Stability of Fredholm properties on interpolation scales

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1. Introduction

We begin with a review of the basic notions of interpolation theory. Let $A=(A_0, A_1)$ be an interpolation couple of Banach spaces, i.e. A_0 and A_1 are Banach spaces which are continuously embedded in a Hausdorff topological vector space. The vector spaces $\Delta(A)=A_0\cap A_1$ and $\sum(A)=A_0+A_1$ are also Banach spaces with respect to the norms $\|\cdot\|_{A_0\cap A_1}$ and $\|\cdot\|_{A_0+A_1}$ given by:

$$\|a\|_{A_0\cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$$

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0}+\|a_1\|_{A_1} \mid a_0\in A_0,\ a_1\in A_1,\ a = a_0+a_1\}.$$

Let $S=\{z\in C \mid 0\leqq \operatorname{Re} z\leqq 1\}$, $S_0=\{z\in C \mid 0<\operatorname{Re} z<1\}$. Given an interpolation couple A, $F(A)$ is defined as the space of all $\sum(A)$-valued analytic functions $f(z)$ on S_0 which are bounded and continuous on S, and which also satisfy for $j=0, 1: f(j+it)$: $R\to A_j$ are continuous and $\lim_{|t|\to\infty} f(j+it)=0$.

We define a norm on $F(A)$:

$$\|f\|_{F(A)} = \max\{\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1}\}.$$

$(F(A); \|\cdot\|_{F(A)})$ is a Banach space.

Definition 1.1. The space A_θ consists of all $a\in\sum(A)$ such that $a=f(\theta)$ for some $f\in F(A)$. Define a norm on A_θ:

$$\|a\|_{A_\theta} = \inf\{\|f\|_{F(A)} \mid f(\theta) = a,\ f\in F(A)\}.$$

Next, one defines another space of analytic functions $H(A)$ as follows. Functions g in $H(A)$ are defined on the strip S with values in $\sum(A)$. Moreover they have the following properties:

(1) $\|g(z)\|_{A_0+A_1} \leqq c(1+|z|)$,

(2) g is continuous on S and analytic on S_0,

(3) For $j=0, 1$, $g(j+it_1)-g(j+it_2) \in A_j$ for all real values of t_1 and t_2 and

$$\|g\|_{H(A)} = \max\left\{\sup_{t_1,t_2}\left\|\frac{g(it_1)-g(it_2)}{t_1-t_2}\right\|_{A_0},\ \sup_{t_1,t_2}\left\|\frac{g(1+it_1)-g(1+it_2)}{t_1-t_2}\right\|_{A_1}\right\}$$

is finite.

The space $H(A)$, reduced modulo constant functions and provided with the norm $\|g\|_{H(A)}$, is a Banach space.

Definition 1.2. The space A^θ consists of all $a \in \sum(A)$, such that $a=g'(\theta)$ for some $g \in H(A)$. The norm on A^θ is

$$\|a\|_{A^\theta} = \inf\{\|g\|_{H(A)} \mid g'(\theta) = a,\ g \in H(A)\}.$$

Definition 1.3. For each $t>0$, define on $\sum(A)$:

$$K(t, a) = K(t, a; A) = \inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1} \mid a_0 \in A_0,\ a_1 \in A_1,\ a = a_0+a_1\}.$$

Definition 1.4. For each $0<\theta<1$, $0<p\leqq\infty$, we define the real interpolation space $A_{\theta,p}$ as the space of all $a \in \sum(A)$ for which

$$\|a\|_{A_{\theta,p}} = \left(\int_0^\infty (t^{-\theta}K(t, a))^p\, dt/t\right)^{1/p} < \infty.$$

In this paper we consider only $1 \leqq p \leqq \infty$.

The following facts are well-known:

(1) $(A_{\theta,p};\ \|\cdot\|_{A_{\theta,p}})$ are Banach spaces. These are the real interpolation spaces of Lions and Peetre.

(2) The spaces A_θ and A^θ are Banach spaces. These are the complex interpolation spaces of Calderón.

(3) $(A_0, A_1)_\theta^* = (A_0^*, A_1^*)^\theta$.

