Vanishing cycles and $\mathscr{D}$-modules

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0. Introduction

This paper arose while we were working on the problem of classifying regular holonomic $\mathscr{D}_X$-modules (X a complex manifold) with a prescribed singular support i.e., the projection under $\pi\colon T^*X \to X$ of the characteristic variety. In [2] we treated the normal crossings case. We believe that such classifications should be done by "$\mathscr{D}$-module theoretic" methods. In other words one should not start to translate the problem into one on classifying perverse sheaves.

The theorem on extensions of perverse sheaves of Verdier [12] has of course an analogue in the framework of $\mathscr{D}$-modules. This analogy arises by means of the Riemann—Hilbert correspondence (cf. [9] or [6]). The main goal of this paper is to give a "$\mathscr{D}$-module theoretic" proof of the analogue of the extension theorem (cf. Thm. 3.2). Meanwhile we establish some results (Prop. 1.5.1 and Prop. 2.4.3) which might be important on their own.

As a preliminary task one is forced to find analogous versions of the nearby cycle functor Ψ_f and the vanishing cycle functor Φ_f of Deligne [1], (where $f\colon X \to \mathbf{C}$ is a non-constant holomorphic function) and the natural morphisms can: $\Psi_f \to \Phi_f$, var: $\Phi_f \to \Psi_f$. In [8] Malgrange considered the structure sheaf $\mathscr{O}_X$ and defined $\mathscr{D}_X$-modules corresponding with $\Psi_f \underline{\mathbf{C}}_X$ and $\Phi_f \underline{\mathbf{C}}_X$. In [3] Kashiwara treats the general case; he defines functors φ and ψ such that for every regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$, $\varphi\mathscr{M}$ resp. $\psi\mathscr{M}$ agree with $\Phi_f \mathscr{F}$ resp. $\Psi_f \mathscr{F}$, where $\mathscr{F} = R\,\mathscr{hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Furthermore there are natural morphisms $c(\mathscr{M})\colon \varphi\mathscr{M} \to \psi\mathscr{M}$, $v(\mathscr{M})\colon \psi\mathscr{M} \to \varphi\mathscr{M}$ corresponding with can, var. The main result (Theorem 3.2) is then as follows (cf. [12], Cor. 1).

Theorem. *The functor*

$$F\colon \mathscr{M} \mapsto \left(\mathscr{M}[f^{-1}],\ \varphi\mathscr{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathscr{M},\ \psi(\pi)\right)$$

defines an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules and the category of triples $(\mathcal{N}, \mathcal{N}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathcal{N}_2, \alpha)$. *Here* $\pi\colon \mathcal{M} \to \mathcal{M}[f^{-1}]$ *denotes the canonical map and* $\mathcal{N}$, $\mathcal{N}_1$, $\mathcal{N}_2$ *are regular holonomic* $\mathcal{D}_X$*-modules such that:*

$\mathcal{N} \cong \mathcal{N}[f^{-1}]$;
$\mathcal{N}_1$, $\mathcal{N}_2$ *are supported by* $X_0 = f^{-1}(0)$;
U, V *are* $\mathcal{D}_X$*-morphisms;*
$\alpha\colon \mathcal{N}_2 \xrightarrow{\cong} \psi\mathcal{N}$ *is an isomorphism satisfying* $\alpha UV = c(\mathcal{N})v(\mathcal{N})\alpha$.

In § 1 we introduce Kashiwara's filtration and state the main properties (Thm. 1.4). Moreover we put forward a nice description of this filtration (Prop. 1.5.1). This enables a rather easy proof of the Artin—Rees property (cf. 1.6.3).

In § 2 following Kashiwara (cf. [3] and also [8]) we introduce functors φ, ψ on $\mathrm{mod}(\mathcal{D}_X)_{\mathrm{hr}}$. We list some properties. Using material from § 1 we deduce the existence of a distinguished triangle in $D_{\mathrm{hr}}(\mathcal{D}_X)$

$$\varphi\mathcal{M} \to \psi\mathcal{M} \to R\Gamma_{[X_0]}\mathcal{M}[1] \to \varphi\mathcal{M}[1].$$

We obtain some corollaries to be used later. We make some comment on relations with Deligne's functors and add a remark concerning why one should restrict attention to regular holonomic $\mathcal{D}_X$-modules.

In § 3 we formulate the main Theorem 3.2. This section is rather technical. By then it is obvious that the functor F is exact and faithful. However the difficulty is to show that F is essentially surjective. To solve this problem we introduce an inverse functor G which does the reconstruction for us. The details are in 3.2.3.

For a moment we return to the classification problem mentioned at the beginning. Suppose one wants to classify holonomic $\mathcal{D}_X$-modules with regular singularities along X_0. The main theorem reduces this to a problem of classifying pairs $\mathcal{N}_1 \rightleftarrows \mathcal{N}_2$ of regular holonomic $\mathcal{D}_X$-modules with support contained in X_0. In a subsequent paper we will return to this question.

The author is indebted to Prof. A. H. M. Levelt and A. R. P. van den Essen for the stimulating discussions with them when studying [3] and [8].

N.B. If we write "module" we always mean "left module".

$\mathcal{mod}(\mathcal{D}_X)_{\mathrm{hr}}$ denotes the category of regular holonomic $\mathcal{D}_X$-modules.

1. The canonical good filtration

1.1. Definition

Let Y be a complex manifold and let Z be a closed submanifold of Y. Let $\mathscr{I}$ be the defining ideal of $Z \subset Y$ i.e., the sections of $\mathscr{O}_Y$ vanishing on Z. Following Kashiwara [3], (see also [1]), we define a descending filtration $F^{\cdot}\mathscr{D}_Y$ on $\mathscr{D}_Y$ by

$$F^k\mathscr{D}_Y := \{P \in \mathscr{D}_Y | P\mathscr{I}^j \subset \mathscr{I}^{j+k}, \text{ all } j \in \mathbf{N}\}, \quad \text{for all } k \in \mathbf{Z}.$$

In local coordinates $(y_1, \dots, y_m, z_1, \dots, z_n)$ on Y such that Z is given by $y_1=0, \dots, y_m=0$, one has

$$z_i, \frac{\partial}{\partial z_i} \in F^0\mathscr{D}_Y, \quad y_j \in F^1\mathscr{D}_Y, \quad \frac{\partial}{\partial y_j} \in F^{-1}\mathscr{D}_Y;$$

$F^0\mathscr{D}_Y$ is the subring of $\mathscr{D}_Y$ generated over $\mathscr{O}_Y$ by

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, y_1\frac{\partial}{\partial z_1}, \dots, y_m\frac{\partial}{\partial y_m}.$$

$F^0\mathscr{D}_Y$ is a noetherian sheaf of rings. (Cf. [11], Ch. III § 1.4 and appendix C. 5; cf. also [6], Ch. I § 1.1.) The $F^k\mathscr{D}_Y$ are coherent modules over $F^0\mathscr{D}_Y$. $F^0\mathscr{D}_Y/F^1\mathscr{D}_Y$ is a coherent sheaf of rings.

1.2. Definition of good filtration

Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module. A descending filtration $F^{\cdot}\mathscr{M}$ on $\mathscr{M}$ is called a good filtration if

(1) $F^l\mathscr{D}_Y F^k\mathscr{M} \subset F^{k+l}\mathscr{M}$, for all $k, l \in \mathbf{Z}$.

(2) $\mathscr{M} = \bigcup_{k \in \mathbf{Z}} F^k\mathscr{M}$.

(3) $F^k\mathscr{M}$ is a coherent $F^0\mathscr{D}_Y$-module, for all $k \in \mathbf{Z}$.

(4) Locally one has:

$$F^l\mathscr{D}_Y F^k\mathscr{M} = F^{k+l}\mathscr{M} \quad \text{if} \quad (l \geqq 0,\ k \gg 0) \quad \text{or} \quad (l \leqq 0,\ k \ll 0).$$

If this is the case then for any $k \in \mathbf{Z}$, $\mathrm{gr}^k\mathscr{M} := F^k\mathscr{M}/F^{k+1}\mathscr{M}$ is a coherent $F^0\mathscr{D}_Y/F^1\mathscr{D}_Y$-module. Notice that a coherent $\mathscr{D}_Y$-module has locally a good filtration. Moreover if $\mathscr{M}$ is a regular holonomic $\mathscr{D}_Y$-module, such a filtration exists globally.

1.3. Definition of canonical good filtration

From now on we assume $Y=X\times\mathbf{C}$, X a complex manifold. Let t be a coordinate on $\mathbf{C}$. The ideal $\mathscr{X}=t\mathcal{O}_Y$ defines the closed submanifold $X\times\{0\}$, which we identify with X. Let θ denote the vectorfield $t\partial$ on Y, where $\partial=\frac{\partial}{\partial t}$. Clearly we have

$$F^k\mathscr{D}_Y=\begin{cases}F^0\mathscr{D}_Y t^k, & \text{if} \quad k\in\mathbf{N}\\ \sum_{j=0}^{-k}F^0\mathscr{D}_Y\partial^j, & \text{if} \quad -k\in\mathbf{N}.\end{cases}$$

The coherent sheaf of rings $F^0\mathscr{D}_Y/F^1\mathscr{D}_Y$ may be identified with $\mathscr{D}_X[\theta]$.

Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module with a good filtration $F^{\bullet}\mathscr{M}$. The filtration is called a canonical good filtration if

(5) there exists a non-zero polynomial $b\in\mathbf{C}[\Theta]$ such that

(i) $b(\theta-k)F^k\mathscr{M}\subset F^{k+1}\mathscr{M}$, for all $k\in\mathbf{Z}$,

(ii) $b^{-1}(0)\subset\{z\in\mathbf{C}|0\leqq\operatorname{Re}z<1\}$.

1.4. Theorem. (Cf. [3], Thm. 1 and [7], Thm. 3.1.) *Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module. Then:*

(i) *$\mathscr{M}$ admits at most one canonical good filtration.*
(ii) *If $\mathscr{M}$ is holonomic, then $\mathscr{M}$ carries locally a canonical good filtration.*
(iii) *If $\mathscr{M}$ is regular holonomic, then $\mathscr{M}$ has a canonical good filtration.*
(iv) *If $\mathscr{M}$ is regular holonomic, then for all $k\in\mathbf{Z}$, $\operatorname{gr}^k\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, where $F^{\bullet}\mathscr{M}$ denotes the canonical good filtration. [Notice that $\mathscr{D}_X\subset\mathscr{D}_X[t\partial]=F^0\mathscr{D}_Y/F^1\mathscr{D}_Y$ and in general $\operatorname{gr}^k\mathscr{M}$ is only coherent over $\mathscr{D}_X[t\partial]$.] In that case $\operatorname{gr}^k\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module.*

1.5. Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module equipped with a canonical filtration $F^{\bullet}\mathscr{M}$. We want to give a more explicit description of this filtration. Therefore we introduce the following notation: for a linear subspace $\mathscr{L}\subset\mathscr{M}$ and any $k\in\mathbf{N}$ put

$$t^{-k}\mathscr{L}:=\{m\in\mathscr{M}|t^km\in\mathscr{L}\}.$$

Consider the descending chain of subspaces of $\mathscr{M}$

$$\ldots\supset t^{-2}F^0\mathscr{M}\supset t^{-1}F^0\mathscr{M}\supset F^0\mathscr{M}\supset tF^0\mathscr{M}\supset t^2F^0\mathscr{M}\supset\ldots.$$

We claim that this is just the canonical good filtration.

1.5.1. Proposition. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module carrying a canonical good filtration $F^{\bullet}\mathscr{M}$. Then for all $k\in\mathbf{Z}$*

$$F^k\mathscr{M}=t^kF^0\mathscr{M}.$$

Proof. Let $b \in \mathbf{C}[\Theta]$ be a non-zero polynomial satisfying condition (5) of 1.3. Observe that for $k \in \mathbf{N}^*$,

$$F^{-k}\mathscr{M} \subset t^{-k}F^0\mathscr{M}, \quad t^kF^0\mathscr{M} \subset F^k\mathscr{M}.$$

Let us prove the other inclusions.

