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# Polynomials on stable spaces

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Let $E$ be a locally convex space. The space $\mathcal{L}(^nE)$ of continuous $n$-linear forms and the space $\mathcal{L}_s(^nE)$ of symmetric $n$-linear forms (which is canonically isomorphic, by the polarization formula, to the space of continuous $n$-homogeneous polynomials on $E$) have been widely used and considered in several branches of functional analysis, in particular in infinite dimensional holomorphy. Our main result leads to the possibility of the immediate application of results from the well developed theory of tensor products and multilinear mappings to the theory of polynomials on locally convex spaces. For example, the results in [6] can be applied to recover easily special cases of results in [5] and at the same time shows (see remark (ii) below) that not all results in [5] follow from [6]. We refer the reader to [7], [9], [11] for further details on these topics.

A locally convex space is said to be *stable* if it is topologically isomorphic to its cartesian square. This note is devoted to proving the following results.

**Theorem 1.** *Let $E$ denote a stable locally convex space. For each $n$ the space $\mathcal{L}(^nE)$ is algebraically isomorphic to $\mathcal{L}_s(^nE)$.*

**Theorem 2.** *Let $E$ denote a stable locally convex space. For each $n$, the spaces $(\mathcal{L}(^nE),\tau_1)$ and $(\mathcal{P}(^nE),\tau_2)$ are topologically isomorphic in the following cases:*

(i) *$\tau_1$ and $\tau_2$ are the compact open topologies,*

(ii) *$\tau_1$ and $\tau_2$ are the topologies of uniform convergence on bounded sets,*

(iii) *$\tau_1$ is the inductive dual topology on $\mathcal{L}(^nE)$ arising from the predual $\widehat{\otimes}_{n,\pi}E$ and $\tau_2$ is the $\tau_\omega$ topology on $\mathcal{P}(^nE)$.*

**Theorem 3.** *Let $E$ denote a stable locally convex space. For each $n$, $\widehat{\otimes}_{n,\pi}E$ is topologically isomorphic to $\widehat{\otimes}_{n,s,\pi}E$.*

The space $\widehat{\otimes}_{n,\pi}E$ is the $n$-fold tensor product of $E$ into itself endowed with the

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projective topology $\pi$ and $\widehat{\otimes}_{n,s,\pi}E$ is the closed subspace of symmetric $n$-tensors on $E$.

The following remarks clarify the scope of our result:

(i) It follows from [1, Lemma 8] and [3, proof of Proposition 1] that $\mathcal{L}(^nE)$ and $\mathcal{L}_s(^nE)$ contain each other as complemented subspaces whenever $E$ is stable. However this property does not imply, by itself, that they are isomorphic. Indeed Gowers [8] has constructed Banach spaces $X$ and $Y$ containing each other complementably (moreover $X$ is isomorphic to $X^2$ and $Y$ is isomorphic to $Y^3$) though they are not isomorphic.

(ii) The first author [4] gave examples of non-stable Banach and Fréchet spaces $F$ such that $\mathcal{L}(^2F)$ is not isomorphic to $\mathcal{L}_s(^2F)$. On the other hand, we provide at the end of this note a non-stable Banach space $X$ such that $\mathcal{L}(^2X)$ is topologically isomorphic to $\mathcal{L}_s(^2X)$.

(iii) We have confined ourselves in this note to continuous forms and polynomials but it is clear from the proof that the result of Theorem 1 is valid for various other spaces (e.g. the set of all $n$-linear forms, the set of hypocontinuous $n$-linear forms, etc.) and some of these cases may also be carried over to the topological setting, e.g. Theorem 2 is valid for the hypocontinuous forms with the compact open topologies.

(iv) The isomorphism established in Theorems 1–3 is as 'canonical' as the isomorphism we suppose exists between $E$ and its square.

