A kind of multilinear operator and the Schatten—von Neumann classes

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1. Introduction

Let $H^l(\mathbf{R}^d)$ denote the collection of all distributions m satisfying

(i) $m\in C^\infty(\mathbf{R}^d\setminus\{0\})$,

(ii) m is homogeneous of degree l, $l\geqq 0$.

Let R^N denote the operator which maps a function m to its Taylor remainder of order N, i.e.

$$R^N m(\eta, \Delta\eta) = m(\eta+\Delta\eta)-\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}D^\alpha m(\eta)(\Delta\eta)^\alpha. \tag{1.1}$$

In general we consider

$$R^{N_1,\dots,N_n}m(\eta, \Delta\eta_1, \dots, \Delta\eta_n) = R^{N_n}R^{N_1,\dots,N_{n-1}}m(\eta, \Delta\eta_1, \dots, \Delta\eta_{n-1}, \Delta\eta_n).$$

In this paper we study the operator $T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)$ defined by

$$\begin{aligned}&[T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)f]\hat{}\,(\xi)\\ &= (2\pi)^{-nd}\int_{\mathbf{R}^{nd}}\prod_{j=1}^n \hat{b}_j(\eta_{j-1}-\eta_j)R^{N_1,\dots,N_n}m(\eta_n, \eta_{n-1}-\eta_n, \dots, \eta_0-\eta_1)\hat{f}(\eta_n)\,d\eta\end{aligned} \tag{1.2}$$

where $d\eta=d\eta_1, \dots, d\eta_n$, $\eta_0=\xi$.

In fact many multilinear singular integrals have the form (1.2). Let $d=1$, $m(\xi)=|\xi|$, then $[b, |D|]=[b, HD]=T_b(R^1 m)$, where H is the Hilbert transform. According to Janson and Peetre [5], $[b, |D|]$ is a paracommutator of the Toeplitz type, it is bounded on $L^2(\mathbf{R})$ if and only if $b'\in L^\infty$, and it is never compact unless $b'=0$. But $D[b, H]=T_b(R^2 m)$ is a paracommutator of the Hankel type; it is bounded on $L^2(\mathbf{R})$ if and only if $b'\in\mathrm{BMO}$, and $D[b, H]\in S_p$ (the Schatten—von Neumann class) if and only if $b\in B_p^{1+(1/p)}$ $(1\leqq p<\infty$, the Besov space). This is the motivation for studying the multilinear operator (1.2) using the Taylor remainder $R^N m$ instead of the difference $m(\xi)-m(\eta)$. Several authors have studied the boundedness of the multilinear operator (1.2) and obtained the BMO-results (direct results), e.g. Cohen [1, 2], Coifman and Meyer [3], Hu [4], Qian [9, 10], Qian and Li [11]. In this paper we study, in the framework of paracommutators (Janson and Peetre [5], Peng [6], [7]) and multi-fold paracommutators (Peng [8]), the boundedness, compactness, and the Schatten—von Neumann properties of the multilinear operator (1.2).

We adopt the notation for the Schatten—von Neumann class S_p, the Besov space B_p^s, the assumptions $A0$, $A1$, $A2$, $A3(\alpha)$, $A4$, $A4\frac{1}{2}$, $A5$, $A10(\alpha)$, A^* of the Fourier kernel $A(\xi,\eta)$, fractional integration or differentiation $I^l, \dots$, in [5, 6, 7, 8].

In § 2, we study the direct results. In § 3, we study the converse results and the Janson—Wolff phenomena. In § 4, we discuss some examples.

2. Direct results

First of all, we study the case $n=1$, i.e. the bilinear operator.

Let $\varphi\in C_0^\infty(0,\infty)$ with $\varphi(t)=1$ on $[\delta^2,\delta^{-2}]$ for some small δ and define

$$A_1(\xi,\eta)=\left(1-\varphi\left(\frac{|\eta|}{|\xi|}\right)\right)\frac{R^N m(\eta,\xi-\eta)}{|\xi-\eta|^l}, \tag{2.1}$$

$$A_2(\xi,\eta)=\varphi\left(\frac{|\eta|}{|\xi|}\right)\frac{R^N m(\eta,\xi-\eta)}{|\xi|^l}. \tag{2.2}$$

Thus

$$T_b(R^N m)=T_{I^{-l}b}(A_1)+T_b^{l,0}(A_2). \tag{2.3}$$

By Lemma 3.1, 3.2 and 3.4 of Janson and Peetre [5],

$$T_b(R^N m)\in S_p \quad \text{if and only if both} \quad T_{I^{-l}b}(A_1) \quad \text{and} \quad T_b^{l,0}(A_2)\in S_p,$$

for $1\leqq p\leqq\infty$,

$$T_b(R^N m) \quad \text{is compact if and only if both} \quad T_{I^{-l}b}(A_1) \quad \text{and} \quad T_b^{l,0}(A_2)$$

are compact.