(1) Let $k \in \mathbf{N}^*$. Suppose $m \in t^{-k}F^0\mathscr{M}$ i.e., $t^k m \in F^0\mathscr{M}$. Hence $\partial^k t^k m \in F^{-k}\mathscr{M}$; thus $(\theta+k)\dots(\theta+1)m \in F^{-k}\mathscr{M}$. On the other hand there exists $N \in \mathbf{N}$, $N \geqq k+1$, such that $m \in F^{-N}\mathscr{M}$; thus $b(\theta+k+1)\dots b(\theta+N)m \in F^{-k}\mathscr{M}$. Because of condition (ii) of 1.3 (5) $(\theta+k)\dots(\theta+1)$ and $b(\theta+k+1)\dots b(\theta+N)$ are relatively prime. This yields $m \in F^{-k}\mathscr{M}$.

(2) Locally there exists $j_0 \in \mathbf{N}$ such that $F^j\mathscr{D}_Y F^{j_0}\mathscr{M} = F^{j+j_0}\mathscr{M}$, for all $j \geqq 0$. It follows that $F^{j_0+j}\mathscr{M} = t^j F^{j_0}\mathscr{M}$, for all $j \geqq 0$. If $j_0 = 0$ we are done, so assume $j_0 > 0$. We will derive that $F^{j_0}\mathscr{M} = tF^{j_0-1}\mathscr{M}$. Let $m \in F^{j_0}\mathscr{M}$. Then $b(\theta-j_0)m \in F^{j_0+1}\mathscr{M}$. Hence there exists $m_0 \in F^{j_0}\mathscr{M} \subset F^{j_0-1}\mathscr{M}$ such that $b(\theta-j_0)m = tm_0$. Writing $b(\theta-j_0) = \theta\tilde{b}(\theta) + b(-j_0)$, with $\tilde{b} \in \mathbf{C}[\Theta]$, we have $b(-j_0)m = t(m_0 - \partial\tilde{b}(\theta)m)$. Note that $b(-j_0) \in \mathbf{C}^*$, $m_0 - \partial\tilde{b}(\theta)m \in F^{j_0-1}\mathscr{M}$ and thus $m \in tF^{j_0-1}\mathscr{M}$. This yields $F^{j_0}\mathscr{M} \subset tF^{j_0-1}\mathscr{M}$. The other inclusion is obvious. It follows that already $F^{j_0-1+j}\mathscr{M} = t^j F^{j_0-1}\mathscr{M}$, for all $j \geqq 0$. By descending induction we arrive at $F^j\mathscr{M} = t^j F^0\mathscr{M}$, for all $j \geqq 0$.

1.6. Using this description of the canonical good filtration we will derive that a morphism $\varphi\colon \mathscr{M}_1 \to \mathscr{M}_2$ of coherent $\mathscr{D}_Y$-modules, carrying a canonical good filtration, is a strict morphism between the filtered modules. By this we mean that

$$\operatorname{Im}\varphi \cap F^k\mathscr{M}_2 = \varphi(F^k\mathscr{M}_1), \quad \text{for all} \quad k \in \mathbf{Z}.$$

This will be done by proving an Artin—Rees lemma for canonical good filtrations. As a preliminary step we have:

1.6.1. The canonical filtration on $\mathscr{H}^0_{[X]}\mathscr{M}$

Let us first recall the following. Let $Z \subset Y$ be a subvariety defined by an ideal $\mathscr{I}$. For a $\mathscr{D}_Y$-module $\mathscr{M}$ one defines

$$\Gamma_{[Z]}\mathscr{M} = \varinjlim \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y/\mathscr{I}^n, \mathscr{M}).$$

This is a $\mathscr{D}_Y$-module with support contained in Z. (Cf. [4], § 1 or [9].) Let $\mathscr{H}^k_{[Z]}$ denote the k-th derived functor of $\Gamma_{[Z]}$. If $\mathscr{M}$ is coherent it is not necessarily the case that $\mathscr{H}^k_{[Z]}\mathscr{M}$ is coherent. However Kashiwara proved:

— If $\mathscr{M}$ is holonomic, then $\mathscr{H}^k_{[Z]}\mathscr{M}$ is holonomic. ([4], Thm. 1.4.)

— If $\mathscr{M}$ is regular holonomic, then also $\mathscr{H}^k_{[Z]}\mathscr{M}$. ([6], Thm. 5.4.1.)

The ideal $t\mathcal{O}_Y$ defines the closed submanifold $X\times\{0\}=X$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and assume that $\mathcal{H}^0_{[X]}\mathcal{M}$ is also coherent. We make the following observations:

(1) $$\mathcal{H}^0_{[X]}\mathcal{M} = \{m\in\mathcal{M}\,|\,\text{there exists } N\in\mathbf{N} \text{ such that } t^N m = 0\}$$
$$= \bigcup_{k\in\mathbf{N}} \operatorname{Ker}(t^k, \mathcal{M}).$$

(2) $\mathcal{H}^0_{[X]}\mathcal{M}$ carries a descending filtration given by

$$F^k\mathcal{H}^0_{[X]}\mathcal{M} := \begin{cases} \operatorname{Ker}(t^{-k}, \mathcal{M}) & \text{if} \quad -k\in\mathbf{N} \\ 0 & \text{if} \quad k\in\mathbf{N}. \end{cases}$$

(3) This is a good filtration, because:

(i) Conditions (1) and (2) of 1.2 are trivially satisfied.
(ii) 1.2 (4) is true, because $\operatorname{Ker}(t^l, \mathcal{M})\cong\bigoplus_{j=0}^{l-1}\partial^j \operatorname{Ker}(t, \mathcal{M})$.
(iii) $\operatorname{Ker}(t, \mathcal{M})$ is a coherent $\mathcal{D}_X$-module ([4], Prop. 4.2), and thus a coherent $F^0\mathcal{D}_Y$-module. This implies 1.2 (3).

(4) Furthermore $(t\partial-k)F^k\mathcal{H}^0_{[X]}\mathcal{M}\subset F^{k+1}\mathcal{H}^0_{[X]}\mathcal{M}$, for all $k\in\mathbf{Z}$ as one easily verifies. Thus the filtration given by (2) on $\mathcal{H}^0_{[X]}\mathcal{M}$ is the canonical good filtration.

1.6.2. The induced filtration on a quotient

Let $\mathcal{M}$ be a coherent $\mathcal{D}_Y$-module and let $\mathcal{N}\subset\mathcal{M}$ be a coherent $\mathcal{D}_Y$-submodule. Suppose $\mathcal{M}$ is equipped with a filtration $F^\bullet\mathcal{M}$ which is canonical good. There are induced filtrations on $\mathcal{N}$ and $\mathcal{M}/\mathcal{N}$ defined by

$$F^k\mathcal{N} := \mathcal{N}\cap F^k\mathcal{M}, \qquad \text{for all} \quad k\in\mathbf{Z},$$

$$F^k(\mathcal{M}/\mathcal{N}) := F^k\mathcal{M}/F^k\mathcal{N}, \quad \text{for all} \quad k\in\mathbf{Z}.$$

Proposition. *The induced filtration $F^\bullet(\mathcal{M}/\mathcal{N})$ is canonical good.*

Proof. Clearly the induced filtration satisfies properties (1), (2), (4) and (5) of the definition of canonical good filtration. Condition (3) is fulfilled, because locally $F^k(\mathcal{M}/\mathcal{N})$ is a $F^0\mathcal{D}_Y$-module of finite type and it is a $F^0\mathcal{D}_Y$-submodule of the coherent $\mathcal{D}_Y$-module $\mathcal{M}/\mathcal{N}$. By [11], Prop. 1.4.2 and the last lines of 1.1 $F^k(\mathcal{M}/\mathcal{N})$ is a coherent $F^0\mathcal{D}_Y$-module, for all $k\in\mathbf{Z}$.

Note. By the last line $F^k\mathcal{N}$ is a coherent $F^0\mathcal{D}_Y$-module, all $k\in\mathbf{Z}$.

1.6.3. The induced filtration on a holonomic submodule

Proposition. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module and let $\mathscr{N}\subset\mathscr{M}$ be a coherent submodule. Assume $\mathscr{M}$ carries a canonical good filtration $F^{\cdot}\mathscr{M}$. Then:*

(i) $\operatorname{Ker}(t, \mathscr{M})\cap F^0\mathscr{M}=0$;
(ii) *the induced filtration on $\mathscr{N}$ is canonical good.*

Proof. We begin the proof of (ii) and meanwhile obtain (i) as a special case. The induced filtration satisfies conditions (1), (2), (3) and (5). The problem is to show that $F^{\cdot}\mathscr{N}$ satisfies condition (4) i.e., locally

$$F^l\mathscr{D}_Y F^k\mathscr{N} = F^{k+l}\mathscr{N} \quad \text{if} \quad (l\geqq 0,\ k\gg 0) \quad \text{or} \quad (l\leqq 0,\ k\ll 0).$$

We prove this in two steps.

1. $F^l\mathscr{D}_Y F^k\mathscr{N} = F^{k+l}\mathscr{N}$ for all $l\leqq 0$, $k\leqq -1$.

It is enough to prove

$$F^{-k-1}\mathscr{N} = F^{-1}\mathscr{D}_Y F^{-k}\mathscr{N}, \quad \text{for all} \quad k\geqq 1.$$

Therefore let $k\in\mathbf{N}^*$ and $n\in F^{-k-1}\mathscr{N}$. Let $b\in\mathbf{C}[\Theta]$ be a non-zero polynomial belonging to the canonical good filtration $F^{\cdot}\mathscr{M}$ (cf. 1.3 (5)). Then $b(\theta+k+1)n\in\mathscr{N}\cap F^{-k}\mathscr{M}=F^{-k}\mathscr{N}$. Write $b(\theta+k+1)=\partial t\tilde{b}(\theta+1)+b(k)$, $\tilde{b}\in\mathbf{C}[\Theta]$ and $b(k)\in\mathbf{C}^*$. Further $t\tilde{b}(\theta+1)n\in F^{-k}\mathscr{N}$, yielding that $n\in F^{-k}\mathscr{N}+\partial F^{-k}\mathscr{N}=F^{-1}\mathscr{D}_Y F^{-k}\mathscr{N}$.

2. The problem seems to be in the tail of the filtration. We shall derive that

$$F^l\mathscr{D}_Y F^k\mathscr{N} = F^{k+l}\mathscr{N}, \quad \text{for all} \quad k, l\in\mathbf{N}.$$

It suffices to show that $F^{k+1}\mathscr{N}\subset tF^k\mathscr{N}$, for all $k\in\mathbf{N}$.
Let us first treat the special case

2a. $\mathscr{N}=\mathscr{H}^0_{[X]}\mathscr{M}$.

Let $k\in\mathbf{N}$ and $n\in F^{k+1}\mathscr{N}=\mathscr{N}\cap F^{k+1}\mathscr{M}$. By Proposition 1.5.1 there exists $m\in F^k\mathscr{M}$ such that $n=tm$. Because $\mathscr{N}=\mathscr{H}^0_{[X]}\mathscr{M}$ there exists $N\in\mathbf{N}$ such that $t^N n=0$. It follows also $t^{N+1}m=0$, hence $m\in\mathscr{H}^0_{[X]}\mathscr{M}=\mathscr{N}$. So $n=tm$ with $m\in F^k\mathscr{N}$. This yields $F^{k+1}\mathscr{N}\subset tF^k\mathscr{N}$.

Hence in the particular case $\mathscr{N}=\mathscr{H}^0_{[X]}\mathscr{M}$ we have established that the induced filtration is canonical good and by unicity (Th. 1.4 (i)) equals the filtration given in 1.6.1. In particular $\operatorname{Ker}(t, \mathscr{M})\cap F^0\mathscr{M}\subset\mathscr{N}\cap F^0\mathscr{M}=0$, so this yields part (i).