Given locally convex spaces $F$ and $E_i$, $i=1,\dots,n$ we denote by $\mathcal{L}(E_1,\dots,E_n)$ the space of $\mathbf{K}$-valued (continuous) $n$-linear mappings on $E_1\times\dots\times E_n$. If $E_1=E_2=\dots=E_n=E$ we write $\mathcal{L}(^nE)$; in this case we denote by $\mathcal{L}_s(^nE)$ the subspace of symmetric $n$-linear mappings. We also write $\mathcal{L}(^{n-1}E,F)$ in place of $\mathcal{L}(E,\overset{(n-1)}{\dots},E,F)$. Finally $\mathcal{L}(^nE;F)$ is the space of (continuous) $n$-linear mappings with values in $F$. The proof of Theorem 1 is by induction. In Proposition 1 we prove the case $n=2$ (the case $n=1$ is trivial) and after some technical preliminaries we complete the proof in Proposition 3.

*Remark* 1. Fundamental systems of compact sets, bounded sets and neighbourhood bases at the origin are all stable under linear topological isomorphisms; moreover, each has the property that set theoretical products in a product space yield a fundamental system in the product, e.g. if $\mathcal{K}_1$ and $\mathcal{K}_2$ are fundamental systems of compact sets in the locally convex spaces $E$ and $F$ then

$$\{k_1\times k_2\,;k_1\in\mathcal{K}_1,\ k_2\in\mathcal{K}_2\}$$

forms a fundamental system of compact sets in $E\times F$. In the proof of Theorem 1 we construct a certain algebraic isomorphism $\omega_n$ between $\mathcal{L}(^nE)$ and $\mathcal{P}(^nE)(\cong\mathcal{L}_s(^nE))$.

An examination of the proof and the above property shows that $\omega_n$ establishes a one to one correspondence between the following sets:

(a) subsets of $\mathcal{P}(^nE)$ which are uniformly bounded on compact subsets of $E$ and subsets of $\mathcal{L}(^nE)$ which are uniformly bounded on compact subsets of $E^n$;

(b) subsets of $\mathcal{P}(^nE)$ which are uniformly bounded on bounded subsets of $E$ and subsets of $\mathcal{L}(^nE)$ which are uniformly bounded on bounded subsets of $E^n$;

(c) locally bounded subsets of $\mathcal{P}(^nE)$ (a subset of $\mathcal{P}(^nE)$ is locally bounded if it takes some neighbourhood of 0 into a bounded set) and locally bounded subsets of $\mathcal{L}(^nE)$ (a subset of $\mathcal{L}(^nE)$ is locally bounded if it takes $V^n$ into a bounded set for some neighbourhood $V$ of 0).

These correspondences are important in order to obtain the topological isomorphisms stated in Theorems 2 and 3.

By $E$ we will always denote a locally convex space topologically isomorphic to its square, but for convenience and in order to identify clearly certain mappings we write $E$ as $F_1\times F_2$ where $F_1\equiv F_2\equiv E$.

**Lemma 1.** *The space $\mathcal{L}(^nE)$ is isomorphic to $\mathcal{L}(^nE)^2$.*

*Proof.* We have $\mathcal{L}(^nE)\cong\mathcal{L}(^{n-1}E, F_1\times F_2)$. Let

$$\alpha_1\colon \mathcal{L}(^{n-1}E, F_1\times F_2)\longrightarrow \mathcal{L}(^{n-1}E, F_1)\times\mathcal{L}(^{n-1}E, F_2),$$
$$\alpha_1(T) := (T|_{E^{n-1}\times F_1}, T|_{E^{n-1}\times F_2}).$$

