval linear equation $[-1, 1] \cdot x = [-2, 3]$ has no algebraic solutions, but $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b}) = [-2, 2] \neq \emptyset$.

However, the numerical procedures ([30, 31, 33] etc) devised so far to implement the approach to inner estimation of the tolerable solution set based on Proposition 1 (the *algebraic approach*) are cumbersome and inefficient, the cause of this being bad algebraic properties of the classical interval arithmetic $\mathbb{IR}$, that is, more precisely, the absence of additive and multiplicative inverses for most intervals. Under these conditions, it is reasonable, as Shary proposed in [25], to embed $\mathbb{IR}$ into a wider algebraic system that would have better algebraic properties, that would be richer in manipulation technique, with the more powerful analytical tools, and then to seek the solution in it. If the interval vector so obtained proves to lie in $\mathbb{IR}$, then it will be a solution to the original problem.

The algebraic completion of $\mathbb{IR}$ has been performed by Kaucher (see, for instance, [11, 12] and references there), who called the algebraic system so constructed "the extended interval arithmetic $\mathbb{IR}$". The elements of $\mathbb{IR}$ are real pairs $[\underline{x}, \overline{x}]$ that are not necessarily subject to the condition $\underline{x} \leq \overline{x}$. Thus, $\mathbb{IR}$ is made up by adding *improper* intervals $[\underline{x}, \overline{x}]$, $\underline{x} > \overline{x}$, to the set $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \leq \overline{x}\}$ of *proper* intervals and real numbers. $\mathbb{IR}$ is an additive group, but the multiplicative group of $\mathbb{IR}$ is formed only by intervals $[\underline{x}, \overline{x}]$ with $\underline{x}\overline{x} > 0$. Also, recall the definition of a basic involution of Kaucher extended arithmetic, namely, the map

$$\mathrm{dual} : \mathbb{IR} \to \mathbb{IR}$$

way exhausted by the mathematical formulations (11) and (9). We suggest that in the context of this paper one should consider the phrases "identification problem", "control problem" merely as labels for the mathematical formulations (11) and (9) respectively.

with action

$$\text{dual}\,[\underline{x},\overline{x}] = [\overline{x},\underline{x}].$$

The inclusion order relation is naturally carried over to $\mathbb{IR}$:

$$[\underline{x},\overline{x}] \subseteq [\underline{y},\overline{y}] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \underline{x} \geq \underline{y} \ \text{ and } \ \overline{x} \leq \overline{y}.$$

The inclusion monotonicity of the interval arithmetic operations also remains valid in Kaucher extended arithmetic:

$$\mathbf{x} \subseteq \mathbf{x}', \mathbf{y} \subseteq \mathbf{y}' \quad \Rightarrow \quad \mathbf{x} * \mathbf{y} \subseteq \mathbf{x}' * \mathbf{y}'$$

for $* \in \{+,-,\cdot,/\}$ and any $\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}' \in \mathbb{IR}$.

The most wonderful fact is that the algebraic solution in Kaucher arithmetic can help in the inner approximation of the controllable and united solution sets too!

Proposition 2. *If the interval vector* $\mathbf{x}_a$ *is an algebraic solution to the system* $\mathbf{A}x = \mathbf{b}$ *and all its components are improper, then* $\text{dual}\,\mathbf{x}_a \subseteq \Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$, *that is,* $\text{dual}\,\mathbf{x}_a$ *is a solution to the interval linear control problem (9).*

Proof. Indeed, suppose an algebraic solution $\mathbf{x}_a$ to (3) has only improper components. Then we have $x \supseteq \mathbf{x}_a$ for any $x \in \text{dual}\,\mathbf{x}_a$ and

$$\mathbf{A}x \supseteq \mathbf{A}\mathbf{x}_a = \mathbf{b}$$

due to the inclusion monotonicity. Therefore, $x \in \Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$, and so $\text{dual}\,\mathbf{x}_a \subseteq \Sigma_{\exists\forall}(\mathbf{A},\mathbf{b})$. □

Proposition 3. *If the interval vector* $\mathbf{x}_a$ *is an algebraic solution to the system* $\mathbf{A}x = \text{dual}\,\mathbf{b}$ *and all its components are improper, then* $\text{dual}\,\mathbf{x}_a \subseteq \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$, *that is,* $\text{dual}\,\mathbf{x}_a$ *is a solution to the interval linear identification problem (11).*

Proof. One can readily see that, for an improper interval $\mathbf{x}$ and a proper interval $\mathbf{y}$,

$$\mathbf{y} \supseteq \mathbf{x} \quad \Longleftrightarrow \quad \mathbf{y} \cap \text{dual}\,\mathbf{x} \neq \emptyset. \tag{12}$$

Furthermore, if $\mathbf{x}_a$ is an algebraic solution to the equation

$$\mathbf{A}x = \text{dual}\,\mathbf{b}$$

and all its components are improper, then

$$\mathbf{A}x \supseteq \mathbf{A}\mathbf{x}_a = \text{dual}\,\mathbf{b}$$

for any $x \in \text{dual}\,\mathbf{x}_a$, that is,

$$\mathbf{A}x \cap \text{dual}\,(\text{dual}\,\mathbf{b}) = \mathbf{A}x \cap \mathbf{b} \neq \emptyset$$

because of (12). Recalling Beeck's characterization of the united solution set [4, 19], we may conclude that $x \in \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$, and so $\text{dual}\,\mathbf{x}_a \subseteq \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$. □

The one-dimensional equation

$$\mathbf{a} * x = \mathbf{b}$$

$* \in \{+,-,\cdot,/\}$, with proper intervals $\mathbf{a}$ and $\mathbf{b}$ provides a major illustration of what has been stated above and the remarkable interpretation of Kaucher interval arithmetic. Suppose that its solvability condition holds, that is, there exists an algebraic solution $\mathbf{x}_a$ to this equation and

$$\mathbf{x}_a = \mathbf{b} *^{-1} \mathbf{a}$$

in Kaucher arithmetic ("$*^{-1}$" denotes the inverse operation to "$*$"). The interval $\mathbf{x}_a$ may be either proper or improper, but the following exact equalities hold in the both cases:

$$\begin{aligned} \mathbf{x}_a &= \Sigma_{\forall\exists}(\mathbf{a},\mathbf{b}) \quad \text{if } \mathbf{x}_a \text{ is proper} \quad \text{and} \\ \text{dual}\,\mathbf{x}_a &= \Sigma_{\exists\forall}(\mathbf{a},\mathbf{b}) \quad \text{if } \mathbf{x}_a \text{ is improper!} \end{aligned}$$

1.4. Discussion

As far as the author knows, the above calculations have not appeared earlier in explicit form, although the possibility of reducing the linear tolerance problem to computation of the algebraic solution was pointed out in a very vague form by Zyuzin [31] not long ago. The algebraic approach to the inner estimation of the united solution set (Proposition 3) has been advanced simultaneously and independently by the author and by Kupriyanova [15] (which was revealed at the international conference INTERVAL'94, St.Petersburg, Russia, March 7–10, 1994; see [27]). Proposition 2 about the inner estimation of the controllable solution set is due to the author.

Nevertheless, our algebraic approach seems not to be completely original. The beginnings of the basic ideas for above theory are contained in some extremely general results by Gardeñes and Trepat [9, 10], namely, in the theorems on the analytical foundation of extended arithmetic semantics. Unfortunately, these results were both so general and formulated so briefly and without necessary explanations that they had been hardly understood and recognized by specialists in interval mathematics. We are going to revive and enrich that approach on a new powerful computational basis. A key property of the algebraic approach is that *it almost always gives inclusion-maximal inner approximations of the solution sets considered.*

Our task is thereby to find algebraic solution to the system (3) in Kaucher interval arithmetic. Taking into account the original statements (8), (9), and (11), it will suffice to restrict our attention to the case of proper interval matrix $\mathbf{A}$. What ways can we offer to tackle that problem?

The field of numerical analysis has amassed a rich arsenal of both theoretical approaches and practical algorithms, but an overwhelming majority of them has to do with operator equations in *linear spaces*. Formally, these methods are not directly applicable to computation of the algebraic solution to the system (3), since $\mathbb{IR}^n$ is not a linear space. We avoid this difficulty by embedding $\mathbb{IR}^n$ into the standard linear space $\mathbb{R}^{2n}$. Further, we assert existence and uniqueness results for algebraic solutions: if the proper interval matrix $\mathbf{A}$ contains a special kind of nonsingular point matrix and is sufficiently narrow (i.e., if $\|\operatorname{rad} \mathbf{A}\|$ is sufficiently small), then the interval linear system $\mathbf{A}x = \mathbf{b}$ has an algebraic solution in Kaucher arithmetic and it is unique. The "embedded" nonlinear equation obtained in $\mathbb{R}^{2n}$ corresponds to an *order convex* operator, and to solve it we use the *subdifferential Newton method*, which is shown to converge to an algebraic solution of the system (3) if the proper interval matrix $\mathbf{A}$ is "sufficiently narrow".

By and large, one may characterize the algebraic approach to the linear tolerance problem (8), the control problem (9) and the identification problem (11) as extremely efficient from the computational viewpoint (in practice it converges in a few iterations), but not sufficiently sensitive to examine the problem comprehensively, since an algebraic solution with the desired properties does not need to always exist. In particular, the technique developed will be useful for very quick computation of solutions to (8), (9), and (11) (in real time devices, for example) provided that their "good solvability" is given *a priori*.

2. Interval arithmetics

Classical interval arithmetic is the algebraic structure $\langle \mathbb{IR}, +, -, \cdot, / \rangle$ whose support is formed by intervals $[\underline{x}, \overline{x}]$, $\underline{x} \le \overline{x}$, of the real axis, while the binary operations—addition, subtraction,

multiplication and division—are defined so that the fundamental property

$$\mathbf{x} * \mathbf{y} = \{x * y \mid x \in \mathbf{x}, y \in \mathbf{y}\} \tag{13}$$

holds for intervals $\mathbf{x}, \mathbf{y}$ such that $(x * y)$, $* \in \{+, -, \cdot, /\}$, makes sense for all $x \in \mathbf{x}$, $y \in \mathbf{y}$. Hence, we have

$$\begin{aligned}
[\underline{x}, \overline{x}] \;+\; [\underline{y}, \overline{y}] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}], \\
[\underline{x}, \overline{x}] \;-\; [\underline{y}, \overline{y}] &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}], \\
[\underline{x}, \overline{x}] \;\cdot\; [\underline{y}, \overline{y}] &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}], \\
[\underline{x}, \overline{x}] \;/\; [\underline{y}, \overline{y}] &= [\underline{x}, \overline{x}] \cdot [1/\overline{y}, 1/\underline{y}] \qquad \text{for } [\underline{y}, \overline{y}] \not\ni 0.
\end{aligned}$$

The classical interval arithmetic $\mathbb{IR}$ is known to be a commutative semigroup with respect to the addition and the multiplication, and is not even a lattice with respect to the natural inclusion ordering, since not every two-element subset of $\mathbb{IR}$ has infimum. "Incompleteness" both of the algebraic and of the order structures of $\mathbb{IR}$ naturally stimulated attempts to create a "more convenient" interval arithmetic based on it. As mentioned above, the algebraic completion of $\mathbb{IR}$ carried out in the works by Kaucher [11, 12] resulted in the algebraic system called "the extended interval arithmetic $\mathbb{IR}$". We shall also use this term as well as the more academic one—"Kaucher interval arithmetic". Afterward, Gardeñes and Trepat studied this arithmetic and established some its helpful properties and important applications [9, 10].