So we can treat the two pieces separately.

Lemma 2.1. *Suppose that $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$. Then A_1 satisfies $A0$, $A1$, $A2$, $A3(\infty)$ and A_2 satisfies $A0$, $A1$, $A2$, $A3(\infty)$ of [5]. Also A_2 satisfies $A0$, $A1$, $A2$, $A3(N)$ of [5] and vanishes on $\Delta_j\times\Delta_k$ when $|j-k|$ is large.*

Proof. It is obvious that A_1 and A_2 satisfy $A0$. If $|j-k|$ is small, $A_1=0$; if $|j-k|$ is large, e.g. $j\gg k$, $\eta\in\Delta_k$, $\zeta\in\Delta_j$, then $|\eta|<\delta|\xi|$. By Lemma 3.6 of [5], we have

$$\|A_1(\zeta,\eta)\|_{M(\Delta_j\times\Delta_k)}$$

$$\leqq \left\|1-\varphi\left(\frac{|\eta|}{|\zeta|}\right)\right\|_{M(\mathbf{R}^d\times\mathbf{R}^d)} \left\|\frac{|\zeta|^l}{|\zeta-\eta|^l}\right\|_{M(\Delta_j\times\Delta_k)} \left\|\frac{R^N m(\eta,\xi-\eta)}{|\zeta|^l}\right\|_{M(\Delta_j\times\Delta_k)}$$

$$\leqq c\left(\left\|\frac{m(\xi)}{|\xi|^l}\right\|_{M(\Delta_j\times\Delta_k)}+\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}\left\|\frac{D^\alpha m(\eta)(\xi-\eta)^\alpha}{|\xi|^l}\right\|_{M(\Delta_j\times\Delta_k)}\right)$$

$$\leqq c\left(\left\|\frac{m(\xi)}{|\xi|^l}\right\|_{L^\infty(\Delta_j)}+\sum_{|\alpha|\leqq N-1}C_\alpha \sup_{\alpha_1+\alpha_2=\alpha}\||\xi|^{\alpha_1-l}\|_{L^\infty(\Delta_j)}\|D^\alpha m(\eta)|\eta|^{\alpha_2}\|_{L^\infty(\Delta_k)}\right)$$

$$\leqq c(1+2^{(k-j)(l+1-N)})\leqq c.$$

So A_1 satisfies $A1$.

It is similar to show that A_1 satisfies $A2$ and A_2 satisfies $A1$. Notice that A_1 vanishes on a neighbourhood of $\{\xi=\eta\}$, it follows that A_1 satisfies $A3(\infty)$.

Let us show that A_2 satisfies $A3(N)$. For any $B=B(\xi_0,r)$ with $r<\delta|\xi_0|$, by Lemma 3.10 of [5], we have

$$\|A_2(\xi,\eta)\|_{M(B\times B)}\leqq c\left(\frac{r}{|\xi_0|}\right)^N \sup_{|\alpha|\leqq m}\ \sup_{\xi,\eta\in B(\xi_0,2r)}|\xi_0|^{|\alpha|}|D^\alpha A_2(\xi,\eta)|\leqq c\left(\frac{r}{|\xi_0|}\right)^N.$$

It is obvious that A_2 vanishes on $\Delta_j\times\Delta_k$ when $|k-j|$ is large. □

Remark. By the definitions of A_p1, A_p3 of Peng [7], we can also show that A_1 satisfies A_p1, $A_p3(\infty)$ and that A_2 satisfies A_p1, $A_p3(N)$, for $0<p\leqq 1$.

Combining Lemma (2.1), Theorems 7.3, 8.1, 13.1, 13.3 (and its extension) of [5], and Theorem 1 of [7], we get the following.