It is easily checked that $\alpha_1$ is an isomorphism. □

**Lemma 2.** *The space $\mathcal{L}_s(^2E)$ is isomorphic to $\mathcal{L}_s(^2E)^2\times\mathcal{L}(^2E)$.*

*Proof.* We have $\mathcal{L}_s(^2E)\cong\mathcal{L}_s(^2(F_1\times F_2))$. Then define

$$\alpha_2\colon \mathcal{L}_s(^2(F_1\times F_2)) = \mathcal{L}_s(F_1\times F_2, F_1\times F_2)\longrightarrow \mathcal{L}_s(^2F_1)\times\mathcal{L}_s(^2F_2)\times\mathcal{L}(F_1, F_2),$$
$$\alpha_2(T) := (T|_{F_1\times F_1}, T|_{F_2\times F_2}, T|_{F_1\times F_2}).$$

If $\alpha_2(T)=0$ then

$$T(x_1+y_1, x_2+y_2) = T(x_1, x_2)+T(y_1, y_2)+T(x_1, y_2)+T(y_1, x_2) = 0+T(x_2, y_1) = 0,$$

so $\alpha_2$ is one to one. Let $T_1\in\mathcal{L}_s(^2F_1)$, $T_2\in\mathcal{L}_s(^2F_2)$, $T_3\in\mathcal{L}(F_1, F_2)$. Define $T\in\mathcal{L}(F_1\times F_2, F_1\times F_2)$ by

$$\begin{aligned} T(x_1+y_1, x_2+y_2) &:= T_1(x_1, x_2)+T_2(y_1, y_2)+T_3(x_1, y_2)+T_3(x_2, y_1)\\ &= T_1(x_2, x_1)+T_2(y_2, y_1)+T_3(x_2, y_1)+T_3(x_1, y_2). \end{aligned}$$

Thus $T\in\mathcal{L}_s(^2F_1\times F_2)$ and clearly $\alpha_2(T)=(T_1, T_2, T_3)$. □

We now prove the case $n=2$ of Theorem 1.

**Proposition 1.** *The space $\mathcal{L}_s(^2E)$ is isomorphic to $\mathcal{L}(^2E)$.*

*Proof.* It is known that $\mathcal{L}_s(^2E)$ is complemented in $\mathcal{L}(^2E)$ [7, Chapter 1]. Hence there exists $V$ such that $\mathcal{L}_s(^2E)\times V\cong\mathcal{L}(^2E)$. By Lemma 2 and Lemma 1 we have

$$\mathcal{L}(^2E)\cong\mathcal{L}_s(^2E)\times V\cong\mathcal{L}_s(^2E)^2\times\mathcal{L}(^2E)\times V\cong\mathcal{L}_s(^2E)\times\mathcal{L}(^2E)^2\cong\mathcal{L}_s(^2E)\times\mathcal{L}(^2E).$$

Hence, by Lemmata 1 and 2 again,

$$\begin{aligned}\mathcal{L}(^2E)&\cong\mathcal{L}(^2E)^2\cong(\mathcal{L}_s(^2E)\times\mathcal{L}(^2E))^2\cong\mathcal{L}_s(^2E)^2\times\mathcal{L}(^2E)^2\\&\cong\mathcal{L}_s(^2E)^2\times\mathcal{L}(^2E)\cong\mathcal{L}_s(^2E).\quad\square\end{aligned}$$

Let $\{I,J\}$ denote a partition of $\{1,\dots,n\}$ into two non-empty sets (so $1\le|I|<n$ and $1\le|J|<n$). Let $\mathcal{L}_{I,J}(^nE)$ denote the set of all $T\in\mathcal{L}(^nE)$ which are invariant under all permutations $\sigma$ of $\{1,\dots,n\}$ for which $\sigma(I)=I$ and $\sigma(J)=J$ (i.e. we can rearrange the $x_i$, $i\in I$, among themselves and the $x_j$, $j\in J$, among themselves without changing the value of the $n$-linear form). Since clearly $\mathcal{L}_{I,J}(^nE)\cong\mathcal{L}_{I',J'}(^nE)$ as long as $|I|=|I'|$ (and hence $|J|=|J'|$), we have

$$\mathcal{L}_{I,J}(^nE)\equiv\mathcal{L}_k(^nE),$$

where $|I|=k$ and $\mathcal{L}_k(^nE)=\mathcal{L}_{\{1,\dots,k\},\{k+1,\dots,n\}}(^nE)$, i.e. we can take $I=\{1,\dots,k\}$, $J=\{k+1,\dots,n\}$.

