Ark. Mat., **36** (1998), 41–71

On the solvability of linear partial differential equations in spaces of hyperfunctions

Paulo D. Cordaro(1) and Jean-Marie Trépreau

It is well known from the theory of linear partial differential equations in spaces of smooth functions and distributions, see Hörmander [11], [12], that the solvability of a differential equation is related to the non-existence of a solution of the homogeneous adjoint equation with compact singular support, and that this may be used to obtain semi-global existence results from the microlocal study of the adjoint operator. In this paper we show that a similar strategy is possible in the framework of hyperfunctions. Actually, we shall consider in this paper the more general case of a system of differential equations without compatibility conditions in the framework of hyperfunctions on a maximally real manifold in $\mathbf{C}^n$ with low regularity.

The first section of the paper may be considered as a continuation of Schapira [26], [27], in which it was shown how functional analysis can be used in the hyperfunction theory of differential operators. We first recall the fact that hyperfunction solvability is insensitive to the geometry of the boundary of the domain (Theorem 1.2) and show that finite dimensional obstruction to solvability never occurs (Theorem 1.3). Then we characterize the hyperfunction solvability of a differential operator in terms of the validity of an *a priori* inequality for the adjoint operator (Theorem 1.4). The main result of this section is perhaps Theorem 1.6 which states that the non-confinement of analytic singularities for the adjoint operator is a sufficient condition for the hyperfunction solvability. This is similar to Theorem 1.2.4 of Hörmander [11].

In Section 2 we give several examples of how the functional analysis statements of Section 1 apply to obtain seemingly new existence theorems or new proofs of classical existence theorems, as corollaries of already available, sometimes deep, microlocal results. Such topics as holonomic systems, hypo-analytic structures or analytic differential equations of principal type on $\mathbf{R}^n$ are touched on. Theorem 2.2

(1) Partially supported by CNPq, Brasil.

gives a very simple proof of the local solvability of the last "compatibility equation" of a maximally overdetermined system. Theorem 2.6 states the solvability of an analytic differential operator of principal type on $\mathbf{R}^n$, satisfying the Nirenberg–Treves condition (P). Theorem 2.7 establishes a weak maximum principle for the hypo-analytic functions, when the hypo-analytic complex is solvable in top degree. The converse assertion is conjectured and discussed in some special cases.

In Section 3 we establish the solvability of an analytic differential operator of principal type satisfying the Nirenberg–Treves condition (P), in the framework of hyperfunctions on a maximally real manifold in $\mathbf{C}^n$ with low regularity. The proof follows the strategy introduced in Hörmander [12] for the C^∞ solvability, that is we prove the non-confinement of analytic singularities for the adjoint operator. However, the needed microlocal results are not available and we use the microlocal transformation theory of Kashiwara and Schapira [16] to obtain them. It allows us to reduce the problem to the analysis of the concrete operator $\partial/\partial z_1$ acting at the boundary of a strictly pseudoconvex domain in $\mathbf{C}^n$. This is similar to what was done in Trépreau [31] to prove the microlocal solvability of an operator satisfying the weaker condition (Ψ). Unfortunately, it is not clear how to get local from microlocal solvability, so we shall rely on the method but not on the main result in [31].

1. Local solvability and non-confinement of singularities

1.1. Notation

For any $n\in\mathbf{N}$, we denote by $z=(z_1,\dots,z_n)$ the variable in $\mathbf{C}^n$ with norm $|z|=\max_{i=1}^n|z_i|$, and we define $dz=dz_1\wedge\ldots\wedge dz_n$. If $K\subset\mathbf{C}^n$ and $\varepsilon>0$, then K_ε denotes the set of all $z\in\mathbf{C}^n$ which lie at a distance $<\varepsilon$ from K; if h is a function $K\to\mathbf{C}^d$, we set