For detailed information on these spaces see [2], [3], [6].

Definition 1.5. Let A and B be Banach spaces and let T be a linear bounded operator mapping A into B. T is a Fredholm operator if and only if its kernel has a finite dimension and $B=TA\oplus M$, with $\dim M<\infty$. $\dim M$ is the codimension of T, $\operatorname{codim} T$. The index of T is defined by $i(T)=\dim\ker T-\operatorname{codim} T$.

Remark 1.6. Let A and B be Banach spaces and let T be a linear bounded operator from A to B. We will denote by $\tilde{T}$ the reduced operator of T, i.e. $\tilde{T}$: $A/\ker T \to B$. If T has a finite codimension, i.e. $B=TA\oplus M$ with $\dim M<\infty$, then by the open mapping theorem, $\tilde{T}^{-1}$: $TA \to A/\ker T$ is bounded, and TA is closed. See e.g. [4], [9].

Let T: $A_j \to B_j$ $(j=0, 1)$ be a linear bounded operator. We will denote by T_s the restriction of T to A_s, and by $T_{s,p}$ the restriction of T to $A_{s,p}$. Let $\|T_0\|$, $\|T_1\|$,

$\|T_s\|$ and $\|T_{s,p}\|$ be the norms of the restrictions of T to A_0, A_1, A_s and $A_{s,p}$ respectively. If we denote $\|T\|=\max\{\|T_0\|, \|T_1\|\}$, then $\|T_s\|\leqq\|T\|$, and $\|T_{s,p}\|\leqq\|T\|$.

Assume that the restriction of T to an interpolation space is a Fredholm operator. The question of the stability of this property when one changes the parameters which determine the interpolation space has been considered by several authors. See [1], [8], [10]. Here we prove: if $T_{s,p}$ is a Fredholm operator, then there exist $\delta>0$, $\varepsilon>0$ such that if $|\theta-s|<\delta$ and $|q-p|<\varepsilon$ then $T_{\theta,q}$ also is a Fredholm operator and $i(T_{\theta,q})=i(T_{s,p})$. We prove this result by proving a corresponding result for complex interpolation and then using a reiteration theorem:

$$A_{s,p}=(A_{s_0,p}, A_{s_1,p})_\zeta \quad A_{\theta,p}=(A_{\theta,p_0}, A_{\theta,p_1})_\eta$$

for appropriate ζ and η.

Sneĭberg in [8] has shown that if $T\colon (A_0, A_1)_\eta\to(B_0, B_1)_\eta$ is a Fredholm operator then there exists $\delta>0$ so that $|\eta-\zeta|<\delta$ implies that $T_\zeta\colon (A_0, A_1)_\zeta\to(B_0, B_1)_\zeta$ is a Fredholm operator (and with the same index). To apply the complex interpolation result to real interpolation scales, however, we need control on the length of the interval of η's for which the stability of the Fredholm property holds. The point is that we need to iterate twice and in Sneĭberg's result the length of the interval of stability of the Fredholm property depends on a certain projection operator and the norm of $\tilde{T}_{s,p}^{-1}$. Sneĭberg's proof does not give control either on the projection operator or on $\|\tilde{T}_{\theta,q}^{-1}\|$ in terms of the behaviour of T on $A_{s,p}$. Therefore if we use Sneĭberg's proof we cannot iterate the result to get a neighborhood of (s,p) of the form $|\theta-s|<\delta$, $|q-p|<\varepsilon$. We therefore give a different proof which gives the required control on the interval of stability of the Fredholm property.

Definition 1.7. Let A, B be Banach spaces and let T be a linear bounded operator from A into B. Then T has property W with constants (k, d) if there exists a subspace M of B, $\dim M=d<\infty$, and for any $y\in B$, there exist $x\in A$, $z\in M$, so that $y=Tx+z$, and

$$\|x\|_A\leqq k\|y\|_B, \quad \|z\|_B\leqq k\|y\|_B.$$

Remark 1.8. Let $A=(A_0, A_1)$, $B=(B_0, B_1)$ be Banach interpolation couples and $T\colon A_j\to B_j$ $(j=0, 1)$ be a linear bounded operator. If T_s has property W with constants (k, d), then there exists a subspace M of B_s so that $\forall y\in B_s$, there exist $f\in F(A)$ and $z\in M$ such that $y=Tf(s)+z$, $\|f\|_{F(A)}\leqq(k+1)\|y\|_{B_s}$, and $\|z\|_{B_s}\leqq k\|y\|_{B_s}$.