**Proposition 2.** *If for each $k<n$ there is an isomorphism*

$$\theta_k:\mathcal{L}(^kE)\longrightarrow\mathcal{L}_s(^kE),$$

*then $\mathcal{L}_{I,J}(^nE)$ is isomorphic to $\mathcal{L}(^nE)$.*

*Proof.* It suffices to show that $\mathcal{L}_k(^nE)$ is isomorphic to $\mathcal{L}(^nE)$. Let

$$\begin{aligned}&\Phi_k:\mathcal{L}(^nE)\longrightarrow\mathcal{L}(^kE;\mathcal{L}(^{n-k}E)),\\&(\Phi_k(T)(x_1,\dots,x_k))(x_{k+1},\dots,x_n):=T(x_1,\dots,x_n).\end{aligned}$$

Then $\Phi_k$ is an isomorphism. Now let

$$\begin{aligned}&\beta_1:\mathcal{L}(^nE)\longrightarrow\mathcal{L}(^kE;\mathcal{L}_s(^{n-k}E)),\\&(\beta_1(T)(x_1,\dots,x_k))(x_{k+1},\dots,x_n):=(\theta_{n-k}(\Phi_k(T)(x_1,\dots,x_k)))(x_{k+1},\dots,x_n)\end{aligned}$$

and let

$$\beta_2:\mathcal{L}(^nE)\longrightarrow\mathcal{L}(^nE),\quad \beta_2(T)=\Phi_k^{-1}(\beta_1(T)).$$

Clearly $\beta_1$ and $\beta_2$ are isomorphisms onto their ranges. We show

$$\beta_2(\mathcal{L}(^nE)) = \{T \in \mathcal{L}(^nE)\,; T(x_1, \dots, x_n) = T(x_1, \dots, x_k, x_{\sigma(k+1)}, \dots, x_{\sigma(n)}) \text{ for all permutations } \sigma \text{ of } \{k+1, \dots, n\}\} := \mathcal{L}_o(^nE).$$

(a) $\beta_2(\mathcal{L}(^nE)) \subset \mathcal{L}_o(^nE)$: If $S = \beta_2(T)$ and $\sigma$ is a permutation of $\{k+1, \dots, n\}$ then

$$\begin{aligned} S(x_1, \dots, x_n) &= (\Phi_k^{-1}(\beta_1(T)))(x_1, \dots, x_n) = (\beta_1(T)(x_1, \dots, x_k))(x_{k+1}, \dots, x_n) \\ &= (\theta_{n-k}(\Phi_k(T)(x_1, \dots, x_k)))(x_{k+1}, \dots, x_n) \\ &= (\theta_{n-k}(\Phi_k(T)(x_1, \dots, x_k)))(x_{\sigma(k+1)}, \dots, x_{\sigma(n)}) \\ &= S(x_1, \dots, x_k, x_{\sigma(k+1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

(b) $\mathcal{L}_o(^nE) \subset \beta_2(\mathcal{L}(^nE))$: Let $S \in \mathcal{L}_o(^nE)$. Then $\Phi_k(S)(x_1, \dots, x_k) \in \mathcal{L}_s(^{n-k}E)$. Let

$$T(x_1, \dots, x_n) = (\theta_{n-k}^{-1}(\Phi_k(S)(x_1, \dots, x_k)))(x_{k+1}, \dots, x_n),$$

so

$$\begin{aligned} \Phi_k(T)(x_1, \dots, x_k) &= \theta_{n-k}^{-1}(\Phi_k(S)(x_1, \dots, x_k)), \\ \theta_{n-k}(\Phi_k(T)(x_1, \dots, x_k)) &= \Phi_k(S)(x_1, \dots, x_k), \\ (\beta_1(T)(x_1, \dots, x_k))(x_{k+1}, \dots, x_n) &= (\theta_{n-k}(\Phi_k(T)(x_1, \dots, x_k)))(x_{k+1}, \dots, x_n) \\ &= (\Phi_k(S)(x_1, \dots, x_k))(x_{k+1}, \dots, x_n) \\ &= S(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \end{aligned}$$

and $\beta_2(T)(x_1, \dots, x_n) = S(x_1, \dots, x_n)$, i.e. $\beta_2(T) = S$.