Taken as a whole the extended interval arithmetic $\mathbb{IR}$ is a quite nontrivially arranged algebraic system. In this section we describe only those aspects of it that are necessary for our future consideration. In particular, for this reason we do not at all dwell on extended interval division. The complete description of $\mathbb{IR}$ may be found in [9, 11, 12].

The elements of $\mathbb{IR}$ are the pairs $[\underline{x}, \overline{x}]$ of reals, that are not connected by the obligatory condition $\underline{x} \le \overline{x}$. Thus, $\mathbb{IR}$ is obtained by adjoining *improper* intervals $[\underline{x}, \overline{x}]$, $\underline{x} > \overline{x}$, to the set $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \le \overline{x}\}$ of the *proper* intervals and the real numbers. The proper and improper intervals, the two major parts of $\mathbb{IR}$, may change places as the result of the duality mapping

$$\mathrm{dual} : \mathbb{IR} \to \mathbb{IR}$$

such that $\mathrm{dual}\,[\underline{x}, \overline{x}] = [\overline{x}, \underline{x}]$. As in classical interval arithmetic,

$$\mathbf{x} \subseteq \mathbf{y} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \underline{\mathbf{x}} \ge \underline{\mathbf{y}} \;\text{ and }\; \overline{\mathbf{x}} \le \overline{\mathbf{y}}$$

but the extended interval arithmetic is a conditionally complete lattice with respect to this inclusion order relation [5], in contrast to $\mathbb{IR}$. In other words,

$$\bigvee_{\gamma \in \Gamma} \mathbf{x}_\gamma := \sup\nolimits_{\subseteq}\{\mathbf{x}_\gamma \mid \gamma \in \Gamma\} = \left[\inf\nolimits_{\le}\{\underline{\mathbf{x}}_\gamma \mid \gamma \in \Gamma\}, \sup\nolimits_{\le}\{\overline{\mathbf{x}}_\gamma \mid \gamma \in \Gamma\}\right]$$

(maximum with respect to the inclusion ordering) and

$$\bigwedge_{\gamma \in \Gamma} \mathbf{x}_\gamma := \inf\nolimits_{\subseteq}\{\mathbf{x}_\gamma \mid \gamma \in \Gamma\} = \left[\sup\nolimits_{\le}\{\underline{\mathbf{x}}_\gamma \mid \gamma \in \Gamma\}, \inf\nolimits_{\le}\{\overline{\mathbf{x}}_\gamma \mid \gamma \in \Gamma\}\right]$$

(minimum with respect to the inclusion ordering) are elements from $\mathbb{IR}$ now, if $\{\mathbf{x}_\gamma \mid \gamma \in$ index set $\Gamma\}$ is a bounded family of "extended intervals".

Addition and multiplication by real numbers are defined upon $\mathbb{IR}$ by

$$\begin{aligned} [\underline{x},\overline{x}] + [\underline{y},\overline{y}] &:= [\underline{x}+\underline{y}, \overline{x}+\overline{y}], \\ \lambda \cdot [\underline{x},\overline{x}] &:= \begin{cases} [\lambda\underline{x}, \lambda\overline{x}], & \text{if } \lambda \in \mathbb{R}_+, \\ [\lambda\overline{x}, \lambda\underline{x}], & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, each element $\mathbf{x}$ of $\mathbb{IR}$ has a unique additive inverse, denoted "opp $\mathbf{x}$", and

$$\begin{aligned} &\mathbf{x} + \operatorname{opp}\mathbf{x} = 0 \quad \Rightarrow \quad \operatorname{opp}[\underline{x},\overline{x}] = [-\underline{x}, -\overline{x}], \\ &\operatorname{opp}(\lambda\mathbf{x}) = \lambda \operatorname{opp}\mathbf{x}, && \lambda \in \mathbb{R}, \\ &\operatorname{opp} x = -x, && x \in \mathbb{R}. \end{aligned}$$

Inclusion monotonicity for addition:

$$\mathbf{x} \subseteq \mathbf{x}', \ \mathbf{y} \subseteq \mathbf{y}' \quad \Rightarrow \quad \mathbf{x}+\mathbf{y} \subseteq \mathbf{x}'+\mathbf{y}'.$$

It follows directly from the definition that, with respect to addition, $\mathbb{IR}$ is a commutative group, which is isomorphic to the additive group of the standard linear space $\mathbb{R}^2$. Sometimes, we denote for brevity the inverse operation for addition, i.e., the inner (algebraic) difference of $\mathbb{IR}$, by $\ominus$, so that

$$\mathbf{x} \ominus \mathbf{y} := \mathbf{x} + \operatorname{opp}\mathbf{y}.$$

The following fundamental formula generalizes the property (13):

$$\mathbf{x}+\mathbf{y} = \bigvee\!\!\!\!\bigwedge_{x \in \operatorname{pro}\mathbf{x}}^{\mathbf{x}} \ \bigvee\!\!\!\!\bigwedge_{y \in \operatorname{pro}\mathbf{y}}^{\mathbf{y}} (x+y) \tag{14}$$

where

$$\bigvee\!\!\!\!\bigwedge^{\mathbf{x}} := \begin{cases} \vee & \text{if } \mathbf{x} \text{ is proper,} \\ \wedge & \text{otherwise} \end{cases}$$

$$\operatorname{pro}\mathbf{x} := \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ is proper,} \\ \operatorname{dual}\mathbf{x} & \text{otherwise} \end{cases} \qquad \text{(proper projection of the interval)}$$

It expresses the connection between the result of the interval addition $\mathbf{x}+\mathbf{y}$ and the results of separate point additions $x+y$ for $x \in \operatorname{pro}\mathbf{x}$ and $y \in \operatorname{pro}\mathbf{y}$.

The set of basic involutions $\operatorname{id}(\cdot)$, $-(\cdot)$, $\operatorname{opp}(\cdot)$, $\operatorname{dual}(\cdot)$ multiply (compose) according to the following Cayley table:

o	id	−	opp	dual
id	id	−	opp	dual
−	−	id	dual	opp
opp	opp	dual	id	−
dual	dual	opp	−	id

Table 1.

In other words, their set's multiplicative structure coincides with the well-known Klein's four-group.

The following inclusion properties:

$$\begin{aligned} \mathbf{x} \subseteq \mathbf{y} \;\Rightarrow\; & -\mathbf{x} \subseteq -\mathbf{y}, \\ & \operatorname{dual} \mathbf{x} \supseteq \operatorname{dual} \mathbf{y}, \\ & \operatorname{opp} \mathbf{x} \supseteq \operatorname{opp} \mathbf{y}, \\ & \lambda\mathbf{x} \subseteq \lambda\mathbf{y}, \qquad \lambda \in \mathbb{R}. \end{aligned}$$

Multiplication in Kaucher arithmetic is also defined on the basis of the representation (14):

$$\mathbf{x} \cdot \mathbf{y} = \bigvee_{x \in \operatorname{pro} \mathbf{x}}^{\mathbf{x}} \bigvee_{y \in \operatorname{pro} \mathbf{y}}^{\mathbf{y}} (x \cdot y). \tag{16}$$

To write this definition out in the explicit form, put

$$\begin{aligned} \mathcal{P} &:= \{\mathbf{x} \in \mathbb{IR} \mid (\underline{\mathbf{x}} \geq 0) \& (\overline{\mathbf{x}} \geq 0)\}, & -\mathcal{P} &:= \{-\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}, \\ \mathcal{U} &:= \{\mathbf{x} \in \mathbb{IR} \mid \underline{\mathbf{x}} \leq 0 \leq \overline{\mathbf{x}}\}, & \mathcal{V} &:= \{\mathbf{x} \in \mathbb{IR} \mid \operatorname{dual} \mathbf{x} \in \mathcal{U}\} \end{aligned}$$

so that $\mathbb{IR} = \mathcal{P} \cup (-\mathcal{P}) \cup \mathcal{U} \cup \mathcal{V}$. Then we have (see [12]):

	$\mathbf{y} \in \mathcal{P}$	$\mathbf{y} \in \mathcal{U}$	$\mathbf{y} \in -\mathcal{P}$	$\mathbf{y} \in \mathcal{V}$
$\mathbf{x} \in \mathcal{P}$	$[\underline{xy}, \overline{xy}]$	$[\overline{xy}, \overline{xy}]$	$[\overline{xy}, \underline{xy}]$	$[\underline{xy}, \underline{xy}]$
$\mathbf{x} \in \mathcal{U}$	$[\underline{xy}, \overline{xy}]$	$[\min\{\underline{xy}, \overline{xy}\}, \max\{\underline{xy}, \overline{xy}\}]$	$[\overline{xy}, \underline{xy}]$	0
$\mathbf{x} \in -\mathcal{P}$	$[\underline{xy}, \overline{xy}]$	$[\underline{xy}, \underline{xy}]$	$[\overline{xy}, \underline{xy}]$	$[\overline{xy}, \overline{xy}]$
$\mathbf{x} \in \mathcal{V}$	$[\underline{xy}, \overline{xy}]$	0	$[\overline{xy}, \underline{xy}]$	$[\max\{\underline{xy}, \overline{xy}\}, \min\{\underline{xy}, \overline{xy}\}]$

Table 2. Multiplication in Kaucher arithmetic

Though it is not evident from definition (16), extended interval multiplication turns out to be commutative and associative [9, 11, 12]. But the multiplicative group of $\mathbb{IR}$ is formed only by intervals $[\underline{x}, \overline{x}]$ with $\underline{x}\overline{x} > 0$, since the cancellation law does not hold on any wider subset of $\mathbb{IR}$.

