Approximation in the mean by polynomials on non-Carathéodory domains

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1. Introduction

Let Ω be a bounded simply connected domain in the complex plane, let dA denote two-dimensional Lebesgue measure and let w be a positive measurable function defined on Ω. Assume, moreover, that w is essentially bounded. The purpose of this investigation is to study the spaces $H^p(\Omega, wdA)$ and $L^p_a(\Omega, wdA)$ which for each p, $1 \leqq p < \infty$, are defined as follows: $H^p(\Omega, wdA)$ consists of those functions that can be approximated arbitrarily closely in the $L^p(\Omega, wdA)$ norm by a sequence of polynomials; $L^p_a(\Omega, wdA)$ denotes the set of functions in $L^p(\Omega, wdA)$ which are analytic in Ω. If w is bounded away from zero locally or, more generally, if $\log w$ is locally integrable then $L^p_a(\Omega, wdA)$ is norm closed and $H^p(\Omega, wdA)$ is contained in $L^p_a(\Omega, wdA)$ (cf. [6], p. 175). It is an old problem to find conditions on Ω and w which imply that $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$. When this occurs the polynomials are said to be complete in $L^p_a(\Omega, wdA)$. In this paper we consider the completeness problem for certain special domains Ω.

Questions of this kind were first considered by Carleman [9] in 1923. He proved that the polynomials are complete in $L^p_a(\Omega, wdA)$ if Ω is a Jordan domain and $w \equiv 1$. A decade later Markuševič and Farrell (cf. [18], [39] & [40, p. 112]) obtained, independently, the corresponding theorem for Carathéodory domains and Sinanjan [51] subsequently extended it to closed Carathéodory sets. A Carathéodory domain (or set) is by definition a domain (or set) whose boundary coincides with the boundary of the unbounded complementary component of its closure. By itself, the Carathéodory property is not sufficient to ensure that the polynomials are complete in $L^p_a(\Omega, wdA)$ for an arbitrary weight w (cf. [36, pp. 3—4] & [40, p. 134]). Weighted polynomial approximation on Carathéodory domains has been studied extensively by many authors and will not be of principal concern to us. For a more detailed description and historical account of this aspect of the completeness problem the reader is referred to the survey article of Mergeljan [40] and the text of Smirnov and Lebedev [53]. The most recent work in this area is that of Hedberg [30] & [31].

Our main objective here is to study weighted polynomial approximation on certain non-Carathéodory domains. The following are more or less typical of the kind of regions we intend to consider:

(i) a Jordan domain with a cut or incision in the form of a simple arc from an interior point to a boundary point;

(ii) a "crescent", i.e. a region which is topologically equivalent to one bounded by two internally tangent circles.

In the first case (i) completeness depends on several factors, one of which is the behavior of the weight w in a neighborhood of the cut. If for example, w is bounded away from zero in any open set which meets the cut then it is easy to see that the polynomials cannot be complete with respect to any of the $L^p(wdA)$ norms, $1 \leqq p < \infty$. On the other hand, if w is allowed to approach zero rapidly at each point of the cut then completeness may occur. This phenomenon was discovered by Keldyš (cf. [36] & [40]) in 1939 and since then has been studied by Džrbašjan [40], Mergeljan, Tamadjan [54] and others. Our principal contribution in this area is the material in Section 3. Here we obtain a sufficient condition (cf. Theorem 3.4) for the polynomials to be complete in $L^p_a(\Omega, wdA)$ for a domain Ω having an arbitrary number of rectifiable cuts. This condition is "best possible" (cf. Theorem 3.5).

Topics concerning approximation on domains of the second kind (ii) are the subject matter of Section 5. In contrast to the situation in case (i) we shall consider here only the weight $w \equiv 1$. Under these circumstances completeness depends entirely on the region Ω and, more specifically, on certain of its metric properties near multiple boundary points. This too was discovered by Keldyš (cf. [40, p. 116]). Later and with additional restrictions on the boundary of the crescent Ω, Džrbašjan and Šaginjan (cf. [40, p. 158]) obtained a necessary and sufficient condition for completeness in this setting. In recent years the problem has been taken up by Havin [23], Havin and Maz'ja [25] & [28], Šaginjan [49], Shapiro [50] and the present author [6]. In Section 5 we obtain extensions and improvements to much of this work. The principal results are Theorems 5.7 & 5.11 and their associated corollaries. As an added dividend we obtain through the ideas of Section 3 a general metric criterion for completeness in $L^p_a(\Omega, dA)$ (cf. Theorem 5.2) which complements a theorem of Mergeljan [42]. Again the condition is very nearly sharp (cf. Theorem 5.4).

Section 6 is devoted to an unpublished result of Carleson which is used in proving Theorem 5.11. In Section 7 we discuss briefly the completeness problem for harmonic polynomials on higher dimensional crescents. This study was begun by Havin and Maz'ja [26] and is still in the very early stage of development.

In the succeeding pages we shall adhere to the following notation: Ω will always be a domain and $\bar{\Omega}$ will denote its closure. For each $p \geqq 1$ the conjugate index $p/(p-1)$ will be denoted by q. The letters C and K will be used to denote various constants which may differ from one formula to the next, even within a single string of estimates.

2. Sobolev spaces and the Cauchy transform

In order to carry out the program described in the introduction we shall have to verify, in certain instances, that $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$. To accomplish this it is sufficient to prove that if $g \in L^q(\Omega, wdA)$ and if $\int QgwdA = 0$ for every polynomial Q then $\int FgwdA = 0$ for every $F \in L^p_a(\Omega, wdA)$. In the cases we consider this is done by a careful analysis of the Cauchy transform $\int \frac{gw(\zeta)}{\zeta - z}\, dA_\zeta$, which is denoted $\widehat{gw}(z)$. As a distribution $\partial\widehat{gw}/\partial\bar{z} = -\pi gw$ and, since $\int QgwdA = 0$ for every polynomial Q, $\widehat{gw}$ vanishes identically in the unbounded complementary component of $\bar{\Omega}$. Our task will be to prove that

(i) $\widehat{gw} = 0$ everywhere in $\mathbf{C} \setminus \bar{\Omega}$ and almost everywhere on $\partial\Omega$ in an appropriate sense.

(ii) there exists a sequence $\varphi_j \in C_0^\infty(\Omega)$ such that $\left\| \frac{\partial \varphi_j}{\partial \bar{z}} - gw \right\|_{L^q(\Omega,\, dA)} \to 0$ as $j \to \infty$.

Once this has been done it follows from property (ii) that if $F \in L^p_a(\Omega, wdA) \cap L^p(\Omega, dA)$ then

$$\lim_{j\to\infty} \int_\Omega F \frac{\partial \varphi_j}{\partial \bar{z}}\, dA = \int_\Omega Fgw\, dA.$$

On the other hand, integrating by parts

$$\int_\Omega F \frac{\partial \varphi_j}{\partial \bar{z}}\, dA = -\int_\Omega \frac{\partial F}{\partial \bar{z}} \varphi_j\, dA = 0$$

for all j, since F is analytic in Ω. Hence, $\int FgwdA = 0$ and so $F \in H^p(\Omega, wdA)$. This argument originated with Havin [22] and Bers [4]. Whenever $L^p_a(\Omega, wdA) \cap L^p(\Omega, dA)$ is dense in $L^p_a(\Omega, wdA)$ we can conclude that the polynomials are complete in $L^p_a(\Omega, wdA)$. For those weights which we consider this will always be the case. In particular, if $w \equiv 1$ the assertion is obvious.

In establishing (i) we shall invariably make use of one of the following two lemmas. Both are variations of a lemma of Carleson [11, Lemma 1] and they are restated here solely for the convenience of the reader. A proof of Lemma 2.1 can be found in the author's paper [6, p. 169] and Lemma 2.2 is in Hedberg [32, p. 164].

Lemma 2.1. *Let E be a compact subset of the plane having connected complement and let $k \in L^q(E, dA)$ for some $q > 1$. If $\hat{k} = 0$ identically in $\mathbf{C} \setminus E$ then $\hat{k}(z_0) = 0$ at every point $z_0 \in \partial E$ where*

$$\int_E \frac{|k(z)|^q}{|z - z_0|} \, dA < \infty.$$

Lemma 2.2. *Let E be a compact set with connected complement and let $k \in L^q(E, dA)$, $1 < q \leqq 2$. If $\hat{k} = 0$ identically in $\mathbf{C} \setminus E$ and ζ_0 is a point of E^0 (the interior of E) at a distance $\delta < 1/e$ from ∂E then*

$$|\hat{k}(\zeta_0)| \leqq C\left\{k^*(\zeta_0)\delta \log 1/\delta + \left(\Gamma_q(\delta)\int_{|z-\zeta_0| \leqq 4\delta} |k(z)|^q \, dA\right)^{1/q}\right\},$$

where $k^*(\zeta) = \sup_r (\pi r^2)^{-1} \int_{|z-\zeta|<r} |k(z)| dA$ is the Hardy—Littlewood maximal function, $\Gamma_q(\delta)$ is equal to $\log 1/\delta$ or δ^{q-2} according to whether $q = 2$ or $q < 2$ and, C is a constant depending only on q and the diameter of E.

In order to describe accurately the meaning of the expression "almost everywhere" as it relates to our problem it is necessary to introduce the notion of capacity. If $1 < q < \infty$ the q-capacity of a compact set $E \subset \mathbf{C}$ is defined by

$$\Gamma_q(E) = \inf \int |\nabla u|^q \, dA,$$

where the infimum is taken over all functions $u \in C_0^\infty(\mathbf{C})$ such that $u \equiv 1$ on E. Here ∇u denotes the gradient of u and $C_0^\infty(\mathbf{C})$ is the set of all infinitely differentiable functions with compact support. If $q \geqq 2$ each function u is required to have its support in some fixed disk. The q-capacity of an arbitrary set X is defined by

$$\Gamma_q(X) = \sup \Gamma_q(E),$$

the supremum being taken over all compact sets $E \subset X$. A property is said to hold Γ_q quasi-everywhere if the set where it fails has Γ_q capacity zero.

We shall denote by W_1^q the Banach space of all functions $u \in L^q(\mathbf{R}^2)$ whose first partial derivatives (taken in the sense of distribution theory) also belong to $L^q(\mathbf{R}^2)$. The norm is customarily defined as follows:

$$\|u\|_{W_1^q} = \left\{\int (|u|^2 + |\nabla u|^2)^{q/2} \, dA\right\}^{1/q}.$$

For a fixed open set $\Omega \subset \mathbf{R}^2$ we denote by $\mathring{W}_1^q(\Omega)$ the closure of $C_0^\infty(\Omega)$ in W_1^q.

Functions of class W_1^q arise naturally in connection with approximation problems. If, for example, $k \in L^q(\Omega)$ and if $\hat{k}$ has compact support then $\hat{k} \in W_1^q$. In particular, by a theorem of Calderón and Zygmund

$$\|\nabla \hat{k}\|_q \leqq C \left\| \frac{\partial \hat{k}}{\partial \bar{z}} \right\|_q = \pi C \|k\|_q,$$

where C depends only on q (cf. [28, p. 564]). The Cauchy transform $\hat{k}$ of an

L^q function also has the following remarkable continuity property (cf. [33, p. 306]): Given any $\varepsilon>0$ there exists an open set U such that $\Gamma_q(U)<\varepsilon$ and $\hat{k}$ is continuous in the complement of U. A function with this property is said to be quasi-continuous. It can be shown that every W_1^q function agrees almost everywhere with one that is quasi-continuous. Of course, if $q>2$ then $\hat{k}$ is actually continuous and likewise each W_1^q function has a continuous representative.

The next lemma is due to Bagby [1, p. 264] (cf. also [33, p. 313]) and can be used to establish assertions such as (ii) when $1<q<\infty$. In case $q=\infty$, however, this must be replaced by an argument of Ahlfors (cf. [4, p. 3] & [32, p. 168]). Consequently, we shall hereafter state all results for $p\geqq 1$, give the proofs for $p>1$ and allow the reader to make appropriate modifications for $p=1$ whenever necessary.

Lemma 2.3. *Let Ω be an open set in the complex plane and let $u\in W_1^q$ be quasi-continuous. In order that $u\in \mathring{W}_1^q(\Omega)$ it is necessary and sufficient that $u=0$ quasi-everywhere with respect to Γ_q on $\mathbf{C}\setminus\Omega$.*

There is another kind of continuity associated with Sobolev functions that can sometimes be useful. A function g which is defined Γ_q quasi-everywhere is said to be Γ_q pseudo-continuous at x_0 if for every $\lambda>0$ the set $\{x\colon |g(x)-g(x_0)|\geqq\lambda\}$ is suitably thin at x_0. For a precise description of thinness the reader is referred to [33]. It will be sufficient for our purpose to know that a Borel set E is not Γ_q-thin at x_0 if $1<q\leqq 2$ and

$$\int_0 \left(\frac{\Gamma_q(E\cap\Delta_r)}{r^{2-q}}\right)^{p-1}\frac{dr}{r}=\infty, \tag{2.1}$$

where $\Delta_r=\Delta(x_0;r)$ denotes the disk of radius r with center at x_0 (cf. [33, p. 302]). Suppose for example, that $k\in L^q(\Omega)$ and that $\hat{k}\equiv 0$ in $\mathbf{C}\setminus\bar{\Omega}$. By a theorem of Fuglede (cf. [33, p. 306]), $\hat{k}$ is pseudo-continuous Γ_q quasi-everywhere. Thus, in order to prove that $\hat{k}=0$ q.e. in $\mathbf{C}\setminus\Omega$ it is enough to verify that (2.1) is satisfied at all points of $\partial\Omega$ with $E=\mathbf{C}\setminus\bar{\Omega}$. We shall make repeated use of this idea in connection with assertion (i). Note that $\Gamma_q(\Delta_r)\approx r^{2-q}$ if $1<q<2$ and $\Gamma_2(\Delta_r)\approx$ $\approx(\log 1/r)^{-1}$ (cf. [61, Lemmas 7, 8] & [33, p. 312]).

3. Approximation with respect to a weight

Let Ω be a bounded simply connected domain and let w be a positive weight defined on Ω. Throughout this section φ will denote a conformal map of Ω onto the open unit disk D and $\psi=\varphi^{-1}$ will be the inverse mapping. We shall confine our attention to those weights w which, when composed with ψ, have one of the following approximation properties A_p, $1 \leqq p < \infty$:

$$H^p(D, w(\psi)\,dA) = L^p_a(D, w(\psi)\,dA) \qquad (A_p)$$

Unfortunately, it is difficult to tell in any given instance if property A_p is satisfied and our results are, therefore necessarily limited. We remark only that w has property A_p for every p if $w(\psi)$ is constant on each circle $|z|=r(r<1)$ (cf. Mergeljan [40, p. 131] for $p=2$). What little additional information is available can also be found in [40].

Most of the results in this section are based on the following elementary fact (cf. [31, p. 121] & [40, p. 136]).

Lemma 3.1. *If, for a fixed p, the weight w has property A_p and if $\varphi^n \cdot (\varphi')^{2/p} \in H^p(\Omega, wdA)$ for $n=0, 1, 2, \ldots$ then $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$.*

Proof. Suppose that $F \in L^p_a(\Omega, wdA)$. For any polynomial Q we have

$$\int_\Omega |F - Q(\varphi)\cdot(\varphi')^{2/p}|^p w\,dA = \int_\Omega |F\cdot(\varphi')^{-2/p} - Q(\varphi)|^p |\varphi'|^2 w\,dA =$$

$$= \int_D |F(\psi)\cdot(\psi')^{2/p} - Q|^p w(\psi)\,dA.$$

Since w has property A_p, the last integral can be made arbitrarily small by a suitable choice of Q. Therefore, $F \in H^p(\Omega, wdA)$, since by hypothesis $Q(\varphi)\cdot(\varphi')^{2/p} \in H^p(\Omega, wdA)$. Q.E.D.

It is important to recognize here that property A_p is not, by itself, sufficient to ensure that $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$. The weight $w \equiv 1$, for example, always has property A_p but, if Ω_0 is the disk with a single radial cut then $H^p(\Omega_0, dA) \neq L^p_a(\Omega_0, dA)$ for any p. The difficulty arises from the fact that Lebesgue measure dA does not "see" the cut. Thus our task will be to find those weights w on a given region Ω for which the measure wdA respects all boundary cuts. In particular, we shall be interested in finding conditions on w which imply property A_p and the remaining hypothesis of Lemma 3.1, namely, that $\varphi^n(\varphi')^{2/p}$ belongs to $H^p(\Omega, wdA)$ for $n=0, 1, 2, \ldots$.

As a rule, we shall also be interested in rather general domains. In this context the part of the boundary which plays the role of the cut in Ω_0 is called the inner boundary. We now make that notion precise.

Definition. For an arbitrary domain Ω, the inner boundary is that part of $\partial\Omega$ which is contained in the interior of $\bar{\Omega}$. The inner boundary will be denoted $\partial^*\Omega$.

In this section we shall restrict our attention to domains Ω for which $\partial\Omega \setminus \partial^*\Omega = \partial\Omega_\infty$, where Ω_∞ denotes the unbounded component of $\mathbf{C} \setminus \bar{\Omega}$. This would exclude the crescent, for instance. We shall further assume, at least initially, that each point of $\partial^*\Omega$ can be joined to $\partial\Omega_\infty$ by a rectifiable arc which, except for one end-point, lies entirely in $\partial^* \Omega$. A typical example would be a Jordan region with one or more rectifiable cuts in it.

One of the basic tools in our investigation is the Denjoy—Carleman theorem for quasi-analytic classes on a rectifiable arc γ. If f is a complex valued function defined on γ and $z_0 \in \gamma$ the derivative $f'(z_0)$ is defined in the usual manner:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}, \quad z \in \gamma.$$

The set of functions having derivatives of all orders along γ is denoted $C^\infty(\gamma)$. A family in $C^\infty(\gamma)$ is quasi-analytic if each of its members is completely determined by its value and the value of its derivatives at a single point. The following version of the well known Denjoy—Carleman theorem is due to A. M. Davie (unpublished, cf. also [13, p. 34]) and will be used by us in subsequent discussions. The proof, which we outline, is based on an argument of Bang [2] (cf. [38, pp. 107—113]).

Theorem 3.2. *Let γ be a rectifiable arc and suppose that $f \in C^\infty(\gamma)$. Assume that there exists a sequence $\{A_n\}_{n=0}^\infty$ of non-negative real numbers such that*

(i) $\sup_\gamma |f^{(n)}| \leqq A_n$, $n=0, 1, 2, \ldots$;
(ii) *the sequence $\{A_n\}$ is logarithmically convex, i.e. $A_n^2 \leqq A_{n-1} A_{n+1}$ for all n;*
(iii) $\sum_{n=0}^\infty (1/A_n)^{1/n} = \infty$.