Therefore $\mathcal{L}(^nE) \cong \mathcal{L}_o(^nE)$ from (a) and (b). On the other hand it is clear that

$$\mathcal{L}_o(^nE) \cong \{T \in \mathcal{L}(^nE)\,; T(x_1, \dots, x_n) = T(x_{\sigma(1)}, \dots, x_{\sigma(n-k)}, x_{n-k+1}, \dots, x_n)\} := \mathcal{L}_{oo}(^nE).$$

To finish we show $\mathcal{L}_{oo}(^nE) \cong \mathcal{L}_k(^nE)$. Let

$$\begin{aligned} &\beta_3 : \mathcal{L}_{oo}(^nE) \longrightarrow \mathcal{L}(^{n-k}E; \mathcal{L}(^kE)), \\ &(\beta_3(T)(x_1, \dots, x_{n-k}))(x_{n-k+1}, \dots, x_n) \\ &\qquad := (\theta_k(\Phi_{n-k}(T)(x_1, \dots, x_{n-k})))(x_{n-k+1}, \dots, x_n) \end{aligned}$$

and let

$$\beta_4 : \mathcal{L}_{oo}(^nE) \longrightarrow \mathcal{L}(^nE), \quad \beta_4(T) = \Phi_{n-k}^{-1}(\beta_3(T)).$$

By going over (a) and (b) for this new situation we see that $\beta_4$ is an isomorphism onto its image and $T$ belongs to $\beta_4(\mathcal{L}_{oo}({}^nE))$ if and only if

$$T(x_1, \dots, x_{n-k}, x_{n-k+1}, \dots, x_n)$$

is equal both to

$$T(x_1, \dots, x_{n-k}, x_{\sigma(n-k+1)}, \dots, x_{\sigma(n)})$$

and to

$$T(x_{\tau(1)}, \dots, x_{\tau(n-k)}, x_{n-k+1}, \dots, x_n),$$

for permutations $\sigma$ and $\tau$, thus if and only if $T$ belongs to $\mathcal{L}_k({}^nE)$. $\square$

**Proposition 3.** *For any positive integer $n$ we have:*
(a) $\mathcal{L}_s({}^nE)$ *is isomorphic to* $\mathcal{L}_s({}^nE)^2 \times \mathcal{L}({}^nE)$,
(b) $\mathcal{L}_s({}^nE)$ *is isomorphic to* $\mathcal{L}({}^nE)$.

*Proof.* The proof is by induction. If $n=2$ then (a) is Lemma 2 and (b) is Proposition 1. We suppose (a) and (b) hold for $k<n$. Thus we can apply Proposition 2 and assume that $\mathcal{L}_k({}^nE)$ is isomorphic to $\mathcal{L}({}^nE)$ for every $k<n$. We recall $E \cong F_1 \times F_2$ where $F_1 \cong F_2 \cong E$ and define the mapping

$$\Psi: \mathcal{L}_s({}^n(F_1 \times F_2)) \longrightarrow \prod_{j=0}^{n} \mathcal{L}(F_1, \overset{(j)}{\dots}, F_1, F_2, \overset{(n-j)}{\dots}, F_2), \quad \Psi(T) = (T_0, \dots, T_n),$$

where

$$T_j(x_1, \dots, x_j, y_{j+1}, \dots, y_n) := T(x_1+0, \dots, x_j+0, 0+y_{j+1}, \dots, 0+y_n)$$