Theorem 2.1. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, $s,t>\max[-d/2,-d/p]$, $s+t+l+d/p<N$, $1<p\leqq\infty$. *Then*

(i) $b\in I^l(\mathrm{BMO})$ *implies that* $T_b(R^N m)\in S_\infty$,

(ii) $b\in I^l(\mathrm{CMO}$ *implies that* $T_b(R^N m)$ *is compact,*

(iii) $b\in B_p^{s+t+l+(d/p)}$ *implies that* $T_b^{s,t}(R^N m)\in S_p$,

(iv) $b\in b_\infty^{s+t+l}$ *implies that* $T_b^{s,t}(R^N m)$ *is compact.* □

Now we study the case $n\geqq 2$. Let X_p denote the space $B_p^{1/p}$ (if $p<\infty$) or the space BMO (if $p=\infty$).

Theorem 2.2. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $0<\alpha_i\leqq 1$, $N_i\in\mathbf{N}$, $d/\alpha_i<p_i\leqq\infty$, *for* $i=1,\dots,n$, *and that* $\sum_{i=1}^n(N_i-\alpha_i)=l$, $1/p=\sum_{i=1}^n 1/p_i$, $1\leqq p\leqq\infty$. *Then*

$$\|T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)\|_{S_p}\leqq C\prod_{i=1}^n\|b_i\|_{I^{N_i-\alpha_i}(X_{p_i})}. \tag{2.4}$$

Proof. If $l=0$, $N_i=\alpha_i=1$, for $i=1,\dots,n$, then Theorem 2.2 implies Theorem 3 of [8]. We prove this theorem using the procedure of the proof of Theorem 3 in [8].

Let $\varphi\in C^\infty(0,\infty)$ be such that $\varphi\equiv 1$ on $(0,n+1)$ and $\varphi\equiv 0$ on $(n+2,\infty)$, $\psi=1-\varphi$. Then we have

$$R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)$$
$$=R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)\,\Pi_{i=1}^n\left[\psi\left(\frac{|\eta_0|}{|\eta_i-\eta_{i-1}|}\right)+\varphi\left(\frac{|\eta_0|}{|\eta_i-\eta_{i-1}|}\right)\right]$$
$$=\sum_{J\in G_n}A_J(\eta_0,\eta_1,\dots,\eta_n)$$

where G_n is the set of subsets J of $\{1,\dots,n\}$,

$$A_J(\eta_0,\eta_1,\dots,\eta_n)=R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)$$
$$\cdot\,\Pi_{j\in J}\psi\left(\frac{|\eta_0|}{|\eta_j-\eta_{j-1}|}\right)\Pi_{j'\in J'}\varphi\left(\frac{|\eta_0|}{|\eta'_j-\eta'_{j-1}|}\right),$$

J' is the complement of J in $\{1,\dots,n\}$.

It suffices to show (2.4) for each A_J.

Let $\bar A_J=R^{N_1,\dots,N_n}m(\eta_n,\eta_{n-1}-\eta_n,\dots,\eta_0-\eta_1)$

$$\cdot\,\Pi_{j\in J}\psi\left(\frac{|\eta_0|}{|\eta_j-\eta_{j-1}|}\right)\frac{1}{|\eta_j|^{N_j-\alpha_j}}\Pi_{j'\in J'}\varphi\left(\frac{|\eta_0|}{|\eta'_j-\eta'_{j-1}|}\right)\frac{1}{|\eta'_j-\eta'_{j-1}|^{N'_j-\alpha'_j}},$$

then

$$T_{b_1,\dots,b_n}(R^{N_1,\dots,N_n}m)=T^{s_0,s_1,\dots,s_n}_{I^{\beta_1}b_1,\dots,I^{\beta_n}b_n}(\bar A_J),$$

where $\beta_j=0$ if $j\in J$, $\beta_{j'}=N_{j'}-\alpha_{j'}$ if $j'\in J'$,

$$s_j=N_{j+1}-\alpha_{j+1}\quad\text{if}\quad j+1\in J,\quad s_{j'}=0\quad\text{if}\quad j'+1\in J',\quad s_n=0.$$

It is not too hard to check $\bar A_J$ satisfies the assumption $A^*(N_1-\alpha_1,\dots,N_n-\alpha_n)$ in Theorem 2 of [8]. So Theorem 2 of [8] shows that

$$\|T_{b_1,\dots,b_n}(A_J)\|_{S_p}\le C\,\Pi_{i=1}^n\|b_i\|_{I^{N_i-\alpha_i}(X_{p_i})}.\quad\square$$

3. Converse results and the Janson—Wolff phenomena

We need some non-degeneracy assumptions on m.