If $a \in \gamma$ and $f^{(n)}(a)=0$ for $n=0, 1, 2, \ldots$ then f vanishes identically on γ.

Proof. The proof makes use of a generalized Taylor formula for functions defined on a rectifiable arc. Let $\xi_0, \xi_1, \ldots, \xi_n$ be points taken in order along γ with $\xi_n = a$ and ξ_0 an arbitrary point of γ. Define polynomials $G_0, G_1, \ldots, G_n$ inductively as follows:

(1) $$G_0(z) = 1;$$

(2) $$G_k(\xi_0, \ldots, \xi_{k-1}, z) = \int_{\xi_{k-1}}^{z} G_{k-1}(\xi_0, \ldots, \xi_{k-2}, t)\, dt \quad \text{for} \quad k \geqq 1.$$

As in [38, p. 108] it can be shown by induction on n and integration by parts that

(3) $$f(\xi_0) = \sum_{k=0}^{n-1} (-1)^k f^{(k)}(\xi_n) G_k(\xi_n) + R_n$$

where the remainder R_n is given by

(4) $$R_n = \sum_{k=1}^{n} (-1)^k \int_{\xi_{k-1}}^{\xi_k} f^{(k)}(t) G_{k-1}(\xi_1, \ldots, \xi_{k-1}, t)\, dt.$$

By choosing $\xi_0 = \xi_1 = \ldots = \xi_{n-1}$ we obtain from (3) and (4) the usual Taylor formula

(3') $$f(\xi_0) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\xi_n)}{k!} (\xi_0 - \xi_n)^k + R_n$$

and, in this case, the remainder has the particularly simple form

$$R_n = \frac{-1}{(n-1)!} \int_{\xi_0}^{\xi_n} f^{(n)}(t)(\xi_0 - t)^{n-1}\, dt. \tag{4'}$$

Verification of the preceding remarks is, at least formally, the same as in the classical case. One must check, however, that the integrations involved can be carried out in the usual way. In particular, the following is needed:

(5) If $F \in C^1(\gamma)$ then $\int_{z_1}^{z_2} F'(z)\, dz = F(z_2) - F(z_1)$ whenever $z_1, z_2 \in \gamma$.

To establish this we assume that γ has a parametric representation $z = z(s)$, where s denotes arc-length on γ. Note that $z'(s)$ exists for almost every s since γ is rectifiable. Thus, if $g(s) = F(z(s))$ and if we write

$$\frac{g(s) - g(s_0)}{s - s_0} = \frac{F(z(s)) - F(z(s_0))}{z(s) - z(s_0)} \cdot \frac{z(s) - z(s_0)}{s - s_0}$$

it is easily seen that

$$\limsup_{s \to s_0} \left| \frac{g(s) - g(s_0)}{s - s_0} \right| \leqq M = \sup_{\gamma} |F'| \quad \text{for every} \quad s_0; \tag{6}$$

$$g'(s_0) = F'(z(s_0)) z'(s_0) \quad \text{for almost every} \quad s_0. \tag{7}$$

We conclude from (6) that g is absolutely continuous and hence if $z_1 = z(s_1)$ and $z_2 = z(s_2)$ then

$$\begin{aligned} F(z_2) - F(z_1) &= g(s_2) - g(s_1) \\ &= \int_{s_1}^{s_2} g'(s)\, ds \\ &= \int_{z_1}^{z_2} F'(z(s)) z'(s)\, ds \\ &= \int_{z_1}^{z_2} F'(z)\, dz. \end{aligned}$$

Suppose now that $f^{(k)}(\xi_n) = 0$ for $k = 0, 1, 2, \ldots$. Our objective is to prove that $f(\xi_0) = 0$. In this endeavor we consider the following two possibilities:

$$\liminf_{k \to \infty} \left(\frac{A_k}{k!} \right)^{1/k} = R < \infty. \tag{8}$$

$$\lim_{k \to \infty} A_k^{1/k} = \infty. \tag{9}$$

If the sequence $\{A_k\}$ happens to satisfy condition (8) we choose $\xi_1 = \ldots = \xi_{n-1} = \xi_0$ and estimate $|f(\xi_0)|$ by means of (3′) and (4′). In this way we obtain another sequence $k_j \to +\infty$ with the property that

$$|f(\xi_0)| \leqq (R|\xi_0 - \xi_n|)^{k_j}$$

for every k_j. Evidently, then, $f(\xi_0)=0$ if $|\xi_0-\xi_n|<1/R$. By repeating the argument a finite number of times we can infer that $f\equiv 0$ on γ.

If (8) fails then (9) holds and in that case the points $\xi_1, \ldots, \xi_{n-1}$ must be chosen more carefully. In order to do this we recall that, by an inequality of Carleman (cf. [21, p. 249] & [38, pp. 22—24]),

$$\sum_{k=2}^{\infty}\left(\frac{1}{A_k}\right)^{1/k} \leqq e\sum_{k=1}^{\infty}\frac{A_k}{A_{k+1}},$$

where e is the base of the natural logarithm. Consequently, both series diverge. Define $\mu_1=1$, $\mu_k=A_{k-1}/A_k$ for $k\geqq 2$ and, with d standing for distance along γ, put $\alpha_n=d(\xi_0, a)/\sum_{k=1}^{n}\mu_k$. By virtue of (ii) the sequence $\{\mu_k\}$ is nonincreasing and, according to our previous remarks, $\alpha_n\to 0$ as $n\to\infty$. Now choose points $\xi_1, \ldots, \xi_{n-1}$ on γ in such a way that

$$d(\xi_{k-1}, \xi_k) = \alpha_n\mu_k \quad (k = 1, \ldots, n).$$

As in [13, p. 35] and [38, pp. 111—113] it follows from (3) and (4) that

$$|f(\xi_0)| \leqq A_1\alpha_n(1-\alpha_n e)^{-1}.$$

Since $\alpha_n\to 0$ as $n\to\infty$, once again $f(\xi_0)=0$. Q.E.D.

We can now state and prove our first result and in so doing we introduce a new concept. A rectifiable arc γ will be called strongly rectifiable if

$$\limsup_{z\to z_0,\, z\in\gamma}\frac{d(z, z_0)}{|z-z_0|} < \infty$$

for every $z_0\in\gamma$. Here, as before, $d(z, z_0)$ denotes the distance from z to z_0 along γ. Since γ is rectifiable, the limit supremum is actually equal to 1 almost everywhere with respect to Hausdorff measure.

Theorem 3.3. *Let Ω be a bounded simply connected domain with $\partial\Omega\setminus\partial^*\Omega=\partial\Omega_\infty$. Assume that each point of $\partial^*\Omega$ can be joined to $\partial\Omega_\infty$ by a strongly rectifiable arc in $\partial^*\Omega$. Let $\delta(z)=\operatorname{dist}(z, \partial^*\Omega)$. If $w\geqq 0$ is bounded and if it has property A_p then $H^p(\Omega, wdA)=L_a^p(\Omega, wdA)$ whenever the following two conditions are satisfied:*

$$\text{(1)}\qquad \left(\int_\Omega\frac{w(z)}{\delta(z)^{np}}\,dA\right)^{1/p} = M_n < \infty, \quad n = 0, 1, 2, \ldots;$$

$$\text{(2)}\qquad \sum_{n=1}^{\infty}\frac{1}{n(M_n)^{1/n}} = \infty.$$

Remark. Theorem 3.3 is somewhat reminiscent of a theorem of Bernštein [3] concerning weighted polynomial approximation on the real line R. Let $h\geqq 0$ be a bounded measurable function defined on R and put $C_h=\{f\colon \lim_{|x|\to\infty}h(x)f(x)=0,$ f continuous$\}$. Bernštein's problem was first posed in 1924 and it asks: For what weights h are the polynomials dense in C_h in the sense that given $f \in C_h$ there exists a sequence of polynomials Q_j such that

$$\sup_x h(x)\,|f(x)-Q_j(x)| \to 0 \quad \text{as} \quad j \to \infty?$$

In 1951 Bernštein himself obtained the following result (since then, of course, a complete solution has been given by Mergeljan [41]):

Theorem (Bernštein). *The polynomials are dense in* C_h *if*

$$\sup_x |x|^n h(x) = M_n < \infty, \quad n = 0, 1, 2, \ldots; \tag{1'}$$

$$\sum_{n=1}^{\infty} \frac{1}{(M_n)^{1/n}} = \infty. \tag{2'}$$

In relation to Theorem 3.3 condition (1′) corresponds to (1), condition (2′) to (2) and the point at ∞ to $\partial^* \Omega$.

Proof of Theorem 3.3. Let $\varphi\colon \Omega \to D$ be a conformal map of Ω onto the open unit disk D. We will verify that $\varphi^n(\varphi')^{2/p} \in H^p(\Omega, w dA)$, $n=0, 1, 2, \ldots$.

Suppose, therefore, that $g \in L^q(\Omega, w dA)$ and $\int Q g w dA = 0$ for every polynomial Q. We must prove that $\int \varphi^n(\varphi')^{2/p} g w dA = 0$ for $n=0, 1, 2, \ldots$. To accomplish this we consider the Cauchy transform $G(z) = \int g w(\zeta)/(\zeta - z) dA_\zeta$. By hypothesis $G \equiv 0$ in Ω_∞. We shall prove that $G=0$ quasi-everywhere with respect to Γ_q in $\mathbf{C} \setminus \Omega$ and then use the argument outlined at the beginning of Section 2.

The first step is to verify that G vanishes identically on $\partial^* \Omega$. To this end, fix $x_1 \in \partial^* \Omega$ and choose a strongly rectifiable arc γ lying in $\partial^* \Omega$ and joining x_1 to a point $x_0 \in \partial \Omega_\infty$. Now apply Lemma 2.1 to G with $E = \bar{\Omega}$, $k = g w$ and $z_0 = x_0$. It is a simple matter to check that the hypotheses are satisfied:

(i) $\int_\Omega |g w|^q dA \leqq (\operatorname{ess\,sup} w)^{q-1} \int_\Omega |g|^q w\, dA < \infty$;

(ii) $\int_\Omega \dfrac{|g w(z)|^\lambda}{|z - x_0|} dA < \infty$ for every $\lambda < q$ and hence for some $\lambda > 1$.

The first is obvious; the second follows from Hölder's inequality and assumption (1). Thus, according to Lemma 2.1, $G(x_0) = 0$. By the same reasoning $\int \dfrac{Q(z)}{z - x_0} g w(z) dA = 0$ for every polynomial Q and so

$$\int_\Omega \frac{1}{z-\zeta} \left(\frac{g w(z)}{z - x_0} \right) dA = 0$$

whenever $\zeta \notin \bar{\Omega}$. Hence, again by Lemma 2.1

$$\int_\Omega \frac{g w(z)}{(z - x_0)^2} dA = 0.$$

Continuing in this manner, we conclude that

$$\int_\Omega \frac{gw(z)}{(z-x_0)^n}\,dA = 0, \quad n = 0, 1, 2, \dots.$$

Now, by virtue of hypothesis (1), $G \in C^\infty(\gamma)$ and for each $x \in \gamma$

$$G^{(n)}(x) = n!\int_\Omega \frac{gw(z)}{(z-x)^{n+1}}\,dA, \quad n = 0, 1, 2, \dots.$$

Here, of course, differentiation is carried out along γ. From this and our earlier remarks it follows that

(iii) $G^{(n)}(x_0) = 0, \quad n = 0, 1, 2, \dots;$
(iv) $\sup_\gamma |G^{(n)}| \leqq n!\, M_{n+1} \|g\|_{L^q(wdA)}, \quad n = 0, 1, 2, \dots.$

We would like to invoke Theorem 3.2 and conclude that $G \equiv 0$ on γ. Unfortunately, Theorem 3.2 cannot be applied to G directly. We can, however, do the following: For each $x \in \gamma$ define $F(x) = \int_{x_0}^x G(z)\,dz$, integration being along γ. Since γ is strongly rectifiable, it is easy to see that $F' = G$ and consequently

(v) $F^{(n)}(x_0) = 0, \quad n = 0, 1, 2, \dots;$
(vi) $\sup_\gamma |F^{(n)}| = \sup_\gamma |G^{(n-1)}| \leqq n!\, M_n \|g\|_{L^q(wdA)}, \quad n = 1, 2, \dots.$

Theorem 3.2 can now be applied to F with $A_n = n!M_n\|g\|$. First, the sequence $\{A_n\}$ is logarithmically convex, since the same is true of $\{M_n\}$. The latter can easily be seen by writing $\delta^{-np} = \delta^{-p(n-1)/2} \cdot \delta^{-p(n+1)/2}$ and applying Schwarz's inequality with respect to the measure wdA. Secondly, since $(n!)^{1/n}$ is asymptotic to n as $n \to \infty$, it follows from hypothesis (2) that $\sum_{n=1}^\infty A_n^{-1/n} = \infty$. Therefore, $F = G = 0$ on γ which implies that $G \equiv 0$ on $\partial^* \Omega$.

We wish now to conclude that $G = 0$ quasi-everywhere on $\partial \Omega_\infty$ and hence quasi-everywhere in $\mathbf{C} \setminus \Omega$. The following facts are known:

(a) G is pseudo-continuous at Γ_q quasi-every point of $\mathbf{C}$ (in fact, actually continuous everywhere if $q > 2$);
(b) G vanishes identically in Ω_∞.

Since Ω_∞ is "thick" in the potential theoretic sense (i.e. (2.1) is satisfied with $E = \Omega_\infty$) at every point of $\partial \Omega_\infty$, the desired conclusion is obvious. Consequently, by Lemma 2.3, $G \in \mathring{W}_1^q(\Omega)$ and $\int FgwdA = 0$ for every $F \in L_a^p(\Omega, wdA) \cap L^p(\Omega, dA)$. Since we can take $F = \varphi^n (\varphi')^{2/p}$, the proof is complete. Q.E.D.

We have seen (Theorem 3.3) that in $L_a^p(\Omega, wdA)$ completeness is subject to the behavior of w near $\partial^* \Omega$. As a general principle, of course, this was discovered by Keldyš and has been known for many years. It is illustrated even more graphically in the following two theorems. The first, Theorem 3.4, was obtained earlier by M. M. Džrbašjan (cf. [40, p. 144]) in the case of a disk with a single radial cut. His argument, however, does not extend to the more general setting to be considered by us. The second, Theorem 3.5, indicates that Theorem 3.4 is "best possible". It is closely related to a theorem of N. Levinson concerning the normality of certain classes of analytic functions and its proof is based on recent work of E. M. Dyn'kin [16].

Theorem 3.4. *Let Ω be bounded simply connected domain with $\partial\Omega\setminus\partial^*\Omega = \partial\Omega_\infty$. Assume that each point of $\partial^*\Omega$ can be joined to $\partial\Omega_\infty$ by a rectifiable arc in $\partial^*\Omega$. Let $\delta(z) = \operatorname{dist}(z, \partial^*\Omega)$. Suppose further that $w(z) \leqq W(\delta(z))$, where W is continuously differentiable and*

(1) $\chi(t) = t\dfrac{W'(t)}{W(t)} \uparrow +\infty$ as $t \downarrow 0$;

(2) $\int_0 \log\log \dfrac{1}{W(t)}\, dt = +\infty$.

Then, $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$ if w has property A_p.

Remark. Hereafter, a majorant W satisfying condition (1) above will be called a regular majorant.

Theorem 3.5. *Let Ω_0 be the region obtained by deleting the positive real axis from the open unit disk. Let $\delta(z)$ be the distance from z to the cut. Suppose that $w(z) = W(\delta(z))$ where*

(3) $W(t) \downarrow 0$ as $t \downarrow 0$;

(4) $\int_0 \log\log \dfrac{1}{W(t)}\, dt < +\infty$.

Then, $H^p(\Omega_0, wdA) \neq L^p_a(\Omega_0, wdA)$ for any p.

Historical Notes: Results of the kind described in Theorems 3.4 & 3.5 have been obtained by several authors. It seems appropriate, therefore, to comment briefly on the work of these men. The following is a summary of what was previously known:

(a) The first and perhaps the most comprehensive result was obtained by Keldyš [36] in 1941 (cf. also [40]). Assuming that Ω is an arbitrary bounded simply connected domain and w has property A_2, he proved that $H^2(\Omega, wdA) = L^2_a(\Omega, wdA)$ if, for some $\varepsilon > 0$,

$$\liminf_{\delta(z)\to 0} [\delta(z)]^{2+\varepsilon} \log\log \frac{1}{w(z)} > 0, \tag{3.1}$$

where $\delta(z)=\operatorname{dist}(z, \partial\Omega)$. His proof consisted in modifying the technique used in proving the classical Runge theorem. He also observed that (3.1) is very nearly sharp in the sense that there exists a domain Ω_1 such that $H^2(\Omega_1, wdA) \neq L^2_a(\Omega_1 wdA)$ for any weight w satisfying

$$\limsup_{\delta(z)\to 0} [\delta(z)]^{2-\varepsilon} \log\log \frac{1}{w(z)} = 0. \tag{3.2}$$

(b) In 1953 Tamadjan [54] noticed that condition (3.1) could be relaxed somewhat without destroying completeness. By modifying Keldyš' technique he was able to replace (3.1) by the weaker estimate

$$\liminf_{\delta(z)\to 0} [\delta(z)]^2 \log\log \frac{1}{w(z)} > 0. \tag{3.3}$$

(c) At the same time that he obtained (3.1) Keldyš noted that this rate of decay for w could be substantially reduced if $\partial\Omega$ were sufficiently nice. In particular, for Ω_0, the disk with a single radial cut, he showed that (3.1) could be replaced by the condition

$$\liminf_{\delta(z)\to 0} [\delta(z)]^{1+\varepsilon} \log\log \frac{1}{w(z)} > 0. \tag{3.4}$$

Again, he observed that this was nearly sharp. That is, $H^2(\Omega_0, wdA) \neq L^2_a(\Omega_0, wdA)$ for any weight w satisfying

$$\limsup_{\delta(z)\to 0} [\delta(z)]^{1-\varepsilon} \log\log \frac{1}{w(z)} = 0. \tag{3.5}$$

(d) Around 1949 Džrbašjan (cf. [40, p. 143]) was able to reduce the rate of decay required for completeness in $L^2_a(\Omega_0, wdA)$ still further, provided w satisfied a certain regularity condition. If $w(z)=W(\delta(z))$ and $W(t)$ decreases regularly (cf. (1), Th. 3.4) as $t\downarrow 0$ then $H^2(\Omega_0, wdA)=L^2_a(\Omega_0, wdA)$ if

$$\int_0 \log\log \frac{1}{W(t)}\, dt = +\infty. \tag{3.6}$$

Of course, he assumed property A_2 as usual. Our Theorem 3.4 includes this as a special case. Moreover, to our knowledge it was not previously known to what extent (3.6) was sharp. This question is taken care of in Theorem 3.5.