Lemma 1.9. *Let A, B be two Banach spaces and let T be a linear bounded operator from A into B. Then T is a Fredholm operator if and only if T and its dual T^* have property W.*

Proof. Let T be a Fredholm operator. Since T has finite codimension d, by remark 1.6, $\tilde{T}^{-1}$ is a linear bounded operator defined on TA. There exists a bounded projection P of B onto TA with $\|P\| \leqq d+1$ (see e.g. [7]). If we denote $(I-P)B=M$, then $\dim M=d$.

For any $b \in B$, $b=Pb+(I-P)b=y+z$ with $y \in TA$ and $z \in M$ such that $\|y\|_B \leqq (d+1)\|b\|_B$, $\|z\|_B \leqq (d+2)\|b\|_B$. There exists $x \in A$, so that $Tx=y$ and $\|x\|_A \leqq 2\|\tilde{T}^{-1}\| \, \|y\|_B \leqq 2(d+1)\|\tilde{T}^{-1}\| \, \|b\|_B$. Hence T has property W with constants $(2(d+1)(\|\tilde{T}^{-1}\|+1), d)$.

T^* also is a Fredholm operator with $\dim\ker T^* = \operatorname{codim} T$, $\operatorname{codim} T^* = \dim\ker T = d^*$ and $\|T^*\| = \|T\|$. Therefore T^* has property W with constants $(2(d^*+1)(\|(\tilde{T}^*)^{-1}\|+1), d^*)$.

Now we assume T and T^* have property W. Then $B=TA+M$ with $\dim M = d < \infty$. Hence we can find M_0 with finite dimension such that $B=TA \oplus M_0$.

Using duality, we have $\dim\ker T < \infty$, so that T is a Fredholm operator, and the theorem is proved.

An important tool in handling finite dimensional subspaces is Auerbach's lemma:

Lemma 1.10. *Let M be a Banach space with dimension $d<\infty$. Then there exist $\{e_1, \ldots, e_d\} \subset M$ and $\{f_1, \ldots, f_d\} \subset M^*$, such that*

(1) $\|e_i\|_M = 1$ *and* $\|f_i\|_{M^*} = 1$, $i=1, \ldots, d$.

(2) $f_i(e_j) = \delta_{ij}$.

See [5] for a proof.

Given a finite set of vectors $\{e_1, \ldots, e_d\}$ in a Banach space A, we define:

$$b(e_1, \ldots, e_d) = \min\left\{\left\|\sum_{i=1}^d c_i e_i\right\|_A \,\middle|\, \max|c_i| = 1\right\}.$$

Clearly $\{e_1, \ldots, e_d\}$ are independent if and only if $b(e_1, \ldots, e_d) > 0$.

We will need the following consequence of Auerbach's lemma.

Lemma 1.11. *Let M be a Banach space with dimension $d<\infty$. Then $\{e_1, \ldots, e_d\}$ can be found so that $\|e_i\|_M = 1$, and $b(e_1, \ldots, e_d) = 1$.*

Proof. Let $\{e_1, \ldots, e_d\}$ be a basis for M, $\{f_1, \ldots, f_d\}$ a basis for M^*, so that $\|e_i\|_M = 1$, $\|f_i\|_{M^*} = 1$, and $f_i(e_j) = \delta_{ij}$. We then have

$$\left\|\sum_{i=1}^d c_i e_i\right\|_M = \sup_{\|f\|_{M^*} \leqq 1} \left|f\left(\sum_{i=1}^d c_i e_i\right)\right| \geqq \sup_{\sum_{i=1}^d |\alpha_i| \leqq 1} \left|\left(\sum_{i=1}^d \alpha_i f_i\right)\left(\sum_{i=1}^d c_i e_i\right)\right|$$

$$= \sup_{\sum_{i=1}^d |\alpha_i| \leqq 1} \left|\sum_{i=1}^d \alpha_i c_i\right| = \max_i \{|c_i|\}.$$

We have shown $b(e_1, \ldots, e_d) \geqq 1$. The opposite inequality is clear, and the proof is complete.