for all $0 \le j \le n$. Since $T$ is symmetric it follows that

$$T_j(x_1, \dots, x_j, y_{j+1}, \dots, y_n) = T_j(x_{\sigma(1)}, \dots, x_{\sigma(j)}, y_{\sigma(j+1)}, \dots, y_{\sigma(n)})$$

for any $\sigma$, permutation of $\{1, \dots, n\}$ such that both $\sigma\{1, \dots, j\} = \{1, \dots, j\}$ and $\sigma\{j+1, \dots, n\} = \{j+1, \dots, n\}$. If $j=0$ or $j=n$ then $T_j \in \mathcal{L}_s({}^nE)$. If $j \ne 0, n$ then $T_j \in \mathcal{L}_j({}^nE)$. We claim $(*)$ that $\Psi$ is an isomorphism onto $\mathcal{L}_s({}^nE)^2 \times \prod_{k=1}^{n-1} \mathcal{L}_k({}^nE)$. Assuming that $(*)$ is true then we get

$$\mathcal{L}_s({}^nE) \cong \mathcal{L}_s({}^nE)^2 \times \prod_{k=1}^{n-1} \mathcal{L}_k({}^nE) \cong \mathcal{L}_s({}^nE)^2 \times \mathcal{L}({}^nE)^{n-1} \cong \mathcal{L}_s({}^nE)^2 \times \mathcal{L}({}^nE)$$

the last equivalence by Lemma 1. This establishes (a) for $n$.

To prove (b) we note that since $\mathcal{L}_s({}^nE)$ is complemented in $\mathcal{L}({}^nE)$ we have $V_n$ such that $\mathcal{L}({}^nE)\cong\mathcal{L}_s({}^nE)\times V_n$. Therefore, by using (a) and Lemma 1 again,

$$\begin{aligned}\mathcal{L}({}^nE) &\cong \mathcal{L}_s({}^nE)\times V_n \cong \mathcal{L}_s({}^nE)^2\times\mathcal{L}({}^nE)\times V_n\\ &\cong \mathcal{L}_s({}^nE)\times\mathcal{L}({}^nE)^2 \cong \mathcal{L}_s({}^nE)\times\mathcal{L}({}^nE),\end{aligned}$$

hence,

$$\mathcal{L}({}^nE)\cong\mathcal{L}({}^nE)^2\cong\mathcal{L}_s({}^nE)^2\times\mathcal{L}({}^nE)^2\cong\mathcal{L}_s({}^nE)^2\times\mathcal{L}({}^nE)\cong\mathcal{L}_s({}^nE)$$

and this gives (b) and completes the proof by induction once we have established $(*)$.

(i) $\Psi$ *is injective*: If $\Psi(T)=0$ then $T_j=0$ for every $j$. Given any element $(x_i+y_i)_{i=1}^n\in(F_1\times F_2)^n$ we can write $T(x_1+y_1,\dots,x_n+y_n)=\sum T(z_1,\dots,z_n)$ where $z_i=x_i$ or $y_i$ and the sum is taken over all possible choices. Since $T$ is symmetric we can rearrange in each term the $z_i$'s so that the $x_i$'s precede the $y_i$'s and so each term has the form

$$T(x_1,\dots,x_j,y_{j+1},\dots,y_n)=T_j(x_1,\dots,x_j,y_{j+1}\dots,y_n)=0.$$

Hence $T\equiv 0$ and this implies $\Psi$ is one to one.