(e) In a recent paper, [59], Beurling has studied uniform weighted approximation and in that setting he has obtained results which complement Theorems 3.4 & 3.5. This work is based on ideas contained in [60] and can be used to give an alternate proof of Theorem 3.5. It is not unlikely that his method can be adapted to other problems encountered in L^p-approximation.

He considers the following situation: Let Ω be the rectangle $\{x+iy: |x|<a, |y|<b\}$ and $\Omega^{\pm}$ its intersection with the upper and lower half planes, respectively. Assume

that $w(y)$ is a continuous function defined on the interval $[-b, b]$ satisfying: $w(y)>0$ for $y\neq 0$, $w(0)=0$ and $w(y)\downarrow 0$ as $|y|\downarrow 0$. Let $C_w(\Omega)$ be the Banach space of all complex valued functions f for which the product $w(y)f(x+iy)$ is continuous on $\bar{\Omega}$ and equal to zero on $[-a, a]$, the norm being

$$\|f\|_w = \sup_{x+iy\in\Omega} w(y)|f(x+iy)|.$$

Define

$$A_w(\Omega) = \{f: f\in C_w(\Omega),\ f \text{ analytic in } \Omega\},$$
$$A_w(\Omega^{\pm}) = \{f: f\in C_w(\Omega),\ f \text{ analytic in } \Omega^+\cup\Omega^-\}.$$

$A_w(\Omega^{\pm})$ is a closed subspace of $C_w(\Omega)$ and the problem is to determine when $A_w(\Omega)$ is dense in $A_w(\Omega^{\pm})$. Beurling proved that the polynomials, and hence $A_w(\Omega)$, are dense in $A_w(\Omega^{\pm})$ if and only if

$$\int_{-\delta}^{\delta} \log\log\frac{1}{w(y)}\,dy = +\infty.$$

The proof of necessity depends on the solvability of a certain generalized Dirichlet problem. This replaces the result of Dyn'kin used by us in establishing Theorem 3.5.

Proof of Theorem 3.4. We shall obtain a bound for the integral $\int_\Omega w(z)\delta(z)^{-np}\,dA$ and then argue as in Theorem 3.3. To do this we simply estimate the maximum of the function $W(t)t^{-np}$, $t>0$, in the obvious manner. Differentiating and setting the derivative equal to zero we find that the maximum occurs at the unique point t which satisfies the equation

$$-npW(t)+tW'(t) = 0,$$

that is, when $t=\chi^{-1}(np)$. Since, in particular, $W(t)\leqq \text{const}\cdot t^p$ it follows that

$$M_n = \left(\int_\Omega \frac{w(z)}{\delta(z)^{np}}\,dA\right)^{1/p} \leqq \frac{C}{\chi^{-1}(np)^{n-1}},$$

where C is independent of n. Thus, if $g\in L^q(\Omega, w\,dA)$ and if γ is any arc lying in $\partial^*\Omega$ the Cauchy transform $G(z)=\int gw(\zeta)/(\zeta-z)\,dA_\zeta$ belongs to $C^\infty(\gamma)$ and

$$\sup_\gamma |G^{(n)}| \leqq n!\,M_{n+1}\|g\|_{L^q(\Omega, w\,dA)} \leqq \frac{Cn!}{\chi^{-1}(np)^n}\|g\|_{L^q(\Omega, w\,dA)}.$$

In order, then, to apply the Denjoy—Carleman theorem as in Theorem 3.3 we must prove that

$$\sum_{n=1}^{\infty} \frac{\chi^{-1}(np)}{n} = \infty. \tag{3.7}$$

Since χ^{-1} is a decreasing function, this series will diverge provided

$$\int_1^\infty \frac{\chi^{-1}(sp)}{s}\,ds = \infty$$

or, equivalently, if $\int_p^\infty \frac{\chi^{-1}(s)}{s}\,ds=\infty$. To verify that the latter diverges we integrate by parts:

$$\int_p^\infty \frac{\chi^{-1}(s)}{s}\,ds = [\chi^{-1}(s)\log s]_p^\infty - \int_p^\infty \log s\, d\chi^{-1}(s).$$

The first term on the right hand side is bounded below and the second is equal to $+\infty$. In particular,

$$\begin{aligned}-\int^\infty \log s\, d\chi^{-1}(s) &= \int_0 \log\chi(t)\,dt \\ &= \int_0 \log t\,dt + \int_0 \log\left(\frac{d}{dt}\log W\right)dt \\ &\geqq \mathrm{const}\cdot\int_0 \log\log\frac{1}{W(t)}\,dt \\ &= +\infty.\end{aligned}$$

We have used here the fact that $d\log W/dt \geqq C\log(1/W)$ (cf. [16, p. 186]). Therefore, the series (3.7) diverges and we can conclude as in Theorem 3.3 that $H^p(\Omega, wdA)=L_a^p(\Omega, wdA)$ if w has property A_p. Q.E.D.

Proof of Theorem 3.5. Since W is monotonic and since $\int_0 \log\log\frac{1}{W(t)}\,dt < +\infty$, there exists a function W_1 with the following properties:

(a) $W_1(t)\leqq W(t)$ for $0<t<1$;

(b) W_1 has a continuous derivative and $t\frac{W_1'(t)}{W_1(t)}\uparrow+\infty$ as $t\downarrow 0$;

(c) $\int_0 \log\log\frac{1}{W_1(t)}\,dt<+\infty$.

The existence of W_1 was demonstrated by Dyn'kin [16, Lemma 3].

Now fix a point ξ in the inner boundary of Ω_0. Since $(W_1)^2$ also has properties (a), (b) and (c), we can find a continuously differentiable function $\mu(z)$ defined for $|z|<1$ and satisfying

(i) $\mu(\xi)=1$;

(ii) $\mu\equiv 0$ off a compact subset of the open unit disk;

(iii) $\left|\frac{\partial\mu}{\partial\bar z}\right|\leqq \mathrm{const}\cdot W_1(\delta(z))^2$.

This is again a consequence of Dyn'kin's work [16] (see, e.g., the proof of Theorem 3, p. 188). We shall write $w_1(z)=W_1(\delta(z))$. If Q is any polynomial then, according to Green's theorem,

$$Q(\xi)=Q(\xi)\mu(\xi)=-\frac{1}{\pi}\int_{\Omega_0} Q(z)\frac{\partial\mu}{\partial\bar z}\frac{dA_z}{(z-\xi)}.$$

From this, property (iii) and our choice of ξ it follows that

$$|Q(\xi)| \leqq C \int_{\Omega_0} |Q(z)| \frac{w_1(z)^2}{\delta(z)} dA$$

for every polynomial Q; the constant C does not depend on Q. Because of the convexity condition (b), $w_1(z)/\delta(z)$ is bounded on Ω_0 (see Theorem 3.4) and so we have

$$|Q(\xi)| \leqq C \|Q\|_{L^1(\Omega_0, w_1 dA)}$$

for every polynomial Q. In other words, the mapping $Q \to Q(\xi)$ is a bounded linear functional on the polynomials in the $L^1(\Omega_0, w_1 dA)$ norm. Hence, it is also a bounded mapping in each of the $L^p(\Omega_0, wdA)$ norms, $p \geqq 1$.

From this we can conclude that $H^p(\Omega_0, wdA) \neq L^p_a(\Omega_0, wdA)$ for any p. If we assume the opposite for some p then we can find a function $g \in L^q(\Omega_0, wdA)$ such that

$$Q(\xi) = \int_{\Omega_0} QgwdA \tag{3.8}$$

for every polynomial Q. Hence, $\int (z-\xi) QgwdA = 0$. Since Ω_0 is simply connected and $\xi \notin \Omega_0$, we can define in Ω_0 an analytic branch of $(z-\xi)^\beta$ for each real β. By choosing a positive integer n so that $1/n < 2/p$ we have $(z-\xi)^{-1/n}$ in $L^p_a(\Omega_0, wdA)$. Consequently, for every polynomial Q

$$\int_{\Omega_0} \frac{Q}{(z-\xi)^{1/n}} (z-\xi) gwdA = 0.$$

That is, $\int (z-\xi)^{1-1/n} QgwdA = 0$. Continuing in this way, we see that

$$\int_{\Omega_0} (z-\xi)^{1-k/n} QgwdA = 0$$

for $k = 1, 2, \ldots, n$ and every polynomial Q. When $k = n$ this reduces to

$$\int QgwdA = 0$$

for every Q, contradicting the reproducing property (3.8). Therefore, we must conclude that $H^p(\Omega_0 wdA) \neq L^p_a(\Omega_0, wdA)$. Q.E.D.

In the proof of Theorem 3.5 the boundedness of the mapping $Q \to Q(\xi)$ played a special role. A point ξ for which this map is bounded on the polynomials in the $L^p(\Omega, wdA)$ norm is called a bounded point evaluation for $H^p(\Omega, wdA)$ or, more precisely, $H^p(\Omega, wdA)$ is said to have a bounded point evaluation at ξ. These bounded point evaluations have been studied extensively in connection with several problems in approximation theory (cf. [5], [6], [7], [52]). What was actually proved in the second part of Theorem 3.5 is the following:

Theorem 3.6. *Let Ω be a bounded simply connected domain and w a positive weight function defined on Ω. Assume that $w \in L^\infty(dA)$. In order for $H^p(\Omega, wdA) = L_a^p(\Omega, wdA)$ it is necessary that H^p has no bounded point evaluations outside Ω.*

The short proof given here was discovered in a dual form by Sinanjan [52] (cf. also [62, pp. 204—206]).

It is natural now to ask if the necessary condition in Theorem 3.6 is also sufficient. As early as 1955 Mergeljan [42, p. 904] had conjectured that indeed it is. Although the conjecture has never been settled, results of this kind can be established in certain cases. In this regard we have found it necessary to insist on more restrictive hypotheses than those suggested by Theorem 3.6. The following is a summary of what we have been able to obtain.

Theorem 3.7. *Let Ω be a bounded simply connected domain and let $w \in L^\infty(\Omega, dA)$ be a weight function. If w has property A_p then $H^p(\Omega, wdA) = L_a^p(\Omega, wdA)$ whenever either of the following two conditions is satisfied:*

(1) *for some $\beta > 2p$ the polynomials have no bounded point evaluations on $\partial^* \Omega$ in the $L^\beta(\Omega, wdA)$ norm.*

(2) *$\int_\Omega w(z)/\delta(z)^n \, dA < \infty$ for every $n \geqq 0$ and, for some $\beta > p$, the polynomials have no bounded point evaluations on $\partial^* \Omega$ in the $L^\beta(\Omega, wdA)$ norm.*

Remark 1. The author is indebted to L. I. Hedberg for pointing our that a more appropriate capacity could be used in place of Γ_q in order to remove an unnecessary restriction that had been placed on $\partial^* \Omega$ in part (1) of an earlier version of Theorem 3.7 (cf. (3.9), (3.10) and (3.12)).

Remark 2. The requirement that the integral in (2) be finite for all n does not, by itself, imply completeness. Consider, for example, the region Ω_0 of Theorem 3.5 and with $W(t) = e^{-1/t}$ put $w(z) = W(\delta(z))$. Then $\int_{\Omega_0} w(z)/\delta(z)^n \, dA < \infty$, since the integrand is bounded. On the other hand, $\int_0 \log \log \frac{1}{W(t)} \, dt < \infty$ and so, by Theorem 3.5, completeness cannot occur.

Before proving Theorem 3.7 we shall state two corollaries. The first follows easily from the theorem and needs no further amplification. The second is a theorem of Keldyš and was described earlier in our historical remarks. It is restated here for the purpose of establishing a connection between our results and those which were previously known. We assume in both cases that $\partial\Omega \setminus \partial\Omega_\infty = \partial^* \Omega$.

Corollary 3.8. *Let $w \in L^\infty(\Omega, dA)$ be a weight having property A_p for all $p \geqq 1$. If, for each p, $H^p(\Omega, wdA)$ fails to have a bounded point evaluation in $\partial^* \Omega$ then $H^p(\Omega, wdA) = L_a^p(\Omega, wdA)$ for every p.*

Corollary 3.9 (Keldyš). *Let* $w \in L^\infty(\Omega, dA)$ *be a weight having property* A_p *for every* p *and let* $\delta(z) = \mathrm{dist}\,(z, \partial^* \Omega)$. *If, for some* $\varepsilon > 0$,

$$\liminf_{\delta(z)\to 0} [\delta(z)]^{2+\varepsilon} \log\log \frac{1}{w(z)} > 0$$

then $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$, *for all* p.

Proof of 3.9. Let $x_0 \in \partial^* \Omega$ and fix $\varrho > 0$. By a lemma of Keldyš (cf. [40, pp. 125—128 & 136—139]) there exists for each p a polynomial Q such that

(i) $Q(x_0) = 1$;
(ii) $\|Q\|_{L^p(\Omega, wdA)} \leqq \varrho$.

Thus, $H^p(\Omega, wdA) = L^p_a(\Omega, wdA)$ by Corollary 3.8. Q.E.D.

In order to prove Theorem 3.7 we must introduce some additional terminology and notation.

If ν is a positive (Borel) measure in the plane and $0 < \alpha < 2$ we define

$$U^\nu_\alpha(x) = \int |x-y|^{\alpha-2} d\nu(y)$$

and if $d\nu = f dA$ we write $U^f_\alpha(x)$ instead. For an arbitrary set $E \subset \mathbf{R}^2$ the (α, q)-capacity of E, denoted $C_{\alpha,q}(E)$, is defined by

$$C_{\alpha,q}(E) = \inf_f \int |f|^q dA, \tag{3.9}$$

where the infimum is taken over all non negative functions $f \in L^q(\mathbf{R}^2, dA)$ such that $U^f_\alpha(x) \geqq 1$ on E. If E happens to be a Borel set it can be shown that

$$C_{\alpha,q}(E)^{1/q} = \sup_\nu \nu(E), \tag{3.10}$$

where the supremum is taken over all positive measures concentrated on E and satisfying $\|U^\nu_\alpha\|_p \leqq 1$. It can also be shown that $C_{1,q}$ is equivalent to Γ_q in the sense that there exists a constant K so that

$$K^{-1} C_{1,q}(E) \leqq \Gamma_q(E) \leqq K C_{1,q}(E) \tag{3.11}$$

for every E. A property will be said to hold (α, q) quasi-everywhere if the set where it fails has (α, q)-capacity zero. For more information concerning these capacities the reader should consult [27], [33] and [61]. The notation Λ_1 will be used to designate 1-dimensional Hausdorff measure.

We shall also need the following lemma on potentials (cf. [6, p. 169], [32, p. 160], [37, p. 80] etc.).

Lemma 3.10. *Let* ν *be a positive measure in the plane of total mass 1. Then*

(a) $\int |U^\nu_1|^p dA \leqq K\{\sup_C U^\nu_q(z)\}^{p-1}$ *if* $1 < q < 2$ *and* K *depends only on* q;

(b) $\int |U^\nu_1|^2 dA \leqq K \sup_C U^\nu_l(z)$, *where* $U^\nu_l(z)$ *denotes the usual logarithmic potential.*

Proof of Theorem 3.7. We shall give the proof based on the first assumption (1); the proof in the second case is similar and will be omitted.

Suppose that $g\in L^q(\Omega, wdA)$ and that $\int QgwdA=0$ for every polynomial Q. Thus, G, the Cauchy transform of gw, vanishes identically in Ω_∞ and, as previously, we must prove that $G=0$ quasi-everywhere on $\partial\Omega$ with respect to the capacity Γ_q. There is no problem in verifying this for $\partial\Omega_\infty$, since G is quasi-continuous andidentically zero in Ω_∞. Thus, we have only to determine the behavior of G on $\partial\Omega\diagdown\partial\Omega_\infty$, i.e. on $\partial^*\Omega$.

Here is where we make use of the capacities introduced above. Since $g\in L^q$ and $w\in L^\infty$, the integral

$$\int_\Omega \left|\frac{g(z)}{z-\xi}\right|^{1+\varepsilon} w(z)\,dA_z < \infty \tag{3.12}$$

$(1-\varepsilon, q/(1+\varepsilon))$ quasi-everywhere (cf. [61, p. 260]). Hence, the integral is finite Λ_1 almost everywhere if $(1-\varepsilon)q/(1+\varepsilon)>1$, that is if $1+\varepsilon<2q/(q+1)$ (cf. [27, p. 133] & [61, p. 290]). We conclude that $(z-\xi)^{-1}g\in L^r(wdA)$ for Λ_1 almost every $\xi\in\partial^*\Omega$ whenever $r<2q/(q+1)$ and we note that the bound on r is the index conjugate to $2p$.

Suppose now that $\xi\in\partial^*\Omega$ and that $(z-\xi)^{-1}g\in L^r(wdA)$ for every $r<2q/(q+1)$. If we assume that $G(\xi)\neq 0$ it follows that

$$Q(\xi) = \frac{1}{G(\xi)}\int_\Omega Q(z)\frac{gw(z)}{z-\xi}\,dA_z$$

for every polynomial Q. But, since the polynomials have no bounded point evaluations on $\partial^*\Omega$ in the L^β-norm, $\beta>2p$, this entails a contradiction. The only alternative is to conclude that $G(\xi)=0$. Therefore, according to our remarks in the preceding paragraph, G vanishes Λ_1 almost everywhere on $\partial^*\Omega$.

In order to see that G actually vanishes on a much larger portion of $\partial^*\Omega$ we appeal to its continuity properties. In particular, we shall prove that $E=\{\xi\in\partial\Omega\colon G(\xi)=0\}$ is thick (i.e. not thin) with respect to Γ_q at every point of $\partial^*\Omega$, from which it follows that $G=0$ q.e. $-\Gamma_q$ on $\partial^*\Omega$. The argument given here is based on the assumption that $1<q<2$. The case $q=2$ can be treated in similar fashion and is left to the reader. If $q>2$ then G is continuous everywhere and there is little to prove.