ND1. If l is an integer, $m\in H^l(\mathbf{R}^d)$, for any $\xi_0\in S_{d-1}$, there exists $0\ne\eta_0\in\mathbf{R}^d$ such that

$$m(\xi_0)-\sum_{|\alpha|=l}\frac{1}{\alpha!}D^\alpha m(\eta_0)\xi_0^\alpha\ne 0.$$

ND2. If l is a non-integer, $m\in H^l(\mathbf{R}^d)$, for any $\xi_0\in S_{d-1}$,

$$m(\xi_0)\ne 0.$$

ND3. If $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, for any $\xi_0\in S_{d-1}$. There exists $0\neq\eta_0\in\mathbf{R}^d$ such that

$$D^N_{\xi_0} m(\eta_0)\neq 0,$$

where $D^N_{\xi_0} m(\eta_0)$ denote the direction derivative of order N along $\xi_0\in S_{d-1}$.

We consider the converse results and the Janson—Wolff phenomena only for the case $n=1$.

Lemma 3.1. *If* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *m satisfies* ND1 (when l is an integer) *or* ND2 (when l is a non-integer), *then* A_1 *in* (2.1) *satisfies* $A4\frac{1}{2}$ *and* $A5$. (For $A4\frac{1}{2}$, see Peng [6].)

Proof. When l is an integer, $N=l+1$. For any $\xi_0\in S_{d-1}$, by ND1, we can take $0\neq\eta_0'\in\mathbf{R}^d$ such that

$$k=\left|m(\xi_0)-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(\eta_0')\xi_0^\alpha\right|>0.$$

By the homogeneity of degree 0 of $D^\alpha m(\eta)$, for any $t\in(0,\infty)$,

$$\left|m(\xi_0)-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(t\eta_0')\xi_0^\alpha\right|=k.$$

Thus, if δ is small enough, we have

$$\begin{aligned}
&\left\| m(\xi_0)-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(t\eta_0')\xi_0^\alpha-\frac{R^N m(\eta,\xi-\eta)}{|\xi|^l}\right\|_{M(U\times V)}\\
&=\left\|\frac{m(\xi)}{|\xi|^l}-m(\xi_0)\sum_{|\alpha|\leqq N-2}\frac{1}{\alpha!}D^\alpha m(\eta)(\xi-\eta)^\alpha/|\xi|^l\right.\\
&\quad-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(\eta)\sum_{\substack{\alpha_1+\alpha_2=\alpha\\|\alpha_2|<0}}C_\alpha\xi^{\alpha_1}\eta^{\alpha_2}/|\xi|^l\\
&\left.-\sum_{|\alpha|=N-1}\frac{1}{\alpha!}D^\alpha m(\eta)\frac{\xi^\alpha}{|\xi|^l}-D^\alpha m(t\eta_0')\xi_0^\alpha\right\|_{M(U\times V)}\leqq\left\|\frac{m(\xi)}{|\xi|^l}-m(\xi_0)\right\|_{L^\infty(U)}\\
&\quad+\sum_{|\alpha|\leqq N-2}\frac{1}{\alpha!}\|D^\alpha m(\eta)\|_{L^\infty(V)}\sum_{\alpha_1+\alpha_2=\alpha}|C_{\alpha_1}|\left\|\frac{\xi^{\alpha_1}}{|\xi|^l}\right\|_{L^\infty(U)}\|\eta^{\alpha_2}\|_{L^\infty(V)}\\
&\quad+\sum_{|\alpha|=N-1}\frac{1}{\alpha!}\sum_{\substack{\alpha_1+\alpha_2=\alpha\\|\alpha_2|>0}}|C_{\alpha_1}|\left\|\frac{\xi^{\alpha_1}}{|\xi|^l}\right\|_{L^\infty(U)}\|D^\alpha m(\eta)\eta^{\alpha_2}\|_{L^\infty(V)}\\
&\quad+\sum_{|\alpha|=N-1}\frac{1}{\alpha!}\|D^\alpha m(\eta)-D^\alpha m(t\eta_0')\|_{L^\infty(V)}\left\|\frac{\xi^\alpha}{|\xi|^l}\right\|_{L^\infty(U)}\\
&\quad+\sum_{|\alpha|=N-1}\frac{1}{\alpha!}|D^\alpha m(t\eta_0')|\left\|\frac{\xi^\alpha}{|\xi|^l}-\xi_0^\alpha\right\|_{L^\infty(U)}\\
&\leqq c\delta^{\frac{1}{2}}\ \text{(choose } t \text{ so that } |t\eta_0'|=|\eta_0|=\delta^{\frac{1}{2}})<k
\end{aligned}$$