Other properties include:

$$\begin{aligned} & \operatorname{dual}(\mathbf{xy}) = \operatorname{dual} \mathbf{x} \cdot \operatorname{dual} \mathbf{y}, \\ & \lambda(\mathbf{xy}) = (\lambda\mathbf{x})\mathbf{y} = \mathbf{x}(\lambda\mathbf{y}), \qquad \lambda \in \mathbb{R}, \\ & \mathbf{x} \subseteq \mathbf{x}', \ \mathbf{y} \subseteq \mathbf{y}' \;\Rightarrow\; \mathbf{x} \cdot \mathbf{y} \subseteq \mathbf{x}' \cdot \mathbf{y}'. \end{aligned}$$

Extended arithmetic subtraction and division are defined:

$$\begin{aligned} \mathbf{x} - \mathbf{y} &= \mathbf{x} + (-1) \cdot \mathbf{y}, \\ \mathbf{x}/\mathbf{y} &= \mathbf{x} \cdot [1/\overline{\mathbf{y}}, 1/\underline{\mathbf{y}}] \qquad \text{for } 0 \notin \operatorname{pro} \mathbf{y}. \end{aligned}$$

Thus, these operations are also inclusion monotone.

The interrelation between the multiplication and the addition is:

$$\begin{aligned} &\text{if } \mathbf{x} \text{ is proper,} && \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}, && \text{(subdistributivity)} \\ &\text{if } \mathbf{x} \text{ is improper,} && \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) \supseteq \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z} && \text{(superdistributivity)} \end{aligned}$$

and these inclusions turn into exact equalities for thin x.

By induction, the above semantic interpretation (14) and (16) may be generalized to some rational expressions. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_p)$ be a proper interval vector, $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_q)$ be an improper one and $f(x, y)$ $(= f(x_1, \ldots, x_p, y_1, \ldots, y_q))$ be a rational expression *having only one occurrence of each variable* (if at all) *to the first power*. We designate by $f(\mathbf{x}, \mathbf{y})$ the result of substitution of the vectors $\mathbf{x}, \mathbf{y}$ in f and execution of all extended interval arithmetic operations, that is, the corresponding "natural interval extension". Then (see [9])

$$f(\mathbf{x}, \mathbf{y}) = \bigvee_{x \in \mathbf{x}} \bigwedge_{y \in \mathrm{pro}\, \mathbf{y}} f(x, y) = \bigwedge_{y \in \mathrm{pro}\, \mathbf{y}} \bigvee_{x \in \mathbf{x}} f(x, y). \tag{17}$$

In the extended interval arithmetic, the operations with vectors and matrices are defined similar to those in $\mathbb{IR}$. The sum (difference) of two interval matrices of identical size is an interval matrix of that same size formed by elementwise sums (differences) of operands. If $\mathbf{X} = (\mathbf{x}_{ij}) \in \mathbb{IR}^{m \times l}$ and $\mathbf{Y} = (\mathbf{y}_{ij}) \in \mathbb{IR}^{l \times n}$, then the product of the matrices $\mathbf{X}$ and $\mathbf{Y}$ is a matrix $\mathbf{Z} = (\mathbf{z}_{ij}) \in \mathbb{R}^{m \times n}$ such that

$$\mathbf{z}_{ij} = \sum_{k=1}^{l} \mathbf{x}_{ik} \mathbf{y}_{kj}.$$

The above semantic interpretation is not always the case for interval matrix operations, remaining valid only for addition and subtraction. In multiplication, we have merely their weak forms. Even for the proper $\mathbf{X}, \mathbf{Y}$, in contrast to the pure equality (13), there holds [19]

$$\mathbf{XY} = \text{ the interval hull of } \{XY \mid X \in \mathbf{X}, Y \in \mathbf{Y}\}$$

where the "interval hull" is the smallest (with respect to inclusion) interval matrix containing the original set. Still some semantic conclusions can be done too. Further, we will consider the extended interval matrix-vector multiplication at length. In particular, if $\mathbf{Y}$ is a proper interval matrix, $\mathbf{v}$ is an improper interval vector and $\vee$, $\wedge$ are componentwise extensions of the corresponding lattice operations, then

$$\mathbf{Av} = \bigvee_{A \in \mathbf{A}} \bigwedge_{v \in \mathrm{pro}\, \mathbf{v}} Av = \bigwedge_{v \in \mathrm{pro}\, \mathbf{v}} \bigvee_{A \in \mathbf{A}} Av = \bigwedge_{v \in \mathrm{pro}\, \mathbf{v}} \mathbf{A}v. \tag{18}$$

The definitions of the midpoint, the absolute value, etc. are carried over to the entire arithmetic $\mathbb{IR}$ in a straightforward way and, in the componentwise manner, over the interval space $\mathbb{IR}^n$.

Finally, the norm $\|\cdot\|$ is defined on $\mathbb{IR}^n$ as in (2), while the topology on the extended interval space $\mathbb{IR}^n$ is set, similar to (1), by the metric

$$\varrho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus \mathbf{y}\|. \tag{19}$$

All the extended arithmetical operations, the matrix-vector operations in $\mathbb{IR}^n$ as well as the operations $\vee$, $\wedge$ and the basic involutions of $\mathbb{IR}^n$ are continuous in the metric (19) (see [12]).

3. Immersion into linear space

3.1. Definition and basic properties

The problem we have thus arrived at is to find an algebraic solution to the interval linear system

$$\mathbf{A}x = \mathbf{b}.$$

Essentially, it is the ordinary problem of the solution of an operator equation, which much of the traditional numerical analysis deals with. The peculiarity of the situation is that the main set $\mathbb{IR}^n$ on which the equation is considered is not a linear space at all, which is due to the lack of distributivity. So most of the existing computational approaches are not directly applicable to our problem.

In actual fact, we can easily avoid this difficulty by making use of an *embedding* of $\mathbb{IR}^n$ in the common and well studied Euclidean space $\mathbb{R}^{2n}$. It is fairly simple to realize that each bijection $\iota : \mathbb{IR}^n \to \mathbb{R}^{2n}$ induces the bijection

$$\iota^\natural : (\mathbb{IR}^n)^{\mathbb{IR}^n} \to (\mathbb{R}^{2n})^{\mathbb{R}^{2n}}$$

from the set of all mappings over $\mathbb{IR}^n$ to the set of all mappings over $\mathbb{R}^{2n}$, such that each $\psi : \mathbb{IR}^n \to \mathbb{IR}^n$ is matched to the *induced mapping*

$$\psi^\iota = \iota \circ \psi \circ \iota^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \tag{20}$$

where "$\circ$" stands for composition. We can thereby change the problem of solution of the equation in $\mathbb{IR}^n$ to the problem of solution of the equation in $\mathbb{R}^{2n}$, a situation that modern numerical analysts are much used to. The major question about the construction of the embedding is to provide a reasonable compromise between its simplicity and convenient form of the induced mappings ψ^ι. We take the following

Definition 2. *A bijective mapping $\iota : \mathbb{IR}^n \to \mathbb{R}^{2n}$ is said to be an immersion of $\mathbb{IR}^n$ into $\mathbb{R}^{2n}$ provided that it satisfies the properties*

(i) ι is an isomorphism of the additive groups $\mathbb{IR}^n$ and $\mathbb{R}^{2n}$,
(ii) ι is a homeomorphism of the topological spaces $\mathbb{IR}^n$ and $\mathbb{R}^{2n}$.

It follows immediately from this definition that

$$\begin{aligned} \iota(0_{\mathbb{IR}^n}) &= 0_{\mathbb{R}^{2n}}, \\ \iota(\operatorname{opp} \mathbf{x}) &= -\iota(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{IR}^n. \end{aligned}$$

In addition, the inverse mapping $\iota^{-1} : \mathbb{R}^{2n} \to \mathbb{IR}^n$ also satisfies conditions (i)–(ii) from the definition of ι, and

$$\begin{aligned} \iota^{-1}(0_{\mathbb{R}^{2n}}) &= 0_{\mathbb{IR}^n}, \\ \iota^{-1}(-x) &= \operatorname{opp} \iota^{-1}(x), \qquad x \in \mathbb{R}^{2n}. \end{aligned}$$

Proposition 4. *An immersion is a positive-homogeneous mapping.*

Proof. It is standard. If $\mathbf{x} \in \mathbb{IR}^n$ and k is a positive integer, then

$$\iota(k\mathbf{x}) = \iota(\underbrace{\mathbf{x} + \mathbf{x} + \cdots + \mathbf{x}}_{k}) = k\iota(\mathbf{x}).$$

If $k = 1/l$ for some positive integer l, then

$$\underbrace{\iota(k\mathbf{x}) + \iota(k\mathbf{x}) + \cdots + \iota(k\mathbf{x})}_{l} = \iota(\mathbf{x}) \quad \Rightarrow \quad \iota(k\mathbf{x}) = l^{-1}\iota(\mathbf{x}) = k\iota(\mathbf{x}).$$

Thus, the equality $\iota(k\mathbf{x}) = k\iota(\mathbf{x})$ is valid for each positive rational k. Extension to all nonnegative reals is performed by passage to the limit, making use of the continuity of ι. □

In particular,

$$\begin{aligned} \iota(\lambda\mathbf{x} + \mu\mathbf{y}) &= \lambda\iota(\mathbf{x}) + \mu\iota(\mathbf{y}), \\ \iota^{-1}(\lambda x + \mu y) &= \lambda\iota^{-1}(x) + \mu\iota^{-1}(y) \end{aligned}$$

for any $\lambda, \mu \in \mathbb{R}^+$ and $\mathbf{x}, \mathbf{y} \in \mathbb{IR}^n$, $x, y \in \mathbb{R}^{2n}$.

Proposition 5. *If $\iota : \mathbb{IR}^n \to \mathbb{R}^{2n}$ is an immersion and T is a nonsingular linear transformation of $\mathbb{R}^{2n}$, then $(T \circ \iota)$ is also an immersion. Conversely, any other immersion κ is represented in the form $(T \circ \iota)$ for some nonsingular linear transformation $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.*

Proof. The first statement is substantiated trivially. To prove the second one, let us consider the mapping $(\kappa^{-1} \circ \iota)$. Evidently, as the composition of two isomorphisms, it is an automorphism of the additive group $\mathbb{R}^{2n}$, that is, a nonsingular linear transformation of the space $\mathbb{R}^{2n}$. We may take $T = \kappa^{-1} \circ \iota$. □

3.2. Standard immersion

The significance of Proposition 5 is that it asserts the equivalence of all immersions of $\mathbb{IR}^n$ into $\mathbb{R}^{2n}$. Any two of them are the same to within a nonsingular linear transformation of $\mathbb{R}^{2n}$, and so, when constructing a specific immersion, we may be guided only by convenience of its representation in the standard bases of $\mathbb{IR}^n$ and $\mathbb{R}^{2n}$. We recommend for practical realization the embedding of $\mathbb{IR}^n$ in $\mathbb{R}^{2n}$ that acts as follows:

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \mapsto (-\underline{\mathbf{x}}_1, -\underline{\mathbf{x}}_2, \ldots, -\underline{\mathbf{x}}_n, \overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2, \ldots, \overline{\mathbf{x}}_n) \tag{21}$$

i.e., when the 1st, 2nd, ..., n-th components of the vector $\iota(\mathbf{x})$ are set to equal to the left endpoints of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ with opposite signs, and the $(n+1)$-th, ..., $2n$-th components of $\iota(\mathbf{x})$ are set to equal to the right endpoints of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, respectively. We shall refer to the mapping (21) as the *standard immersion* σ.