Suppose that $\xi_0\in\partial^*\Omega$ and let $\Delta_r=\Delta(\xi_0; r)$ be the disk of radius r with center at ξ_0. Because Ω is connected and simply connected, each circle $|z-\xi_0|=r$ meets $\partial\Omega$ provided r is sufficiently small ($r\leqq r_0$, say). We can therefore find a Borel set $X\subset(\partial\Omega\cap\Delta_r)$ which projects in a one-to-one manner along circles centered at ξ_0 onto a fixed radial segment of Δ_r (cf. [37, p. 159] for a more detailed exposé of a similar assertion). By setting $\mu(B)=\Lambda_1(\mathrm{Proj}\,(B\cap X))$ for each Borel set B, we obtain a measure on X. On the other hand, circular projection is a length decreasing map and so μ is actually concentrated on $E \cap \Delta_r$, i.e. $\Lambda_1(\mathrm{Proj}\,(\partial\Omega \setminus E) \cap \Delta_r) = 0$. If we now apply Lemma 3.10 with $\nu = r^{-1}\mu$ we have

$$\begin{aligned}\int_C \left\{\int \frac{d\nu(\zeta)}{|\zeta - z|}\right\}^p dA_z &\leqq K\left\{\sup_{z\in C} \int \frac{d\nu(\zeta)}{|\zeta - z|^{2-q}}\right\}^{p-1} \\ &\leqq K\left\{\frac{1}{r}\int_0^r \frac{ds}{\big|s - |z|\big|^{2-q}}\right\}^{p-1} \\ &\leqq K\left\{\frac{1}{r}\int_0^r \frac{ds}{s^{2-q}}\right\}^{p-1} \\ &\leqq K r^{(q-2)p/q}.\end{aligned}$$

Thus, $\sigma = K^{-1} r^{(2-q)/q}\nu$ is a measure on $E \cap \Delta_r$ with the property that $\|U_1^\sigma\|_p \leqq 1$. Consequently, since σ has total mass equal to $K^{-1} r^{(2-q)/q}$, it follows from (3.10) and (3.11) that

$$\Gamma_q(E \cap \Delta_r) \geqq C r^{2-q}$$

and this holds for all $r \leqq r_0$. We conclude from this and (2.1) that E is thick at ξ_0 and hence thick at every point of $\partial^*\Omega$. Q.E.D.

4. Some weights with property A_p.

Having discussed weighted polynomial approximation at some length, one fact stands out. In every instance where we were able to prove that $H^p(\Omega, w dA) = L_a^p(\Omega, w dA)$ we found it necessary to insist the weight w possess a certain approximation property, called property A_p. Our intention here is to describe a method for constructing weights which have this property and which satisfy the various decay estimates at $\partial^*\Omega$ required by the theorems of Section 3.

The method is due to Keldyš [36, p. 14] and is based on the following lemma concerning the distortion of sets under a conformal mapping. Although the lemma is old and quite well known, the most convenient reference seems to be a fairly recent paper of Hedberg, [30, p. 542]. We shall make use of this result again in Section 5 in connection with the completeness problem for crescents (cf. Th. 5.7).

Lemma 4.1. *Let Ω be a bounded simply connected domain and let φ be a conformal map of Ω onto the open unit disk. If $d(z) = \mathrm{dist}\,(z, \partial\Omega)$ there exists a constant $C > 0$, which depends only on the diameter of Ω* and on $d(\varphi^{-1}(0))$, *such that*

$$1 - |\varphi(z)| \leqq C\sqrt{d(z)}$$

for every $z \in \Omega$.

Suppose now that $k(t)$ is a function defined for $0<t<\infty$. With Ω and φ as in the lemma put $\psi=\varphi^{-1}$ and consider a weight w of the form

$$w(z) = \exp\{-\exp k(1-|\varphi(z)|)\}.$$

Since $w(\psi)$ is constant on every circle $|\xi|=r(r<1)$, w has property A_p for every p. Thus, if each point of $\partial^*\Omega$ can be joined to $\partial\Omega_\infty$ by a rectifiable arc lying entirely in $\partial^*\Omega$ it follows from Theorem 3.4 and Lemma 4.1 that $H^p(\Omega, wdA)=L_a^p(\Omega, wdA)$ if $k(t)=1/t^2$. Earlier Keldyš, [36] obtained the analogous result for an arbitrary Ω with $k(t)=1/t^{4+\varepsilon}$, $\varepsilon>0$.

By making use of a theorem of Džrbašjan (cf. [17] & [40, p. 133]) one can also construct examples in which the weight approaches zero only at points in the closure of the inner boundary and still completeness occurs.

5. Approximation with respect to area

We have, thus far, been concerned with only one aspect of the completeness problem, namely, with weighted approximation. In this context the domain Ω is fixed and one asks: for which weights w is $H^p(\Omega, wdA)=L_a^p(\Omega, wdA)$? We now abandon that point of view and shift our attention to the domain Ω and we ask: for which domains Ω is $H^p(\Omega, dA)=L_a^p(\Omega, dA)$? In particular, we shall want to know which non-Carathéodory domains have this property. At first we shall consider a rather general setting and return later to the subject of crescents.

We begin by describing a general metric criterion for completeness, which generalizes a theorem of Mergeljan [42]. This criterion will allow us, for instance, to consider domains with boundary cuts (cf. Cor. 5.3). Mergeljan's theorem and our generalization of it are as follows:

Theorem 5.1 (Mergeljan). *Let Ω be a bounded simply connected domain and let S be a circle containing $\bar{\Omega}$ in its interior. The polynomials are complete in $L_a^p(\Omega, dA)$ for every p if there exists a sequence of points $\{\zeta_n\}_{n=1}^\infty$ having the following properties:*

(1) the closure of the point set $\{\zeta_n\}$ contains $\partial\Omega$;

(2) each point ζ_n can be joined to S by a rectifiable arc Γ of length L such that

$$\operatorname{meas}\Omega_t(\Gamma) < \exp\{-\exp(5L/t)\}, \tag{5.1}$$

where $\Omega_t(\Gamma)=\{z\in\Omega : \operatorname{dist}(z, \Gamma)\leqq t\}$.

Theorem 5.2. *Let Ω be a bounded simply connected domain. The polynomials are complete in $L_a^p(\Omega, dA)$ for every p if there exists a sequence of points $\{\zeta_n\}_{n=1}^\infty$ having the following properties:*

(1) *the closure of the point set* $\{\zeta_n\}$ *contains* $\partial\Omega$;

(2′) *each point* ζ_n *can be joined to* $\partial\Omega_\infty$ *by a rectifiable arc* Γ *such that*

(a) meas $\Omega_t(\Gamma) \leqq V(t)$ *for some regular majorant* V;

(b) $\int_0 \log\log \frac{1}{V(t)}\, dt = +\infty$.

Remark 1. By requiring that the estimate (5.1) be satisfied along curves which actually protrude into Ω_∞, Mergeljan has put an unnecessary restriction on $\partial\Omega_\infty$. We are able to avoid this assumption by making use of potential theoretic properties of the Cauchy transform such as those described in Section 2. In addition we can replace (5.1) by the weaker estimate (b) of Theorem 5.2.

Remark 2. Prior to the publication of Theorem 5.1, Mergeljan and Tamadjan jointly [44, p. 85] obtained a result which suggested that, in order for completeness to occur, the estimate (5.1) is nearly sharp. They showed that the right hand side of (5.1) cannot be replaced by $\exp\{-\exp(c/t^\lambda)\}$ for any $\lambda<1$. Their example is fairly complicated and for that reason we shall give another (cf. Th. 5.4) which is stronger and somewhat easier to visualize. Both examples, however, require precise estimates for the harmonic measure of certain sets.

Before giving the proof of Theorem 5.2 it will be instructive to indicate its relationship to the completeness problem for domains with boundary cuts. In so doing, we obtain a generalization of still another theorem of Mergeljan and Tamadjan [44, p. 80]: Let E be a perfect nowhere dense set of points on the circle $|z|=1$. For each $x\in E$ let $S_x=\{z : \arg z=\arg x,\ 1-\varrho\leqq|z|\leqq 1\ (0<\varrho<1)\}$ and put $S_E=\bigcup_{x\in E} S_x$. Assuming ϱ is fixed, set $\Omega_E=(|z|<1)\setminus S_E$. Thus, Ω_E is a bounded simply connected domain and $\partial\Omega_E$ consists almost entirely of cuts. Put $\complement E=(|z|=1)\setminus E$, $\Delta_t(x)=\{e^{i\theta} : |\theta-\arg x|\leqq t\}$ and denote 1-dimensional Lebesgue measure by Λ_1. The following is an immediate consequence of Theorem 5.2:

Corollary 5.3. *The polynomials are complete in* $L^p_a(\Omega_E, dA)$ *for every* p *if there exists a countable set* E', *everywhere dense in* E, *such that for each* $x\in E'$

(1) $\Lambda_1(\Delta_t(x)\cap\complement E)\leqq V(t)$ *for* $t\leqq t(x)$ *and some regular majorant* V;

(2) $\int_0 \log\log \frac{1}{V(t)}\, dt = +\infty$.

Proof of Theorem 5.2. Let $g\in L^q(\Omega, dA)$ and suppose that $\int Qg\,dA=0$ for every polynomial Q.

To achieve our objective we shall first prove that $\hat g\equiv 0$ on each arc Γ described in the statement of the theorem. Since the points ζ_n are dense in $\partial\Omega$, this implies that the Borel set $E=\{x : \hat g(x)=0\}$ cannot be thin at any point of $\partial\Omega$. In particular, if $x\in\partial\Omega$ and $\Delta_r=\Delta(x;r)$ is the disk with center at x and radius r then $\Gamma_q(\Delta_r\cap E)\geqq$

$\geqq C\Gamma_q(\Delta_r)$ and

$$\int_0 \left(\frac{\Gamma_q(\Delta_r \cap E)}{r^{2-q}}\right)^{p-1} \frac{dr}{r} = +\infty.$$

On the other hand, $\hat{g}$ is pseudo-continuous at Γ_q quasi-every point of $\partial\Omega$ and therefore $\hat{g}=0$ q.e. on $\partial\Omega$.

To see that $\hat{g}\equiv 0$ on a fixed Γ we proceed as in Theorem 3.4. We set $\delta(z)= =\operatorname{dist}(z, \Gamma)$ and we prove

(i) $\left(\int_\Omega \frac{dA}{\delta(z)^{np}}\right)^{1/p} = M_n < \infty, \quad n = 0, 1, 2, \ldots;$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n(M_n)^{1/(n+1)}} = \infty.$

The proof consists in writing (i) as a Stieltjes integral and then integrating by parts: If $\mu(t)=\operatorname{meas}\Omega_t(\Gamma)$,

$$\int_\Omega \frac{dA}{\delta(z)^{np}} = \int_0^\infty \frac{1}{t^{np}}\, d\mu(t) = np \int_0^\infty \frac{\mu(t)}{t^{np+1}}\, dt.$$

Since $\mu(t)\leqq V(t)$, it follows that

$$\int_\Omega \frac{dA}{\delta(z)^{np}} \leqq Cn \max\left(\frac{V(t)}{t^{(n+1)p}}\right).$$

Furthermore, if $\chi(t)=t\dfrac{V'(t)}{V(t)}$ the maximum on the right hand side occurs when $t=\chi^{-1}(p(n+1))$ and

$$\left(\int_\Omega \frac{dA}{\delta(z)^{np}}\right)^{1/p} \leqq C\frac{n}{\chi^{-1}(p(n+1))^{n+1}}.$$

Therefore, condition (ii) will be satisfied if we can prove that

$$\sum_{n=1}^{\infty} \frac{\chi^{-1}(p(n+1))}{n} = \infty.$$

This is done exactly as in Theorem 3.4.

As a consequence of (i) we conclude that $\hat{g}\in C^\infty(\Gamma)$ and that $\hat{g}^{(n)}(x_0)=0$ for all n at the point x_0 where Γ meets $\partial\Omega_\infty$. Then, by virtue of (ii), $\hat{g}\equiv 0$ on Γ.

Q.E.D.

The fact that Theorem 5.2 establishes very nearly the correct order of magnitude on $\operatorname{meas}\Omega_t(\Gamma)$ in order for completeness to occur is confirmed by the following theorem.

Theorem 5.4. *If in Theorem 5.2 the majorant* $V(t)$ *is replaced by*

$$\exp\left\{-\exp\frac{1}{t(\log 1/t)^{\lambda}}\right\}$$

for any $\lambda>1$ *then the polynomials may fail to be complete in* $L^p_a(\Omega, dA)$.

Proof. Fix $\lambda>1$. In order to establish the theorem we shall construct a crescent shaped domain Ω having the following properties:

(i) Each point of the bounded component of $\mathbf{C}\setminus\bar{\Omega}$ can be joined to $\partial\Omega_\infty$ by a rectifiable arc Γ in such a way that

$$\operatorname{meas}\Omega_t(\Gamma) = \exp\left\{-\exp\frac{1}{t(\log 1/t)^{\lambda}}\right\} \tag{5.2}$$

for all t sufficiently small;

(ii) $H^p(\Omega, dA)\neq L^p_a(\Omega, dA)$ for any p.

The construction is actually quite simple: Let $\psi(t)$ denote the expression on the right side of (5.2). Choose an integer k so that $\lambda>1+1/k$ and put $\Theta(x)= =x^{k+1}e^{-1/x^k}$. For $0\leqq x\leqq x_0$ consider the two curves $y=\pm\Theta(x)$ which together form a single arc or cusp in the (x, y)-plane with vertex at the origin. Join the ends of the latter so as to form a simple closed rectifiable curve γ lying in the right half plane and containing the line segment $0<x\leqq x_0$ in its interior. Let Ω be any crescent satisfying the following conditions:

(a) Ω has a unique multiple boundary point at the origin and $\bar{\Omega}$ is otherwise contained entirely in the half plane $\operatorname{Re} z>0$;

(b) For $|t|$ sufficiently small the horizontal lines $y=\pm t$ meet Ω in two segments each of which is bisected by γ and each of which has length $\frac{1}{2}\psi'(t)$;

(c) Except for the origin, γ is entirely contained in Ω and separates x_0 from ∞;

(d) x_0 belongs to the bounded component of $\mathbf{C}\setminus\bar{\Omega}$.

With Ω chosen in this way property (i) is clearly satisfied: If ζ_0 is a point in the bounded component of $\mathbf{C}\setminus\bar{\Omega}$ choose an arc Γ, also in that component, so that Γ joins ζ_0 to $(0, 0)$. We may assume that Γ coincides with the positive real axis in a neighborhood of the origin. Then, by virtue of (b),

$$\operatorname{meas}\Omega_t(\Gamma) = \int_0^t \psi'(t)\,dt = \psi(t)$$

for t sufficiently small.

Verification of property (ii) is a more delicate matter. With this as our goal let $\mu=\mu_{x_0}$ be the harmonic measure on γ which represents the point x_0 and for $z\in\gamma$ let $\delta(z)$ be the distance from z to $\partial\Omega_\infty$. We shall prove that

$$\int_\gamma \log\delta(z)\,d\mu(z) > -\infty. \tag{5.3}$$

Having done so, we can then infer as in [6, p. 184] that every sequence of polynomials which is bounded in the $L^p(\Omega, dA)$ norm is also a normal family inside γ. In fact, if $\{f_j\}_{j=1}^{\infty}$ is such a sequence, i.e. if $\|f_j\|_{L^p(\Omega)} \leqq K$ for all j, then by the area mean value theorem

$$|f_j(z)| \leqq \frac{C_1}{\delta(z)^{2-2/q}} \|f_j\|_{L^p(\Omega, dA)} \leqq \frac{C_2}{\delta(z)^{2-2/q}}$$

for all j and all $z \in \gamma$. Here we have made use of the fact that near the origin γ is approximately midway between the two components of $\mathbf{C} \setminus \bar{\Omega}$. The constants C_1 and C_2 are, of course, independent of j. Since $\int_\gamma \log \delta(z) d\mu(z) > -\infty$, we can find a bounded nonvanishing function h, analytic in the region bounded by γ, and satisfying $|h(z)| = \delta(z)^{2-2/q}$ almost everywhere with respect to arc length and hence almost everywhere with respect to harmonic measure on γ. Thus, $|f_j h| \leqq K$ a.e. on γ. It follows that $|f_j h| \leqq K$ everywhere inside γ and so $f_j h$, $j=1, 2, \ldots$, is a normal family there. Because h is nowhere zero, the sequence f_j, $j=1, 2, \ldots$, is itself a normal family in that region and we have established our claim. This clearly implies that $H^p(\Omega, dA) \neq L^p_a(\Omega, dA)$ (cf. [6, p. 182]).

To complete the proof of the theorem it remains to show that the integral in (5.3) converges and, since the integrand is bounded on sets away from the origin, it is enough to prove that

$$\int_{\gamma_0} \log \delta(z)\, d\mu(z) > -\infty,$$

where γ_0 is the intersection of γ with $\{z\colon \operatorname{Re} z < x_0\}$. We proceed as follows: Let $\omega(x)$ denote the harmonic measure of that part of γ which lies to the left of the line $\operatorname{Re} z = x$. For each point $z = (x, \pm\Theta(x))$ in γ let $\delta(x)$ be equal to $\delta(z)$. Inasmuch as $\delta(x) \geqq \text{const} \cdot \psi'(\Theta(x)) \geqq \text{const} \cdot \psi(\Theta(x))$,

$$\int_{\gamma_0} \log \frac{1}{\delta(z)}\, d\mu(z) = \int_0^{x_0} \log \frac{1}{\delta(x)}\, d\omega(x) \leqq C \int_0^{x_0} \log \frac{1}{\psi(\Theta(x))}\, d\omega(x).$$

If we now integrate by parts on the right we see that

$$\int_{\gamma_0} \log \frac{1}{\delta(z)}\, d\mu(z) \leqq C_3 + C_4 \int_0^{x_0} \omega(x) \frac{\psi'(\Theta(x))}{\psi(\Theta(x))}\, dx.$$

A bound for the latter can then be obtained by utilizing these two facts:

(iii) $\omega(x) \leqq K \exp\left\{-\frac{2}{\pi} \int_x^{x_0} \frac{1}{\Theta(r)}\, dr\right\} \leqq K \exp\{-C \exp 1/x^k\}$;

(iv) $\dfrac{\psi'(\Theta(x))}{\psi(\Theta(x))} \leqq \dfrac{K}{\Theta(x)^3} \exp\{C x^{k\lambda - k - 1} \exp 1/x^k\}$.

Estimate (iii) follows from a well known theorem of Carleman [45, p. 76]; (iv) is just a straightforward calculation. It is easy to see that $\omega(x)\dfrac{\psi'(\Theta(x))}{\psi(\Theta(x))}$ is bounded as $x\to 0$, since $k\lambda-k-1>0$ by choice of k. Therefore $\int_{\gamma_0}\log(1/\delta(z))\,d\mu(z)<+\infty$.
Q.E.D.