2b. The general case.

Let $k\in\mathbf{N}$ and let $n\in F^{k+1}\mathscr{N}$. There exists $m\in F^k\mathscr{M}$ such that $n=tm$ (Prop. 1.5.1). Denote with $\bar{m}$ the image of m in $\mathscr{M}/\mathscr{N}$. Then $t\bar{m}=0$ in $\mathscr{M}/\mathscr{N}$. Hence $\bar{m}\in\operatorname{Ker}(t, \mathscr{M}/\mathscr{N})\cap F^k(\mathscr{M}/\mathscr{N})\subset\operatorname{Ker}(t, \mathscr{M}/\mathscr{N})\cap F^0(\mathscr{M}/\mathscr{N})$. By Proposition

1.6.2 the induced filtration on $\mathscr{M}/\mathscr{N}$ is canonical good, hence by (a) above $\bar{m}=0$. It follows that $m\in\mathscr{N}\cap F^k\mathscr{M}=F^k\mathscr{N}$ and $n=tm\in tF^k\mathscr{N}$. This yields $F^{k+1}\mathscr{N}\subset tF^k\mathscr{N}$ for all $k\in\mathbf{N}$.

1.6.4. Corollary. *Let $\varphi\colon \mathscr{M}_1\to\mathscr{M}_2$ be a morphism of coherent $\mathscr{D}_Y$-modules. Assume $\mathscr{M}_1$ and $\mathscr{M}_2$ carry canonical good filtrations $F^\bullet\mathscr{M}_1$, $F^\bullet\mathscr{M}_2$. Then φ is a strict morphism of filtered modules.*

Proof. Put $\mathscr{N}=\operatorname{Im}\varphi\subset\mathscr{M}_2$. Then $F^\bullet\mathscr{M}_1$ induces a canonical good filtration on $\mathscr{N}$ (Prop. 1.6.2). Also $F^\bullet\mathscr{M}_2$ induces a canonical good filtration on $\mathscr{N}$ (Prop. 1.6.3). But there can be only one canonical good filtration on $\mathscr{N}$, hence $\varphi(F^k\mathscr{M}_1)=\mathscr{N}\cap F^k\mathscr{M}_2$, for all $k\in\mathbf{Z}$.

1.6.5. Corollary. *Let $\mathscr{M}_1\hookrightarrow\mathscr{M}_2\twoheadrightarrow\mathscr{M}_3$ be a short exact sequence of coherent $\mathscr{D}_Y$-modules with a canonical good filtration. Then for all $k\in\mathbf{Z}$ we have exact sequences:*

(i) $F^k\mathscr{M}_1\hookrightarrow F^k\mathscr{M}_2\twoheadrightarrow F^k\mathscr{M}_3$ *of $F^0\mathscr{D}_Y$-modules;*

(ii) $\operatorname{gr}^k\mathscr{M}_1\hookrightarrow \operatorname{gr}^k\mathscr{M}_2\twoheadrightarrow \operatorname{gr}^k\mathscr{M}_3$ *of $\mathscr{D}_X[\theta]$-modules.*

1.6.6. Remark. Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module admitting a canonical good filtration $F^\bullet\mathscr{M}$. The multiplication with t induces, for all $k\in\mathbf{N}$, a bijection $\operatorname{gr}^k\mathscr{M}\xrightarrow{\cong}\operatorname{gr}^{k+1}\mathscr{M}$. This follows from 1.6.3 (i) and 1.5.1.

2. Vanishing cycles and nearby cycles

Let X, Y be as in 1.3. Let $f\colon X\to\mathbf{C}$ be a non-constant holomorphic function on X.

Let

$$i\colon X\to Y=X\times\mathbf{C},\quad x\mapsto(x,f(x))$$

be the embedding on the graph of f.

Finally put $X_0:=f^{-1}(0)$.

2.1. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then

$$i_*\mathscr{M}=(\mathscr{D}_Y/\mathscr{D}_Y(t-f))\otimes_{\mathscr{D}_X}\mathscr{M}$$

(where we have identified X with the graph of f) is a coherent $\mathscr{D}_Y$-module supported on the graph of f. If $\mathscr{M}$ is holonomic, then $i_*\mathscr{M}$ is holonomic (cf. [6], Lemma 5.1.9). If $\mathscr{M}$ has regular singularities, then $i_*\mathscr{M}$ has regular singularities (ibid.). In fact i_* is an exact functor and establishes an equivalence between the category of coherent $\mathscr{D}_X$-modules and the category of coherent $\mathscr{D}_Y$-modules with support contained in the graph of f (cf. [4], Prop. 4.2). The inverse functor of i_* is given by

$$\operatorname{Ker}(t-f,\ \cdot) = \mathscr{hom}_{\mathcal{O}_Y}(\mathcal{O}_Y/(t-f)\mathcal{O}_Y,\ \cdot).$$

In fact one has the identification

$$i_*\mathscr{M} \cong \mathbf{C}[\partial]\otimes\mathscr{M},$$

where the $\mathscr{D}_Y$-structure on the right-hand side is determined by (in local coordinates $x_1, \ldots, x_d$, t on Y): for all $m\in\mathscr{M}$, $i\in\mathbf{N}$:

$$\begin{aligned} t(\partial^i\otimes m) &= -i\partial^{i-1}\otimes m+\partial^i\otimes fm \\ \partial(\partial^i\otimes m) &= \partial^{i+1}\otimes m \\ \partial_\alpha(\partial^i\otimes m) &= \partial^i\otimes\partial_\alpha m-\partial^{i+1}\otimes\partial_\alpha(f)m, \end{aligned}$$

$$\text{for all } \alpha\in\{1, \ldots, d\},\ \left(\partial_\alpha = \frac{\partial}{\partial x_\alpha}\right).$$

2.2. Definitions

2.2.1. The category of coherent $\mathscr{D}_X$-modules $\mathscr{M}$ satisfying the requirement that $i_*\mathscr{M}$ admits a canonical good filtration is by 1.6.2, 1.6.3 and 1.6.4 an abelian category. Let us denote this category by $\mathscr{R}$. Following Kashiwara [3] (see also Malgrange [8] and M. Saito [10]) we define for any $\mathscr{M}\in\mathscr{R}$

$$\begin{aligned} \psi\mathscr{M} &:= F^0(i_*\mathscr{M})/F^1(i_*\mathscr{M}) \\ \varphi\mathscr{M} &:= F^{-1}(i_*\mathscr{M})/F^0(i_*\mathscr{M}), \end{aligned}$$

where $F^\bullet(i_*\mathscr{M})$ denotes the canonical good filtration on $i_*\mathscr{M}$.

2.2.2. Left multiplication with t resp. ∂ induces maps

$$\begin{aligned} c(\mathscr{M})&\colon \varphi\mathscr{M} \to \psi\mathscr{M}, \\ v(\mathscr{M})&\colon \psi\mathscr{M} \to \varphi\mathscr{M}. \end{aligned}$$

2.2.3. If we make the identification $X=X\times\{0\}$ (see 1.3), then $\psi\mathscr{M}$ and $\varphi\mathscr{M}$ have the structure of a module over $F^0\mathscr{D}_Y/F^1\mathscr{D}_Y=\mathscr{D}_X[t\partial]$. Moreover $\psi\mathscr{M}$ and $\varphi\mathscr{M}$ are coherent $\mathscr{D}_X[t\partial]$-modules. The mappings $c(\mathscr{M})$ and $v(\mathscr{M})$ are $\mathscr{D}_X$-linear and the action of $t\partial$ on $\varphi\mathscr{M}$ (resp. $\psi\mathscr{M}$) is given by $v(\mathscr{M})\circ c(\mathscr{M})-1_{\varphi\mathscr{M}}$ (resp. $c(\mathscr{M})\circ v(\mathscr{M})$). The $\mathscr{D}_X$-modules $\psi\mathscr{M}$ and $\varphi\mathscr{M}$ have their support contained in

$$i(X)\cap(X\times\{0\}) = \operatorname{graph}(f)\cap(X\times\{0\}) = X_0\times\{0\}.$$

2.3. Restriction to regular holonomic modules

By Thm. 1.4 (iii) the category $\mathscr{R}$ contains the category of the regular holonomic modules $\mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$. According to Theorem 1.4 (iv) $\psi\mathscr{M}$ and $\varphi\mathscr{M}$ are regular holonomic $\mathscr{D}_X$-modules if $\mathscr{M}$ is regular holonomic. The restrictions of ψ and φ to $\mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ are still denoted ψ and φ. We view then as functors from $\mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ to itself. There exist natural transformations $c\colon \varphi\to\psi$, $v\colon \psi\to\varphi$. These satisfy the condition that for any $\mathscr{M}\in \mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ there exists a non-zero polynomial $b\in \mathbf{C}[\Theta]$ such that:

(i) the $\psi\mathscr{M}$-endomorphism $c(\mathscr{M})v(\mathscr{M})$ satisfies $b(c(\mathscr{M})v(\mathscr{M}))=0$,
(ii) $b^{-1}(0)\subset\{z\in\mathbf{C}|0\leqq \operatorname{Re} z<1\}$.

2.4. A distinguished triangle

Our goal in this subsection is to show the existence of a distinguished triangle in $D_{\mathrm{hr}}(\mathscr{D}_X)$, the derived category of bounded complexes of $\mathscr{D}_X$-modules with regular holonomic cohomology. For any $\mathscr{M}\in \mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ there exists a distinguished triangle

$$\varphi\mathscr{M}\to\psi\mathscr{M}\to \mathrm{R}\Gamma_{[X_0]}\mathscr{M}[1]\to\varphi\mathscr{M}[1].$$

Or in more down to earth terms, there exists an exact sequence of regular holonomic $\mathscr{D}_X$-modules

$$\mathscr{H}^0_{[X_0]}\mathscr{M}\hookrightarrow\varphi\mathscr{M}\xrightarrow{c}\psi\mathscr{M}\twoheadrightarrow\mathscr{H}^1_{[X_0]}\mathscr{M}.$$

We start with a lemma; its proof is a bit technical.

2.4.1. Lemma. *Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. There exists a natural isomorphism of $\mathscr{D}_X$-modules*

$$\mathscr{H}^0_{[X_0]}\mathscr{M}\xrightarrow{\cong}\operatorname{Ker}(t, i_*\mathscr{M}) = \mathit{hom}_{\mathscr{O}_Y}(\mathscr{O}_Y/t\mathscr{O}_Y, i_*\mathscr{M}).$$

Proof. Recall that (cf. 1.6.1, 2.1)

$$\mathscr{H}^0_{[X_0]}\mathscr{M}=\bigcup_{n\in\mathbf{N}}\operatorname{Ker}(f^n,\mathscr{M})\quad i_*\mathscr{M}=\mathbf{C}[\partial]\otimes\mathscr{M}.$$

Let $p=\sum_{j=0}^n\partial^j\otimes m_j\in i_*\mathscr{M}$.
Then $tp=\sum_{j=0}^{n-1}\partial^j\otimes(-(j+1)m_{j+1}+fm_j)+\partial^n\otimes fm_n$. Hence

$$\begin{aligned} p\in\operatorname{Ker}(t,i_*\mathscr{M}) \quad &\text{iff}\quad fm_n=0,\quad nm_n=fm_{n-1},\ \dots,\ m_1=fm_0\\ &\text{iff}\quad f^{n+1}m_0=0,\\ &\qquad j!\,m_j=f^jm_0\quad\text{for all}\quad j\in\{1,\dots,n\}.\end{aligned}$$

This clearly implies that the injective maps

$$\varepsilon_n(\mathscr{M})\colon \operatorname{Ker}(f^n, \mathscr{M}) \hookrightarrow \operatorname{Ker}(t, i_*\mathscr{M}),$$

$$m \mapsto \sum_{j=0}^{n} \frac{1}{j!}\partial^j \otimes f^j m, \quad \text{for all } m,$$

induce a bijective $\mathscr{O}_X$-linear map

$$\varepsilon(\mathscr{M})\colon \mathscr{H}^0_{[X_0]}\mathscr{M} \to \operatorname{Ker}(t, i_*\mathscr{M}).$$