$$|h|_K=\sup_{z\in K}|h(z)|.$$

We shall use the terminology FS and DFS to refer to the class of Fréchet–Schwartz spaces and to the class of all strong duals of Fréchet–Schwartz spaces, see Grothendieck [6], Köthe [19]. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbf{C}^n$. If $\Omega\subset\mathbf{C}^n$ is open, then $\mathcal{O}(\Omega)$, endowed with the semi-norms $|\cdot|_K$, $K\subset\subset\Omega$, is an FS space. An analytic functional ϕ on Ω is an element of the dual space $\mathcal{O}'(\Omega)$; it is carried by a compact set $K\subset\Omega$ if for every $\varepsilon>0$ there exists C such that $|\phi(h)|\le C|h|_{K_\varepsilon}$ for all $h\in\mathcal{O}(\Omega)$. Let $K\subset\mathbf{C}^n$ be a compact set; the space $\mathcal{O}(K)$ of germs of holomorphic functions at K, endowed with the locally convex limit topology, is a DFS space with the FS space $\mathcal{O}'(K)$ as strong dual, $\phi\in\mathcal{O}'(K)$ acts on every space $\mathcal{O}(K_\varepsilon)$ and the topology of $\mathcal{O}'(K)$ is induced by the semi-norms

$$\|\phi\|_{K_\varepsilon} = \sup_{h\in\mathcal{O}(K_\varepsilon)} \frac{|\phi(h)|}{|h|_{K_\varepsilon}}.$$

If Ω is pseudoconvex and the compact set $K\subset\Omega$ is $\mathcal{O}(\Omega)$-convex, then $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(K)$ and $\mathcal{O}'(K)$ can be identified with the space of analytic functionals on Ω which are carried by K.

1.2. Hyperfunctions and analytic functionals

Let M be a maximally real manifold in $\mathbf{C}^n$ (actually we might replace $\mathbf{C}^n$ by a Stein manifold), that is a totally real n-dimensional submanifold of $\mathbf{C}^n$, of class C^1. Sato's theory of hyperfunctions extends to this situation (see Harvey [8], Harvey–Wells [9]) and so does the microlocal theory of Sato–Kawai–Kashiwara [25] (see Kashiwara–Schapira [16]). We denote by $\mathcal{B}$ the sheaf of hyperfunctions on M. For the sake of simplicity, a section of $\mathcal{A}=\mathcal{O}|_M$ will be called analytic even if M is not real analytic. Though this is not essential, we shall assume that M is orientable, and in fact oriented, in order to avoid difficulties in identifying $\mathcal{A}$ with a subsheaf of $\mathcal{B}$.

We shall adopt the point of view of Martineau [20] about Sato's theory by identifying compactly supported hyperfunctions with analytic functionals carried by M. Martineau assumed M real analytic but it was proved by Harvey and Wells [9] that his results remain valid when M is of class C^1. Let us recall the content of Theorem 2.2 in [9], which is important in this respect:

There exists a fundamental neighborhood system $\mathcal{F}$ of M with the following properties: (i) *each $U\in\mathcal{F}$ is pseudoconvex,* (ii) *$\mathcal{O}(V)$ is dense in $\mathcal{O}(U)$ if $U,V\in\mathcal{F}$ and $U\subset V$,* (iii) *each compact set $K\subset M$ is $\mathcal{O}(U)$-convex for each $U\in\mathcal{F}$.*

To summarize, we shall use the following notation.

Notation 1.1. The set $M\subset\mathbf{C}^n$ is an oriented maximally real manifold of class C^1, with $0\in M$. The set Ω denotes a pseudoconvex neighborhood of M with the property that every compact set $K\subset M$ is $\mathcal{O}(\Omega)$-convex.

With this notation $\mathcal{O}'(K)$ is identified with the space of analytic functionals on Ω carried by K, if $K\subset M$ is compact. Let us recall that $\mathcal{B}$ is a flabby sheaf on M and that, if $U\subset\subset M$ is open in M (the notations $\overline{U}$ and ∂U always refer to the closure and the boundary of U relative to M), the identity

$$\mathcal{B}(U)=\mathcal{O}'(\overline{U})/\mathcal{O}'(\partial U)$$

holds.

Some of our results are local near $0\in M$; then we may shrink M and take $\Omega=\mathbf{C}^n$ in Notation 1.1. Some other results, like Theorem 1.4 and Theorem 1.6, are global and concern an open subset $U\subset\subset M$. Let us already emphasize the fact that these results do not apply to compact manifolds: it will be assumed that U has no compact connected component, with the consequence that the restriction map $\mathcal{O}(\overline{U})\to\mathcal{O}(\partial U)$ is injective by the uniqueness of analytic continuation, and as a result that $\mathcal{O}'(\partial U)$ is dense in $\mathcal{O}'(\overline{U})$.