The crescent is the simplest and the first non-Carathéodory set to be studied in connection with the completeness problem for the polynomials. Traditionally it was understood to be a domain Ω topologically equivalent to the region bounded by two internally tangent circles. As we have seen in Theorems 5.2 & 5.4, the polynomials may or may not be complete in $L_a^p(\Omega, dA)$ depending on the thickness of Ω near the multiple boundary point. That fact, of course, was discovered by Keldyš in 1939. About ten years later and with additional restrictions on the domain Ω a condition was found that is both necessary and sufficient for completeness to occur. That was due to the combined efforts of M. M. Džrbašjan, who established sufficiency, and A. L. Šaginjan, who established necessity. Their contribution was this (cf. [40, p. 158]):

Theorem 5.5. *Let Ω be a crescent with multiple boundary point at the origin such that Ω is situated between the two circles $|z-1|=1$ and $|z-1/2|=1/2$. Denote by $l(r)$ the linear measure of $(|z|=r)\cap\Omega$ and assume that $r\dfrac{l'(r)}{l(r)}\uparrow+\infty$ as $r\downarrow 0$. Then in order for $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ for any p it is necessary and sufficient that*

$$\int_0 \log l(r)\,dr=-\infty. \tag{5.4}$$

By requiring Ω to lie between the two circles $|z-1|=1$ and $|z-1/2|=1/2$ one precludes the possibility of a cusp at the multiple boundary point. This is essential to the theorem and cannot be omitted. If, for example, Ω is the region described in Theorem 5.4 then

(a) Ω has a cusp at the origin,
(b) $\int_0 \log l(r)\,dr=-\infty$,

but $H^p(\Omega, dA)\neq L_a^p(\Omega, dA)$ for any p. The difficulty can, of course, be circumvented by replacing (5.4) with a more severe restriction. The following is a corollary of Theorem 5.2 and is in that spirit. We shall return to an idea of this kind later in Section 7 when we take up the question of harmonic approximation on higher dimensional crescents.

Theorem 5.6. *Let Ω be a crescent with multiple boundary point x_0. Suppose further that there exists a line segment L from x_0 into the bounded component of $\mathbf{C}\setminus\bar\Omega$ so that* $\operatorname{dist}(z, L)\geqq C\operatorname{dist}(z, x_0)^k$ *for some $k>0$ and every $z\in\Omega$. Denote* *by* $l(r)$ *the linear measure of* $(|z-x_0|=r)\cap\Omega$. *If*

(1) $r\dfrac{l'(r)}{l(r)}\uparrow+\infty$ *as* $r\downarrow 0$;

(2) $\int_0 r^{k-1}\log\log\dfrac{1}{l(r)}\,dr=+\infty$

then $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ *for every* p.

Remark. Although Theorem 5.6 applies to situations in which Ω has a cusp at the multiple boundary point the hypothesis $\operatorname{dist}(z, L)\geqq C\operatorname{dist}(z, x_0)^k$ restricts the rate at which such a cusp may pinch together.

During the past few years Havin and Maz'ja [25] & [28] and the present author [6] have examined the question of polynomial completeness in $L_a^p(\Omega, dA)$ for domains similar to, but more general than, the traditional crescent. It is our intention here to continue that study. The setting is as follows: X will denote a compact set with connected complement and U will be a domain (also connected) which is contained in X. We assume that $\mathbf{C}\setminus\bar{U}$ is connected and we put $\Omega=(X\setminus U)^0$, the interior of $X\setminus U$. For lack of a better terminology Ω will once again be called a crescent and, unless otherwise stated, our subsequent usage of that nomenclature will be in this generalized sense.

We shall state and prove two theorems concerning completeness in this setting. The first is quite general and includes the sufficiency portion of both Theorems 5.5 & 5.6 as special cases, provided $1\leqq p<3$. The second is less general, but provides greater information in those cases to which it applies.

Theorem 5.7. *Let* $\Omega=(X\setminus U)^0$ *be a crescent with* X *and* U *as above. Fix a point* $x_0\in U$ *and let* μ *be the harmonic measure on* ∂U *representing* x_0. $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ *for* $1\leqq p<3$ *if*

$$\int_{\partial U}\log\delta(z)\,d\mu(z)=-\infty, \tag{5.5}$$

where $\delta(z)=\operatorname{dist}(z, \partial\Omega_\infty)$.

Theorem 5.8. *Let* $\Omega=(X\setminus U)^0$ *and put* $\delta(z)=\operatorname{dist}(z, \partial\Omega_\infty)$. *Assume that* ∂U *is a Jordan curve of at least class* C^1 *and for each* $z\in\partial U$ *let* $n(z)$ *be the unit exterior normal to* ∂U *at* z.

(1) *If* $|n(z_1)-n(z_2)|\leqq C|z_1-z_2|^\alpha$ *for all* $z_1, z_2\in\partial U$ *and some* $\alpha>0$ *then* $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ *for every* p *if* $\int_{\partial U}\log\delta(z)\,|dz|=-\infty$.

(2) *If* $\alpha=1$ *in (1) then in order for* $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ *for any* p *it is necessary and sufficient that* $\int_{\partial U}\log\delta(z)\,|dz|=-\infty$.

Remark 1. The divergence of the integral in (5.5) does not depend on the particular choice of $x_0\in U$, since if a, b are any two points in U the corresponding harmonic measures μ_a, μ_b are mutually boundedly absolutely continuous (cf. [11, p. 172]).

Remark 2. Results similar to Theorem 5.7 were discovered several years ago by Havin and Maz'ja ([25, p. 66] & [28, p. 562]). Assuming that ∂U is a Jordan curve they proved that $H^p(\Omega, dA)=L^p_a(\Omega, dA)$ whenever $1\leqq p\leqq 2$ and $\int_{\partial U}\log|\varphi|\,d\mu = -\infty$ for every φ belonging to a certain infinite family $\mathcal{F}$. Only by imposing additional smoothness restrictions on ∂U were they able to translate this into a single "metric criterion" such as (5.5) (cf. [28, p. 563]). Thus, Theorem 5.7 has two main advantages: (i) It express completeness in terms of the divergence of a single integral; (ii) It applies to more general domains.

Remark 3. Suppose that $\Omega=(X\setminus U)^0$ is a crescent of the kind described in Theorem 5.6. If the remaining hypotheses of that theorem are satisfied and, in particular, if

$$\int_0 r^{k-1}\log\log\frac{1}{l(r)}\,dr = +\infty$$

then necessarily $\int_{\partial U}\log\delta(z)\,d\mu(z) = -\infty$. To see this it suffices to consider a domain Ω' obtained by doubling Ω across ∂U in the vicinity of the multiple boundary point x_0. Thus, $\Omega\subset\Omega'$ and $\partial\Omega$ lies approximately midway between the two boundary components of Ω' near x_0. Relative to Ω' the assumptions of Theorem 5.6 are still satisfied. Hence, $H^p(\Omega', dA)=L^p_a(\Omega', dA)$. If we were to assume that $\int_{\partial U}\log\delta(z)\,d\mu(z) > -\infty$ we could then argue as in Theorem 5.4 to obtain the opposite conclusion. Therefore, that assumption cannot be made. In this connection one should also see the example of Havin and Maz'ja [25, p. 67] concerning the case $k=1$.

Proof of Theorem 5.7. Assume that $\int_{\partial U}\log\delta(z)\,d\mu(z) = -\infty$. Fix $p<3$ and let $k\in L^q(\Omega)$ have the property that $\int_\Omega Qk\,dA=0$ for every polynomial Q. Thus, $\hat{k}\equiv 0$ in Ω_∞. To prove that $H^p(\Omega, dA)=L^p_a(\Omega, dA)$ it suffices to show that $\hat{k}\equiv 0$ in U.

For this purpose we choose a sequence of smoothly bounded domains U_j, $j=1, 2, \ldots$, so that

(i) $x_0\in U_j$ and $\overline{U}_j\subset U_{j+1}$ for every j;

(ii) $U=\bigcup_{j=1}^{\infty} U_j$.

We shall prove that $\int_{\partial U_j}\log|\hat{k}|\,d\mu_j\to -\infty$ as $j\to\infty$, where μ_j is the harmonic measure for x_0 on ∂U_j. Since $\log|\hat{k}|$ is subharmonic in U, these integrals form a nondecreasing sequence in j and therefore

$$\log|\hat{k}(x_0)| \leqq \int_{\partial U_j}\log|\hat{k}|\,d\mu_j = -\infty$$

for all j. Consequently, $\hat{k}(x_0)=0$. Hence $\hat{k}\equiv 0$ in U_j, $j=1, 2, \dots$, since x_0 is an arbitrary point of U.

In order to relate the integrals of the preceding paragraph to the one in the statement of the theorem we multiply and divide $|\hat{k}(z)|$ by $\delta(z)^\varepsilon$, where $\varepsilon>0$ is to be specified later. This yields the identity

$$\int_{\partial U_j} \log|\hat{k}(z)|\, d\mu_j = \varepsilon \int_{\partial U_j} \log\delta(z)\, d\mu_j + \int_{\partial U_j} \log\left(\frac{|\hat{k}(z)|}{\delta(z)^\varepsilon}\right) d\mu_j.$$

As $j\to\infty$ we shall see that the first integral on the right approaches $-\infty$ and that, for suitable ε, the second is uniformly bounded above. The result is that $\lim_{j\to\infty} \int_{\partial U_j} \log|\hat{k}|\, d\mu_j = -\infty$ as claimed.

The first assertion is the easiest to prove. Since $\|\mu_j\|_{\mathrm{var}}=1, j=1, 2, \dots$, we may, by passing to a subsequence if necessary, assume that the sequence μ_j converges weakly to a measure which is evidently supported on ∂U. That measure is easily seen to be μ, the unique harmonic measure for x_0 on ∂U. Suppose now that $N>0$ is fixed. By hypothesis, we can choose $\eta>0$ so that

$$\int_{\partial U} \log(\delta(z)+\eta)\, d\mu < -N.$$

On the other hand, $\log(\delta(z)+\eta)$ is continuous on $\overline{U}$ and so according to our remarks

$$\int_{\partial U_j} \log(\delta(z)+\eta)\, d\mu_j \to \int_{\partial U} \log(\delta(z)+\eta)\, d\mu$$

as $j\to\infty$. Therefore, $\int \log(\delta(z)+\eta) d\mu_j < -N$ and hence $\int \log\delta(z) d\mu_j < -N$ for $j\geqq j_0$. In other words, $\int \log\delta(z) d\mu_j \to -\infty$ as $j\to\infty$.

The proof can now be completed by indicating the choice of ε that will make

$$\sup_j \int_{\partial U_j} \log\left(\frac{|\hat{k}(z)|}{\delta(z)^\varepsilon}\right) d\mu_j < \infty. \tag{5.6}$$

In this regard we shall consider the cases $1\leqq p<2$ and $2\leqq p<3$ separately.

If $1\leqq p<2$ then $k\in L^q(\Omega, dA)$, $q>2$, and $\hat{k}$ satisfies a Hölder condition

$$|\hat{k}(z_1)-\hat{k}(z_2)| \leqq C|z_1-z_2|^\alpha,$$

where $0<\alpha<1$ and z_1, z_2 are any two complex numbers. If $q=\infty$ any $\alpha<1$ will do; otherwise we take $\alpha=(q-2)/q$. Hence, $|\hat{k}(z)|\leqq C\delta(z)^\alpha$ for all z and the supremum in (5.6) will be finite as soon as $\varepsilon\leqq\alpha$.

If $2\leqq p<3$ then $k\in L^q(\Omega, dA)$, $q\leqq 2$, and $\hat{k}$ need not satisfy a Hölder condition. However, there is a substitute, namely, the estimate of Lemma 2.2. Also,

$$\int_{\partial U_j} \log\left(\frac{|\hat{k}(z)|}{\delta(z)^\varepsilon}\right) d\mu_j < \int_{\partial U_j} \frac{|\hat{k}(z)|}{\delta(z)^\varepsilon}\, d\mu_j$$

and so we may direct out efforts toward finding a bound for the larger of the two integrals. In that endeavor some additional notation will be used: G_j will denote Green's function for U_j with pole at x_0, ∇ will be the gradient operator and $\partial/\partial n$ will stand for differentiation with respect to the outward normal on appropriate boundary curves. Since ∂U_j is smooth, $d\mu_j = -\dfrac{\partial G_j}{\partial n}|dz|$.

By removing a small disk $|z-x_0| \leqq \varrho$ from U_j we obtain a domain U_j' on which $|\hat{k}(z)|\delta(z)^{-\varepsilon}$ is Lipschitz and on which G_j is uniformly well behaved. We shall assume hereafter that ϱ is fixed and that $|z-x_0| \leqq \varrho$ is contained in every U_j. The divergence theorem can then be applied in the following form:

$$\int_{\partial U_j'} \frac{|\hat{k}(z)|}{\delta(z)^{\varepsilon}} \frac{\partial G_j}{\partial n} |dz| = \int_{U_j'} \nabla \left(\frac{|\hat{k}(z)|}{\delta(z)^{\varepsilon}} \right) \cdot \nabla G_j \, dA.$$

In this way we obtain constants C_1 and C_2 for which

$$\left| \int_{\partial U_j'} \frac{|\hat{k}(z)|}{\delta(z)^{\varepsilon}} \frac{\partial G_j}{\partial n} |dz| \right| \leqq C_1 \int_{U_j'} \frac{|\hat{k}(z)|}{\delta(z)^{1+\varepsilon}} |\nabla G_j| \, dA + C_2 \int_{U_j'} |\nabla(|\hat{k}|)| \frac{|\nabla G_j|}{\delta(z)^{\varepsilon}} \, dA =$$
$$= C_1 I_1 + C_2 I_2.$$

Thus we have the problem of finding bounds for I_1 and I_2 that are independent of j. We shall carry this out for I_2 and then indicate the necessary modifications in the case of I_1.

It is a consequence of Hölder's inequality and the Calderón—Zygmund theorem on the continuity of singular integral operators (cf. [8] & [28, p. 564]) that

$$I_2 \leqq C \|k\|_q \left(\int_{U_j'} \left| \frac{\nabla G_j}{\delta(z)^{\varepsilon}} \right|^p dA \right)^{1/p}.$$

If $\delta(z)$ is replaced by $\delta_j(z) = \operatorname{dist}(z, \partial U_j)$ in the expression on the right we retain an upper bound for I_2 and the resulting integral is estimated as follows: For each j let φ_j be a conformal map of U_j onto $|w|<1$ with $\varphi_j(x_0)=0$; put $\psi_j = \varphi_j^{-1}$. The following inequalities are satisfied with all constants independent of j.

(a) $1-|\varphi_j(z)| \leqq C\sqrt{\delta_j(z)}$ for all $z \in U_j$;
(b) $|\nabla G_j| \leqq C|\varphi_j'|$ on U_j';
(c) $|\psi_j'(w)| \geqq C(1-|w|)$.

(a) is simply a reaffirmation of Lemma 4.1 and (c) is a well known estimate, due to Pick and Nevanlinna [45, p. 91], for the distortion associated to a conformal mapping of the open unit disk onto a bounded domain. The remaining inequality (b) can be derived from the equation $G_j = -\log|\varphi_j|$. Since G_j is real,

$$|\nabla G_j| = 2\left|\frac{\partial G_j}{\partial z}\right| = \left|\frac{\varphi_j'}{\varphi_j}\right|, \quad j = 1, 2, \ldots.$$

We must, therefore, find a lower bound for $|\varphi_j|$ on U_j' which is independent of j. As an aid in that undertaking we let φ be a conformal map of U onto $|w|<1$ with $\varphi(x_0)=0$. When properly defined at x_0, the function φ/φ_j is analytic in U_j, continuous on $\overline{U}_j$ and $|\varphi/\varphi_j|\leqq 1$ on ∂U_j. Thus, by the maximum principle, $|\varphi/\varphi_j|\leqq 1$ throughout U_j and

$$|\varphi_j(z)| \geqq \min_{U_j'} |\varphi| > 0$$

for every $z\in U_j'$. This is the desired lower bound. Taken in concert, inequalities (a), (b) and (c) have the following implication:

$$\begin{aligned} |I_2|^p &\leqq C_1 \int_{U_j'} \left|\frac{\nabla G_j}{\delta_j(z)^\varepsilon}\right|^p dA \\ &\leqq C_2 \int_{U_j} \frac{|\varphi_j'|^{p-2}}{(1-|\varphi_j|)^{2p\varepsilon}} |\varphi_j'|^2 \, dA \\ &= C_2 \int_{|w|<1} \frac{1}{|\psi_j'|^{p-2}} \frac{1}{(1-|w|)^{2p\varepsilon}} \, dA \qquad (5.7) \\ &\leqq C_3 \int_{|w|<1} \frac{1}{(1-|w|)^{p-2+2p\varepsilon}} \, dA. \end{aligned}$$

The last integral is evidently independent of j and, moreover, can be made finite by choosing ε so that $p-2+2p\varepsilon<1$. Since $p<3$, that choice is possible in a manner consistent with the requirement that $\varepsilon>0$.

The estimation of I_1 can be carried out along similar lines. In this case we write

$$I_1 = \int_{U_j'} \frac{|\hat{k}(z)|}{\delta(z)^{1-\varepsilon}} \frac{|\nabla G_j|}{\delta(z)^{2\varepsilon}} \, dA$$

and apply Hölder's inequality to obtain

$$I_1 \leqq \left(\int_{U_j'} \frac{|\hat{k}(z)|^q}{\delta(z)^{(1-\varepsilon)q}} \, dA\right)^{1/q} \left(\int_{U_j'} \frac{|\nabla G_j|^p}{\delta(z)^{2p\varepsilon}} \, dA\right)^{1/p}.$$

By the reasoning used earlier, we can choose $\varepsilon>0$ in such a way that the second factor admits a bound which is independent of j. With ε fixed in this way it follows from Lemma 2.2 and a short calculation (cf. [6, p. 178]) that the first factor is majorized by a quantity of the form $C(\varepsilon)\|k\|_q$.

Thus, we have shown that

$$-\int_{\partial U_j} \frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} \frac{\partial G_j}{\partial n} |dz| + \int_{|z-x_0|=\rho} \frac{|\hat{k}(z)|}{\delta(z)^\varepsilon} \frac{\partial G_j}{\partial n} |dz| \leqq K$$

where K is a constant that is independent of j. On the other hand, it is easily seen that the integrals over $|z-x_0|=\varrho$ are also uniformly bounded for $j=1, 2, \ldots$. Hence, the same is true of the remaining integrals and the theorem follows. Q.E.D.