2. Fredholm operators and complex interpolation spaces

Lemma 2.1. *Let* $A=(A_0, A_1)$ *be an interpolation couple of Banach spaces with* $A_0\cap A_1$ *dense in* A_j, $j=0, 1$ *and* $f\in F(A)$. *Let* $0<\theta, s<1$ *and define:*

$$q(z, s) = \left|\frac{d(z)-d(s)}{1-\overline{d(z)}\,d(s)}\right|$$

where $d(\cdot)$ *is a conformal map of the strip* S_0 *onto the open unit disc* D. *Then:*

$$\|f(\theta)\|_{A_\theta} \geqq \|f\|_{F(A)} \frac{\|f(s)\|_{A_s}-q(\theta, s)\|f\|_{F(A)}}{\|f\|_{F(A)}-q(\theta, s)\|f(s)\|_{A_s}}, \tag{2.1}$$

$$\|f(\theta)\|_{A_\theta} \leqq \|f(s)\|_{A_s}+q(\theta, s)\|f\|_{F(A)} \tag{2.2}$$

$$\|h'(\theta)\|_{A^\theta} \geqq \|h\|_{H(A)} \frac{\|h'(s)\|_{A^s}-q(\theta, s)\|h\|_{H(A)}}{\|h\|_{H(A)}-q(\theta, s)\|h'(s)\|_{A^s}} \tag{2.3}$$

$$\|h'(\theta)\|_{A^\theta} \leqq \|h'(s)\|_{A^s}+q(\theta, s)\|h\|_{H(A)}. \tag{2.4}$$

For the proofs of (2.1) and (2.3) see [8]. (2.2) and (2.4) are easy consequences of (2.1) and (2.3).

Remark 2.2. Suppose $u\in F(A)$ and $\|u(\theta)\|_{A_\theta}\leqq\|u\|_{F(A)}\leqq 2\|u(\theta)\|_{A_\theta}$. By (2.2), if s is close to θ so that $q(\theta, s)\leqq 1/2$, then

$$\|u(s)\|_{A_s} \geqq \|u(\theta)\|_{A_\theta}/2.$$

Lemma 2.3. *Let* A, B *be Banach spaces, and let* T *be a linear bounded operator from* A *into* B. *Assume that there exist a subspace* M *of* B *with* $\dim M=d<\infty$ *and constants* k, r *with* $r<1$, *so that* $\forall y\in B$ *there exists a decomposition* $y=Tx+z+y_1$ *with* $y_1\in B$, $x\in A$, $z\in M$, *so that*

$$\|x\|_A \leqq k\|y\|_B, \quad \|z\|_B \leqq k\|y\|_B, \quad \|y_1\|_B \leqq r\|y\|_B,$$

then T *has property* W *with constants* $(k/(1-r), d)$.

Proof. By repeated applications of the hypotheses we have $y=T\sum_{i=1}^n x_i+\sum_{i=1}^n z_i+y_n$, with $z_i\in M$, $y_n\in B$, $x_i\in A$, and

$$\|y_n\|_B \leqq r^n\|y\|_B$$

$$\|x_i\|_A \leqq k\|y_{i-1}\|_B \leqq kr^{i-1}\|y\|_B$$

$$\|z_i\|_B \leqq k\|y_{i-1}\|_B \leqq kr^{i-1}\|y\|_B.$$

Since A and B are Banach spaces and M is a closed subspace of B, we have

$$y_n \to 0, \quad \sum_{i=1}^n x_i \to x\in A, \quad \sum_{i=1}^n z_i \to z\in M, \quad \text{as} \quad n\to\infty.$$

Therefore $y=Tx+z$, with $\|x\|_A\leqq\frac{k}{1-r}\|y\|_B$ and $\|z\|_B\leqq\frac{k}{1-r}\|y\|_B$, and the proof is complete.