Clearly ε is functorial in $\mathscr{M}$, so it remains to check that $\varepsilon(\mathscr{M})$ is $\mathscr{D}_X$-linear. Therefore let $\xi \in \mathscr{D}er(\mathscr{O}_X)$, $m \in \operatorname{Ker}(f^n, \mathscr{M})$. Then $\xi m \in \operatorname{Ker}(f^{n+1}, \mathscr{M})$. Using the description of the $\mathscr{D}_Y$-structure on $\mathbf{C}[\partial]\otimes\mathscr{M}$ in 2.1 one obtains

$$\begin{aligned}
\xi\varepsilon(\mathscr{M})(m) &= \xi \sum_{j=0}^{n} \frac{1}{j!}\partial^j \otimes f^j m \\
&= \sum_{j=0}^{n} \frac{1}{j!}\partial^j \otimes \xi f^j m - \sum_{j=0}^{n} \frac{1}{j!}\partial^{j+1} \otimes \xi(f) f^j m \\
&= \sum_{j=0}^{n} \frac{1}{j!}\partial^j \otimes (f^j\xi + j\xi(f)f^{j-1})m \\
&\qquad - \sum_{j=1}^{n+1} \frac{1}{(j-1)!}\partial^j \otimes \xi(f) f^{j-1} m \\
&= \sum_{j=0}^{n} \frac{1}{j!}\partial^j \otimes f^j \xi m = \varepsilon(\mathscr{M})(\xi m).
\end{aligned}$$

2.4.2. Corollary. *Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. Then*

$$\mathrm{R}\Gamma_{[X_0]}\mathscr{M} \xrightarrow{\cong} \mathrm{R}\,\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y/t\mathscr{O}_Y, i_*\mathscr{M}).$$

Proof. The result follows once we have checked that

$$\mathscr{M} \text{ injective } \mathscr{D}_X\text{-module} \Rightarrow i_*\mathscr{M} \text{ is acyclic for } \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y/t\mathscr{O}_Y, -).$$

But this is clear, because an injective $\mathscr{D}_X$-module $\mathscr{M}$ is injective when considered as an $\mathscr{O}_X$-module. Hence $\mathscr{M}$ is divisible by f i.e., multiplication by f on $\mathscr{M}$ is surjective. This implies that the multiplication with t on $i_*\mathscr{M}$ is surjective i.e.,

$$\mathscr{E}xt^i_{\mathscr{O}_Y}(\mathscr{O}_Y/\mathscr{O}_Y t, i_*\mathscr{M}) = 0,$$

for all $i>0$. The argument is as follows: Let $m \in \mathscr{M}$. There exists $n \in \mathscr{M}$ such that $fn=m$. Proceed by induction on j, using $t(\partial^j \otimes n) = -j\partial^{j-1}\otimes n + \partial^j \otimes m$.

2.4.3. Proposition. *Let $\mathscr{M} \in \mathscr{mod}(\mathscr{D}_X)_{\mathrm{hr}}$. In $D_{\mathrm{hr}}(\mathscr{D}_X)$ we have a distinguished triangle, functorial in $\mathscr{M}$,*

$$\varphi\mathscr{M} \to \psi\mathscr{M} \to \mathrm{R}\Gamma_{[X_0]}\mathscr{M}[1] \to \varphi\mathscr{M}[1].$$

Proof. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Consider the short exact sequence of $F^0\mathcal{D}_Y$-modules

$$F^0 i_*\mathcal{M} \hookrightarrow i_*\mathcal{M} \twoheadrightarrow i_*\mathcal{M}/F^0 i_*\mathcal{M},$$

where $F^\cdot i_*\mathcal{M}$ denotes the canonical good filtration on $i_*\mathcal{M}$. The functor $\mathrm{R}\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, -)$ applied to this sequence yields a distinguished triangle

$$\begin{tikzcd}[column sep=tiny] & \mathrm{R}\,\mathcal{h}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}/F^0 i_*\mathcal{M}) \arrow[dl, "+"'] & \\ \mathrm{R}\,\mathcal{h}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, F^0 i_*\mathcal{M}) \arrow[rr] & & \mathrm{R}\,\mathcal{h}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}). \arrow[ul] \end{tikzcd}$$

By 2.4.2 we have

$$\mathrm{R}\,\mathcal{h}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, i_*\mathcal{M}) = \mathrm{R}\Gamma_{[X_0]}\mathcal{M}.$$

By Proposition 1.6.3 (i) and Proposition 1.5.1 it follows

$$\mathrm{R}\,\mathcal{h}om_{\mathcal{O}_Y}(\mathcal{O}_Y/t\mathcal{O}_Y, F^0 i_*\mathcal{M}) = F^0 i_*\mathcal{M}/tF^0 i_*\mathcal{M}[-1] = \psi\mathcal{M}[-1].$$

By Proposition 1.5.1

$$\mathrm{Ker}\,(t, i_*\mathcal{M}/F^0 i_*\mathcal{M}) = t^{-1}F^0 i_*\mathcal{M}/F^0 i_*\mathcal{M} = \varphi\mathcal{M}.$$

Let us investigate $\mathrm{Coker}\,(t, i_*\mathcal{M}/F^0 i_*\mathcal{M})$. Therefore let $m \in i_*\mathcal{M}$. For some $N \in \mathbf{N}^*$, $m \in F^{-N} i_*\mathcal{M}$. It follows that $a(\theta)m \in F^0 i_*\mathcal{M}$, where $a(\theta) = b(\theta+1)\dots b(\theta+N)$ and $b \in \mathbf{C}[\Theta]$ is a non-zero polynomial satisfying (5) of Subsection 1.3. Then $a(\theta) = t\partial\tilde{a}(\theta) + a(0)$ with $\tilde{a} \in \mathbf{C}[\Theta]$ and $a(0) \in \mathbf{C}^*$. Thus $a(0)m + t\partial\tilde{a}(\theta)m \in F^0 i_*\mathcal{M}$. We conclude that $\mathrm{Coker}\,(t, i_*\mathcal{M}/F^0 i_*\mathcal{M}) = 0$.

Finally collecting things we end up with a distinguished triangle

$$\varphi\mathcal{M} \to \psi\mathcal{M} \to \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \to \varphi\mathcal{M}[1].$$

2.4.4. Some easy consequences

Let $\mathcal{M} \in \mathcal{mod}\,(\mathcal{D}_X)_{\mathrm{hr}}$. There are two natural distinguished triangles in $\mathrm{D}_{\mathrm{hr}}(\mathcal{D}_X)$

$$\varphi\mathcal{M} \to \psi\mathcal{M} \to \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \to \varphi\mathcal{M}[1] \quad \mathcal{M} \to \mathcal{M}[f^{-1}] \to \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \to \mathcal{M}[1].$$

It follows immediately from these triangles.

2.4.4.1. Proposition. *For every $\mathcal{M} \in \mathcal{mod}\,(\mathcal{D}_X)_{hr}$ the following are equivalent*

(i) $\mathcal{M} \xrightarrow{\simeq} \mathcal{M}[f^{-1}]$;
(ii) $\mathrm{R}\Gamma_{[X_0]}\mathcal{M} = 0$;
(iii) $c(\mathcal{M})$: $\varphi\mathcal{M} \to \psi\mathcal{M}$ *is an isomorphism.*

2.4.4.2. Proposition. *For every* $\mathscr{M} \in \mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ *the following are equivalent:*

(i) $\operatorname{supp}(\mathscr{M}) \subset X_0$;

(ii) $\mathscr{H}^0_{[X_0]}\mathscr{M} = \mathscr{M}$;

(iii) $\mathscr{M}[f^{-1}] = 0$;

(iv) $\varphi\mathscr{M} = \mathscr{M}$.

Furthermore any of these conditions implies $\psi\mathscr{M} = 0$.

Proof. The equivalence of (i), (ii) and (iii) is well-known. (iv)⇒(i) is clear. Now (ii) implies $i_*\mathscr{M} = \mathscr{H}^0_{[X \times \{0\}]} i_*\mathscr{M}$, hence $\psi\mathscr{M} = 0$ (cf. 1.6.1) and thus $\varphi\mathscr{M} = \mathrm{R}\Gamma_{[X_0]}\mathscr{M} = \mathscr{M}$.

Remark. With a little more effort one can show that $\psi\mathscr{M} = 0$ implies $\varphi\mathscr{M} = \mathscr{M}$.

2.4.4.3. Corollary. *Let* $\mathscr{M} \in \mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ *and let* $\pi\colon \mathscr{M} \to \mathscr{M}[f^{-1}]$ *be the canonical map. Then:*

(i) $\psi(\pi)\colon \psi\mathscr{M} \to \psi(\mathscr{M}[f^{-1}])$ *is an isomorphism;*

(ii) *there exists an exact sequence*

$$\mathscr{H}^0_{[X_0]}\mathscr{M} \hookrightarrow \varphi\mathscr{M} \xrightarrow{\varphi(\pi)} \varphi(\mathscr{M}[f^{-1}]) \twoheadrightarrow \mathscr{H}^1_{[X_0]}\mathscr{M};$$

(iii) $c(\mathscr{M}[f^{-1}]) \circ \varphi(\pi) = \psi(\pi) \circ c(\mathscr{M})$.

2.4.5. An alternative proof of 2.4.3

The reader who is not happy with the given proof of 2.4.3 is offered a different approach. We give a derivation, in the category $\mathit{mod}(\mathscr{D}_X)_{\mathrm{hr}}$ of an equivalent formulation of 2.4.3. We avoid the use of Corollary 2.4.2. First we need some preliminary results.

2.4.5.1. Sublemma. *Let* $\mathscr{M}$ *be a coherent* $\mathscr{D}_X$*-module with support contained in* X_0 *(thus* $\mathscr{M} = \mathscr{H}^0_{[X_0]}\mathscr{M}$*).*
Then $\varphi\mathscr{M} = \mathscr{M}$ *and* $\psi\mathscr{M} = 0$.

Proof. Note that $i_*\mathscr{M}$ is coherent $\mathscr{D}_Y$-module supported on $X = X \times \{0\}$ i.e., $i_*\mathscr{M} = \mathscr{H}^0_{[X]} i_*\mathscr{M}$. Thus the canonical good filtration $F^{\bullet} i_*\mathscr{M}$ satisfies (cf. 1.6.1) $F^{-1} i_*\mathscr{M} = \operatorname{Ker}(t, i_*\mathscr{M})$ and $F^k i_*\mathscr{M} = 0$, for all $k \in \mathbf{N}$. Thus $\psi\mathscr{M} = 0$ and by Lemma 2.4.1 $\varphi\mathscr{M} = \mathscr{M}$.

2.4.5.2. Sublemma. *Let* $\mathscr{M}$ *be a* $\mathscr{D}_X$*-module. Then the map "multiplication by* t*"*

$$i_*(\mathscr{M}[f^{-1}]) \xrightarrow{t\cdot} i_*(\mathscr{M}[f^{-1}])$$

is bijective.

Proof. The injectivity follows using Lemma 2.4.1. The surjectivity follows (as in the proof of Corollary 2.4.2) by induction on j, using $t(\partial^j \otimes f^{-1}m) = -j\partial^{j-1} \otimes f^{-1}m + \partial^j \otimes m$ for all $m \in \mathcal{M}[f^{-1}]$.