The apparent breakdown of Theorem 5.7 for $p\geqq 3$ raises several questions. One wonders, for instance, if $p=3$ is the correct upper bound or if the theorem is perhaps true for all values of p. Of course, if ∂U is sufficiently smooth these questions are answered by Theorem 5.8. However, before commenting on the proof of that result we prefer to mention two others which are essentially corollaries of Theorem 5.7. In connection with these the reader should also consult the paper of Havin and Maz'ja [28, Lemma 3.3].

We assume that Ω, X, U and $\delta(z)$ have the same meaning as before; μ will again be harmonic measure on ∂U.

Corollary 5.9. *Let p be fixed and assume that $\int_{\partial U} \log \delta(z)\, d\mu(z)=-\infty$. If ψ is a conformal map of $|w|<1$ onto U and if*

$$\int_{|w|<1} \frac{1}{|\psi'|^{p-2+\lambda}}\, dA <\infty$$

for some $\lambda>0$, then $H^p(\Omega, dA)=L^p_a(\Omega, dA)$.

Remark. The fact that $\int |\psi'|^{2-p} dA<\infty$ for $p<3$ seems to have been first noticed by T. A. Metzger (Amer. Math. Soc. Proc. 37 (1973), 468—470), who used it in connection with another approximation problem.

Proof. We may assume that $p\geqq 2$. Also, it is helpful to choose a sequence of domains $\{U_j\}$ which invade U by letting U_j be the image of the disk $|w|<1-1/j$ under the map ψ, $j=2, 3, \ldots$. Evidently, $\psi_j(z)=\psi((1-1/j)z)$ maps $|w|<1$ onto U_j conformally. Repeating the argument in the proof of Theorem 5.7 we eventually reach a point where we must estimate an expression of the form (5.7). In this case it is easy to see that

$$\int_{|w|<1} |\psi_j'|^{2-p}(1-|w|)^{-2p\varepsilon}\, dA \leqq C\int_{|w|<1} |\psi'|^{2-p}(1-|w|)^{-2p\varepsilon}\, dA,$$

where C is independent of j. For any pair of conjugate indices r and s the expression on the right is $\leqq C\|(\psi')^{2-p}\|_r \|(1-|w|)^{-2p\varepsilon}\|_s$. The latter can be made finite by selecting r so that $r(p-2)\leqq p-2+\lambda$ and then adjusting ε accordingly. Q.E.D

Corollary 5.10. *If ∂U is a class C^1 Jordan curve and if $\int_{\partial U} \log \delta(z)\, d\mu(z)=-\infty$ then $H^p(\Omega, dA)=L^p_a(\Omega, dA)$ for every p.*

Proof. Let ψ: $(|w|<1)\to U$ conformally. By a theorem of Warschawski [56, p. 254] $\int_{|w|<1} |\psi'|^{-k} dA<\infty$ for all $k\geqq 0$, since $\partial U\in C^1$. Hence, the assertion follows from Corollary 5.9. Q.E.D.

Proof of Theorem 5.8. Part (1) follows from Corollary 5.10. Under the restrictions imposed on ∂U harmonic measure $d\mu$ is boundedly equivalent to $|dz|$. Part (2) was proven in [6]. Q.E.D.

Our efforts thus far have been largely directed at establishing the completeness of the polynomials in $L_a^p(\Omega, dA)$ for various domains Ω. Another line of inquiry consists in starting with a domain for which it is known at the outset that $H^p(\Omega, dA) \neq L_a^p(\Omega, dA)$. The object then is to give a complete description of the functions in $L_a^p(\Omega, dA)$ which admit approximation by polynomials. That approach was taken by Havin in [23] and he succeeded there in carrying out such a program for a certain special class of crescents. Using his ideas, some unpublished work of Carleson (cf. Section 6) and results described earlier in this section it is now possible to obtain a similar characterization for a wider class of domains.

For the remainder of this discussion $\Omega=(X\setminus U)^0$ will be a crescent. We assume that ∂U is a Jordan curve of at least class C^2 so that the harmonic measure μ_x on ∂U for $x \in U$ is boundedly equivalent to $|dz|$. We further assume that

$$\int_{\partial U} \log \delta(z)\,|dz| > -\infty, \tag{5.8}$$

where $\delta(z)=\operatorname{dist}(z, \partial\Omega_\infty)$. Thus, according to Theorem 5.8, $H^p(\Omega, dA) \neq L_a^p(\Omega, dA)$ for any p. With some additional assumptions on Ω (principally on ∂X) we shall characterize, by means of a certain auxiliary function Q, those functions in $L_a^p(\Omega, dA)$ which belong to $H^p(\Omega, dA)$. Q is defined as follows: Let

$$v(z) = \int_{\partial U} \log \delta(t)\, d\mu_z(t),$$

let $w(z)$ be the harmonic conjugate of v and put $Q=e^{v+iw}$. It is easy to check that Q is analytic in U, continuous on $\overline{U}$ and $|Q(z)|=\delta(z)$ for each $z \in \partial U$. A function F analytic in U is said to belong to the class $E^p(U)$ if there exists a sequence of rectifiable Jordan curves $C_1, C_2, \ldots$ in U, tending to ∂U, such that

$$\sup_n \int_{C_n} |F(z)|^p\,|dz| < \infty.$$

Additional information concerning the space $E^p(U)$ can be found in [15].

Theorem 5.11. *Let $\Omega=(X\setminus U)^0$ be a crescent whose interior and exterior boundaries (i.e. ∂U and ∂X, respectively) are Jordan curves of class $C^{2+\alpha}$. Assume that (5.8) is satisfied. If $f \in L_a^p(\Omega, dA)$ the following are equivalent:*

(1) *$f \in H^p(\Omega, dA)$;*

(2) *f can be extended to a function $\tilde{f}$ which is analytic in X^0 and so that $\tilde{f}Q^{1/p} \in E^p(U)$.*

Remark 1. A curve γ is said to belong to the class $C^{2+\alpha}$, $0<\alpha<1$, if it has a parametric representation $\gamma(s)$ in terms of arc length s such that $\gamma''(s) \in \mathrm{Lip}_\alpha$.

Remark 2. Suppose that the hypotheses of Theorem 5.11, and hence those of Theorem 5.8, are satisfied. One of the implications of the earlier theorem (cf. [6, pp. 182—184]) is that every $f \in H^p(\Omega, dA)$ extends analytically to X^0; the proof is based on an idea of Šaginjan (cf. [40, p. 121]). What is being asserted now is that the extension $\tilde{f}(z)$ is subject to a specific growth restriction as z approaches a multiple boundary point from within U. In certain cases results of this kind were known to Keldyš (cf. [36, p. 4] & [40, p. 135]) as early as 1939. He proved that if Ω_0 is the region bounded by the two tangent circles $|z|=1$ and $|z-1/2|=1/2$ then every $f \in H^2(\Omega_0, dA)$ has an extension $\tilde{f}$ which satisfies an inequality of the form

$$|\tilde{f}(x)| \leqq \frac{K}{(1-x)^{3/2}} \quad \text{for} \quad 1/2 < x < 1.$$

More recently, Shapiro [50, p. 293] extended this by proving $\tilde{f} \in E^\lambda(|z-1/2|<1/2)$ for every $\lambda < 1/2$. He also raised the question of determining precisely those functions which can be approximated by polynomials in the $L^2(\Omega_0, dA)$ norm. The answer was given by Havin in [23] and is also contained in Theorem 5.11. Moreover, as Havin pointed out, every $f \in H^2(\Omega_0, dA)$ has, as a consequence of these results, an extension $\tilde{f}$ which belongs to $E^\lambda(|z-1/2|<1/2)$ for $\lambda < 2/3$. This is because we may take $Q(z)=(z-1)^2$ and

$$\int |\tilde{f}|^\lambda |dz| \leqq \left(\int |\tilde{f}|^{3\lambda} |z-1|^{3\lambda} |dz|\right)^{1/3} \left(\int \frac{1}{|z-1|^{3\lambda/2}} |dz|\right)^{2/3} < \infty$$

provided $3\lambda/2 < 1$. The integration is over $|z-1/2|=1/2$.

The proof of Theorem 5.11 proceeds generally along the same lines as the proof of the corresponding theorem of Havin, [23, Th. 1]. In this case, however, there are more serious technical difficulties which must be overcome and that sometimes required the use of quite sophisticated tools. Among these are two theorems of Carleson. The first is a generalization of the Privalov—Zygmund theorem [58, p. 121] on the modulus of continuity of the conjugate function and is unpublished. Since it is used here in an essential way, we include a proof in Section 6. The second concerns the following problem: For which measures ν carried by the open unit disk $|w|<1$ does there exist a constant $C>0$ such that

$$\left\{\int_{|w|<1} |f|^p \, d\nu\right\}^{1/p} \leqq C \left\{\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta\right\}^{1/p}$$

for every f belonging to the usual Hardy space $H^p(d\theta)$? A complete description was given by Carleson, [10, Th. 1] (cf. also [15, p. 157]). In short, it is necessary and sufficient that there be a constant $A>0$ with the property that

$$\nu(S) \leqq A \cdot h \tag{5.9}$$

for every set S of the form $S=\{re^{i\theta}\colon 1-h\leqq r<1,\ \theta_0\leqq\theta\leqq\theta_0+h\}$. If $A\geqq 1$ one may take $C=4(80)^4A^2$ (cf. [15, p. 163]).

A function F analytic in a Jordan region, such as U, is by definition in the Nevanlinna class $N(U)$ if there exists a sequence of Jordan curves $\gamma_1, \gamma_2, \ldots$ in U, tending to ∂U, such that

$$\sup_n \int_{\gamma_n} \log^+|F|\, d\mu_n < \infty,$$

where each μ_n is the harmonic measure on γ_n for a fixed point $x_0\in U$. One can easily check that the definition is independent of the particular invading sequence $\gamma_1, \gamma_2, \ldots$. If $F\in N(U)$ and

$$\log|F(z)| = \int_{\partial U} \log|F|\, d\mu_z$$

for every $z\in U$ then F is called outer. In this sense the function Q of Theorem 5.11 is outer. We shall make use of the following fact: If F is an outer function in a region U with a class C^2 boundary and if $\int_{\partial U} |F(z)|^p\,|dz|<\infty$, then $F\in E^p(U)$. The proof is left to the reader.

Proof of Theorem 5.11. (1)$\Rightarrow$(2). Since $\int_{\partial U} \log\delta(z)|dz|>-\infty$, it follows that f extends to a function $\hat{f}$ which is analytic in X^0. Thus, in order to verify (2) we need only prove that $\hat{f}Q^{1/p}\in E^p(U)$.

We begin by constructing a family of Jordan curves $\{\gamma_c\}$, $0\leqq c\leqq\varepsilon$, in the following way: Let $\gamma(t)$, $0\leqq t\leqq 1$, be a parametric representation for ∂U such that $\gamma(t)\in C^{2+\alpha}$ and $\gamma'(t)$ is nowhere zero. For each $z\in\partial U$ let $n(z)$ be the unit outward pointing normal. As in [6, p. 183] select a field of unit vectors $N(z)$ which is C^∞ along ∂U and has the property $n(z)\cdot N(z)\geqq 1/2$. The notation here denotes the usual inner product and the bound on $n(z)\cdot N(z)$ implies that, at each $z\in\partial U$, the vectors $n(z)$ and $N(z)$ make an angle of not more than $\pi/6$ radians. Since the field N is Lipschitz and transverse along ∂U, the vectors $\varepsilon N(z)$, attached to ∂U at z, fill out a tubular neighborhood T_ε around ∂U in a one-to-one manner, provided ε is sufficiently small (cf. [57, Th. 1.5, p. 157]). Thus, the curves γ_c, $0\leqq c\leqq\varepsilon$, parameterized by

$$\gamma_c(t) = \gamma(t)+c\delta(\gamma(t))N(\gamma(t)), \quad 0\leqq t\leqq 1,$$

are simple closed Jordan curves lying in Ω. Since $\partial X\in C^{2+\alpha}$, it is a consequence of the implicit function theorem that $\delta(z)$ is a function of class $C^{2+\alpha}$ in $\Omega\cap W$ for some neighborhood W of ∂X. We may assume without loss of generality that $\partial U\subset W$. Each γ_c is then a curve of class $C^{2+\alpha}$.

For reasons that will become apparent in the next paragraph it is advantageous to express the γ_c's as the level sets of a single function u. This can best be done by considering the map $\xi: T_\varepsilon\to\partial U$ which associates to each $z\in T_\varepsilon$ its projection $\xi(z)$

onto ∂U along the unique vector $N(\xi(z))$ passing through z. It is easily checked that ξ satisfies a Lipschitz condition of order 1 in a neighborhood of ∂U. The function u is defined as follows:

$$u(z) = \frac{|z-\xi(z)|}{\delta(\xi(z))}.$$

Evidently, $u \in \mathrm{Lip}_1$ away from the zeros of δ and $\gamma_c = \{z\colon u(z)=c\}$. Also, for almost every c sufficiently small, $c \leqq \varepsilon$ say, $|\nabla u| \leqq K/\delta(z)$ along γ_c with K independent of c.

Suppose now that $f \in H^p(\Omega, dA)$. We must prove that $\tilde{f}Q^{1/p} \in E^p(U)$. By hypothesis, there exists a sequence of polynomials f_j, $j=1, 2, \ldots$, with

$$\int_\Omega |f_j - f|^p \, dA \to 0 \quad \text{as} \quad j \to \infty.$$

Let Q_c be an outer function in the region bounded by γ_c with $|Q_c(z)| = \delta(z)$ for all $z \in \gamma_c$. The construction is the same as for Q. If $T = \bigcup_{0 \leqq c \leqq \varepsilon} \gamma_c$ then according to the co-area formula (cf. [19, pp. 426—427] & [20])

$$\int_0^\varepsilon \Big(\int_{\gamma_c} |f_j - f|^p |Q_c| \, |dz| \Big) dc = \int_T |f_j - f|^p \delta(z) |\nabla u| \, dA \leqq \text{const} \cdot \int_\Omega |f_j - f|^p \, dA$$

and the latter tends to zero as $j \to \infty$. Hence, for almost every $c \leqq \varepsilon$,

$$\int_{\gamma_c} |f_j - f|^p |Q_c| \, |dz| \to 0 \quad \text{as} \quad j \to \infty.$$

We shall prove that for a sufficiently small fixed c

$$\int_{\partial U} |f_j - \tilde{f}|^p |Q| \, |dz| \overset{\text{(i)}}{\leqq} K_1 \int_{\partial U} |f_j - \tilde{f}|^p |Q_c| \, |dz| \overset{\text{(ii)}}{\leqq} K_2 \int_{\gamma_c} |f_j - f|^p |Q_c| \, |dz|, \tag{5.10}$$

where K_1 and K_2 are constants that do not depend on j or c. If we assume for the moment that this has been done it follows by an appropriate choice of c that $f_j Q^{1/p} \to \tilde{f} Q^{1/p}$ in $L^p(\partial U, |dz|)$. On the other hand, U is a Smirnov domain and $Q^{1/p} \in E^p(U)$ and so $Q^{1/p}$ can be approximated by a sequence of polynomials in the $L^p(\partial U, |dz|)$ norm (cf. [15, p. 173]). Consequently, $\tilde{f}Q^{1/p}$ is the limit of a sequence of polynomials in $L^p(\partial U, |dz|)$ and therefore $\tilde{f}Q^{1/p} \in E^p(U)$.

The proof of the implication (1)$\Rightarrow$(2) can now be completed by verifying the inequalities (i) and (ii) of (5.10). The first of these follows from Theorem 6.1. We assume that c is small enough so that γ_c lies entirely in T_ε. Thus, for each $x \in \partial U$ the vector $N(x)$ meets γ_c exactly once before leaving T_ε. Call that point of intersection x^*. Since $|Q_c| \in C^{2+\alpha}(\gamma_c)$, it follows from Theorem 6.1 via a conformal mapping that $Q_c \in \mathrm{Lip}_1$ with respect to the interior of γ_c. Here we use the fact that a conformal map of $|w|<1$ onto $\operatorname{int}(\gamma_c)$ is necessarily bi-Lipschitzian and has a $C^{2+\alpha}$ extension to $|w| \leqq 1$, since $\gamma_c \in C^{2+\alpha}$ (cf. [55, p. 73]). In any case, we have

$$|Q_c(x) - Q_c(x^*)| \leqq K|x - x^*|$$

for a suitable constant K. Moreover, K is easily seen to be independent of c. Thus, when c is sufficiently small we obtain the following:

$$|Q_c(x)| \geqq |Q_c(x^*)| - K|x-x^*| \geqq \delta(x^*) - 1/2\delta(x^*).$$

Therefore, $|Q_c(x)| \geqq \text{const}\, |Q(x)|$ if c is small and inequality (i) is clear.

Suppose now that c is fixed. The second inequality (ii) can be obtained from our remarks concerning Carleson measures. Because of the smoothness of γ_c and ∂U the problem can be transferred to the unit disk by mapping $\operatorname{int}(\gamma_c)$ onto $|w|<1$. Condition (5.9) can then be verified directly with v equal to arc length on the image of ∂U. There is no difficulty in making this transition, since the derivative of the mapping function is bounded away from 0 and ∞.

Proof of Theorem 5.11. (2)$\Rightarrow$(1). The proof will be given in two stages. In particular, we shall prove the following two assertions:

(iii) If $\tilde{f}Q^{\alpha/p} \in E^p(U)$ for some $\alpha<1$ then $f \in H^p(\Omega, dA)$;

(iv) If $\tilde{f}Q^{1/p} \in E^p(U)$ then f is the limit in $L^p(\Omega)$ of functions belonging to $L^p_a(\Omega, dA)$ and satisfying the assumptions of (iii).

For the purpose of this discussion it will again be assumed that ∂X and ∂U are of class $C^{2+\alpha}$. The distance from a point z to ∂U will be denoted by $\varrho(z)$. Since $\partial U \in C^{2+\alpha}$, the function $\varrho \in C^{2+\alpha}$ at points of Ω near ∂U. We assume without loss of generality that ∂X lies close enough to ∂U so that ϱ is class $C^{2+\alpha}$ along the entire length of ∂X.

Suppose now that $f \in L^p_a(\Omega, dA)$ and that $\tilde{f}Q^{1/p} \in E^p(U)$. Since

$$\int_{\partial U} \log \delta(z)|dz| > -\infty,$$

it is easy to see that

$$\int_{\partial X} \log \varrho(z)\,|dz| > -\infty.$$

It is possible, therefore, to define in X^0 an outer function G so that $|G(z)| = \varrho(z)$ on ∂X. This function has two important properties:

(a) $fG^{\varepsilon/p} \in L^p_a(\Omega, dA)$;

(b) $(\tilde{f}G^{\varepsilon/p})Q^{(1-\varepsilon)/p} \in E^p(U)$ for $\varepsilon < 1$.