(ii) $\Psi$ *is surjective*: Given $(T_0,T_1,\dots,T_n)$, with

$$T_0\in\mathcal{L}_s({}^nF_2),\quad T_j\in\mathcal{L}_j(F_1,\overset{(j)}{\dots},F_1,F_2,\overset{(n-j)}{\dots},F_2),\quad T_n\in\mathcal{L}_s({}^nF_1),$$

we define $S$ by

$$\begin{aligned}S(x_1+y_1,\dots,x_n+y_n)&=T_0(y_1,\dots,y_n)+\sum_{j=1}^n T_1(x_j,y_2,\dots,y_{j-1},y_1,y_{j+1},\dots,y_n)\\ &+\sum_{j,k}T_2(x_j,x_k,y_3,\dots,y_{j-1},y_1,y_{j+1},\dots,y_{k-1},y_2,y_{k+1},\dots,y_n)\\ &+\sum_{i,j,k}T_3(x_i,x_j,x_k,y_4,\dots,y_{i-1},y_1,y_{i+1},\dots,y_{j-1},y_2,y_{j+1},\dots,y_{k-1},\\ &\qquad y_3,y_{k+1},\dots,y_n)+\dots .\end{aligned}$$

Note that $S$ is symmetric. Indeed $T_0$ is symmetric. It is readily checked, since $T_1\in\mathcal{L}_1(F_1,F_2,\dots,F_2)$, that

$$\sum_{j=1}^n T_1(x_j,y_2,\dots,y_{j-1},y_1,\dots,y_n)$$

is symmetric with respect to the arguments $(x_1+y_1, \dots, x_n+y_n)$. In the same way it is shown that each other summand is also symmetric. On the other hand

$$\begin{aligned}\Psi(S)_j(x_1, \dots, x_j, y_{j+1}, \dots, y_n) &= S(x_1+0, \dots, x_j+0, 0+y_{j+1}, \dots) \\ &= T_j(x_1, \dots, x_j, y_{j+1}, \dots, y_n),\end{aligned}$$

so $\Psi(S)=(T_0, T_1, \dots, T_n)$, hence $\Psi$ is an isomorphism which establishes $(*)$ and completes the proof. $\square$

Proposition 3(b) establishes Theorem 1. We now prove Theorems 2 and 3. For each $n$ we let $\omega_n$ denote the algebraic isomorphism between $\mathcal{L}(^nE)$ and $\mathcal{P}(^nE)$ ($\cong \mathcal{L}_s(^nE)$). The correspondence in (a) and (b) of Remark 1 establishes (i) and (ii) of Theorem 2, respectively. We now show Theorem 3. Let

$$Q_1(^nE) = \{\phi \in \mathcal{P}(^nE)' ; \phi \text{ restricted to the locally bounded subsets of } \mathcal{P}(^nE) \text{ is } \tau_0\text{-continuous}\},$$

and let

$$Q_2(^nE) = \{\phi \in \mathcal{L}(^nE)' ; \phi \text{ restricted to the locally bounded subsets of } \mathcal{L}(^nE) \text{ is } \tau_0\text{-continuous}\}.$$

Using results of Ryan [13], Mazet [10] and Mujica and Nachbin [12], Boyd proved (Proposition 1 and subsequent comments of [2]) that $Q_1(^nE)$ endowed with the topology of uniform convergence on the locally bounded subsets of $\mathcal{L}_s(^nE)$ is topologically isomorphic to $\widehat{\otimes}_{n,s,\pi}E$. The method of proof in [12] and [2] extends readily to the $n$-linear forms and we see that $Q_2(^nE)$, endowed with the topology of uniform convergence on the locally bounded subsets of $\mathcal{L}(^nE)$ is topologically isomorphic to $\widehat{\otimes}_{n,\pi}E$. Since we noted in Remark 1 that the locally bounded subsets of $\mathcal{L}_s(^nE)$ and $\mathcal{L}(^nE)$ can be identified under $\omega_n$ and that $\omega_n$ is a topological isomorphism for the $\tau_0$ topologies we conclude that $\widehat{\otimes}_{n,s,\pi}E$ and $\widehat{\otimes}_{n,\pi}E$ are topologically isomorphic. This proves Theorem 3.