2.4.5.3. Corollary. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module such that $i_*\mathcal{M}$ admits a canonical good filtration. Assume that the canonical map $\pi\colon \mathcal{M} \to \mathcal{M}[f^{-1}]$ is an isomorphism. Then $c(\mathcal{M})\colon \varphi\mathcal{M} \to \psi\mathcal{M}$ is an isomorphism.*

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd} F^0 i_*\mathcal{M} \arrow[r, hook] \arrow[d, "\cong"', "t\cdot"] & F^{-1} i_*\mathcal{M} \arrow[r, two heads] \arrow[d, "t\cdot"] & \varphi\mathcal{M} \arrow[d, "c"] \\ F^1 i_*\mathcal{M} \arrow[r, hook] & F^0 i_*\mathcal{M} \arrow[r, two heads] & \psi\mathcal{M}. \end{tikzcd}$$

The left arrow is bijective by Prop. 1.5.1 and Prop. 1.6.3 (i). By Sublemma 2.4.5.2 the middle arrow is injective. This Lemma together with the fact that $F^{-1}i_*\mathcal{M} = t^{-1}F^0 i_*\mathcal{M}$ (cf. Prop. 1.5.1) yield the surjectivity of the middle arrow. Hence c is bijective.

2.4.5.4. Proposition. *Let $\mathcal{M} \in \mathcal{M}od\,(\mathcal{D}_X)_{\mathrm{hr}}$. There exists a natural exact sequence of regular holonomic $\mathcal{D}_X$-modules*

$$\mathcal{H}^0_{[X_0]}\mathcal{M} \hookrightarrow \varphi\mathcal{M} \xrightarrow{c} \psi\mathcal{M} \twoheadrightarrow \mathcal{H}^1_{[X_0]}\mathcal{M}.$$

Proof. Consider the short exact sequences of $\mathcal{D}_X$-modules

$$\mathcal{H}^0_{[X_0]}\mathcal{M} \hookrightarrow \mathcal{M} \twoheadrightarrow \tilde{\mathcal{M}}$$

$$\tilde{\mathcal{M}} \hookrightarrow \mathcal{M}[f^{-1}] \twoheadrightarrow \mathcal{H}^1_{[X_0]}\mathcal{M}$$

where $\tilde{\mathcal{M}} = \mathrm{Im}\,(\mathcal{M} \to \mathcal{M}[f^{-1}])$. In view of Sublemma 2.4.5.1 these give rise to two commutative diagrams with exact rows

$$\begin{tikzcd} \mathcal{H}^0_{[X_0]}\mathcal{M} \arrow[r, hook] \arrow[d] & \varphi\mathcal{M} \arrow[r, two heads] \arrow[d, "c"] & \varphi\tilde{\mathcal{M}} \arrow[d, "c"] \\ 0 \arrow[r, hook] & \psi\mathcal{M} \arrow[r, two heads] & \psi\tilde{\mathcal{M}} \end{tikzcd}$$

$$\begin{tikzcd} \varphi\tilde{\mathcal{M}} \arrow[r, hook] \arrow[d, "c"] & \varphi(\mathcal{M}[f^{-1}]) \arrow[r, two heads] \arrow[d, "\cong"', "c"] & \mathcal{H}^1_{[X_0]}\mathcal{M} \arrow[d] \\ \psi\tilde{\mathcal{M}} \arrow[r, hook] & \psi(\mathcal{M}[f^{-1}]) \arrow[r, two heads] & 0. \end{tikzcd}$$

By Corollary 2.4.5.3 $c(\mathcal{M}[f^{-1}])$ is bijective. Hence the second diagram gives $\mathrm{Ker}\, c(\tilde{\mathcal{M}}) = 0$, $\mathrm{Coker}\, c(\tilde{\mathcal{M}}) = \mathcal{H}^1_{[X_0]}\mathcal{M}$. Now using the first diagram it follows that

$$\mathrm{Ker}\, c(\mathcal{M}) = \mathcal{H}^0_{[X_0]}\mathcal{M}, \quad \mathrm{Coker}\, c(\mathcal{M}) = \mathcal{H}^1_{[X_0]}\mathcal{M}.$$

2.5. Relation with Deligne's functors

2.5.1. It is well-known that the contravariant "solution" functor

$$S\colon \mathscr{M}od\,(\mathscr{D}_X)_{\mathrm{hr}} \to \mathrm{Perv}\,(X)$$

defined by

$$S(\mathscr{M}) = \mathrm{R}\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X),$$

establishes an (anti-)equivalence of categories. Here $\mathrm{Perv}\,(X)$ denotes the category of perverse sheaves on X. This is known as the Riemann—Hilbert correspondence (cf. [9] or [6]). Via this equivalence the functors φ resp. ψ correspond to the vanishing cycle functor Φ_f resp. the nearby cycle functor Ψ_f as introduced by Deligne [1]. More precise for $\mathscr{M}\in\mathscr{M}od\,(\mathscr{D}_X)_{\mathrm{hr}}$ there are natural isomorphisms (cf. [3], Thm. 2)

$$S(\varphi\mathscr{M})|_{X_0} \cong \Phi_f(S\mathscr{M})[-1],$$

$$S(\psi\mathscr{M})|_{X_0} \cong \Psi_f(S\mathscr{M})[-1].$$

c agrees with the canonical map $\mathrm{can}\colon \Psi_f\to\Phi_f$ and $v\,\dfrac{\exp(2\pi i\theta)-1}{\theta}$ agrees with the variation map $\mathrm{var}\colon \Phi_f\to\Psi_f$, where $\theta=cv$. The monodromy on Ψ_f is given by $S(\exp 2\pi i\theta)$ (loc. cit.).

Furthermore

$$S(\mathrm{R}\Gamma_{[X_0]}\mathscr{M}) \cong S(\mathscr{M})|_{X_0}$$

for every regular holonomic $\mathscr{D}_X$-module $\mathscr{M}$. (Cf. [9], Prop. 1.2.1). So our distinguished triangle corresponds to the fundamental distinguished triangle in $\mathrm{Perv}\,(X)$

$$S(\mathscr{M})|_{X_0} \to \Psi_f(S\mathscr{M}) \to \Phi_f(S\mathscr{M}) \to S(\mathscr{M})|_{X_0}[1].$$

2.5.2. We have seen that we might define functors φ and ψ on a somewhat bigger category $\mathscr{R}$ (cf. 2.2.1), the abelian category of coherent $\mathscr{D}_X$-modules $\mathscr{M}$ such that $i_*\mathscr{M}$ carries a canonical good filtration. To assure that $\varphi\mathscr{M}$ and $\psi\mathscr{M}$ are again regular holonomic $\mathscr{D}_X$-modules we restricted those functors to the category $\mathscr{M}od\,(\mathscr{D}_X)_{\mathrm{hr}}$ (cf. Thm. 1.4 (iv)). We shall indicate that in some sense this is necessary too.

Note that, for $\mathscr{M}\in\mathscr{M}od\,(\mathscr{D}_X)_{\mathrm{h}}$, $S(\mathscr{M})\in\mathrm{D}_c(X)$, the derived category of bounded complexes of $\underline{\mathbf{C}}_X$-modules with constructible cohomology. On this level S is not an equivalence. The functors Φ_f and Ψ_f are defined on $\mathrm{D}_c(X)$ and take their values in $\mathrm{D}_c(X_0)$. In $\mathrm{D}_c(X_0)$ there exists, for all $\mathscr{F}\in\mathrm{D}_c(X)$, a distinguished triangle

$$\mathscr{F}|_{X_0} \to \Psi_f\mathscr{F} \to \Phi_f\mathscr{F} \to \mathscr{F}|_{X_0}[1].$$

- **Assume** that for any $\mathscr{M}\in\mathscr{R}_{\mathrm{h}}:=\mathscr{R}\cap\mathscr{M}od\,(\mathscr{D}_X)_{\mathrm{h}}$:

— there exist natural isomorphisms

$$S(\varphi\mathcal{M})|_{X_0} \cong \Phi_f(S\mathcal{M})[-1], \quad S(\psi\mathcal{M})|_{X_0} \cong \Psi_f(S\mathcal{M})[-1].$$

Note that $\varphi\mathcal{M}$ and $\psi\mathcal{M}$ are holonomic $\mathcal{D}_X$-modules.

Under these assumptions 2.4.3 still holds i.e., for any $\mathcal{M}\in\mathcal{R}_h$ we have a distinguished triangle in $\mathcal{Mod}\,(\mathcal{D}_X)_h$

$$\varphi\mathcal{M} \to \psi\mathcal{M} \to \mathrm{R}\Gamma_{[X_0]}\mathcal{M}[1] \to \varphi\mathcal{M}[1].$$

This triangle corresponds via S to the triangle above (take $\mathcal{F}=S(\mathcal{M})$). This yields $S(\mathrm{R}\Gamma_{[X_0]}\mathcal{M})=S(\mathcal{M})|_{X_0}$ i.e., $\mathcal{M}$ is regular along X_0. So in order that (•) holds, we have to limit ourselves to regular holonomic $\mathcal{D}_X$-modules.

3. The main theorem

Let X, Y, f, X_0 be as in § 2. In this section we prove that the mapping, for all $\mathcal{M}\in\mathcal{Mod}\,(\mathcal{D}_X)_{\mathrm{hr}}$:

$$\mathcal{M} \mapsto (\mathcal{M}[f^{-1}],\ \varphi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M},\ \psi(\pi))$$

(with $\pi\colon \mathcal{M}\to\mathcal{M}[f^{-1}]$ the canonical map), defines an equivalence of categories. By the Riemann—Hilbert correspondence (cf. 2.5.1) this corresponds to Verdier's extension theorem of perverse sheaves (cf. [12]). Of course this offers a way to prove the above claim, but we prefer to give a derivation using only the language of $\mathcal{D}$-modules (without an appeal to the Riemann—Hilbert correspondence).

3.1. Definitions and notations

First of all we introduce some notations in order to be able to formulate the theorem correctly. Let $\mathcal{Mod}\,(\mathcal{D}_X)_{X_0,\mathrm{hr}}$ denote the category of regular holonomic $\mathcal{D}_X$-modules with support contained in X_0.

3.1.1. Let $\mathcal{C}(\mathcal{D}_X)_{X_0,\mathrm{hr}}$ denote the category determined as follows:

— Objects: quadruples $(\mathcal{M}_1, \mathcal{M}_2, U, V)$ where

$\mathcal{M}_1, \mathcal{M}_2\in\mathcal{Mod}\,(\mathcal{D}_X)_{X_0,\mathrm{hr}}$,

$U\in\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_2)$, $V\in\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_2, \mathcal{M}_1)$.

— Morphisms: $(\alpha_1, \alpha_2)\in\mathrm{Hom}\,((\mathcal{M}_1, \mathcal{M}_2, U, V), (\mathcal{M}_1', \mathcal{M}_2', U', V'))$ iff

$$\alpha_1\in\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_1, \mathcal{M}_1'), \quad \alpha_2\in\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_2, \mathcal{M}_2')$$

such that

$U'\alpha_1 = \alpha_2 U, \quad \alpha_1 V = V'\alpha_2.$

$\mathscr{C}(\mathscr{D}_X)_{X_0,\mathrm{hr}}$ is an abelian category. For details we refer to [2, § 1]. Note that for all $\mathscr{M}\in\mathscr{M}od(\mathscr{D}_X)_{\mathrm{hr}}$ we have $(\varphi\mathscr{M}, \psi\mathscr{M}, c(\mathscr{M}), v(\mathscr{M}))\in\mathscr{C}(\mathscr{D}_X)$, where for convenience we dropped "X_0, hr". If $\alpha\colon \mathscr{M}\to\mathscr{M}'$ is a morphism, then $(\varphi(\alpha), \psi(\alpha))$ is a morphism in $\mathscr{C}(\mathscr{D}_X)$.

Notation. In the sequel we use the notation $\mathscr{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{M}_2$ to denote the object $(\mathscr{M}_1, \mathscr{M}_2, U, V)\in\mathscr{C}(\mathscr{D}_X)$.