The first is clear since G is bounded. The second can be obtained from Theorem 6.1 by means of conformal mapping. For each $z \in \partial U$ let z^* be the point where $n(z)$ first meets ∂X. By assumption, $|G| \in C^{2+\alpha}(\partial X)$ and therefore by Theorem 6.1,

$$|G(z)| \leqq K|z-z^*| + |G(z^*)| \leqq K|Q(z)|$$

for a suitable constant K. Since G/Q is an outer function in U, it follows from our introductory remarks that G/Q is bounded there. To obtain (b) we simply write

$$\tilde{f}G^{\varepsilon/p}Q^{(1-\varepsilon)/p}=\tilde{f}Q^{1/p}(G/Q)^{\varepsilon/p}$$

and use the fact that $\tilde{f}Q^{1/p}\in E^p(U)$.

Evidently, $fG^{\varepsilon/p}\to f$ pointwise almost everywhere as $\varepsilon\to 0$ and $|fG^{\varepsilon/p}|\leqq C|f|$. Hence, by the Lebesgue dominated convergence theorem

$$\int_\Omega |fG^{\varepsilon/p}-f|^p\,dA\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

If we assume (iii) then, of course, $fG^{\varepsilon/p}\in H^p(\Omega, dA)$ for all $\varepsilon<1$ and so $f\in H^p(\Omega, dA)$ as asserted.

Turning now to the proof of (iii) we suppose that $\tilde{f}Q^{\alpha/p}\in E^p(U)$ for some $\alpha<1$. Let $k\in L^q(\Omega)$ be a function which is orthogonal to the polynomials and put $\varphi=\hat{k}$. By Lemma 2.3, $\varphi\in \mathring{W}_1^q(X^0)$; that is, there exists a sequence $\varphi_n\in C^\infty$, each of which is compactly supported in X^0, and so that

$$\int |\nabla\varphi_n-\nabla\varphi|^q\,dA\to 0 \quad \text{as} \quad n\to\infty. \tag{5.11}$$

If we can prove that $\int_\Omega f\dfrac{\partial\varphi}{\partial\bar{z}}\,dA=0$ then (iii) will follow, since $\dfrac{\partial\varphi}{\partial\bar{z}}=-\pi k$.

In order to accomplish this we utilize Green's theorem in the following way:

$$\int_\Omega f\frac{\partial\varphi_n}{\partial\bar{z}}\,dA=\frac{1}{2i}\int_{\partial U}\tilde{f}\varphi_n\,dz. \tag{5.12}$$

Since the φ_n's were chosen so as to satisfy (5.11),

$$\int_\Omega f\frac{\partial\varphi_n}{\partial\bar{z}}\,dA\to\int_\Omega f\frac{\partial\varphi}{\partial\bar{z}}\,dA$$

as $n\to\infty$. Our goal is to prove that

$$\frac{1}{2i}\int_{\partial U}\tilde{f}\varphi_n\,dz\to\frac{1}{2i}\int_{\partial U}\tilde{f}\varphi\,dz$$

as $n\to\infty$. Here is where we need to assume that $\tilde{f}Q^{\alpha/p}\in E^p(U)$ for some $\alpha<1$. By means of Hölder's inequality we have

$$\left|\int_{\partial U}\tilde{f}\varphi_n\,dz-\int_{\partial U}\tilde{f}\varphi\,dz\right|\leqq\left(\int_{\partial U}|\tilde{f}\delta^{\alpha/p}|^p\,|dz|\right)^{1/p}\left(\int_{\partial U}\frac{|\varphi_n-\varphi|^q}{\delta(z)^{\alpha q/p}}\,|dz|\right)^{1/q}.$$

In view of the aforementioned hypothesis, the first integral on the right is finite. To see that the second approaches zero as $n\to\infty$ we assume that the φ_n's have been selected as in [33, Lemma 4]. Although φ_n will no longer be smooth, it is quasi-continuous and Green's formula (5.12) is still valid, since the mollifiers of φ_n converge pointwise and boundedly quasi-everywhere back to φ_n (cf. [1, p. 262]).

The construction in [33] also ensures that $|\varphi_n| \leqq |\varphi|$ and hence

$$\frac{|\varphi_n - \varphi|^q}{\delta(z)^{\alpha q/p}} \leqq 2 \frac{|\varphi|^q}{\delta(z)^{\alpha q/p}}, \quad n = 1, 2, \ldots.$$

Furthermore, since $\alpha < 1$, the function on the right belongs to $L^1(\partial U, |dz|)$; this follows from the argument in [6, pp. 185—187]. Now, $\varphi_n \to \varphi$ pointwise quasi-everywhere relative to Γ_q and therefore (cf. [27, Th. 7.1]) almost everywhere on ∂U with respect to $|dz|$. As a result, the dominated convergence theorem implies that

$$\int_{\partial U} \frac{|\varphi_n - \varphi|^q}{\delta(z)^{\alpha q/p}} |dz| \to 0 \quad \text{as} \quad n \to \infty$$

and consequently

$$\int_{\Omega} f \frac{\partial \varphi}{\partial \bar{z}} dA = \frac{1}{2i} \int_{\partial U} \tilde{f} \varphi dz.$$

The demonstration of Theorem 5.11 can now be concluded by proving that $\int_{\partial U} \tilde{f}\varphi dz = 0$. Thus, by a theorem of Smirnov (cf. [15, p. 170]), it suffices to show that $\tilde{f}\varphi \in E^1(U)$. This is most easily accomplished by expressing $\tilde{f}\varphi$ in the form

$$\tilde{f}\varphi = \tilde{f} Q^{\alpha/p} \frac{\varphi}{Q^{\alpha/p}}.$$

By hypothesis, $\tilde{f}Q^{\alpha/p} \in E^p(U)$ and, as we have seen, $\varphi \in E^q(U)$. Since $Q^{-\alpha/p}$ is outer, the product $\tilde{f}\varphi$ belongs to a class N^+ which is defined in an obvious way through conformal mapping (cf. [15, pp. 25—28]). Inasmuch as $\int_{\partial U} |\tilde{f}\varphi|\,|dz| < \infty$ by our earlier remarks, $\tilde{f}\varphi \in E^1(U)$ (cf. [15, p. 28]).

We have shown that $\int_{\Omega} f \frac{\partial \varphi}{\partial \bar{z}} dA = 0$ and therefore $f \in H^p(\Omega, dA)$. Q.E.D.

The following is an offshoot of the preceding argument and generalizes a theorem of Havin [23, Th. 2]. Since we are considering regions with a large number (possibly infinitely many) multiple boundary points we must avoid local hypotheses such as condition (28) of Havin's paper.

Corollary 5.12. *Let* $\Omega = (X \setminus U)^0$ *be as in Theorem* 5.11 *and put* $\delta(z) = \operatorname{dist}(z, \partial X)$. *If* F *is analytic in* X^0 *and* $FQ^{1/p} \in E^p(U)$ *then*

$$\int_{\partial U} |F|^p \delta(z) |dz| \leqq K \int_{\Omega} |F|^p dA$$

for some constant K *not depending on* F.

Proof. If $\int_{\Omega} |F|^p dA = \infty$ there is nothing to prove. If, however, this integral is finite then $F \in H^p(\Omega, dA)$ and inequality (5.10) holds with K_1 and K_2 independent of $c \leqq \varepsilon$. If we integrate both sides of this inequality from 0 to ε and apply the co-area formula we obtain the desired conclusion. Q.E.D.

6. Functions whose boundary values have a smooth modulus

Let D denote the open unit disk. Suppose that F is analytic in D and continuous on $\bar{D}$. In the preceding section (cf. Theorem 5.11) we encountered the following problem: To what extent is the behavior of F on $\bar{D}$ influenced by the "smoothness" of $|F|$ on ∂D? The purpose of this section is to describe a result which the author became aware of in a conversation with Lennart Carleson and which provides an answer to this question in case F is an outer function. We recall here that F is outer in D if it can be written in the form

$$F(z) = \lambda \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|F(\theta)|\, d\theta\right\},$$

where λ is a constant with $|\lambda|=1$. It can be assumed here that $\lambda=1$.

For economy in notation we shall make use of the following terminology: A function k (defined on either $\bar{D}$ or ∂D) will be said to belong to the class Λ_α (i.e. $\Lambda_\alpha(\bar{D})$ or $\Lambda_\alpha(\partial D)$, respectively) if

(i) $k \in \mathrm{Lip}_\alpha$ in case $0<\alpha\leqq 1$;

(ii) $k^{[\alpha]} \in \mathrm{Lip}_{\alpha-[\alpha]}$ in case $\alpha>1$ and $[\alpha]<\alpha$.

Here $[\alpha]$ denotes the greatest integer $\leqq\alpha$ and $k^{[\alpha]}$ is the $[\alpha]$-th derivative of k. Note that Λ_1 now denotes the class Lip_1 and not Hausdorff measure as in previous sections.

Theorem 6.1. *Let F be analytic in the open unit disk D and continuous on $\bar{D}$. If F is an outer function and if $|F| \in \Lambda_\alpha(\partial D)$ then*

(1) $F \in \Lambda_{\alpha/2}(\bar{D})$ *if* $0<\alpha<2$;

(2) $F \in \Lambda_1(\bar{D})$ *if* $\alpha>2$.

Remark 1. In case $\alpha>2$ one can actually conclude more than is asserted in Theorem 6.1. However, since we are primarily interested in knowing when $F \in \Lambda_1(\bar{D})$, we have chosen to omit further discussion of this point.

Remark 2. The Lipschitz constant associated to F on $\bar{D}$ depends only on the Lipschitz constants and bounds for the derivatives of $|F|$ on ∂D. This fact was used implicitly in the proof of Theorem 5.11.

Suppose for the moment that $0<\alpha<1$. If we assume that $|F| \in \Lambda_\alpha(\partial D)$ and that F does not vanish on $\bar{D}$ it follows directly from the Privalov—Zygmund theorem [58, p. 121] on the modulus of continuity of the conjugate function that $F \in \Lambda_\alpha(\bar{D})$. However, if F is allowed to have zeros on ∂D then $F \in \Lambda_{\alpha/2}(\bar{D})$ and the exponent $\alpha/2$ is, in general, best possible. This result was first obtained by Jacobs in his thesis [35], unpublished. Later, Havin [24] rediscovered and strengthened it to include functions which vanish in D provided their zeros do not accumulate tangentially at any boundary point. We, of course, are interested in the result for outer functions, but with $\alpha > 1$.

The proof of Theorem 6.1 is patterned after the proof of a similar theorem in [12, p. 223]. As in that paper, we shall work principally in the upper half plane rather than in the disk. By transferring the problem in this way we do not have to distinguish between derivatives with respect to arc length and derivatives with respect to z when studying the behavior of F on the boundary. We shall need the following simple lemma on Cauchy integrals (cf. [12, p. 224]):

Lemma 6.2. *Suppose that* $|k(\pm t)| \leqq \omega(t)$ *a.e. and that* $\int_0^\delta \frac{\omega(t)}{t} dt < \infty$. *If* $\chi(z) = \int_{-\delta}^{\delta} \frac{k(t)}{t-z} dt$, *then for* $z \in S_0(\delta) = \{z = x+iy\colon |x| < y,\ |z| < \delta\}$,

$$|\chi(z) - \chi(0)| \leqq C \int_0^{|z|} \frac{\omega(t)}{t} dt + C|z| \int_{|z|}^{\delta} \frac{\omega(t)}{t^2} dt.$$

If instead $|k(\pm t)| \leqq |t|^n \omega(t)$, *then*

$$|\chi^{(n)}(z) - \chi^{(n)}(0)| \leqq C^n n! \int_0^{|z|} \frac{\omega(t)}{t} dt + C^n n!\, |z| \int_{|z|}^{\delta} \frac{\omega(t)}{t^2} dt$$

for $z \in S_0(\delta)$.

Proof of Theorem 6.1. Suppose that $|F| \in \Lambda_\alpha(\partial D)$. We shall prove that $F \in \Lambda_{\alpha/2}(\partial D)$ if $1 < \alpha < 2$ and $F \in \Lambda_1(\partial D)$ if $\alpha > 2$. Having done so, we can then conclude that F belongs to the same Lipschitz class on $\bar{D}$ as on ∂D. If, for example, $F \in \Lambda_1(\partial D)$ this follows easily from a well known theorem of Privalov (cf. [15, p. 42]). If $F \in \Lambda_{\alpha/2}(\partial D)$ with $\alpha < 2$ we can argue as in [35, p. 31]. In this case for each fixed $\zeta \in \partial D$ the function $(z-\zeta)^{-\alpha/2} \in H^1(\partial D, d\theta)$ and so

$$F_1(z) = \frac{F(z) - F(\zeta)}{(z-\zeta)^{\alpha/2}}$$

also belongs to $H^1(\partial D, d\theta)$. Since $F \in \Lambda_{\alpha/2}(\partial D)$, there is a constant M which is independent of ζ and for which $|F_1| \leqq M$ almost everywhere on ∂D. Thus, $|F_1| \leqq M$ everywhere in the interior of D and it follows that

$$|F(z) - F(\zeta)| \leqq M |z-\zeta|^{\alpha/2} \tag{6.1}$$

for all $z \in \bar{D}$ and $\zeta \in \partial D$. If we now take $\zeta \in D$ the function

$$u(z) = \frac{|F(z) - F(\zeta)|}{|z-\zeta|^{\alpha/2}}$$

is subharmonic and has a removable singularity at ζ. By (6.1), $|u(z)| \leqq M$ for all $z \in \partial D$ and so by the maximum principle $|u| \leqq M$ everywhere on $\bar{D}$. Thus, we have shown that

$$|F(z)-F(\zeta)| \leqq M|z-\zeta|^{\alpha/2}$$

for all $z, \zeta \in \bar{D}$, i.e. $F \in \Lambda_{\alpha/2}(\bar{D})$.

Suppose now that $1<\alpha<2$ and that $|F| \in \Lambda_\alpha(\partial D)$. Fix an arbitrary point $\tau \in \partial D$. If $F(\tau)=0$ then clearly

$$|F(\sigma)-F(\tau)| = \big||F(\sigma)|-|F(\tau)|\big| \leqq K|\sigma-\tau|$$

for all $\sigma \in \partial D$. If $|F(\tau)|=m>0$ we shall prove that there are positive constants K and C which do not depend on m and which have the property that

$$|F(\sigma)-F(\tau)| \leqq K|\sigma-\tau|^{\alpha/2} \tag{6.2}$$

whenever $|\sigma-\tau| \leqq Cm^{2/\alpha}$. Since

$$|F(\sigma)-F(\tau)| \leqq \big||F(\sigma)|-|F(\tau)|\big|+2m \leqq K'|\sigma-\tau|^{\alpha/2}$$

if $|\sigma-\tau| \geqq Cm^{2/\alpha}$, it can then be inferred that $F \in \Lambda_{\alpha/2}(\partial D)$.

In order to establish (6.2) we may assume that $\tau=-1$ (otherwise, we could replace $F(z)$ by $F(\varrho z)$ for a suitable constant ϱ to obtain a function with the desired property). The function $f(z)=F\left(\dfrac{z-i}{z+i}\right)$ is therefore outer in the upper half plane, $|f|$ belongs to Λ_α on the boundary and $|f(0)|=m$. We shall see that

$$|f(x)-f(0)| \leqq K|x|^{\alpha/2} \tag{6.3}$$

for all real x with $|x|<Cm^{2/\alpha}$. This is evidently equivalent to (6.2).

We begin by modifying f in the following way: Let $g(t)=|f(t)|$ and put $a=g'(0)$, differentiation being along the real axis. By assumption

$$\big||f(t)|-m-at\big| \leqq K|t|^\alpha$$

and consequently

$$\left|\log\left|\frac{|f(t)|-at}{m}\right|\right| \leqq \frac{K}{m}|t|^\alpha$$

if $|t| \leqq Cm^{1/\alpha}$; the constants K and C are independent of m. With this C let

$$k(t) = \begin{cases} |f(t)|-at, & |t| \leqq Cm^{1/\alpha} \\ |f(t)|, & |t| \geqq Cm^{1/\alpha} \end{cases}$$

and define a new outer function $f_0(z)$ by setting

$$\log f_0(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z} \log|k(t)| \frac{dt}{1+t^2}.$$

Verification of assertion (6.3) can now be accomplished by proving that

(i) $|f_0(z)-f_0(0)| \leqq K|z|^{\alpha/2}$

(ii) $\left|\frac{f}{f_0}(z)-\frac{f}{f_0}(0)\right| \leqq \frac{K}{m}|z|^{\alpha/2}$

for every z in the sector $S_0(\eta)$, where $\eta = Cm^{2/\alpha}$. From these two facts it follows that

$$|f(z)-f(0)| \leqq |f_0(z)|\left|\frac{f}{f_0}(z)-\frac{f}{f_0}(0)\right|+\left|\frac{f}{f_0}(0)\right||f_0(z)-f_0(0)|$$

$$\leqq K\left(\frac{|f_0(z)|}{m}+1\right)|z|^{\alpha/2} \leqq K'|z|^{\alpha/2}$$

provided $z \in S_0(\eta)$. Likewise, a similar estimate is valid in each sector $S_x(\eta)$ if $|x| < \frac{C}{2}m^{2/\alpha}$. An appropriate choice of z^* in $S_x(\eta) \cap S_0(\eta)$ then yields the inequality $|z^*-x|^{\alpha/2}+|z^*|^{\alpha/2} \leqq 2|x|^{\alpha/2}$ and we see that

$$|f(x)-f(0)| \leqq K|x|^{\alpha/2}$$

for $|x| < \frac{C}{2}m^{2/\alpha}$ with K and C independent of m.

As soon as we have established inequalities (i) and (ii) part (1) of the theorem will follow. We prefer to carry out the proof of (ii), since it is the more difficult of the two. Here we can actually obtain a better estimate than the one stated. This is not true with regard to (i), however.

With $\delta = Cm$ write

$$\log\frac{f}{f_0}(z) = \frac{1}{\pi i}\left(\int_{-\delta}^{\delta}+\int_{|t|\geqq\delta}\right)\frac{1+tz}{t-z}\log\left|\frac{f}{f_0}(t)\right|\frac{dt}{1+t^2} = \chi(z)+\lambda(z), \quad \text{respectively.}$$

Since $||f(t)|-|f_0(t)|| \leqq |at|$ for all real t, there is a constant C with the property that

$$\left|\log\left|\frac{f}{f_0}(t)\right|\right| \leqq \frac{K}{m}|t|$$

provided $|t| < Cm$. Hence, by Lemma 6.2

$$|\chi(z)-\chi(0)| \leqq \frac{K'}{m}|z|\,|\log|z|| \tag{6.4}$$

if $z \in S_0(\eta)$.