Theorem 2.4. *Let $A=(A_0, A_1)$ and $B=(B_0, B_1)$ be Banach interpolation couples and let $T\colon A_j\to B_j$ $(j=0, 1)$ be a linear bounded operator. If T_s has property W with constants (k, d), then there exists $\delta>0$ which depends only on k, d and $\|T\|$ such that $|\theta-s|<\delta$ implies that T_θ has property W with constants $(8(k+1)d, d)$, and* $\operatorname{codim} T_\theta\leqq d$.

Proof. Since $A_\theta=((A_0, A_1)_{s_0}, (A_0, A_1)_{s_1})_\zeta$, where $(1-\zeta)s_0+\zeta s_1=\theta$, and since $(A_0, A_1)_{s_0}\cap(A_0, A_1)_{s_1}$ is dense in $(A_0, A_1)_{s_j}$, we can assume without loss of generality that $A_0\cap A_1$ is dense in A_j, and that $B_0\cap B_1$ is dense in B_j, for $j=0, 1$.

Since T_s has property W with constants (k, d), $B_s=TA_s+M_s$ with $\dim M_s=d$. By lemma 1.11, we can find $\{e_1, ..., e_d\}$, a basis for M_s, so that

(2.5) $$b_s=\min\left\{\left\|\sum_{i=1}^d c_ie_i\right\|_{B_s}\Big|\max|c_i|=1\right\}=1.$$

Let $v_i\in F(B)$ so that $1\leqq\|v_i\|_{F(B)}<2$ and $v_i(s)=e_i$. Denote by M_t the space spanned by $\{v_i(t)\}$.

Choose $\delta>0$ so that $|\theta-s|<\delta$ implies $q(\theta, s)\leqq 1/2$, then by remark 2.2, $\|v_i(\theta)\|_{B_\theta}\geqq 1/2$.

Let $y_\theta\in B_\theta$. There exists $u\in F(B)$ so that $u(\theta)=y_\theta$ and $\|u\|_{F(B)}\leqq 2\|y_\theta\|_{B_\theta}$. Set $y_s=u(s)$. By remark 2.2, we have

(2.6) $$\|y_\theta\|_{B_\theta}/2\leqq\|y_s\|_{B_s}=\|u(s)\|_{B_s}\leqq\|u\|_{F(B)}\leqq 2\|y_\theta\|_{B_\theta}.$$

Since T_s has property W with constants (k, d), we have, by remark 1.8, $\psi\in F(A)$ so that $y_s=T\psi(s)+z_s^1$ with $z_s^1=\sum_{i=1}^d c_ie_i\in M_s$ and

(2.7) $$\|\psi\|_{F(A)}\leqq(k+1)\|y_s\|_{B_s}\leqq 2(k+1)\|y_\theta\|_{B_\theta},$$

(2.8) $$\|z_s^1\|_{B_s}\leqq k\|y_s\|_{B_s}\leqq 2k\|y_\theta\|_{B_\theta}.$$

Set $\phi=\sum_{i=1}^d c_iv_i$, then $z_s^1=\phi(s)$. By (2.5) and (2.8) we have

$$|c_i|\leqq\left\|\sum_{i=1}^d c_ie_i\right\|_{B_s}=\|z_s^1\|_{B_s}\leqq 2k\|y_\theta\|_{B_\theta}.$$

From $\|v_i\|_{F(B)}<2$ it follows that

(2.9) $$\|\phi\|_{F(B)}\leqq\sum_{i=1}^d|c_i|\,\|v_i\|_{F(B)}\leqq 4kd\|y_\theta\|_{B_\theta}.$$