Each immersion $\iota : \mathbb{IR}^n \to \mathbb{R}^{2n}$ naturally induces a partial ordering "$\sqsubseteq$" on the linear space $\mathbb{R}^{2n}$, which is an image of the inclusion order on $\mathbb{IR}^n$. Specifically, for $x, y \in \mathbb{R}^{2n}$ one can say that "x does not exceed y" and write "$x \sqsubseteq y$" if and only if $\iota^{-1}(x) \subseteq \iota^{-1}(y)$ in $\mathbb{IR}^n$. Since for any $x, y, u, v \in \mathbb{R}^{2n}$ there holds

$$\begin{aligned} x \sqsubseteq y,\ \ \alpha \in \mathbb{R}^+ \ &\Rightarrow\ \alpha x \sqsubseteq \alpha y, \\ x \sqsubseteq y,\ \ u \sqsubseteq v \ &\Rightarrow\ x + u \sqsubseteq y + v \end{aligned}$$

then the partial order "$\sqsubseteq$" conforms to the linear structure of $\mathbb{R}^{2n}$ and therefore is set by some *positive cone* $K_{\sqsubseteq} = \{x \in \mathbb{R}^{2n} \mid x \sqsupseteq 0\}$ [5, 14, 20], so that

$$x \sqsubseteq y \quad \Longleftrightarrow \quad y - x \in K_{\sqsubseteq}.$$

Clearly, the concrete formulas for "$\sqsubseteq$" depends on the form of immersion, but for the standard immersion (21) they look especially simple. It is not hard to see that then

$$x \sqsubseteq y \quad \text{if and only if} \quad x \leq y \quad \text{in the componentwise sense} \tag{22}$$

that is, if $x_i \leq y_i$, $i = 1, 2, \ldots, 2n$. The positive cone under the standard immersion is correspondingly

$$K_{\leq} = \{x \in \mathbb{R}^{2n} \mid x_i \geq 0, i = 1, 2, \ldots, 2n\}.$$

Thus the induced partial order on the space $\mathbb{R}^{2n}$ coincides with the common componentwise ordering and this is the main justification of the form (21) for the standard immersion that we have chosen. Furthermore, the above is a sufficiently telling argument for us to treat from now on only the standard immersion σ of the form (21) as well as the componentwise ordering (22) on $\mathbb{R}^{2n}$, which is associated with σ. So shall we.

Consider now the properties of the mappings (20) induced over $\mathbb{R}^{2n}$ by the immersion of the interval space. The simplest and the most elementary of them are those corresponding to the multiplication by thin matrices in $\mathbb{IR}^n$.

Proposition 6. *If $\psi : \mathbb{IR}^n \to \mathbb{IR}^n$ is an operator of multiplication by a thin matrix $Q = (q_{ij})$, $Q \in \mathbb{R}^{n \times n}$, i.e., $\psi(\mathbf{x}) = Q\mathbf{x}$, then ψ^{ι} is a linear transformation of the space $\mathbb{R}^{2n}$. For the standard immersion σ, the matrix of the linear transformation ψ^{σ} has the size $2n \times 2n$ and the following block form*

$$\left(\begin{array}{c|c} Q^+ & Q^- \\ \hline Q^- & Q^+ \end{array}\right) \tag{23}$$

where the matrices $Q^+ = (q_{ij}^+)$ and $Q^- = (q_{ij}^-)$ are the positive and negative parts of Q respectively, that is, such that

$$q_{ij}^+ = \max\{q_{ij}, 0\} \quad \text{and} \quad q_{ij}^- = \max\{-q_{ij}, 0\}.$$

Proof. The first statement immediately follows from the distributivity relation

$$a \cdot (\mathbf{x} + \mathbf{y}) = a \cdot \mathbf{x} + a \cdot \mathbf{y}$$

that is valid for thin a, while the second one is a consequence of the multiplication rule

$$\lambda \cdot [\underline{x}, \overline{x}] = \text{if} \quad \lambda \in \mathbb{R}^+ \quad \text{then} \quad [\lambda\underline{x}, \lambda\overline{x}], \quad \text{else} \quad [\lambda\overline{x}, \lambda\underline{x}]$$

and formula (21). □

The block $2n \times 2n$-matrix from Proposition 6 is so significant in our theory that we shall use a special designation for it.

Definition 3. *For a given $n \times n$-matrix Q, we put*

$$Q^{\sigma} \stackrel{\text{def}}{=} \left(\begin{array}{c|c} Q^+ & Q^- \\ \hline Q^- & Q^+ \end{array}\right).$$

Notice that for any matrices $C, D \in \mathbb{R}^{n \times n}$ the relation

$$C \leq D$$

in the common componentwise sense does not necessarily imply

$$C^{\sigma} \leq D^{\sigma}.$$

The other important feature is that the matrices $Q^{\sigma} \in \mathbb{R}^{2n \times 2n}$ are always *non-negative*: such matrices must represent "$\leq$"-isotone operators on $\mathbb{R}^{2n}$ that correspond to inclusion-isotone multiplication on Q in $\mathbb{IR}^n$.

3.3. ι-nonsingular matrices

Below, the mappings Ψ that satisfy the nonsingularity condition

$$\Psi(u) \neq 0 \iff u \neq 0 \tag{24}$$

will play a leading role. It is vital to note that the nonsingularity of the point matrix Q in the sense of classical linear algebra does not necessarily mean that the corresponding operator of multiplication by Q in $\mathbb{IR}^n$ is also nonsingular in the sense of (24). The matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{25}$$

for instance, has nonzero determinant, but

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} [-1,1] \\ [1,-1] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e., this matrix generates a singular positive-homogeneous operator on $\mathbb{IR}^2$. To distinguish such cases, we give the following

Definition 4. *We say that the matrix $Q \in \mathbb{R}^{n\times n}$ is ι-nonsingular, if multiplication by it meets the nonsingularity requirement (24) on $\mathbb{IR}^n$, that is, if*

$$Q\mathbf{x} = 0 \iff \mathbf{x} = 0 \in \mathbb{IR}^n.$$

Otherwise, we call the matrix Q ι-singular.

Obviously, if a matrix is singular in the common sense, then, *a fortiori*, it is ι-singular. As a corollary of Proposition 6 we get

Proposition 7. *The point matrix $Q \in \mathbb{R}^{n\times n}$ is ι-nonsingular if and only if the matrix $Q^\sigma \in \mathbb{R}^{2n\times 2n}$ is nonsingular in the common sense, i.e., its determinant does not equal to zero.*

For example, the identity matrix

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

is ι-nonsingular, and the matrix (25) is ι-singular. All nonnegative nonsingular matrices are ι-nonsingular.

Our immediate task is to show that there are sufficient ι-nonsingular matrices.

From a more general viewpoint, the operators of multiplication by a matrix (both point and interval) are continuous positive-homogeneous operators on $\mathbb{IR}^n$. The topology on their set is determined by the standard norm

$$\|\Psi\| = \sup_{\|\mathbf{u}\|\neq 0} \frac{\|\Psi\mathbf{u}\|}{\|\mathbf{u}\|}$$

which is equivalent to

$$\|\Psi\| = \sup_{\|\mathbf{u}\|=1} \|\Psi\mathbf{u}\|.$$

Since the unit sphere is compact in $\mathbb{IR}^n$, nonsingularity of Ψ is equivalent to $\inf_{\|\mathbf{u}\|=1} \|\Psi\mathbf{u}\| > 0$.

Proposition 8. *The nonsingular positive-homogeneous operators on $\mathbb{IR}^n$ form an open subset in the set of all positive-homogeneous operators.*

Proof. Let Ψ be a nonsingular positive-homogeneous operator on $\mathbb{IR}^n$ with $\delta = \inf_{\|\mathbf{u}\|=1} \|\Psi\mathbf{u}\| > 0$, and let Θ be a positive-homogeneous operator such that $\|\Psi \ominus \Theta\| < \delta$. For any $\mathbf{u} \in \mathbb{IR}^n$, we have

$$\|\Theta\mathbf{u}\| = \|\Psi\mathbf{u} \ominus (\Psi \ominus \Theta)\mathbf{u}\| \geq \|\Psi\mathbf{u}\| - \|(\Psi \ominus \Theta)\mathbf{u}\|$$

and because of this,

$$\inf_{\|\mathbf{u}\|=1} \|\Theta\mathbf{u}\| \geq \inf_{\|\mathbf{u}\|=1} \|\Psi\mathbf{u}\| - \sup_{\|\mathbf{u}\|=1} \|(\Psi \ominus \Theta)\mathbf{u}\| = \delta - \|\Psi \ominus \Theta\| > 0.$$

Hence, the positive-homogeneous operator Θ is also nonsingular. □

We have thus shown, among other things, that the set of $\imath$-nonsingular matrices is *nonmeager* in the sense of Baire [6] under the natural choice of the topology over the sets of all matrices $\mathbb{R}^{n\times n}$ or $\mathbb{IR}^{n\times n}$, or, in other words, is of the second Baire category. In particular, the set of all $\imath$-nonsingular matrices form an everywhere dense subset in the set of all square matrices.

4. Investigating the equation

As the result of the embedding, we have reduced investigation of the mappings $\mathbb{IR}^n \to \mathbb{IR}^n$ to investigation of the mappings $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of the standard Euclidean space. In particular, we have turned the original problem that concerns us, i.e., that of finding zeros of

$$\varphi(\mathbf{u}) = \mathbf{A}\mathbf{u} \ominus \mathbf{b},$$

into the problem of solution of the equation

$$\Phi(x) = 0 \tag{26}$$

in $\mathbb{R}^{2n}$ such that $\Phi = \sigma \circ \varphi \circ \sigma^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, i.e.,

$$\Phi(x) := \sigma\big(\mathbf{A} \cdot \sigma^{-1}(x) \ominus \mathbf{b}\big) = \sigma\big(\mathbf{A} \cdot \sigma^{-1}(x)\big) - \sigma(\mathbf{b}). \tag{27}$$

On the other hand, theoretical investigation of the original equation is sometimes easier or more convenient in the interval space $\mathbb{IR}^n$, when useful properties of interval arithmetics are invoked. So, we will utilize both above representations of the equation under study.