To obtain a similar estimate for $|\lambda(z)-\lambda(0)|$ we consider the following two possibilities:

(iii) $|f(t)| \geqq m/2$ whenever $|t| < Cm^{1/\alpha}$;

(iv) $|f(t_0)| < m/2$ for some $t_0, |t_0| < Cm^{1/\alpha}$.

In the first case, since $|f_0(t)| \geqq Am$ if $|t| \leqq Cm^{1/\alpha}$, we have

$$|\lambda(z)-\lambda(0)| \leqq K|z| \int_{Cm}^{Cm^{1/\alpha}} \frac{|\log m|}{t^2}\, dt \leqq \frac{K}{m} |\log m|\,|z| \leqq \frac{K}{m} |z|\,\big|\log |z|\big| \tag{6.5}$$

for every $z \in S_0(\eta)$. Thus, we may suppose that $|f(t_0)| < m/2$ with $|t_0| < Cm^{1/\alpha}$ and this enables us to estimate the size of the derivative of $g=|f|$ at the origin. Either $|f(t)|$ is always $<m/2$ when $|t_0|<|t|<Cm^{1/\alpha}$ or it is not. In both cases, however, there is a point t^* such that $|t_0|<|t^*|<Cm^{1/\alpha}$ and so that $|g'(t^*)| \leqq Cm^{(\alpha-1)/\alpha}$. Inasmuch as

$$|g'(0)-g'(t^*)| \leqq C\,|t^*|^{\alpha-1} \leqq Cm^{(\alpha-1)/\alpha},$$

we conclude that $|a|=|g'(0)| \leqq Cm^{(\alpha-1)/\alpha}$. Therefore, if $|t|<Cm^{1/\alpha}$ we have

$$\left|\frac{f}{|f_0|}(t)-1\right| = \frac{|a|}{|f_0(t)|}\,|t| \leqq \frac{C}{m^{1/\alpha}}\,|t|$$

and it follows that

$$\left|\log\left|\frac{f}{f_0}(t)\right|\right| \leqq \frac{C}{m^{1/\alpha}}\,|t|$$

for essentially the same values of t. Consequently, by Lemma 6.2.

$$|\lambda(z)-\lambda(0)| \leqq \frac{K}{m^{1/\alpha}}\,|z| \int_{Cm}^{Cm^{1/\alpha}} \frac{1}{t}\, dt \leqq \frac{K}{m^{1/\alpha}} |\log m|\,|z| \leqq \frac{K}{m^{1/\alpha}} |z|\,\big|\log |z|\big| \tag{6.6}$$

if $z \in S_0(\eta)$.

From the inequalities (6.4), (6.5) and (6.6) we readily conclude that

$$\left|\log\frac{f}{f_0}(z)-\log\frac{f}{f_0}(0)\right| \leqq \frac{K}{m}\,|z|\,\big|\log |z|\big|$$

whenever $z \in S_0(\eta)$. Thus,

$$\left|\frac{f}{f_0}(z)-\frac{f}{f_0}(0)\right| \leqq \frac{K}{m}\,|z|\,\big|\log |z|\big| \leqq \frac{K'}{m}\,|z|^{\alpha/2}$$

for every $z \in S_0(\eta)$ and so we have established assertion (1) of the theorem.

Suppose now that $|F| \in \Lambda_\alpha(\partial D)$ for some $\alpha>2$. In order to prove that $F \in \Lambda_1(\partial D)$ we can proceed as before. In particular, it is sufficient to prove that if

$|F(\tau)|=m>0$ and if f is the corresponding outer function in the upper half plane with $|f(0)|=m$, then

$$|f(x)-f(0)| \leqq K|x|$$

for all real x with $|x|<Cm$. On the other hand, this will follow if we can show that $|f'|\leqq K$ on some open rectangle $R=\{x+iy\colon |x|<Cm, 0<y<\varepsilon\}$. For, in this case, if $|x|<Cm$ we can choose an arc $\gamma\in R$ joining 0 to x and having length $\leqq 2|x|$ and thereby obtain

$$|f(x)-f(0)| \leqq \int_\gamma |f'(z)|\,|dz| \leqq 2K|x|.$$

Let $g=|f|$. Put $a=g'(0)$, $b=g''(0)/2$ and form the polynomial $Q(z)=$ $=m+az+bz^2$. If $B(z)$ is the Blaschke product associated to the zeros of Q in the upper half plane, then

$$\log\frac{Q}{B}(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+tz}{t-z}\log|Q(t)|\frac{dt}{1+t^2}$$

because Q has no singular part. Consequently,

$$\frac{f'(z)}{f(z)}-\frac{Q'(z)}{Q(z)}+\frac{B'(z)}{B(z)} = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{(t-z)^2}\log\left|\frac{f}{Q}(t)\right|dt.$$

We shall prove that $|f'(z)/f(z)|\leqq K/m$ in some rectangle R by estimating all remaining terms in this equation.

Clearly, $|Q'(z)/Q(z)|\leqq K/m$. Also, since $B(z)\neq 0$ if $|z|<Cm$, we see that $|B'(z)/B(z)|\leqq K/m$ if $|z|<\dfrac{C}{2}m$. To obtain a bound for the integral we proceed as follows: with $\delta=Cm^{1/2}$ and with C to be specified later we write

$$\int_{-\infty}^{\infty}\frac{1}{(t-z)^2}\log\left|\frac{f}{Q}(t)\right|dt = \int_{-\delta}^{\delta}\frac{1}{(t-z)^2}\log\left|\frac{f}{Q}(t)\right|dt+\lambda(z) = \chi(z)+\lambda(z). \tag{6.7}$$

The function $\lambda(z)$ is analytic and bounded in a disk $|z|<Cm^{1/2}$ and so $|\lambda(z)|\leqq$ $\leqq K/m$. Furthermore, K can be shown to be independent of m.

Since $|f|\in\Lambda_\alpha$ on the boundary,

$$\big||f(t)|-Q(t)\big| \leqq K|t|^\alpha.$$

If $|f(t)|\geqq m/2$ for every t satisfying $|t|<Cm^{1/\alpha}$ this yields

$$\left|\frac{Q(t)}{|f(t)|}-1\right| \leqq \frac{K}{|f(t)|}|t|^\alpha \leqq \frac{K}{m}|t|^\alpha \tag{6.8}$$

provided $|t|<C'm^{1/\alpha}$. Conversely, if $|f(t_0)|<m/2$ for some point t_0 with $|t_0|<$ $<Cm^{1/\alpha}$, then

$$\left|\frac{|f(t)|}{Q(t)}-1\right| \leqq \frac{K}{|Q(t)|}|t|^\alpha \leqq \frac{K}{m}|t|^\alpha \tag{6.9}$$

if $|t|<Cm^{1/2}$, since $|a|\leqq C'm^{(\alpha-1)/\alpha}$, $|b|\leqq C''m^{(\alpha-2)/2\alpha}$ and $\alpha>2$. If we choose C in (6.7) so that (6.8) and (6.9) are satisfied for $|t|<Cm^{1/2}$ we have in both cases

$$\left|\log\left|\frac{f}{Q}(t)\right|\right| \leqq \frac{K}{m}|t|^\alpha,$$

whenever $|t|<Cm^{1/2}$. It follows that $|\chi(z)|\leqq K/m$ if $z\in S_0(\eta)$ with $\eta=Cm$. Similarly, the same estimate holds in a union of sectors $S_x(\eta)$, $|x|<C'm$, and hence in some rectangle R.

In summary, we have shown: $|f'/f|\leqq K/m$ and so $|f'|\leqq K$ in R and that completes the proof of the theorem. Q.E.D.

Remark. Suppose that F is an outer function in D which is continuous on $\bar{D}$. Let $\omega(\delta)$ denote the modulus of continuity of F on ∂D. If $|F|\in C^2(\partial D)$ the preceding argument shows that $\omega(\delta)=O(\delta\log 1/\delta)$, but it will not give $\omega(\delta)=$ $=O(\delta)$.

7. Approximation by harmonic polynomials

Many of the questions that occupied our attention in previous discussions can easily be rephrased in terms of harmonic polynomials. Let us assume, for instance, that Ω is an arbitrary plane domain. For each p, $1\leqq p<\infty$, the spaces $H^p_\Delta(\Omega, dA)$ and $L^p_\Delta(\Omega, dA)$ are defined as follows: $H^p_\Delta(\Omega, dA)$ consists of those functions that can be approximated in the $L^p(\Omega, dA)$ norm by a sequence of harmonic polynomials and $L^p_\Delta(\Omega, dA)$ denotes the set of all functions in $L^p(\Omega, dA)$ that are harmonic in Ω. By the area mean value theorem, $H^p_\Delta(\Omega, dA)\subset L^p_\Delta(\Omega, dA)$ and so one is led to ask: For which domains Ω is $H^p_\Delta(\Omega, dA)=L^p_\Delta(\Omega, dA)$? Whenever this occurs the harmonic polynomials are said to be complete in $L^p_\Delta(\Omega, dA)$.

The first results which the author is aware of in connection with the completeness problem for harmonic polynomials were obtained by Šaginjan in 1954 (Akad. Nauk Armjan. SSR Dokl. 19, 97—103). He proved that every bounded harmonic function on Ω belongs to $H^p_\Delta(\Omega, dA)$ if Ω satisfies either of the following two conditions:

(i) Ω is a Carathéodory domain;
(ii) Ω is a "crescent" and $H^p(\Omega, dA)=L^p_a(\Omega, dA)$.

There the matter stood until, in 1966, Sinanjan [51] succeeded in proving that

$H_\Delta^p(\Omega, dA)=L_\Delta^p(\Omega, dA)$ for every Carathéodory domain Ω. We shall now state two theorems which follow easily from our earlier results and which together suggest that for crescents there is a close relationship between the completeness of the analytic and harmonic polynomials.

Theorem 7.1. *Let Ω be a crescent. If $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ then $H_\Delta^p(\Omega, dA)=$ $=L_\Delta^p(\Omega, dA)$.*

Theorem 7.2. *Let $\Omega=(X\setminus U)^0$ be a crescent. Assume that ∂U is class C^1 and that $n(z)$, the unit exterior normal to ∂U at z, satisfies a Lipschitz condition $|n(z_1)-n(z_2)|\leqq C|z_1-z_2|$. Let $\delta(z)=\operatorname{dist}(z, \partial\Omega_\infty)$. In order that $H_\Delta^p(\Omega, dA)=$ $=L_\Delta^p(\Omega, dA)$, for any p, it is necessary and sufficient that*

$$\int_{\partial U} \log \delta(z)\,|dz| = -\infty.$$

Remark. If $\Omega=(X\setminus U)^0$ with ∂U sufficiently smooth it follows from Theorem 7.2 that $H_\Delta^p(\Omega, dA)=L_\Delta^p(\Omega, dA)$ if and only if $H^p(\Omega, dA)=L_a^p(\Omega, dA)$. At the present time we do not have an example of a crescent for which the first of these holds, but for which the second fails. In connection with this cf. [63] & [64, p. 157].

Proof of Theorem 7.1. Suppose that $H^p(\Omega, dA)=L_a^p(\Omega, dA)$ and let h be any function in $L^q(\Omega, dA)$ with the property that $\int QhdA=0$ for every harmonic polynomial Q. Thus, the potential

$$H(z) = \int_\Omega \log\frac{1}{|\xi-z|}\,h(\xi)\,dA_\xi$$

is identically zero in Ω_∞. In order to prove that $H_\Delta^p(\Omega, dA)=L_\Delta^p(\Omega, dA)$ it is sufficient to show that H belongs to the Sobolev space $\mathring{W}_2^q(\Omega)$ and, since $\mathbf{C}\setminus\bar{\Omega}$ has only two components, we need only verify that $H\equiv 0$ in the bounded component U. These assertions follow from results of Hedberg and Polking (cf. [34, Thms. 3 & 4] & [46, Thm. 2.9]).

As a distribution $\partial H/\partial z=-\hat{h}/2$, where $\hat{h}$ is the Cauchy transform of h. Evidently, then, $\hat{h}\equiv 0$ in Ω_∞ and, because $H^p(\Omega, dA)=L_a^p(\Omega, dA)$, it follows that $\partial H/\partial z=-\hat{h}/2\equiv 0$ in U. Similarly, $\partial H/\partial\bar{z}\equiv 0$ there, since $\bar{h}$, as well as h, is orthogonal to the harmonic polynomials. This implies that H is constant in U. On the other hand, H is known to be continuous and so $H(z)\to 0$ as z approaches a multiple boundary point from within U. Therefore, $H\equiv 0$ in U and $H_\Delta^p(\Omega, dA)=L_\Delta^p(\Omega, dA)$. Q.E.D.

Proof of Theorem 7.2. The proof here is analogous to that of the corresponding result in the analytic case—Theorem 5.8. We first construct a Jordan curve γ lying in Ω and surrounding the bounded complementary component U. This is done in such a way that there is a fixed constant $\varepsilon>0$ and so that for each $z_0\in\gamma$ the disk with center at z_0 and radius $\varepsilon\delta(z_0)$ is contained in Ω. Thus, if F is harmonic in Ω it follows from the Poisson integral formula that

$$\left|\frac{\partial F}{\partial z}(z_0)\right| \leqq \left(\frac{C}{\delta(z_0)}\right)^{(2/p)+1} \|F\|_{L^p(\Omega, dA)}$$

(cf. [26, Lemma 6, p. 103]). Assuming that $\int_{\partial U} \log \delta(z)\,|dz| > -\infty$, there is a function h which is analytic and nowhere zero inside γ and so that $|h(z)| = \delta(z)^{(2/p)+1}$ for every $z \in \gamma$. Hence, if $\{Q_j\}_{j=1}^{\infty}$ is a sequence of harmonic polynomials and $Q_j \to F$ in $L^p(\Omega, dA)$, then

$$\sup_{\gamma}\left|h\frac{\partial Q_j}{\partial z}\right| \leqq K, \quad j = 1, 2, \ldots.$$

Consequently, $\left\{h\dfrac{\partial Q_j}{\partial z}\right\}_{j=1}^{\infty}$ is a uniformly bounded sequence of analytic functions inside γ and so in that region some subsequence converges uniformly on compact subsets. This implies that $\partial F/\partial z$ extends analytically to U. In particular, F cannot be one of the functions $\log|z-a|$, $a \in U$. Therefore, $H^p_\Delta(\Omega, dA) \neq L^p_\Delta(\Omega, dA)$ if $\int_{\partial U} \log \delta(z)\,|dz| > -\infty$.

If we suppose that $\int_{\partial U} \log \delta(z)\,|dz| = -\infty$ then $H^p(\Omega, dA) = L^p_a(\Omega, dA)$ and $H^p_\Delta(\Omega, dA) = L^p_\Delta(\Omega, dA)$ by Theorem 7.1. Q.E.D.

With regard to the harmonic polynomials questions concerning approximation in the plane have natural analogues in higher dimensions. These often lead, however, to more serious difficulties than those encountered in the planar case and, consequently, very little has been published in this area.

By a crescent in $\mathbf{R}^n$ we shall mean a domain topologically equivalent to the region bounded by two internally tangent balls. Lebesgue measure will be denoted by dV. In keeping with previous notation $L^p_\Delta(\Omega, dV)$ will stand for the set of functions in $L^p(\Omega, dV)$ which are harmonic in Ω and $H^p_\Delta(\Omega, dV)$ will designate the closure of the harmonic polynomials in the $L^p(\Omega, dV)$ norm. The following was announced in [26]:

Theorem 7.3. (Havin & Maz'ja). *Let Ω be a crescent in $\mathbf{R}^n$ with its multiple boundary point at the origin. Assume, moreover, that U, the bounded component complementary to Ω, lies entirely in the half space $\mathbf{R}^n_+ = \{x = (x_1, \ldots, x_n) : x_n > 0\}$ and that $(|x| < \varrho) \cap U = (|x| < \varrho) \cap \mathbf{R}^n_+$ for every sufficiently small but positive ϱ. If the interior and exterior boundaries of Ω are class C^2 surfaces and if there exists a function $W(\varrho)$ such that*

(1) $\varrho \dfrac{W'(\varrho)}{W(\varrho)} \uparrow +\infty$ *as* $\varrho \downarrow 0$;

(2) meas $\{\Omega \cap (|x| < \varrho)\} \leqq W(\varrho)$;

(3) $\int_0 \log W(\varrho)\, d\varrho = -\infty$

then $H^p_\Delta(\Omega, dV) = L^p_\Delta(\Omega, dV)$ for every $p > 1$.

The requirement $(|x|<\varrho)\cap U=(|x|<\varrho)\cap \mathbf{R}^n_+$ simply means that ∂U is flat near the multiple boundary point. One wonders, therefore, to what extent this rather severe topological restriction can be relaxed without destroying compleneteness. This seems to be a very delicate question and it is still far from being settled. However, if (3) is replaced by the stronger assumption $\int_0 \log\log \frac{1}{W(\varrho)}\, d\varrho = +\infty$ then completeness will occur whenever there is a two sided cone with vertex at the multiple boundary point and which, otherwise, lies entirely in $\mathbf{C}\diagdown\bar{\Omega}$. The proof is based on ideas introduced in Sections 3 & 5 and will be omitted. We mention the result only to establish a connection between our work and the more general problem.

The question of approximation on crescents by harmonic polynomials is closely related to an older more important uniqueness problem. The connection arises in the following manner: Suppose that Ω is a crescent in $\mathbf{R}^n$ and fix $p\geqq 1$. Let $k\in L^q(\Omega, dV)$ be any function with the property that $\int_\Omega QkdV=0$ for every harmonic polynomial Q and form the appropriate potential U^k (i.e. the logarithmic potential in the plane and the Newtonian potential in higher dimensions). Since k is orthogonal to the harmonic polynomials, U^k vanishes identically in Ω_∞ (cf. [14, p. 105]). In order to prove that $H^p_\Delta(\Omega, dV)=L^p_\Delta(\Omega, dV)$ one must first verify that $U^k\equiv 0$ in the bounded complementary component U. If we assume that $\partial\Omega$ is smooth, $q>n/2$ and $\int_{\partial U}\log\delta(z)\,dS=-\infty$, where dS denotes surface area, then

(a) $U^k(x)\to 0$ as x approaches any multiple boundary point from inside U;

(b) $\int_{\partial U}\log|\nabla U^k|\,dS=-\infty$.

One is therefore led to ask whether or not (b) is sufficient to imply that U^k is constant in U. The results of Havin and Maz'ja, [26], are based on a partial answer to this question, the details of which recently appeared in [29]. In this connection the reader should also consult the works of Mergeljan [43] and Rao [47].

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