Finally to check (iii) of Theorem 2 we take inductive duals of $\widehat{\otimes}_{n,s,\pi}E$ and $\widehat{\otimes}_{n,\pi}E$. (The inductive dual of a locally convex space $F$ is the inductive limit $F_i' := \lim_{\overrightarrow{V}} (F_V', \|\cdot\|_V)$ where $V$ ranges over a fundamental neighbourhood system at the origin.) Our statement follows since $(\widehat{\otimes}_{n,s,\pi}E)_i' \cong (\mathcal{P}(^nE), \tau_\omega)$ ([2]). $\square$

For a nuclear power series space $E$, the existence of a topological isomorphism between $\mathcal{L}(^2E)$ and $\mathcal{L}_s(^2E)$ implies that $E$ is stable ([4, Theorem 10]). We give an example to show that the analogous result does not hold for Banach spaces.

**Lemma 3.** *Let $X$ be a Banach space such that $X'$ is stable and has the approximation property. If every continuous linear mapping from $X$ to $X'$ is compact then $\mathcal{L}(^2X)$ is topologically isomorphic to $\mathcal{L}_s(^2X)$.*

*Proof.* Compactness of the linear mappings and the approximation property on $X'$ imply $\mathcal{L}(^2X)\cong\mathcal{L}(X;X')\cong X'\widehat{\otimes}_\varepsilon X'$ where $\otimes_\varepsilon$ denotes the injective tensor product. Similarly, $\mathcal{L}_s(^2X)\cong X'\widehat{\otimes}_{s,\varepsilon}X'$. Since $X'$ is isomorphic to its square, the methods used in Lemmata 1, 2 and Proposition 1 can be adapted to prove that $X'\widehat{\otimes}_\varepsilon X'$ is isomorphic to its square, $X'\widehat{\otimes}_{s,\varepsilon}X'$ is isomorphic to its square and $X'\widehat{\otimes}_\varepsilon X'\cong X'\widehat{\otimes}_{s,\varepsilon}X'$. □

*Example* 1. Let $\Omega$ denote the set of all ordinals less than or equal to the first uncountable ordinal. By [15] the Banach space $C(\Omega)$ is not stable. However $C(\Omega)'\cong l_1(\Omega)$ and $l_1(\Omega)$ is stable and has the approximation property and, moreover, every continuous linear map from $C(\Omega)$ to $l_1(\Omega)$ is compact since $C(\Omega)$ does not contain any copy of $l_1$. By Lemma 3, $\mathcal{L}(^2C(\Omega))\cong\mathcal{L}_s(^2C(\Omega))$ and we conclude that stability is not always required in our previous results.

The above example leads naturally to the following question: *If $E$ and $F$ are Banach spaces and $E'\cong F'$ does this imply that $\mathcal{P}(^nE)\cong\mathcal{P}(^nF)$ for all $n$?* Using the arguments of Lemma 3 we give a partial positive answer.

**Proposition 4.** *Let $E$ and $F$ denote Banach spaces such that $E'$ is isomorphic to $F'$. If $E'$ has the Schur property and the approximation property then $\mathcal{P}(^nE)$ is isomorphic to $\mathcal{P}(^nF)$ for every $n\in\mathbf{N}$.*

*Proof.* By our hypothesis $E$ and $F$ do not contain $l_1$ and have the Dunford–Pettis property. Let $G=E$ or $F$. By [14, Corollary 3.4] the $n$-fold projective tensor product $\widehat{\otimes}_{n,\pi}G$ does not contain $l_1$ and hence every continuous linear mapping from $\widehat{\otimes}_{n-1,\pi}G$ to $G'$ is compact. By the method of Lemma 3 we obtain $\mathcal{L}(^nG)\cong\widehat{\otimes}_{n,\varepsilon}G'$ and $\mathcal{L}_s(^nG)\cong\widehat{\otimes}_{n,s,\varepsilon}G'$. Since $E'\cong F'\cong G'$ this implies $\mathcal{P}(^nE)\cong\mathcal{P}(^nF)$ and completes the proof. □

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