We would like to remind the following fundamental definitions [5, 7, 14, 20]:

Definition 5. *Let U be a linear space with a partial order $\preccurlyeq$. An operator $T : U \to U$ is called isotone with respect to the partial ordering "$\preccurlyeq$" if $T(x) \preccurlyeq T(y)$ for any $x, y \in U$, $x \preccurlyeq y$. An operator T is called antitone if $T(x) \succcurlyeq T(y)$ for any $x, y \in U$, $x \preccurlyeq y$. The operator T is called positive if $x \succcurlyeq 0$ implies $T(x) \succcurlyeq 0$.*

For linear mappings, isotonity is known to be equivalent to its positivity.

Proposition 9. *The induced mapping $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, defined by (27), is isotone for any interval matrix $\mathbf{A} \in \mathbb{IR}^{2n\times 2n}$.*

Proof. Let $x, y \in \mathbb{R}^{2n}$, $x \leq y$. Then $\sigma^{-1}(x) \subseteq \sigma^{-1}(y)$, and by the inclusion monotonicity property,

$$\mathbf{A}\sigma^{-1}(x) \ominus \mathbf{b} \subseteq \mathbf{A}\sigma^{-1}(y) \ominus \mathbf{b}$$

that is,

$$\Phi(x) = \sigma\big(\mathbf{A}\sigma^{-1}(x) \ominus \mathbf{b}\big) \;\leq\; \sigma\big(\mathbf{A}\sigma^{-1}(y) \ominus \mathbf{b}\big) = \Phi(y)$$

as required. □

4.1. Existence and uniqueness of algebraic solutions

The next two results—Theorems 1 and 2—are very important, but their comprehensive proofs involve much tedious mathematics, so in this short paper we present only their brief outlines.

Theorem 1. *If the interval matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *is sufficiently narrow (that is, if* $\|\mathrm{rad}\ \mathbf{A}\|$ *is sufficiently small) and* $\mathrm{pro}\,\mathbf{A}$ *contains an* i*-nonsingular point matrix, then the interval linear equation* $\mathbf{A}x = \mathbf{b}$ *has an algebraic solution for any* $\mathbf{b} \in \mathbb{IR}^n$.

Proof. It is based on the theory of the topological degree of a mapping (see, e.g., [20]).

Let $\tilde{A}$ from $\mathbf{A}$ be an i-nonsingular point matrix. If $\mathbf{A}$ is sufficiently narrow, then we may perform a continuous nonsingular homotopy (even the linear homotopy) from the mapping (27) to the mapping

$$\tilde{\Phi}(x) := \sigma\big(\tilde{A} \cdot \sigma^{-1}(x)\big) - \sigma(\mathbf{b}).$$

As the consequence of the Poincare-Bohl theorem the topological degrees of $\Phi(x)$ and $\tilde{\Phi}(x)$ coincide with respect to a sufficiently large ball having the center in the origin of coordinates. The equation $\tilde{A}x = \mathbf{b}$ always has a solution, so does $\mathbf{A}x = \mathbf{b}$ too. □

Theorem 2. *If the interval matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *is sufficiently narrow and* $\mathrm{pro}\,\mathbf{A}$ *contains only* i*-nonsingular point matrices, then the algebraic solution to the interval linear equation* $\mathbf{A}x = \mathbf{b}$ *is unique.*

Proof. First, the equation $\mathbf{A}x = \mathbf{b}$ may have only a finite number of different isolated solutions, if all point matrices $A \in \mathrm{pro}\,\mathbf{A}$ are i-nonsingular. Second, even the above situation is impossible for sufficiently narrow $\mathbf{A}$, when the mapping (27) is "almost linear" in $\mathbb{R}^{2n}$. □

Without uniqueness, algebraic solutions to $\mathbf{A}x = \mathbf{b}$ may constitute entire order segments. From a practical standpoint, inclusion-maximal and inclusion-minimal algebraic solutions are of prime importance, and there can be several noncomparable such solutions. For example, if $\mathbf{A} = [-1, 1]$, $\mathbf{b} = [-1, 1]$, every interval of the form $[\varepsilon, 1]$ and $[-1, \varepsilon]$, $-1 \leq \varepsilon \leq 1$, is an algebraic solution. Hence, the equation has one inclusion-maximal algebraic solution $[-1, 1]$ as well as two noncomparable inclusion-minimal solutions -1 and 1.

If the set of algebraic solutions to (3) is bounded, then each algebraic solution is contained in an inclusion-maximal algebraic solution, and contains an inclusion-minimal algebraic solution to (3).

Indeed, if an algebraic solution $\mathbf{x}_a$ is not contained in a wider algebraic solution, then it is maximal. Otherwise, let us take the maximum $\bigvee\{\mathbf{y} \in \mathbb{IR}^n \mid \mathbf{A}\mathbf{y} = \mathbf{b}, \mathbf{y} \supset \mathbf{x}_a\}$ of all algebraic solutions containing $\mathbf{x}_a$. We may conclude from the continuity properties of the arithmetical and $\vee$-operations on $\mathbb{IR}$ that the interval vector so obtained is also an algebraic solution. The second assertion is proved similarly, taking the inverse partial ordering.

The next interesting example is the interval system

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} [2,-2] \\ [2,-2] \end{pmatrix}$$

which also illustrates Theorem 2 as well as Proposition 3 about the inner estimation of the united solution set. Here, we have the whole parametric family of algebraic solutions

$$\begin{pmatrix} [1+t,-1-t] \\ [1-t,-1+t] \end{pmatrix}$$

with $t \in \mathbb{R}$. All these algebraic solutions are noncomparable and each of them is simultaneously both inclusion-minimal and inclusion-maximal.

Note that Theorems 1, 2 and Proposition 9 hold true for *any* interval matrix $\mathbf{A}$ from $\mathbb{IR}^{n\times n}$ and right-hand side vector $\mathbf{b} \in \mathbb{IR}^n$, not necessarily proper. The rest of this section will be devoted to equations that arise in the implementation of the algebraic approach to the linear tolerance, control and identification problems (8), (9), (11), namely, to the equations with the *proper* interval matrices.

4.2. Maximality of the interval estimates

Kupriyanova was the first to notice that the algebraic approach to the inner approximation of the united solution set almost always gives estimates which are *maximal by inclusion*. In [15], she formulated and proved the corresponding statement, but only for nonsingular interval matrices $\mathbf{A}$. Below, we give the most general formulation of that result and its proof of our own.

Theorem 3. *If the improper interval vector $\mathbf{x}_a$ is an inclusion-minimal algebraic solution to the system $\mathbf{A}x = \operatorname{dual}\mathbf{b}$, then $\operatorname{dual}\mathbf{x}_a$ is an inclusion-maximal interval vector contained in $\Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$, that is, provides an inclusion-maximal solution to the linear identification problem (11).*

Proof. We shall carry it out *ad absurdum*. Assume that a proper interval vector $\mathbf{y}$ exists such that $\mathbf{y} \supset \operatorname{dual}\mathbf{x}_a$, $\mathbf{y} \neq \operatorname{dual}\mathbf{x}_a$. Then $\operatorname{dual}\mathbf{y} \subset \mathbf{x}_a$ and

$$\mathbf{A}\cdot(\operatorname{dual}\mathbf{y}) \subset \mathbf{A}\mathbf{x}_a = \operatorname{dual}\mathbf{b}. \tag{28}$$

It should be recognized that the equality in (28) is impossible for the minimal algebraic solution $\mathbf{x}_a$, by its very definition. In view of the representation (18), we have the strict inclusion

$$\bigwedge_{y\in\mathbf{y}} \mathbf{A}y \subset \operatorname{dual}\mathbf{b}$$

or, more precisely,

$$\begin{cases} \sup_{y\in\mathbf{y}}(\underline{\mathbf{A}y})_i \geq \overline{\mathbf{b}}_i, \\ \inf_{y\in\mathbf{y}}(\overline{\mathbf{A}y})_i \leq \underline{\mathbf{b}}_i, \qquad i = 1,2,\ldots,n, \end{cases}$$

with at least one of these $2n$ inequalities (say, the k-th) being strict.

Let us take the vector $\tilde{y} \in \mathbf{y}$ which provides that strict inequality. We may have

$$\text{either} \quad (\underline{\mathbf{A}y})_k > \overline{\mathbf{b}}_k \quad \text{or} \quad (\overline{\mathbf{A}y})_k < \underline{\mathbf{b}}_k$$

but in both cases $\mathbf{A}\tilde{y} \cap \mathbf{b} = \emptyset$. Hence, $\tilde{y} \notin \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$ due to Beeck's characterization, and so $\mathbf{y} \not\subseteq \Sigma_{\exists\exists}(\mathbf{A},\mathbf{b})$. □

The next two "maximality theorems" have been established by the author recently.