3.1.2. Let $\mathrm{Rc}(X, X_0)$ denote the category determined as follows:

— Objects: triples $(\mathscr{M}, \mathscr{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{M}_2, \alpha)$ where

$\mathscr{M}\in\mathscr{M}od(\mathscr{D}_X)_{\mathrm{hr}}$ such that $\mathscr{M}\xrightarrow{\cong}\mathscr{M}[f^{-1}]$ (canonical map);

$\mathscr{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{M}_2\in\mathscr{C}(\mathscr{D}_X)$;

$\alpha\colon \mathscr{M}_2\xrightarrow{\cong}\psi\mathscr{M}$ is a $\mathscr{D}_X$-linear isomorphism such that

$\alpha UV = c(\mathscr{M})v(\mathscr{M})\alpha$.

— Morphisms:

$(\beta, \beta_1, \beta_2)\in\mathrm{Hom}((\mathscr{M}, \mathscr{M}_1 \rightleftarrows \mathscr{M}_2, \alpha), (\mathscr{M}', \mathscr{M}_1' \rightleftarrows \mathscr{M}_2', \alpha'))$

iff $\beta\colon \mathscr{M}\to\mathscr{M}'$ is $\mathscr{D}_X$-linear and

$(\beta_1, \beta_2)\in\mathrm{Hom}_{\mathscr{C}}(\mathscr{M}_1 \rightleftarrows \mathscr{M}_2, \mathscr{M}_1' \rightleftarrows \mathscr{M}_2')$ such that

$\alpha'\beta_2 = \psi(\beta)\alpha$.

$\mathrm{Rc}(X, X_0)$ is an abelian category. Note that for all $\mathscr{M}\in\mathscr{M}od(\mathscr{D}_X)_{\mathrm{hr}}$

$$(\mathscr{M}, \mathscr{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{M}_2, \alpha)\in\mathrm{Rc}(X, X_0),$$

$c(\mathscr{M})\colon \varphi\mathscr{M}\to\psi\mathscr{M}$ is an isomorphism (by 2.4.4.1 (iii)). Note furthermore that we have a morphism $(c(\mathscr{M})^{-1}\alpha U, \alpha)\in\mathrm{Hom}_{\mathscr{C}}(\mathscr{M}_1\rightleftarrows\mathscr{M}_2, \varphi\mathscr{M}\rightleftarrows\psi\mathscr{M})$.

3.1.2.1. Remark

Let $(\mathscr{M}, \mathscr{M}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{M}_2, \alpha)\in\mathrm{Rc}(X, X_0)$. $\mathscr{M}_1$ (resp. $\mathscr{M}_2$) can be given the structure of a $\mathscr{D}_X[t\partial]$-module by defining the action of $t\partial$ as the $\mathscr{D}_X$-endomorphism $VU-1$ (resp. UV) (1 denotes the identity map on $\mathscr{M}_1$). Let $b\in\mathbf{C}[\Theta]$ be a non-zero polynomial belonging to the canonical good filtration $F^{\cdot}i_*\mathscr{M}$ (cf. (5) of 2.1). This implies:

$b(t\partial)\psi\mathscr{M} = 0$,

$b^{-1}(0)\subset\{z\in\mathbf{C}|0\leqq\mathrm{Re}\, z<1\}$.

Hence $b(t\partial)\mathcal{M}_2=0$. Further $Ub(t\partial+1)\mathcal{M}_1=b(t\partial)U\mathcal{M}_1=0$ i.e., $\partial tb(\partial t)\mathcal{M}_1=0$. So we see that $\mathcal{M}_1 \rightleftarrows \mathcal{M}_2$ satisfies the additional requirement (compare with 2.3): there exists a non-zero polynomial $a\in \mathbf{C}[\Theta]$ with:

$$a(t\partial)\mathcal{M}_2 = 0,$$

$$a(t\partial+1)\mathcal{M}_1 = 0,$$

$$a^{-1}(0) \subset \{z\in \mathbf{C} | 0 \leqq \operatorname{Re} z < 1\}.$$

3.1.2.2. Remark

Let $\mathcal{M}\in \mathcal{M}od\,(\mathcal{D}_X)_{\mathrm{hr}}$ and denote $\pi\colon \mathcal{M}\to\mathcal{M}[f^{-1}]$ the canonical map. By Corollary 2.4.4.3 we have that $\psi(\pi)$ is an isomorphism and that $\psi(\pi)c(\mathcal{M})v(\mathcal{M})= =v(\mathcal{M}[f^{-1}])c(\mathcal{M}[f^{-1}])\psi(\pi)$. From this it follows that

$$\left(\mathcal{M}[f^{-1}],\ \varphi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M},\ \psi(\pi)\right)\in \mathrm{Rc}(X, X_0).$$

Furthermore if $\alpha\colon \mathcal{M}\to\mathcal{M}'$ is a morphism, then $\left(\alpha[f^{-1}], \varphi(\alpha), \psi(\alpha)\right)$ is a morphism in $\mathrm{Rc}\,(X, X_0)$.

3.2. Theorem. *The functor*

$$F\colon \mathcal{M}od\,(\mathcal{D}_X)_{\mathrm{hr}} \to \mathrm{Rc}\,(X, X_0)$$

defined by, for all $\mathcal{M}\in \mathcal{M}od\,(\mathcal{D}_X)_{\mathrm{hr}}$

$$\mathcal{M} \mapsto \left(\mathcal{M}[f^{-1}],\ \varphi\mathcal{M} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathcal{M},\ \psi(\pi)\right)$$

establishes an equivalence of categories.

The rest of this subsection is devoted to the proof of the theorem. As we mentioned already this theorem is an analogue of a theorem on extensions of perverse sheaves due to Verdier [12]. Before we begin with the proof we derive two lemmas. These throw some light on how to reconstruct $\mathcal{M}$ from the data $F(\mathcal{M})$. Needless to say that they will be used in the derivation of 3.2.

The difficulty of the derivation lies in the reconstruction. Given an object $(\mathcal{N}, \mathcal{N}_1 \rightleftarrows \mathcal{N}_2, \alpha)\in \mathrm{Rc}\,(X, X_0)$, find a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ such that $F(\mathcal{M})$ is isomorphic to the given element in $\mathrm{Rc}\,(X, X_0)$. The idea is to recapture the various levels of the filtration from the given data and thereby reconstructing $\mathcal{M}$. By considerations in § 1 $F^0 i_*\mathcal{M}$, the zeroth-level, must equal $F^0 i_*\mathcal{N}$. The first lemma we derive, Lemma 3.2.1, tells us how the (-1)-level can be regained. Successive applications of the second Lemma 3.2.2 take care of the $(-k)$-levels for all $k \geqq 2$.

During this process a lot of things need to be checked. This we plan to do in Subsection 3.2.3. Finally in Subsection 3.2.4 we finish the proof of Theorem 3.2.

3.2.1. Lemma. *Let $\mathscr{M}\in\mathscr{M}od\,(\mathscr{D}_X)_{\mathrm{hr}}$. Denote $F^{\cdot}i_*\mathscr{M}$ the canonical good filtration on $i_*\mathscr{M}$. Denote $\pi\colon \mathscr{M}\to\mathscr{M}[f^{-1}]$ the canonical map. Consider the commutative diagrams*

$$\begin{array}{ccccccc} F^0 i_*\mathscr{M} & \twoheadrightarrow & \psi\mathscr{M} & \qquad & F^{-1}i_*\mathscr{M} & \twoheadrightarrow & \varphi\mathscr{M} \\ \cong\big\downarrow i_*\pi & & \cong\big\downarrow \psi(\pi) & & \big\downarrow i_*\pi & & \big\downarrow \varphi(\pi) \\ F^0 i_*\mathscr{M}[f^{-1}] & \twoheadrightarrow & \psi\mathscr{M}[f^{-1}], & & F^{-1}i_*\mathscr{M}[f^{-1}] & \twoheadrightarrow & \varphi\mathscr{M}[f^{-1}] \end{array}$$

where the horizontal arrows are the obvious projections.

Then these are pull-back diagrams of $F^0\mathscr{D}_Y$-modules.

Proof. By Cor. 2.4.4.3 (i) $\psi(\pi)$ is an isomorphism. It follows from the Corollaries 1.6.4 and 1.6.1 (2) that $i_*\pi\colon F^0 i_*\mathscr{M}\to F^0 i_*\mathscr{M}[f^{-1}]$ is an isomorphism. This settles the diagram on the left. The assertion about the diagram on the right is easily verified by chasing in the following commutative diagram with exact rows and exact columns. We leave it to the reader.

$$\begin{array}{ccccc} & & F^{-1}i_*\mathscr{H}^0_{[X_0]}\mathscr{M} & = & \varphi(\mathscr{H}^0_{[X_0]}\mathscr{M}) \\ & & \downarrow & & \downarrow \\ F^0 i_*\mathscr{M} & \hookrightarrow & F^{-1}i_*\mathscr{M} & \twoheadrightarrow & \varphi\mathscr{M} \\ \cong\big\downarrow i_*\pi & & \big\downarrow i_*\pi & & \big\downarrow \varphi(\pi) \\ F^0 i_*\mathscr{M}[f^{-1}] & \hookrightarrow & F^{-1}i_*\mathscr{M}[f^{-1}] & \twoheadrightarrow & \varphi(\mathscr{M}[f^{-1}]) \end{array}$$

3.2.2. Lemma. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module carrying a canonical good filtration $F^{\cdot}\mathscr{M}$. Then, for any $k\in\mathbf{N}$, $k\neq 0$, we have a push-out diagram of $\mathrm{pr}^{-1}\mathscr{D}_X$-modules ($\mathrm{pr}\colon X\times\mathbf{C}\to X$ denotes the projection)*

$$\begin{array}{ccc} F^{-k+1}\mathscr{M} & \hookrightarrow & F^{-k}\mathscr{M} \\ \big\downarrow \partial & & \big\downarrow \partial \\ F^{-k}\mathscr{M} & \hookrightarrow & F^{-k-1}\mathscr{M}. \end{array}$$

Proof. Let $k\in\mathbf{N}$, $k\neq 0$. We must show that the following sequence is exact (as $\mathrm{pr}^{-1}\mathscr{D}_X$-modules)

$$F^{-k+1}\mathscr{M}\xrightarrow{r} F^{-k}\mathscr{M}\oplus F^{-k}\mathscr{M}\xrightarrow{s} F^{-k-1}\mathscr{M}\to 0.$$

$$m\mapsto(-\partial m, m)\quad (m,m')\mapsto m+\partial m'.$$

Clearly $sr=0$. Now let $m, m'\in F^{-k}\mathscr{M}$ be such that $s(m,m')=m+\partial m'=0$. Hence $t\partial m'=-tm\in F^{-k+1}\mathscr{M}$. But $b(t\partial+k)m'\in F^{-k+1}\mathscr{M}$, where $b\in\mathbf{C}[\Theta]$ is a non-zero polynomial as in (5) of 1.3. As $k\neq 0$, $t\partial$ and $b(t\partial+k)$ are relatively prime, this implies that $m'\in F^{-k+1}\mathscr{M}$ and thus $(m,m')=r(m')$. This establishes the exactness in the middle.

Finally let us show that s is surjective. Let $n \in F^{-k-1}\mathcal{M}$. Then

$$b(t\partial+k+1)n \in F^{-k}\mathcal{M}.$$

As $b(t\partial+k+1)=\partial t\tilde{b}(t\partial+1)+b(k)$, with $b(k)\in \mathbf{C}^*$ (because $k\neq 0$) and some $\tilde{b}\in \mathbf{C}[\Theta]$, it follows that $n\in F^{-k}\mathcal{M}+\partial F^{-k}\mathcal{M}=\operatorname{Im} s$.

Remark. We have seen in 1.1 that $\mathrm{pr}^{-1}\mathcal{D}_X$ is a subring of $F^0\mathcal{D}_Y$. Elements of $\mathrm{pr}^{-1}\mathcal{D}_X$ commute with ∂. In fact $\mathrm{pr}^*\mathcal{D}_X[t\partial]=F^0\mathcal{D}_Y$.