which implies that $R^N m(\eta, \xi-\eta)/|\xi|^l$ is invertible in $M(U\times V)$, moreover by Lemma 3.6 of [5], A_1 is invertible in $M(U\times V)$.

When l is a non-integer, $l>0$, $N=[l]+1$, ND2 implies that, for any $\xi_0\in S_{d-1}$, $|m(\xi_0)|=k>0$. If δ is small enough, we have

$$\left\|m(\xi_0)-\frac{R^N m(\eta, \xi-\eta)}{|\xi|^l}\right\|_{M(U\times V)}$$

$$=\left\|m(\xi_0)-\frac{m(\xi)}{|\xi|^l}+\sum_{|\alpha|\leqq N-1}\frac{1}{\alpha!}D^\alpha m(\eta)\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1}\xi^{\alpha_1}\eta^{\alpha_2}/|\xi|^l\right\|_{M(U\times V)}$$

$$\leqq\left\|\frac{m(\xi)}{|\xi|^l}-m(\xi_0)\right\|_{L^\infty(U)}+\sum_{|\alpha|\leqq N+1}\frac{1}{\alpha!}\sum_{\alpha_1+\alpha_2=\alpha}|C_{\alpha_1}|\left\|\frac{\xi^\alpha}{|\xi|^l}\right\|_{L^\infty(U)}\|D^\alpha(\eta)\eta^{\alpha_2}\|_{L^\infty(V)}$$

$$\leqq c\delta^{l+1-N}\ (\text{choose } |\eta_0|=2\delta)<k.$$

This implies that $R^N m(\eta, \xi-\eta)/|\xi|^l$ is invertible in $M(U\times V)$, again by Lemma 3.6 of [5], A_1 is invertible in $M(U\times V)$.

Because A_1 satisfies $A0$, that A_1 satisfies $A5$ implies that A_1 satisfies $A4\frac{1}{2}$.

Lemma 3.2. *If* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *m satisfies* ND3, *then* A_2 *satisfies* $A10(N)$. (For $A10(N)$, see Peng [7].)

Remark 3.1. It is easy to see from the proof that A_1 satisfies also $A_p4\frac{1}{2}$ of [7] for any $0<p<1$.

Proof. Recall the assumption $A10(N)$: for any $0\neq\theta\in\mathbf{R}^d$, there exist a positive number $\delta<\frac{1}{2}$ and a subset V_θ of $\mathbf{R}^d$ such that if N_r denote the number of integer points contained in $V_\theta\cap B_r$, where $B_r=B(0, r)$, then $\overline{\lim}_{r\to\infty}N_r/r^d>0$, and for every $\underline{n}\in V_\theta$,

$$\left\|\frac{1}{A(\cdot+\underline{n}+\theta, \cdot+\underline{n})}\right\|_{M(B\times B)}\leqq c|\underline{n}|^N, \quad \text{where} \quad B=B(0, \delta).$$

For any $0\neq\theta\in S_{d-1}$, by ND3, there exists $0\neq\eta_0\in\mathbf{R}^d$ such that

$$D_\theta^N m(\eta_0)=\sum_{|\alpha|=N}D^\alpha m(\eta_0)\theta^\alpha\neq 0.$$

We can assume that $|\eta_0|=1$, $k=|D_\theta^N m(\eta_0)|>0$. By the continuity, there exists δ such that if $|\xi-\theta|<\delta$, $|\eta-\eta_0|<\delta$, then

$$\left|\sum_{|\alpha|=N}D^\alpha m(\eta)\xi^\alpha\right|\geqq k/2.$$

Let $V_\theta=\left\{\eta\in\mathbf{R}^d\colon \left|\frac{\eta}{|\eta|}-\eta_0\right|<\delta, |\eta|>23/\delta\right\}$, then V_θ satisfies the condition of $A10(N)$.