From $u(s)=T\psi(s)+\phi(s)$, by (2.2), (2.6), (2.7) and (2.9) we obtain that

$$\begin{aligned}\|T\psi(\theta)+\phi(\theta)-u(\theta)\|_{B_\theta}&\leqq\|T\psi(s)+\phi(s)-u(s)\|_{B_s}+q(\theta, s)\|T\psi+\phi-u\|_{F(B)}\\&\leqq q(\theta, s)(\|T\psi\|_{F(B)}+\|\phi\|_{F(B)}+\|u\|_{F(B)})\\&\leqq q(\theta, s)(2(k+1)\|T\|+4kd+2)\|y_\theta\|_{B_\theta}.\end{aligned}$$

Choosing a smaller δ so that

$$q(\theta, s)\big(2(k+1)\|T\|+4kd+2\big) < 1/2,$$

for $|\theta-s|<\delta$, we then have

$$\|T\psi(\theta)+\phi(\theta)-u(\theta)\|_{B_\theta} \leqq \|y_\theta\|_{B_\theta}/2 \tag{2.10}$$

with δ independent of y_θ. Set $y_\theta^1=u(\theta)-\phi(\theta)-T\psi(\theta)$, $x_\theta^1=\psi(\theta)$ and $z_\theta^1=\phi(\theta)\in M_\theta$. We get $y_\theta=Tx_\theta^1+y_\theta^1+z_\theta^1$, and the following estimates:

$$\|x_\theta^1\|_{A_\theta} \leqq \|\psi\|_{F(A)} \leqq 2(k+1)\|y_\theta\|_{B_\theta}$$

$$\|z_\theta^1\|_{B_\theta} \leqq \|\phi\|_{F(B)} \leqq 4kd\|y_\theta\|_{B_\theta}$$

$$\|y_\theta^1\|_{B_\theta} = \|u(\theta)-\phi(\theta)-T\psi(\theta)\|_{B_\theta} \leqq \|y_\theta\|_{B_\theta}/2.$$

By lemma 2.3, T_θ has property W with constants $\big(8(k+1)d, d\big)$, and the proof is complete.

Remark 2.5. It is not difficult to see that, by (2.3) and (2.4), theorem 2.4 holds for interpolation in Calderón's second (upper) method. By duality, if T_s^* has property W with constants k^* and d^*, then there exists $\delta>0$ which depends only on k^*, d^* and $\|T^*\|$ so that $|\theta-s|<\delta$ implies that T_θ^* has property W with constants $\big(8(k^*+1)d^*, d^*\big)$, and $\operatorname{codim} T_\theta^* \leqq d^*$.

Corollary 2.6. *Let $A=(A_0, A_1)$ and $B=(B_0, B_1)$ be interpolation couples of Banach spaces and let $T\colon A_j\to B_j$ $(j=0, 1)$ be a linear bounded operator. If T_s and T_s^* have property W with constants (k, d) and (k^*, d^*) respectively, then there exists $\delta>0$ which depends only on k, k^*, d, d^*, $\|T\|$ and $\|T^*\|$, such that $|\theta-s|<\delta$ implies that T_θ is a Fredholm operator with* $\dim\ker T_\theta \leqq \dim\ker T_s$, $\operatorname{codim} T_\theta \leqq \operatorname{codim} T_s$ *and $i(T_\theta)=i(T_s)$.*

Proof. By theorem 2.4 and remark 2.5, there exists $\delta>0$ which depends only on k, k^*, d, d^*, $\|T\|$ and $\|T^*\|$ such that $|\theta-s|<\delta$ implies that T_θ and T_θ^* have property W. Hence T_θ are Fredholm operators with $\dim\ker T_\theta \leqq \dim\ker T_s$ and $\operatorname{codim} T_\theta \leqq \operatorname{codim} T_s$. We now apply theorem 2 of [8] which states that if T_θ are Fredholm operators for all $\theta\in(\alpha_0, \alpha_1)$, then the index of T_θ is constant on (α_0, α_1), and get our theorem.