Remark. The lemma implies that ∂: $\mathrm{gr}^{-k}\mathcal{M}\to \mathrm{gr}^{-k-1}\mathcal{M}$ is bijective for $k\geqq 1$. Compare this with 1.6.6.

3.2.3. The reconstruction procedure

The reconstruction is rather technical. We begin with defining an abelian category $\mathcal{A}$ and an additive subcategory $\mathcal{A}^*$. The category $\mathcal{A}^*$ serves as in intermediate in the construction of a $\mathcal{D}_X$-module $\mathcal{M}$ from the given data

$$N=(\mathcal{N}, \mathcal{N}_1 \rightleftarrows \mathcal{N}_2, \alpha)\in \mathrm{Rc}\,(X, X_0),$$

such that $F(\mathcal{M})=N$. We define a functor P: $\mathcal{A}^*\to\mathcal{A}^*$ that takes us from the $(-k)$-level to the $(-k-1)$-level of the filtration to exist on $i_*\mathcal{M}$. Repeated applications of P yield an inductive system. Taking the direct limit gives a functor P^∞: $\mathcal{A}^*\to \mathcal{M}od\,(\mathcal{D}_Y)$ which regains $i_*\mathcal{M}$ from the (-1)-level of the filtration.

Finally in 3.2.3.5 we define a functor Q: $\mathrm{Rc}\,(X, X_0)\to\mathcal{A}^*$ that extracts the (-1)-level from the given element $N\in \mathrm{Rc}\,(X, X_0)$.

In 3.2.3.6 we consider the composition $P^\infty Q$ and introduce the inverse G: $\mathrm{Rc}\,(X, X_0)\to \mathcal{M}od\,(\mathcal{D}_X)_{\mathrm{hr}}$.

3.2.3.1. Denote $\mathcal{A}$ the category defined as follows:
— Objects: $(G, H, \imath, \delta)$ where G and H are $\mathrm{pr}^{-1}\mathcal{D}_X$-modules and $\imath, \delta$: $G\to H$ are $\mathrm{pr}^{-1}\mathcal{D}_X$-morphisms.
— Morphisms: $(\alpha, \beta)\in \mathrm{Hom}_{\mathcal{A}}((G, H, \imath, \delta), (G', H', \imath', \delta'))$ iff α: $G\to G'$, β: $H\to H'$ are $\mathrm{pr}^{-1}\mathcal{D}_X$-morphisms satisfying $\beta\imath=\imath'\alpha$, $\beta\delta=\delta'\alpha$.

$\mathcal{A}$ is an abelian category (because it is a functor category. Cf. [2], § 1). The kernel and the cokernel of a morphism in $\mathcal{A}$ are evident.

Define a map P: $\mathcal{A}\to\mathcal{A}$ as follows: for every $(G, H, \imath, \delta)\in\mathcal{A}$, let $P(G, H, \imath, \delta)=(H, I, \imath_1, \delta_1)\in\mathcal{A}$ be given by the push-out diagram of $\mathrm{pr}^{-1}\mathcal{D}_X$-modules

$$\begin{array}{ccc} G & \xrightarrow{\imath} & H \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_1} \\ H & \xrightarrow{\imath_1} & I \end{array}$$

Because of the universal property of push-out, P is functorial. This yields also that P is right exact.

3.2.3.2. Denote $\mathscr{A}^*$ the subcategory of $\mathscr{A}$ given as follows:

— Objects: $(G, H, \iota, \delta)\in\mathscr{A}$ that satisfy the additional requirements:

(i) the $\mathrm{pr}^{-1}\mathscr{D}_X$-structure on G (resp. H) comes from a $F^0\mathscr{D}_Y$-structure on G (resp. H);

(ii) ι is a $F^0\mathscr{D}_Y$-linear injection;

(iii) for all $h\in\mathscr{O}_Y$: $\delta h - h\delta = \iota\partial(h)$;

(iv) $tH\subset \operatorname{Im}\iota$ i.e., $t\colon H\to H$ factors through ι; [Abusing language we denote the factorisation by $t\colon H\to G$ (thus $\iota t = t$). There will be no ambiguity for ι is injective.];

(v) the action of $t\partial\in F^0\mathscr{D}_Y$ on G (resp. H) is given by the $\mathrm{pr}^{-1}\mathscr{D}_X$-endomorphism $t\delta$ (resp. $\delta t - 1$).

— Morphisms: (α, β) as above but α and β are now supposed to be $F^0\mathscr{D}_Y$-linear.

Certainly $\mathscr{A}^*$ is an additive subcategory of $\mathscr{A}$; it is not abelian, because of the injectivity condition in (ii). But it perfectly makes sense to talk about exact sequences in $\mathscr{A}^*$, namely those sequences which are exact when considered in $\mathscr{A}$.

3.2.3.3. Lemma. *P restricts to an exact functor from $\mathscr{A}^*$ to $\mathscr{A}^*$ which we still denote by P; let $(G, H, \iota, \delta)\in\mathscr{A}^*$ and let us put $(H, I, \iota_1, \delta_1) = P(G, H, \iota, \delta)$. Then there exists a unique structure of a $F^0\mathscr{D}_Y$-module on I that satisfies* (i) *to* (v) *above.*

Proof. Let us first see that $P(\mathscr{A}^*)\subset\mathscr{A}^*$. Let $(G, H, \iota, \delta)\in\mathscr{A}^*$. Then $P(G, H, \iota, \delta)$ is given by the push-out of $\mathrm{pr}^{-1}\mathscr{D}_X$-modules

$$\begin{array}{ccc} G & \overset{\iota}{\hookrightarrow} & H \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_1} \\ H & \xrightarrow{\iota_1} & I \end{array}$$

I is a $\mathrm{pr}^{-1}\mathscr{D}_X$-module. We extend this structure to one over $F^0\mathscr{D}_Y$ as follows. Let $h\in\mathscr{O}_Y$. Consider the $\mathbf{C}$-linear mappings

$$\delta_1 h - \iota_1\partial(h)\colon H\to I,\quad \iota_1 h\colon H\to I.$$

These satisfy

$$\begin{aligned} (\delta_1 h - \iota_1\partial(h))\iota &= \delta_1\iota h - \iota_1\iota\partial(h) \quad (\iota \text{ is } F^0\mathscr{D}_Y\text{-linear}) \\ &= \iota_1(\delta h - \iota\partial(h)) \\ &= \iota_1 h\delta \quad (\text{by (iii) of 3.2.3.2}). \end{aligned}$$

Thus by the universal property of push-out there exists a unique **C**-linear map, denoted h: $I \to I$, satisfying $h\delta_1 = \delta_1 h - \imath_1 \partial(h)$ and $h\imath_1 = \imath_1 h$. This defines the structure of a $\mathrm{pr}^* \mathscr{D}_X$-module on I, extending the $\mathrm{pr}^{-1} \mathscr{D}_X$-structure and satisfying (iii) of 3.2.3.2.

Further $\imath_1$ becomes $\mathrm{pr}^* \mathscr{D}_X$-linear. Observe that $\imath_1$ is injective. Note that $\delta_1 t - \imath_1 = \imath_1(\delta t - 1)$, so t: $I \to I$ factors through $\imath_1$: $H \to I$. This yields 3.2.3.2 (iv).

Denote t: $I \to H$ the factorisation. One has $t\delta_1 = \delta t - 1$ i.e., the action of $t\partial \in F^0 \mathscr{D}_Y$ on H is given by $t\delta_1$. Define the action of $t\partial \in F^0 \mathscr{D}_Y$ on I to be $\delta_1 t - 1$. This gives I the desired structure of a $F^0 \mathscr{D}_Y$-module i.e., 3.2.3.2 (i) and establishes 3.2.3.2 (iv).

Note further that $(t\delta)\imath_1 = (\delta_1 t - 1)\imath_1 = \delta_1 t - \imath_1 = \imath_1(\delta t - 1) = \imath_1(t\partial)$, hence $\imath_1$ is $F^0 \mathscr{D}_Y$-linear. This establishes 3.2.3.2 (ii) and all together $P(\mathscr{A}^*) \subset \mathscr{A}^*$.

The next thing we must check is the functoriality of $P|_{\mathscr{A}^*}$. So let (α, β): $(G, H, \imath, \delta) \to (G', H', \imath', \delta')$ be a morphism in $\mathscr{A}^*$. By the universal property of push-out there exists a unique $\mathrm{pr}^{-1} \mathscr{D}_X$-linear map γ: $I \to I'$ satisfying $\gamma\delta_1 = \delta_1' \beta$, $\gamma\imath_1 = \imath_1' \beta$, where I is as above and $(H', I', \imath_1', \delta_1') = P(G', H', \imath', \delta')$. Thus $P(\alpha, \beta) = (\beta, \gamma)$. We must verify that γ is $F^0 \mathscr{D}_Y$-linear. Therefore let $h \in \mathscr{O}_Y$ and consider the **C**-linear mapping $\gamma h - h\gamma$: $I \to I'$. It satisfies:

$$\begin{aligned}(\gamma h - h\gamma)\delta_1 &= \gamma h\delta_1 - h\gamma\delta_1 \\ &= \gamma(\delta_1 h - \imath_1 \partial(h)) - h\delta_1' \beta \\ &= \delta_1' \beta h - \imath_1' \beta \partial(h) - h\delta_1' \beta \\ &= (\delta_1' h - \imath_1' \partial(h) - h\delta_1')\beta = 0\end{aligned}$$

and

$$(\gamma h - h\gamma)\imath_1 = \gamma\imath_1 h - h\imath_1' \beta = \imath_1'(\beta h - h\beta) = 0.$$

It follows, by the universal property of push-out, that $\gamma h - h\gamma = 0$. Consequently γ is $\mathrm{pr}^* \mathscr{D}_X$-linear. Especially we have $\imath_1' \beta t = \gamma \imath_1 t = \imath_1' t\gamma$, yielding $\beta t = t\gamma$ as $\imath_1'$ is injective. This implies

$$\gamma(t\partial) = \gamma(\delta_1 t - 1) = \gamma\delta_1 t - \gamma = \delta_1' \beta t - \gamma = (\delta_1' t - 1)\gamma = (t\partial)\gamma$$

i.e., γ is $F^0 \mathscr{D}_Y$-linear.

Finally the fact that $\imath$ is injective implies that P is exact.

3.2.3.4. From $\mathscr{A}^*$ to $\mathscr{Mod}\,(\mathscr{D}_Y)$

Let $A \in \mathscr{A}^*$. Lemma 3.2.2 suggests that we should take a series of push-outs, yielding an inductive system $\{((P^k A)_1, \imath_k) | k \in \mathbf{N}\}$ of $F^0 \mathscr{D}_Y$-modules. Here for any $j \in \{1, 2, 3, 4\}$ $(\cdot)_j$ denotes projecting on the j-th factor; for all $k \in \mathbf{N}$, $\imath_k := (P^k A)_3$ is an injective $F^0 \mathscr{D}_Y$-morphism from $(P^k A)_1$ into $(P^{k+1} A)_1$.

Put $P^\infty A := \text{inj.}\lim\, (P^k A)_1 \in \mathscr{Mod}\,(F^0 \mathscr{D}_Y)$.

We give $P^\infty A$ a $\mathscr{D}_Y$-structure as follows. For any $k \in \mathbf{N}$ define $\delta_k := (P^k A)_4$ a $\mathrm{pr}^{-1} \mathscr{D}_X$-linear map from $(P^k A)_1$ to $(P^{k+1} A)_1$. These satisfy $\delta_{k+1} \imath_k = \imath_{k+1} \delta_k$, for all $k \in \mathbf{N}$, yielding a $\mathrm{pr}^{-1}\mathscr{D}_X$-endomorphism δ of $P^\infty A$. For any $h \in \mathcal{O}_Y$ we have

$$\delta h = \mathrm{inj.\,lim}\,(\delta_k h) = \mathrm{inj.\,lim}\,\big(h\delta_k + \iota_k \partial(h)\big) = h\delta + \partial(h).$$

Consequently $P^\infty A$ becomes a $\mathscr{D}_Y$-module by defining the action of $\partial \in \mathscr{D}_Y$ as the endomorphism δ. Obviously $P^\infty A$ is functorial in A i.e., we get an exact functor

$$P^\infty\colon \mathscr{A}^* \to \mathscr{M}od\,(\mathscr{D}_Y).$$

Note that $(P^k A)_1$ may be regarded as a $F^0\mathscr{D}_Y$-submodule of $P^\infty A$. These induce a filtration on $P^\infty A$.