Theorem 4. *If the proper interval vector* $\mathbf{x}_a$ *is an inclusion-maximal algebraic solution to* $\mathbf{A}x = \mathbf{b}$, *then it is an inclusion-maximal interval vector contained in* $\Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, *that is, provides an inclusion-maximal solution to the linear tolerance problem (8).*

Proof. Let us take an interval vector $\mathbf{y} \supset \mathbf{x}_a$, $\mathbf{y} \neq \mathbf{x}_a$. It follows from the definition of the maximal algebraic solution that

$$\mathbf{A}\mathbf{y} \supset \mathbf{b} \tag{29}$$

where the equality is impossible. With respect to interval vectors, such as $\mathbf{A}\mathbf{y}$ and $\mathbf{b}$ are, the inclusion (29) means that a whole face of $\mathbf{A}\mathbf{y}$ exists that does not belong to $\mathbf{b}$. Formally, there exist an index $k \in \{1, 2, \ldots, n\}$ and a real number t, equal either to $(\underline{\mathbf{A}\mathbf{y}})_k$ or to $(\overline{\mathbf{A}\mathbf{y}})_k$, such that

$$\Big((\mathbf{A}\mathbf{y})_1, \ldots, (\mathbf{A}\mathbf{y})_{k-1}, t, (\mathbf{A}\mathbf{y})_{k+1}, \ldots, (\mathbf{A}\mathbf{y})_n\Big) \cap \mathbf{b} = \emptyset.$$

We now recall that

$$\mathbf{A}\mathbf{y} = \text{ the interval hull of } \{Ay \mid A \in \mathbf{A},\ y \in \mathbf{y}\}$$

so $\tilde{A}\tilde{y} \in \Big((\mathbf{A}\mathbf{y})_1, \ldots, (\mathbf{A}\mathbf{y})_{k-1}, t, (\mathbf{A}\mathbf{y})_{k+1}, \ldots, (\mathbf{A}\mathbf{y})_n\Big)$ for some $\tilde{A} \in \mathbf{A}$ and $\tilde{y} \in \mathbf{y}$. In particular, one may conclude that $\tilde{A}\tilde{y} \notin \mathbf{b}$ for some $\tilde{A} \in \mathbf{A}$. The latter just means $\tilde{y} \notin \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$, and $\mathbf{y} \not\subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$. □

Theorem 5. *If the improper interval vector* $\mathbf{x}_a$ *is an inclusion-minimal algebraic solution to* $\mathbf{A}x = \mathbf{b}$, *then* dual $\mathbf{x}_a$ *is an inclusion-maximal interval vector contained in* $\Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$, *that is, provides an inclusion-maximal solution to the control problem (9).*

Proof. If a proper interval vector $\mathbf{y}$ exists such that

$$\text{dual } \mathbf{x}_a \subset \mathbf{y} \subseteq \Sigma_{\forall\exists}(\mathbf{A}, \mathbf{b})$$

then dual $\mathbf{y} \subset \mathbf{x}_a$ and

$$\mathbf{A} \cdot (\text{dual } \mathbf{y}) \subset \mathbf{b}.$$

Here, when dual $y \neq \mathbf{x}_a$, equality is impossible by virtue of minimality of $\mathbf{x}_a$.

Invoking the representation (18), we may rewrite the latter in the form

$$\bigwedge_{y \in \mathbf{y}} \mathbf{A}y \ \subset \ \mathbf{b}.$$

By the very definition of the lattice operation "$\wedge$", the above inclusion means that

$$\mathbf{A}y \not\supseteq \mathbf{b} \tag{30}$$

for at least one point $\tilde{y} \in \mathbf{y}$, since otherwise, if all $\mathbf{A} \cdot y \supseteq \mathbf{b}$, we would have the inverse relation $\bigwedge_{y \in \mathbf{y}} \mathbf{A}y \supseteq \mathbf{b}$. Specifically, (30) implies $\tilde{y} \notin \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$, and so $\mathbf{y} \not\subseteq \Sigma_{\exists\forall}(\mathbf{A}, \mathbf{b})$ as required. □

4.3. Order convexity and subdifferentiability

If $\mathbf{A}$ is a proper interval matrix, then for any $\mathbf{u}, \mathbf{v} \in \mathbb{IR}^n$

$$\mathbf{A}(\mathbf{u}+\mathbf{v}) \subseteq \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}. \tag{31}$$

A consequence of this subdistributivity property is of crucial importance for our further consideration. Really, it follows from (31) that

$$\sigma\Big(\mathbf{A}(\mathbf{u}+\mathbf{v})\Big) \leq \sigma(\mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v})$$

and so we have for $\lambda \in (0,1)$ and any $x, y \in \mathbb{R}^{2n}$ such that $x = \sigma(\mathbf{u})$, $y = \sigma(\mathbf{v})$

$$\begin{aligned}
\Phi\Big(\lambda x + (1-\lambda)y\Big) &= \sigma\Big(\mathbf{A}\sigma^{-1}\Big(\lambda x + (1-\lambda)y\Big) \ominus \mathbf{b}\Big) \\
&= \sigma\Big(\mathbf{A}\Big(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}\Big) \ominus \mathbf{b}\Big) \\
&\leq \sigma\Big(\lambda\mathbf{A}\mathbf{u} + (1-\lambda)\mathbf{A}\mathbf{v} \ominus \mathbf{b}\Big) \\
&= \lambda\sigma(\mathbf{A}\mathbf{u} \ominus \mathbf{b}) + (1-\lambda)\sigma(\mathbf{A}\mathbf{v} \ominus \mathbf{b}) \\
&= \lambda\Phi(x) + (1-\lambda)\Phi(y).
\end{aligned} \tag{32}$$

The relationship obtained is the multidimensional analogue of the well-known convexity inequality. Let us recall the following definitions

Definition 6. *Let U, V be real linear spaces, and V is partially ordered by the order "$\preccurlyeq$". The mapping $F : U \to V$ is called order convex with respect to "$\preccurlyeq$" if*

$$F\Big(\lambda x + (1-\lambda)y\Big) \preccurlyeq \lambda F(x) + (1-\lambda)F(y)$$

for any $x, y \in U$ and $\lambda \in (0,1)$ (see [20]).

Definition 7. *A vector w is called a subgradient of the convex function $f : \mathbb{R}^k \to \mathbb{R}$ at the point x if*

$$f(z) \geq f(x) + (w, z - x)$$

for any z. The set of all subgradients of the function f at the point x is called subdifferential of the function f at x, while the function f itself is said to be subdifferentiable at x if its subdifferential is nonempty at that point. (see [2, 23]).

It is well known [2, 23] that convex bounded continuous functions are always subdifferentiable in the interiority of their domain.

The chain of transformations (32) shows that the mapping under investigation $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is order convex with respect to the common componentwise order on $\mathbb{R}^{2n}$, which is equivalent to the functionals $\Phi_i : \mathbb{R}^{2n} \to \mathbb{R}$—coordinate components of Φ—being convex for $i = 1, 2, \ldots, 2n$. Therefore, for $\Phi_i(x)$, $i = 1, 2, \ldots, 2n$, the subdifferentials are defined at any point $x \in \mathbb{R}^{2n}$, i.e., the sets of vectors $s_i \in \mathbb{R}^{2n}$, $i = 1, 2, \ldots, 2n$, such that

$$\Phi_i(x+v) - \Phi_i(x) \geq (s_i, v) \qquad \text{for all } v \in \mathbb{R}^{2n}.$$

Then, constructing $2n \times 2n$-matrix $S = (s_1, s_2, \ldots, s_{2n})^\top$, we may conclude that at each $x \in \mathbb{R}^{2n}$ the set of linear operators $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that satisfy the inequality

$$\Phi(x+v) - \Phi(x) \geq S(v)$$

for any $v \in \mathbb{R}^{2n}$ is nonempty. We will call this set the *order subdifferential* of the order convex map Φ at the point x and denote it by $\partial\Phi(x)$.[2]

Proposition 10. *All linear operators $S \in \partial\Phi(x)$ are isotone at any point $x \in \mathbb{R}^{2n}$.*

Proof. As was already mentioned, isotonity of a linear mapping is equivalent to its positivity. It is sufficient therefore to prove that $S(v) \geq 0$ for any $x, v \in \mathbb{R}^{2n}$, $S \in \partial\Phi(x)$ and $v \geq 0$.

From the definition of subdifferential we have

$$S(v) \geq \Phi(x) - \Phi(x-v).$$

On the other hand, $x \geq x - v$ and by Proposition 9

$$\Phi(x) - \Phi(x-v) \geq 0.$$

Hence, $S(v)$ is actually ≥ 0 □

4.4. Estimate of the subdifferential

To conclude this section, we consider the practical calculation of the subdifferential $\partial\Phi(x)$ and some its estimates, which is important for the implementation of the algorithm to be developed and will be utilized in the proof of its convergence as well.

The natural componentwise partial order on $\mathbb{R}^{2n}$ is the direct product of the orders "$\leq$" on $\mathbb{R}$. Therefore, the order subdifferential is the direct product of the common subdifferentials of the separate components of $\Phi(x)$. The subdifferential of the convex function $\Phi_i(x)$, $i = 1, 2, \ldots, 2n$, coincides with its gradient at its points of the differentiability [2, 23]. At the points of differentiability x, the subdifferential $\partial\Phi(x)$ thereby consists of the only element, namely, of the Jacobi matrix

$$\begin{pmatrix} \dfrac{\partial\Phi_1(x)}{\partial x_1} & \cdots & \dfrac{\partial\Phi_1(x)}{\partial x_{2n}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial\Phi_{2n}(x)}{\partial x_1} & \cdots & \dfrac{\partial\Phi_{2n}(x)}{\partial x_{2n}} \end{pmatrix} \tag{33}$$

of the mapping $\Phi(x)$.

In the general case, the subdifferential $\partial\Phi(x)$ is known to be a convex polyhedral set, with one-sided directional derivatives

$$\frac{\partial\Phi_i(x)}{\partial g} = \lim_{\alpha \searrow 0} \frac{\Phi_i(x+\alpha g) - \Phi_i(x)}{\alpha}$$

[2] It is more correctly to speak of *order $\leq$-subdifferential* or *$\leq$-order subdifferential*, but we drop the symbol "$\leq$" for brevity, since no other orders on $\mathbb{R}^{2n}$ are considered.

$g \in \mathbb{R}^{2n}$, $\alpha \in \mathbb{R}_+$, being the *support functions* for the component subdifferentials $\partial\Phi_i(x)$ [2, 23]. Let us denote by

$$\frac{\partial\Phi_i(x)}{\partial x_j^-} = \lim_{\alpha \searrow 0} \frac{\Phi_i(x_1, \ldots, x_{j-1}, x_j - \alpha, x_{j+1}, \ldots, x_{2n}) - \Phi_i(x_1, \ldots, x_{2n})}{\alpha}$$

and

$$\frac{\partial\Phi_i(x)}{\partial x_j^+} = \lim_{\alpha \searrow 0} \frac{\Phi_i(x_1, \ldots, x_{j-1}, x_j + \alpha, x_{j+1}, \ldots, x_{2n}) - \Phi_i(x_1, \ldots, x_{2n})}{\alpha}$$

that is, the one-sided partial derivatives, from the left and from the right, with respect to the jth coordinate direction. Assuming differentiability,

$$\frac{\partial\Phi_i(x)}{\partial x_j^-} = \frac{\partial\Phi_i(x)}{\partial x_j^+} = \frac{\partial\Phi_i(x)}{\partial x_j}.$$

For the functions $\Phi_i(x)$, the existence of the one-sided derivative with respect to any coordinate direction in no way affects the existence of the derivative with respect to the other variables. Besides, the matrices of the form (33) composed of the one-sided derivatives obviously belong to the subdifferential $\partial\Phi(x)$. Hence, $\partial\Phi(x)$ is a direct product of the "partial" subdifferentials, that is, an interval matrix whose elements are $[\partial\Phi_i(x)/\partial x_j^-, \partial\Phi_i(x)/\partial x_j^+]$.