3. Applications to real interpolation spaces

Theorem 3.1. *Let $A=(A_0, A_1)$ and $B=(B_0, B_1)$ be Banach interpolation couples, and let $T\colon A_j\to B_j$ $(j=0, 1)$ be a linear bounded operator. Let $T_{s,p}$ be a Fredholm operator with* $\operatorname{codim} T_{s,p}=d<\infty$ *and* $\dim\ker T_{s,p}=d^*<\infty$, *where* $0<s<1$ *and*

$1<p<\infty$. *Then there exist* δ, $\varepsilon>0$ *such that* $|\theta-s|<\delta$ *and* $|q-p|<\varepsilon$ *imply that* $T_{\theta,q}$ *is also a Fredholm operator with* $\dim\ker T_{\theta,q}\leqq d^*$ *and* $\operatorname{codim}(T_{\theta,q})\leqq d$. *Furthermore we have* $i(T_{\theta,q})=i(T_{s,p})$.

If $p=1$, *then there exists* $\delta>0$ *so that* $|\theta-s|<\delta$ *implies that* $T_{\theta,1}$ *is a Fredholm operator with* $\dim\ker T_{\theta,1}\leqq\dim\ker T_{s,1}$, $\operatorname{codim} T_{\theta,1}\leqq\operatorname{codim} T_{s,1}$ *and* $i(T_{\theta,1})=i(T_{s,1})$.

Proof. We will prove the case $1<p<\infty$ and leave the case $p=1$ to the reader.

Let $0<s_0<s<s_1<1$. Let ζ_0 be given by $s=(1-\zeta_0)s_0+\zeta_0 s_1$. By theorem 4.7.2 in [2], we have

(3.1) $$(A_{s_0,p}, A_{s_1,p})_{\zeta_0} = A_{s,p} \quad (B_{s_0,p}, B_{s_1,p})_{\zeta_0} = B_{s,p}$$

and

(3.2) $$(A^*_{s_0,p}, A^*_{s_1,p})^{\zeta_0} = A^*_{sp} \quad (B^*_{s_0,p}, B^*_{s_1,p})^{\zeta_0} = B^*_{s,p}.$$

The constants involved in these norm equivalences remain uniformly bounded when ζ ranges on a compact subinterval of $(0, 1)$.

By assumption, the range of T_{ζ_0} has finite codimension d and the kernel of T_{ζ_0} has finite dimension d^*. Furthermore we have

$$\|\tilde{T}_{\zeta_0}^{-1}\| \leqq m\|\tilde{T}_{s,p}^{-1}\| \quad \text{and} \quad \|(\tilde{T}^*_{\zeta_0})^{-1}\| \leqq m\|(\tilde{T}^*_{s,p})^{-1}\|.$$

(The constant m is needed since the norms in (3.1) and (3.2) are equivalent, not equal.)

By lemma 1.9, T_{ζ_0} and $T^*_{\zeta_0}$ have property W with constants (l, d) and (l^*, d^*) respectively, where $l=2(d+1)(m\|\tilde{T}_{s,p}^{-1}\|+1)$ and $l^*=2(d^*+1)(m\|(\tilde{T}^*_{s,p})^{-1}\|+1)$. By theorem 2.4 and remark 2.5, there exists δ' which depends only on l, l^*, d, d^*, m, $\|T\|$ and $\|T^*\|$ such that $|\zeta_0-\zeta|<\delta'$ implies that T_ζ and T^*_ζ have property W with constants $(8(l+1)d, d)$ and $(8(l^*+1)d^*, d^*)$, respectively. Hence T_ζ is a Fredholm operator, with $\dim\ker T_\zeta\leqq\dim\ker T_{\zeta_0}$, $\operatorname{codim} T_\zeta\leqq\operatorname{codim} T_{\zeta_0}$ and $i(T_\zeta)=i(T_{\zeta_0})$, and so if θ satisfies $|\theta-s|<\delta$, where $\delta=\delta'(s_0+s_1)$, then $T_{\theta,p}$ is a Fredholm operator with $\dim\ker T_{\theta,p}\leqq\dim\ker T_{s,p}$, $\operatorname{codim} T_{\theta,p}\leqq\operatorname{codim} T_{s,p}$ and $i(T_{\theta,p})=i(T_{s,p})$. For the repeated interpolation below we need to observe moreover that $T_{\theta,p}$ and $T^*_{\theta,p}$ have property W with constants $(8m_1(l+1)d, d)$ and $(8m_1(l^*+1)d^*, d^*)$, respectively. Again the constant m_1 is needed since we have norm equivalences in (3.1) and (3.2). As observed before, it is uniform on compact subintervals of $(0, 1)$.