Let $\underline{n}\in V_\theta$, if $u\in B$, $v\in B$, $B=B(0,\delta)$, then

$$\left|R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))\right| = \left|\sum_{|\alpha|=N}\frac{1}{\alpha!}D^\alpha m(\bar{\eta})(u+\theta-v)^\alpha\right| \geqq ck|\underline{n}|^{l-N}.$$

Note that $R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))\in C^\infty(2B\times 2B)$, so

$$1/R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))$$

can be expressed as the absolutely convergent Fourier series:

$$\frac{1}{R^N m(v+\underline{n}, (u+\underline{n}+\theta)-(v+\underline{n}))}\sum_{j,k\in Z^d} a_{\underline{j},\underline{k}}\beta_{\underline{j},\underline{k}}(u)\gamma_{\underline{j},\underline{k}}(v),$$

where

$$\sum |a_{\underline{j},\underline{k}}| \leqq c\sum_{|\alpha|\leqq M}\left\|D^\alpha\frac{1}{R^N m(\cdot+\underline{n}, (\cdot+\underline{n}+\theta)-(\cdot+\underline{n}))}\right\|_{L^\infty(2B\times 2B)} \leqq c|\underline{n}|^{N-l}.$$

Therefore

$$\left\|\frac{1}{A_2(\cdot+\underline{n}+\theta, \cdot+\underline{n})}\right\|_{M(B\times B)} \leqq c\,|n|^N,$$

i.e. $A10(N)$ holds. □

Lemma 3.1, Theorem 10.1 of [5] and Theorem 2 of [6] and its extension give the following converse results.

Theorem 3.1. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and m satisfies* ND1 *(when l is an integer) or* ND2 *(when l is a non-integer). Then* $T_b(R^N m)$ *is bounded on* $L^2(\mathbf{R}^d)$ *implies that* $I^{-l}b\in$ BMO, *and* $T_b(R^N m)$ *is compact implies that* $I^{-l}b\in$ CMO.

Lemma 3.1, Theorem 9.1 of [5] and Theorem 2 of [7] give the following converse results.

Theorem 3.2. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and m satisfies* ND1 *(when l is an integer) or* ND2 *(when l is a non-integer). Then for* $1\leqq p\leqq\infty$, *any* s, t, $T_b^{s,t}(R^N m)\in S_p$ *implies that* $b\in B_p^{s+t+l+d/p}$. *For* $0<p<1$, $s, t>-d/2$, *and* $T_b^{s,t}(R^N m)\in S_p$ *implies that the following a priori inequality holds*

$$\|b\|_{B_p^{s+t+l+d/p}} \leqq c\,\|T_b^{s,t}(R^{N_m})\|_{S_p}.$$

Lemma 3.2 and Theorem 4 of [7] give the following results about the Janson—Wolff phenomena.

Theorem 3.3. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and m satisfies* ND3. *Then for* $1\leqq p\leqq d/N-l-s-t$, $T_b^{s,t}(R^N m)\in S_p$ *implies that b is a polynomial. For* $0<p\leqq\min(d/N-l-s-t, 1)$, $b\in S'(\mathbf{R}^d)$ *with* $\hat{b}$ *with compact support such that* $T_b^{s,t}(R^N m)\in S_p$ *implies that b is a polynomial.*

Applications.

1. Combining Theorem 2.1, 3.1, 3.2 and 3.3, we get the following

Theorem Σ. *Suppose that* $m\in H^l(\mathbf{R}^d)$, $l\geqq 0$, $N=[l]+1$, *and* m *satisfies* ND1 *(when* l *is an integer) or* ND2 *(when* l *is a non-integer) and* ND3. *Then*

(i) $T_b(R^N m)$ *is bounded on* $L^2(\mathbf{R}^d)$ *if and only if* $I^{-l}b\in$ BMO,

(ii) $T_b(R^N m)$ *is compact if and only if* $I^{-l}b\in$ CMO,

(iii) *for* $d/N-l<p<\infty$ *and* $p\geqq 1$, $T_b(R^N m)\in S_p$ *if and only if* $b\in B_p^{l+d/p}$; *for* $0<p<1$, *directly,* $b\in B_p^{l+d/p}$ *implies* $T_b(R^N m)\in S_p$ *and, conversely, an a priori inequality holds.*