Note added in proof. As was kindly pointed out to me by A.H.M. Levelt the above procedure can be simplified by noting that $\big(P^n(G, H, \iota, \delta)\big)_2 = \mathrm{Coker}\,(\alpha_n\colon G^n \to H^{n+1})$ for all $n \in \mathbf{N} - \{0\}$ and that $P^\infty(G, H, \iota, \delta) = \mathrm{Coker}\,(\alpha_\infty\colon G^{(\mathbf{N})} \to H^{(\mathbf{N})})$. Here for all $(g_1, \ldots, g_n) \in G^n$ we have put

$$\alpha_n(g_1, \ldots, g_n) = \big(\delta(g_1), \delta(g_2) - \iota(g_1), \ldots, \delta(g_n) - \iota(g_{n-1}), -\iota(g_n)\big)$$

and $\alpha_\infty(g_1, g_2, \ldots) = \big(\delta(g_1), \delta(g_2) - \iota(g_1), \ldots\big)$ for all $(g_1, g_2, \ldots) \in G^{(\mathbf{N})}$.

3.2.3.5. From $\mathrm{Rc}\,(X, X_0)$ **to** $\mathscr{A}^*$

The final step (that is the first step of the reconstruction) is to define a functor $Q\colon \mathrm{Rc}\,(X, X_0) \to \mathscr{A}^*$. Its definition is suggested by Lemma 3.2.1.

Let $(\mathscr{N}, \mathscr{N}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{N}_2, \alpha) \in \mathrm{Rc}\,(X, X_0)$. One has:

(i) $\mathscr{N} = \mathscr{N}[f^{-1}]$, thus $c(\mathscr{N})$ is an isomorphism (cf. Prop. 2.4.4.1);
(ii) $\mathscr{N}_1$, $\mathscr{N}_2$, $\varphi\mathscr{N}$, $\psi\mathscr{N}$ have the structure of a module over $\mathrm{gr}^0\mathscr{D}_Y = \mathscr{D}_X[t\partial]$ (see 2.2.3) and thus a $F^0\mathscr{D}_Y$-structure;
(iii) α and $c(\mathscr{N})^{-1}\alpha U$ are $F^0\mathscr{D}_Y$-linear;
(iv) U, V, $c(\mathscr{N})$, $v(\mathscr{N})$ commute with $\mathrm{pr}^{-1}\mathscr{D}_X \subset F^0\mathscr{D}_Y$.

Denote $F^\bullet i_*\mathscr{N}$ the canonical good filtration on $i_*\mathscr{N}$ and define $G := F^0 i_*\mathscr{N}$. Consider the diagram of $F^0\mathscr{D}_Y$-modules

$$\begin{tikzcd} & N_1 \arrow[d, "c(\mathscr{N})^{-1}\alpha U"] \\ F^{-1}i_*\mathscr{N} \arrow[r, two heads] & \varphi\mathscr{N}. \end{tikzcd}$$

Define H to be the pull-back (as $F^0\mathscr{D}_Y$-modules). This yields a commutative diagram with exact rows

$$\begin{tikzcd} G \arrow[r, hook, "\iota"] \arrow[d, equal] & H \arrow[r, two heads] \arrow[d] & N_1 \arrow[d, "c(\mathscr{N})^{-1}\alpha U"] \\ F^0 i_*\mathscr{N} \arrow[r, hook] & F^{-1}i_*\mathscr{N} \arrow[r, two heads] & \varphi\mathscr{N}. \end{tikzcd}$$

Evidently we have a pull-back diagram

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[d, equal] & N_2 \arrow[d, "\cong"', "\alpha"] \\ F^0 i_* \mathcal{N} \arrow[r, two heads] & \psi \mathcal{N}. \end{tikzcd}$$

By the universal property of pull-backs $V\colon \mathcal{N}_2 \to \mathcal{N}_1$ induces an unique $\mathrm{pr}^{-1}\mathcal{D}_X$-linear map $\delta\colon G \to H$.

We verify that $(G, H, \iota, \delta) \in \mathcal{A}^*$. Clearly 3.2.3.2 (i), (ii) are true. Also 3.2.3.2 (iii) is easily checked. Of course $U\colon \mathcal{N}_1 \to \mathcal{N}_2$ agrees with $t\colon H \to G$, which establishes 3.2.3.2 (iv). Finally the action of $t\partial$ on G (resp. H) is given by $t\delta$ (resp. $\delta t - 1$), which takes care of 3.2.3.2 (v).

Clearly this construction is functorial and therefore yields a functor

$$Q\colon \mathrm{Rc}(X, X_0) \to \mathcal{A}^*.$$

Note furthermore that Q is exact.

3.2.3.6. The inverse functor G

In this subsection we investigate the effect of the functor $P^\infty Q$ on objects in the image of F, the one in Theorem 3.2. Does the reconstruction work well?

Let $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$ and denote $F^\cdot i_* \mathcal{M}$ the canonical good filtration on $i_* \mathcal{M}$. For all $k \in \mathbf{N}$ denote by $\iota_k\colon F^{-k} i_* \mathcal{M} \to F^{-k-1} i_* \mathcal{M}$ the inclusion. By Lemma 3.2.1 and the definition of Q there exists a natural isomorphism

$$Q(F\mathcal{M}) \cong (F^0 i_* \mathcal{M}, F^{-1} i_* \mathcal{M}, \iota_0, \partial).$$

Applying P^k to both sides and using Lemma 3.2.2 yields a natural isomorphism

$$P^k Q(F\mathcal{M}) \cong (F^{-k} i_* \mathcal{M}, F^{-k-1} i_* \mathcal{M}, \iota_k, \partial), \quad \text{for all} \quad k \in \mathbf{N}.$$

Hence there exists a natural isomorphism

$$P^\infty Q(F\mathcal{M}) \cong i_* \mathcal{M}.$$

Therefore the next definition doesn't come as a surprise. Define

$$G\colon \mathrm{Rc}(X, X_0) \to \mathcal{M}od(\mathcal{D}_X)$$

by putting for all $M \in \mathrm{Rc}(X, X_0)$,

$$G(M) := \mathrm{Ker}(t - f, P^\infty Q(M)).$$

The foregoing can then be restated as: there exists a natural isomorphism $GF(\mathcal{M}) \cong \mathcal{M}$, for all $\mathcal{M} \in \mathcal{M}od(\mathcal{D}_X)_{\mathrm{hr}}$.

3.2.4. Proof of Theorem 3.2

It remains to verify that for all $M \in \mathrm{Rc}(X, X_0)$:

(i) $G(M) \in \mathscr{M}od(\mathscr{D}_X)_{\mathrm{hr}}$;

(ii) $FG(M) \cong M$, functorial in M.

(i) Let $M = (\mathscr{N}, \mathscr{N}_1 \underset{V}{\overset{U}{\rightleftarrows}} \mathscr{N}_2, \alpha) \in \mathrm{Rc}(X, X_0)$. In $\mathscr{C}(\mathscr{D}_X)$ we have an exact sequence

$$\begin{array}{ccccccc}
\mathrm{Ker}\, U & & \mathscr{N}_1 & & \varphi\mathscr{N} & & \mathrm{Coker}\, U \\
\Big\| & \hookrightarrow & V\Big\|U & \xrightarrow{\binom{c^{-1}\alpha U}{\alpha}} & v\Big\|c & \twoheadrightarrow & \Big\| \\
0 & & \mathscr{N}_2 & & \psi\mathscr{N} & & 0
\end{array}$$

This yields an exact sequence in $\mathrm{Rc}(X, X_0)$

$$(*) \qquad F(\mathrm{Ker}\, U) \hookrightarrow M \to F(\mathscr{N}) \twoheadrightarrow F(\mathrm{Coker}\, U).$$

Here

$$F(\mathrm{Ker}\, U) = (0, \mathrm{Ker}\, U \rightleftarrows 0, 1),$$

$$F(\mathrm{Coker}\, U) = (0, \mathrm{Coker}\, U \rightleftarrows 0, 1),$$

because $\mathrm{Ker}\, U$ and $\mathrm{Coker}\, U$ are regular holonomic $\mathscr{D}_X$-modules with support contained in X_0 (Prop. 2.4.4.2);

$$F(\mathscr{N}) = (\mathscr{N}, \varphi\mathscr{N} \underset{v}{\overset{c}{\rightleftarrows}} \psi\mathscr{N}, 1)$$

because $\mathscr{N} = \mathscr{N}[f^{-1}]$. By 3.2.3.6 we obtain an exact sequence (for P^∞ and Q are exact) of $\mathscr{D}_Y$-modules

$$i_* \mathrm{Ker}\, U \hookrightarrow P^\infty Q(M) \to i_*\mathscr{N} \twoheadrightarrow i_* \mathrm{Coker}\, U.$$

It follows that $P^\infty Q(M)$ is supported on $i(X)$, so applying the functor $\mathrm{Ker}\,(t-f, \cdot)$ yields an exact sequence of $\mathscr{D}_X$-modules

$$\mathrm{Ker}\, U \hookrightarrow G(M) \to \mathscr{N} \twoheadrightarrow \mathrm{Coker}\, U.$$

So finally we arrive at the conclusion that $G(M)$ is a regular holonomic $\mathscr{D}_X$-module and $G(M)[f^{-1}] \xrightarrow{\cong} \mathscr{N}$.

(ii) Let $M \in \mathrm{Rc}(X, X_0)$ be as above; then $i_* G(M) = P^\infty Q(M)$. Let $k \in \mathbf{N}$. Applying the exact functor $P^k Q$ to the exact sequence $(*)$ yields (by 3.2.3.6) an exact sequence of $F^0\mathscr{D}_Y$-modules

$$F^{-k} i_* \mathrm{Ker}\, U \hookrightarrow (P^k Q(M))_1 \to F^{-k} i_* \mathscr{N} \twoheadrightarrow F^{-k} i_* \mathrm{Coker}\, U.$$

It follows that $(P^k Q(M))_1$ is a coherent $F^0\mathscr{D}_Y$-module, for every $k \in \mathbf{N}$. By construction $i_*G(M)=P^\infty Q(M)$ carries a filtration $F^\cdot i_*G(M)$, where for $k\in\mathbf{Z}$

$$F^k i_*G(M):=\begin{cases}\operatorname{Im}((P^{-k}Q(M))_1 \hookrightarrow i_*G(M)), & \text{if } -k\in\mathbf{N}\\ t^k F^0 i_*G(M), & \text{if } k\in\mathbf{N}^*.\end{cases}$$

So we have established that this filtration satisfies 1.2 (3). By construction of P^∞ it satisfies 1.2, (2) and (4). The definition of the $\mathscr{D}_Y$-structure on $P^\infty Q(M)$ implies that the filtration fulfils 1.2 (1). Hence it is a good filtration. By 3.1.2.1 there exists a non-zero polynomial $b\in\mathbf{C}[\Theta]$ with: $b^{-1}(0)\subset\{z\in\mathbf{C}|0\leqq \operatorname{Re} z<1\}$, $b(t\partial)\mathscr{N}_2=0$ and $b(t\partial+1)\mathscr{N}_1=0$. Clearly this implies 1.3 (5) i.e., it is the canonical good filtration on $i_*G(M)$. Consequently $FG(M)\cong M$.

We leave it to the reader to verify that the isomorphism is functorial in M.

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Note added in proof. After the preparation of the paper the author was informed that C. Sabbah, M. Saito and B. Malgrange have obtained similar results.

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