We choose $1\leqq p_0<p<p_1<\infty$ and take η so that $1/p=(1-\eta)/p_0+\eta/p_1$. Then:

$$(A_{\theta,p_0}, A_{\theta,p_1})_\eta = A_{\theta,p}, \quad (B_{\theta,p_0}, B_{\theta,p_1})_\eta = B_{\theta,p}$$

and

$$(A^*_{\theta,p_0}, A^*_{\theta,p_1})^\eta = A^*_{\theta,p}, \quad (B^*_{\theta,p_0}, B^*_{\theta,p_1})^\eta = B^*_{\theta,p}.$$

As in the first part of the proof, for any fixed θ satisfying $|\theta-s|<\delta$, we can find $\varepsilon>0$ which depends only on l, l^*, d, d^*, m, m_1, $\|T\|$ and $\|T^*\|$ (and so is the same for all θ, $|\theta-s|<\delta$), such that $|q-p|<\varepsilon$ implies that $T_{\theta,q}$ and $T^*_{\theta,q}$ have property W. Hence $T_{\theta,q}$ are Fredholm operators with $\operatorname{codim} T_{\theta,q}\leqq d$, $\dim\ker T_{\theta,q}\leqq d^*$, and $i(T_{\theta,q})=i(T_{\theta,p})$, and the theorem is proved.

Corollary 3.2. *Let $A=(A_0, A_1)$ and $B=(B_0, B_1)$ be Banach interpolation couples and let T: $A_j\to B_j$ ($j=0, 1$) be a linear bounded operator. Then*

(1) *If* $\ker T_{s,p}=\{0\}$ *and* $\operatorname{codim} T_{s,p}=d<\infty$, *where* $0<s<1$ *and* $1<p<\infty$, *then there exist* $\delta>0$ *and* $\varepsilon>0$ *such that* $|\theta-s|<\delta$ *and* $|q-p|<\varepsilon$ *imply that* $\ker T_{\theta,q}=\{0\}$, $\operatorname{codim} T_{\theta,q}=d$.

If $p=1$, *then there exists* $\delta>0$ *so that* $|\theta-s|<\delta$ *implies that* $T_{\theta,1}$ *is a Fredholm operator with* $\ker T_{\theta,1}=\{0\}$, $\operatorname{codim} T_{\theta,1}=d$.

(2) *If* $\dim\ker T_{s,p}=r<\infty$ *and* $T_{s,p}$ *is surjective, where* $0<s<1$ *and* $1<p<\infty$, *then there exist* $\delta>0$ *and* $\varepsilon>0$ *such that* $|\theta-s|<\delta$ *and* $|q-p|<\varepsilon$ *imply that* $\dim\ker T_{\theta,q}=r$ *and* $T_{\theta,q}$ *is surjective.*

If $p=1$, *then there exists* $\delta>0$ *so that* $|\theta-s|<\delta$ *implies that* $T_{\theta,1}$ *is a Fredholm operator with* $\dim\ker T_{\theta,1}=r$ *and* $T_{\theta,1}$ *is surjective.*

Proof. Assume that $\ker T_{s,p}=\{0\}$ and $\operatorname{codim} T_{s,p}=d<\infty$. By theorem 3.1, there exist $\delta>0$ and $\varepsilon>0$ so that $|\theta-s|<\delta$ and $|q-p|<\varepsilon$ imply that $T_{\theta,q}$ is a Fredholm operator with $i(T_{\theta,q})=i(T_{s,p})$. Furthermore, $\dim\ker T_{\theta,q}\leqq\dim\ker T_{s,p}$ and $\operatorname{codim} T_{\theta,q}\leqq\operatorname{codim} T_{s,p}$. Since $\dim\ker T_{s,p}=0$, we have $\ker T_{\theta,q}=\{0\}$ and $\operatorname{codim} T_{\theta,q}=d$.

The case of $p=1$ and part (2) follow similarly.

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