(iv) *for* $1\leqq p\leqq d/N-l$, $T_b(R^N m)\in S_p$ *if and only if* b *is a polynomial; for* $0\leqq p\leqq \min(d/N-l, 1)$, $b\in S'(\mathbf{R}^d)$ *with* $\hat{b}$ *with compact support implies that* b *is a polynomial.*

2. Higher commutators of fractional integration.

In particular, if $m(\xi)=|\xi|^l$, $l>0$, then $m\in H^l(R^d)$, and m satisfies ND1 (or ND2) and ND3. So Theorem Σ gives a generalization of Example 8 in [5] from the commutators of fractional integration to the higher commutators.

3. Multilinear singular integrals.

Lemma (Qian [10]). *Suppose that* $\Omega\in H^0(\mathbf{R}^d)$, *and* $\int_{S^{d-1}}\Omega(x)x^\beta\,d\sigma(x)=0$, *for* $|\beta|\leqq l$ *and* $l>0$. *Denote, for* $N_1+\ldots+N_n\leqq l+n$,

$$T_{b_1,\ldots,b_n}^{N_1,\ldots,N_n}(\Omega)f(x) = \text{p.v.} \int \prod_{j=1}^n p^{N_j}b_j(x, y-x)\frac{\Omega(x-y)}{|x-y|^{d+l}}f(y)\,dy.$$

Then

$$T_{b_1,\ldots,b_n}^{N_1,\ldots,N_n}(\Omega)f = T_{b_1,\ldots,b_n}(R^{N_1,\ldots,N_n}m)f \quad \text{for every} \quad f\in C_0^\infty(\mathbf{R}^d),$$

where

$$m(\xi) = c|\xi|^l\int_{S_{d-1}}\Omega(y)L(\xi' y)\,d\sigma(y), \quad \xi' = \xi/|\xi|, \quad L = L_1+L_2,$$

$$L_1(t) = \int_0^\infty \frac{e^{itr}}{r^{l+1}}\,dr, \quad L_2(t) = \frac{(it)^{l+1}}{l!}\int_0^1\int_0^1 u^l e^{irt(1-u)}\,du\,dr.$$

(See Qian [10], Theorem 1.) □

Many authors have studied the boundedness (direct results) of $T_{b_1,\ldots,b_n}^{N_1,\ldots,N_n}(\Omega)$. Cohen [2] obtained the result for the case $n=1$, $N_1=1$, Hu [4] obtained the result for the case $N_1=\ldots=N_n=1$. Qian [9] obtained the result for the general case.

Qian and Li [11] obtained the boundedness (direct results) of $T_{b_1,\ldots,b_n}(R^{N_1,\ldots,N_n}m)$.

Theorem 2.2 of this paper gives the characterization of the boundedness and the Schatten—von Neumann properties for $T_{b_1,\ldots,b_n}(R^{N_1,\ldots,N_n}m)$. It includes the result of Qian and Li [11].

Theorem 2.2 and Lemma 4.1 give the characterization of the boundedness and the Schatten—von Neumann properties for $T^{N_1,\dots,N_n}_{b_1,\dots,b_n}(\Omega)$. It includes the results of Cohen [2], Hu [4] and Qian [9].

For the case $n=1$, Theorem Σ and Lemma 4.1 give a perfect characterization of the boundedness, the compactness, the Schatten—von Neumann properties and the Janson—Wolff phenomena for both $T^N_b(\Omega)$ and $T_b(R^N m)$.

Remark. Finally, we say a few words why we deal only with the case $N=[l]+1$. In this case, the operator $T_b(R^N m)$ behaves as a Hankel operator, so we can study its compactness and Schatten—von Neumann properties. For the case $N=[l]$ some results on boundedness are obtained in [4], [10], [11]. But then $T_b(R^N m)$ behaves as a Toeplitz operator and, therefore, cannot be compact in general. We will study this case elsewhere.

Notice also that in the proof of Lemma 3.1, the choice $|\eta_0|=\delta^{1/2}$ guarantees that the fourth term is small; the choice $|\eta_0|=2\delta$ can not do this job.

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