More precisely, if $\pm \in \{+, -\}$, then we have for $i = 1, 2, \ldots, n$

$$\begin{aligned}
\frac{\partial\Phi_i(x)}{\partial x_j^{\pm}} &= \frac{\partial}{\partial x_j^{\pm}} \Big(\sigma\big(\mathbf{A} \cdot \sigma^{-1}(x)\big) - \sigma(\mathbf{b}) \Big)_i \\
&= \frac{\partial}{\partial x_j^{\pm}} \sum_{k=1}^{n} \Big(\underline{-\mathrm{a}_{ik} \big(\sigma^{-1}(x)\big)_k} \Big) \\
&= \begin{cases} -\dfrac{\partial}{\partial x_j^{\pm}} \Big(\underline{\mathrm{a}_{ij} \big(\sigma^{-1}(x)\big)_j} \Big), & \text{if } \ j \in \{1, 2, \ldots, n\}, \\ \dfrac{\partial}{\partial x_j^{\pm}} \Big(\underline{\mathrm{a}_{i,j-n} \big(\sigma^{-1}(x)\big)_{j-n}} \Big), & \text{if } \ j \in \{n+1, \ldots, 2n\} \end{cases} \\
&= \left(\begin{array}{c} \text{modulus of the one-sided derivative} \\ \text{of } \underline{\mathrm{a}_{ij}\mathbf{y}}, \text{ if } j \in \{1, 2, \ldots, n\}, \text{ or} \\ \text{of } \underline{\mathrm{a}_{i,j-n}\mathbf{y}}, \text{ if } j \in \{n+1, \ldots, 2n\}, \\ \text{with respect to an endpoint of } \mathbf{y} \end{array} \right).
\end{aligned} \tag{34}$$

Similarly, for $i = n+1, \ldots, 2n$,

$$\frac{\partial\Phi_i(x)}{\partial x_j^{\pm}} = \left(\begin{array}{c} \text{modulus of the one-sided derivative} \\ \text{of } \overline{\mathrm{a}_{i-n,j}\mathbf{y}}, \text{ if } j \in \{1, 2, \ldots, n\}, \text{ or} \\ \text{of } \overline{\mathrm{a}_{i-n,j-n}\mathbf{y}}, \text{ if } j \in \{n+1, \ldots, 2n\}, \\ \text{with respect to an endpoint of } \mathbf{y} \end{array} \right). \tag{35}$$

Close inspection of Table 2 shows that the derivatives (34)–(35) are equal either to the absolute value of an endpoint of the interval a_{ij} (respectively, $\mathrm{a}_{i,j-n}$, $\mathrm{a}_{i-n,j}$, $\mathrm{a}_{i-n,j-n}$) or, if $0 \in \mathrm{a}_{ij}$ (respectively, $\mathrm{a}_{i,j-n}$, $\mathrm{a}_{i-n,j}$, $\mathrm{a}_{i-n,j-n}$), the derivatives (34)–(35) may equal to zero. In any event,

$$\begin{aligned}
\frac{\partial\Phi_i(x)}{\partial x_j^{\pm}} &\in [\langle \mathbf{a}_{ij}\rangle, |\mathbf{a}_{ij}|], && \text{if } i,j \le n,\\
\frac{\partial\Phi_i(x)}{\partial x_j^{\pm}} &\in [\langle \mathbf{a}_{i,j-n}\rangle, |\mathbf{a}_{i,j-n}|], && \text{if } i \le n < n+1 \le j,\\
\frac{\partial\Phi_i(x)}{\partial x_j^{\pm}} &\in [\langle \mathbf{a}_{i-n,j}\rangle, |\mathbf{a}_{i-n,j}|], && \text{if } j \le n < n+1 \le i,\\
\frac{\partial\Phi_i(x)}{\partial x_j^{\pm}} &\in [\langle \mathbf{a}_{i-n,j-n}\rangle, |\mathbf{a}_{i-n,j-n}|], && \text{if } n+1 \le i,j.
\end{aligned}$$

So, overall, the following estimate

$$\begin{pmatrix} \langle \mathbf{A}\rangle & \langle \mathbf{A}\rangle \\ \langle \mathbf{A}\rangle & \langle \mathbf{A}\rangle \end{pmatrix} \le \partial\Phi(x) \le \begin{pmatrix} |\mathbf{A}| & |\mathbf{A}| \\ |\mathbf{A}| & |\mathbf{A}| \end{pmatrix} \tag{36}$$

is valid.

5. Algorithms

5.1. A brief overview

To solve an operator equation, one often reduces it to the form

$$x = G(x). \tag{37}$$

Then, having chosen some $x^{(0)}$, we iterate

$$x^{(k+1)} = G(x^{(k)}). \tag{38}$$

Under suitable conditions, $x^{(k)}$ converges to the fixed point x^* of the mapping G, that is, to the solution of (37). To obtain more sophisticated numerical algorithms for finding algebraic solutions, it is natural to avail ourselves of the fact that the main space (either $\mathbb{IR}^n$ or $\mathbb{R}^{2n}$) carries the additional partial ordering structure (inclusion order or its induced). The solution of both algebraic and functional equations in partially ordered spaces is a developed branch of modern numerical analysis. Classical results on this subject may be found, for instance, in the well known books by Krasnoselskii [14], Collatz [7], Ortega and Rheinboldt [20] and others. The standard ways to solve equations of the type (37) with monotone and their related operators rely upon the facts that are variations of the widely known *Kantorovich lemma* (see [20] as well as [7, 14] where it appears in the nameless form). This kind of methods for computing the algebraic solution were developed in the works by Zyuzin [30–32] and Kupriyanova [15]. Their main result is a stationary one-step iterative method in $\mathbb{IR}^n$, which can be formulated as follows:

Let the operator $H : \mathbb{IR}^n \to \mathbb{IR}^n$ be defined by

$$\big(H(\mathbf{x})\big)_i = \left(\mathbf{b}_i \ominus \sum_{j=1, j\ne i}^{n} \mathbf{a}_{ij}\mathbf{x}_j \right) \oslash \mathbf{a}_{ii}, \qquad i = 1, 2, \ldots, n \tag{39}$$

and there exist intervals $\mathbf{v}^{(0)}, \mathbf{w}^{(0)} \in \mathbb{IR}^n$, such that $\mathbf{v}^{(0)} \subseteq \mathbf{w}^{(0)}$ and

$$\begin{aligned} \mathbf{v}^{(0)} &\subseteq H(\mathbf{v}^{(0)}) \subseteq \mathbf{w}^{(0)}, \\ \mathbf{v}^{(0)} &\subseteq H(\mathbf{w}^{(0)}) \subseteq \mathbf{w}^{(0)} \end{aligned}$$

(i.e., the order segment $[\mathbf{v}_0, \mathbf{w}_0]$ is H-invariant). Then the sequences

$$\mathbf{v}^{(k)} := H(\mathbf{w}^{(k-1)}), \tag{40}$$

$$\mathbf{w}^{(k)} := H(\mathbf{v}^{(k-1)}), \qquad k = 1, 2, \ldots \tag{41}$$

converges to fixed points $\mathbf{v}^*$ and $\mathbf{w}^*$ of the operator H (which are algebraic solutions to the original equation),

$$\mathbf{v}^{(0)} \subseteq \mathbf{v}^{(1)} \subseteq \cdots \subseteq \mathbf{v}^* \subseteq \mathbf{w}^* \subseteq \cdots \subseteq \mathbf{w}^{(1)} \subseteq \mathbf{w}^{(0)}$$

and any fixed point $\mathbf{u}^* \in [\mathbf{v}^{(0)}, \mathbf{w}^{(0)}]$ of the operator H lies within $[\mathbf{v}^{(0)}, \mathbf{w}^{(0)}]$.

How can an initial approximation for the method (39)–(41) be found? That is a crucial point and, unfortunately, there are no satisfactory prescriptions for such a selection in general. If we seek a proper algebraic solution to the system (3), then, as Zyuzin has shown in [32], the desired choice of invariant order segment for (39) reduces to solution of an auxiliary linear tolerance problem. This result (which is in a sense a conversion to Proposition 1) indicates that finding an invariant order interval for the scheme described above turns out to be at any rate no easier than the original problem.

The other significant disadvantage that is intrinsic to all simple iterative schemes (38) (in particular, to (39)) and their modifications is the low convergence rate, which is merely linear.

5.2. Subdifferential Newton method

We have managed to overcome the above drawbacks (more or less) successfully. One of the main mathematical results of this work is the following iterative algorithm that solves the equation (26)–(27) in the enveloping space $\mathbb{R}^{2n}$:

Algorithm I
(subdifferential Newton method with a special starting approximation)

As the starting vector $x^{(0)}$ take the solution of the "midpoint" system

$$(\operatorname{mid} \mathbf{A})^{\sigma} x = \sigma(\mathbf{b}).$$

If the k-th approximation $x^{(k)}$, $k = 0, 1, \ldots$, has already been found, then compute any $S^{(k)} \in \partial\Phi(x^{(k)})$ and put

$$x^{(k+1)} := x^{(k)} - \tau\left(S^{(k)}\right)^{-1}\left(\Phi(x^{(k)})\right).$$

Here, τ is a damping factor from $(0, 1]$ whose choice should be considered separately. We incline to recommend τ to be equal or close to 1. Our computational experience shows that

then, as a rule, Algorithm I gives *exact* solution to the problem in a small finite number of iterations, which usually does not exceed the dimension of the system. In this kind of method, the damping factor was originally introduced to prevent divergence. However, it is not quite clear for me yet whether taking $\tau < 1$ really improves convergence of the subdifferential Newton method or not. Anyway, the less τ, the slower Algorithm I works.

Complete investigation of the subdifferential Newton method is beyond the scope of the present work. Computer experiments with it demonstrate very interesting and surprising phenomena (see Section 6), which are to be studied thoroughly. Below, we shall prove, based on the standard technique (see, e.g., [20]), a local convergence theorem that amounts to the following:

Theorem 6. *If the proper interval matrix* $\mathbf{A}$ *is sufficiently narrow and all point* $2n \times 2n$*-matrices* C *that satisfy*

$$\begin{pmatrix} \langle \mathbf{A} \rangle & \langle \mathbf{A} \rangle \\ \langle \mathbf{A} \rangle & \langle \mathbf{A} \rangle \end{pmatrix} \le C \le \begin{pmatrix} |\mathbf{A}| & |\mathbf{A}| \\ |\mathbf{A}| & |\mathbf{A}| \end{pmatrix}$$

are nonsingular, then Algorithm I converges to $\sigma(\mathbf{x}_a)$*, where* $\mathbf{x}_a$ *is an algebraic solution of the system* $\mathbf{A}x = \mathbf{b}$.

Proof. Let us specify what is meant by a "sufficiently narrow" interval matrix $\mathbf{A}$. We shall require that

> the convex hull of the set
>
> $$\bigcup \left\{ C^{-1}K_{\le} \;\middle|\; C \in \mathbb{R}^{2n \times 2n}, \begin{pmatrix} \langle \mathbf{A} \rangle \langle \mathbf{A} \rangle \\ \langle \mathbf{A} \rangle \langle \mathbf{A} \rangle \end{pmatrix} \le C \le \begin{pmatrix} |\mathbf{A}||\mathbf{A}| \\ |\mathbf{A}||\mathbf{A}| \end{pmatrix} \right\} \tag{42}$$
>
> (where $C^{-1}K_{\le}$ denotes the preimage of the cone $K_{\le}$ under the linear transformation C) itself is a cone $K_{\trianglelefteq}$ in $\mathbb{R}^{2n}$, that is, defines a partial ordering "$\trianglelefteq$" of $\mathbb{R}^{2n}$.

This is not an arbitrary condition. If the matrix $\mathbf{A}$ is thin, that is, $\mathbf{A} = A$, then $(A^{\sigma})^{-1}K_{\le}$ is actually a cone, being the image of the cone in the linear transformation. If the matrices $C_1, C_2 \in \mathbb{R}^{2n \times 2n}$ are "sufficiently close" then the cones $C_1^{-1}K_{\le}$ and $C_2^{-1}K_{\le}$ are close too, and their convex hull is still a cone. So the condition (42) reflects "narrowness" of the interval matrix $\mathbf{A}$ in some sense indeed.

It is worth noting that if a matrix $C \in \mathbb{R}^{2n \times 2n}$ satisfies

$$\begin{pmatrix} \langle \mathbf{A} \rangle & \langle \mathbf{A} \rangle \\ \langle \mathbf{A} \rangle & \langle \mathbf{A} \rangle \end{pmatrix} \le C \le \begin{pmatrix} |\mathbf{A}| & |\mathbf{A}| \\ |\mathbf{A}| & |\mathbf{A}| \end{pmatrix}$$

then it is nonnegative, i.e.,

$$CK_{\le} \subseteq K_{\le}.$$

This implies

$$C^{-1}K_{\le} \supseteq K_{\le}$$

so that

$$K_{\trianglelefteq} \supseteq K_{\le}$$

(one could say that "the partial order $\trianglelefteq$ is weaker than the common $\le$").

Further, owing to the special choice of the starting vector,

$$\mathbf{A} \cdot \sigma^{-1}(x^{(0)}) \supseteq (\operatorname{mid} \mathbf{A}) \cdot \sigma^{-1}(x^{(0)}) = \mathbf{b}$$

so

$$\Phi(x^{(0)}) \geq 0.$$

Next note that by the definition of subdifferential

$$\Phi(x^{(k+1)}) \geq \Phi(x^{(k)}) + S^{(k)}(x^{(k+1)} - x^{(k)})$$

for $S^{(k)} \in \partial\Phi_i(x^{(k)})$ and any $k = 0, 1, 2, \ldots$, while by virtue of Algorithm I

$$S^{(k)}(x^{(k+1)} - x^{(k)}) = -\tau\Phi(x^{(k)}). \tag{43}$$

Hence, for $0 < \tau \leq 1$ we get by induction

$$\Phi(x^{(k)}) \geq 0, \qquad k = 0, 1, 2, \ldots \tag{44}$$

which, bearing in mind the representation (27) for $\Phi(x)$, is equivalent under our assumption to

$$x^{(k)} \trianglerighteq \sigma^{-1}(\mathbf{b}), \qquad k = 0, 1, 2, \ldots$$

The other important point is that the sequence $\{x^{(k)}\}$ generated by our algorithm is monotonically decreasing with respect to the order "$\trianglelefteq$", that is

$$x^{(k)} \trianglerighteq x^{(k+1)} \tag{45}$$

for all $k = 0, 1, 2, \ldots$. Indeed, combining (43) and (44) one obtains

$$S^{(k)}(x^{(k+1)} - x^{(k)}) \leq 0.$$

As we could see, the inequality (36)

$$\begin{pmatrix} \langle \mathbf{A} \rangle & \langle \mathbf{A} \rangle \\ \langle \mathbf{A} \rangle & \langle \mathbf{A} \rangle \end{pmatrix} \leq S^{(k)} \leq \begin{pmatrix} |\mathbf{A}| & |\mathbf{A}| \\ |\mathbf{A}| & |\mathbf{A}| \end{pmatrix}$$

holds for each $S^{(k)}$, and thereby, in view of (42), (45) follows.

So, we have proved

$$x^{(k)} \trianglerighteq x^{(k+1)} \trianglerighteq \sigma^{-1}(\mathbf{b})$$

and there exists a limit x^* of the sequence $\{x^{(k)}\}$. We can find it by solving the fixed-point equation

$$x^* = x^* - \tau\left(S^{(k)}\right)^{-1}\left(\Phi(x^*)\right).$$

Therefore, $\Phi(x^*) = 0$. □

Note that this result may be taken as yet another way of proving the existence of an algebraic solution to the equation $\mathbf{A}x = \mathbf{b}$ in extended interval arithmetic.

6. Computational tests

In this section we summarize numerical experiments carried out with the subdifferential Newton method on a PC/AT 486. The Algorithm I was implemented using Turbo C in standard double precision floating point arithmetic.

Now let us illustrate the theoretical constructions of the previous sections with the results of numerical tests.

Example 1, the classical one (see Figure 1).

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$

When used $\tau = 1$, Algorithm I gives, in merely 2 iterations, the exact algebraic solution—the vector $([-0.333\ldots.0.333\ldots],[-0.333\ldots.0.333\ldots])^{\top}$—which is easily seen to be the inclusion-maximal interval vector contained in the tolerable solution set to this system. On the other hand, the absolute error of the result is in the order of 10^{-7} only after 22 iterations in the method (39) based on the Kantorovich lemma [32, 33].

Example 2.

$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} x = \begin{pmatrix} [2,-2] \\ [2,-2] \end{pmatrix}$$

As in the previous case, Algorithm I converges to the exact algebraic solution $([1,-1],[1,-1])^{\top}$ in 2 iterations for $\tau = 1$. One can readily see from Figure 1 that the vector dual $([1,-1],[1,-1])^{-}$ provides a good inner approximation for the united solution set of the system (7), even maximal by inclusion.

Example 3 [30].

$$\begin{pmatrix} [2,4] & [-5,-1] & [-2,3] \\ [-3,1] & [5,7] & [4,6] \\ [-1,1] & [-2,1] & [-7,-2] \end{pmatrix} x = \begin{pmatrix} [-28,43] \\ [-60,29] \\ [-11,39] \end{pmatrix}$$

With $\tau = 1$, Algorithm I converges in 4 iterations to the exact algebraic solution $([2,5],[-3,4],[-4,-1])^{\top}$.

Example 4, when Algorithm I diverges.

$$\begin{pmatrix} [3,4] & [-5,-2] & [-2,2] \\ [-3,-1] & [6,7] & [5,6] \\ [-1,0] & [-1,1] & [-4,1] \end{pmatrix} x = \begin{pmatrix} [-28,43] \\ [-60,69] \\ [-11,39] \end{pmatrix}.$$

For this interval linear system and with $\tau = 1$, Algorithm I generates an oscillating sequence that evidently does not converge to any limit. It is interesting to note that the right-hand side of this system is wider than that of the previous example, while all elements of the matrix but a_{33} are thinner. Nevertheless, the method fails.

Example 5. Let us turn to the interval linear systems with the matrix from [29]:

$$\begin{pmatrix} [n-1,N] & [\alpha-1,1-\beta] & \ldots & [\alpha-1,1-\beta] \\ [\alpha-1,1-\beta] & [n-1,N] & \ldots & [\alpha-1,1-\beta] \\ \vdots & \vdots & \ddots & \vdots \\ [\alpha-1,1-\beta] & [\alpha-1,1-\beta] & \ldots & [n-1,N] \end{pmatrix} x = \begin{pmatrix} [1-n,n-1] \\ [1-n,n-1] \\ \vdots \\ [1-n,n-1] \end{pmatrix}$$

where n denotes dimension ($n \geq 2$), $0 < \alpha \leq \beta \leq 1$ and N is a real number $\geq n - 1$. For any tested dimensions $1 \leq n \leq 45$, any pseudorandomly chosen N and arbitrary right-hand interval vector, Algorithm I converged to exact algebraic solution of the system in 2 iterations.

Example 5, mysterious behavior of the subdifferential Newton method.

For the interval linear 7×7-system

$$\begin{pmatrix} [4,6] & [-9,0] & [0,12] & [2,3] & [5,9] & [-23,-9] & [15,23] \\ [0,1] & [6,10] & [-1,1] & [-1,3] & [-5,1] & [1,15] & [-3,-1] \\ [0,3] & [-20,-9] & [12,77] & [-6,30] & [0,3] & [-18,1] & [0,1] \\ [-4,1] & [-1,1] & [-3,1] & [3,5] & [5,9] & [1,2] & [1,4] \\ [0,3] & [0,6] & [0,20] & [-1,5] & [8,14] & [-6,1] & [10,17] \\ [-7,-2] & [1,2] & [7,14] & [-3,1] & [0,2] & [3,5] & [-2,1] \\ [-1,5] & [-3,2] & [0,8] & [1,11] & [-5,10] & [2,7] & [6,82] \end{pmatrix} x = \begin{pmatrix} [-10,95] \\ [35,14] \\ [-6,2] \\ [30,7] \\ [4,95] \\ [-6,46] \\ [-2,65] \end{pmatrix}.$$

Algorithm I converges to the exact algebraic solution

$$\begin{pmatrix} [-1.22474317578, 0.50542987670] \\ [18.26444337096, -9.51750410300] \\ [-0.02818650587, 1.16075521933] \\ [16.40769576636, -14.45553419850] \\ [-1.34356527337, 3.98821848038] \\ [-3.52893852104, 4.54345836822] \\ [5.43086236811, -0.67400838683] \end{pmatrix}$$

in 9 iterations. At the same time, if we replace the element a_{77} in the matrix by the interval $[10.82]$, which is narrower than the original one, the Algorithm I diverges.

An advertisement

Public domain software that implements subdifferential Newton method (as well as its text in C) is available.

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