For the sake of simplicity we embed $L^1_{\text{loc}}(M)$ as a subsheaf of $\mathcal{B}$ in a noncanonical way by identifying f with the analytic functional

$$f(h)=\int_M f(z)h(z)\,dz$$

when f has compact support. This is not invariantly defined, but using another analytic non-vanishing density $a(z)\,dz$ for the identification would not change much to what follows, since $u\mapsto au$ is a sheaf isomorphism of $\mathcal{B}$. In particular $\mathcal{A}$ is identified with a subsheaf of $\mathcal{B}$. Also, if M is of class C^m, there is a canonical injective sheaf mapping from the sheaf $\mathcal{D}'_{m-1}$ of distributions of order $m-1$ on M to the sheaf $\mathcal{B}$, which induces the obvious restriction map on compactly supported sections (see [9, Theorem 3.5]).

By a differential operator on M, we shall always mean an operator

$$P=\sum_{|\alpha|\le m} a_\alpha(z)\frac{\partial^\alpha}{\partial z^\alpha}$$

with analytic coefficients $a_\alpha\in\mathcal{A}(M)$. The adjoint operator ${}^tP$ of P is the differential operator on M defined by

$${}^tPh=\sum_{|\alpha|\le m}(-1)^{|\alpha|}\frac{\partial^\alpha}{\partial z^\alpha}(a_\alpha h).$$

The operator P acts on analytic functionals carried by M by the formula

$$P\phi(h)=\phi({}^tPh),\quad \text{if } h\in\mathcal{O}(\Omega),$$

and this action extends as a sheaf homomorphism of $\mathcal{B}$. On the other hand, P acts on the sheaf $\mathcal{O}$, hence on the sheaf $\mathcal{A}$ by restriction,

$$P(f|_M)=(Pf)|_M.$$

This action can also be described as follows: if M is of class C^m and $1 \leq k \leq n$, then $\partial/\partial z_k$ induces a vector field L_k of class C^{m-1} on M, determined by the property that $\partial/\partial z_k - L_k$ is antiholomorphic. Actually, $dz_1, \dots, dz_n$ induce a basis of 1-forms of class C^{m-1} on M and $L_1, \dots, L_n$ is the dual basis of vector fields on M. Clearly $\partial f/\partial z_k = L_k f$ if f is analytic, so $\partial f/\partial z_k$ is well-defined if f is of class C^1 and the definition agrees with the hyperfunction definition, since if f has compact support

$$\frac{\partial f}{\partial z_k}(h) + f\left(\frac{\partial h}{\partial z_k}\right) = \pm \int_M d(fh\, dz_1 \wedge \dots \wedge \widehat{dz_k} \wedge \dots \wedge dz_n) = 0,$$

due to Stoke's formula (the notation $\widehat{dz_k}$ means that dz_k is omitted in the wedge product). Thus, if M is of class C^m and P is of order m, the action of P on hyperfunctions is compatible with the natural action of P on functions of class C^m and our identification of functions with hyperfunctions.

1.3. Local solvability and a priori inequalities

Let us consider d differential operators $P_1, \dots, P_d$ on M and the following associated "underdetermined" system P:

$$P(u_1, \dots, u_d) = \sum_{i=1}^{d} P_i u_i = f. \tag{1.1}$$

The adjoint system is the "overdetermined system" ${}^tP$ defined by

$${}^tPf = ({}^tP_1 f, \dots, {}^tP_d f) = (u_1, \dots, u_d). \tag{1.2}$$

A main idea in Schapira [26], to circumvent the fact that the topology of $\mathcal{B}(U)$ is not separated, was to notice that, if $A: \mathcal{B} \to \mathcal{B}$ is a sheaf morphism, $A: \mathcal{B}(U) \to \mathcal{B}(U)$ is onto if and only if the map $A': \mathcal{O}'(\overline{U}) \times \mathcal{O}'(\partial U) \to \mathcal{O}'(\overline{U})$ defined by $(\phi, \nu) \mapsto A\phi + \nu$ is onto. This remark is useful, since $\mathcal{O}'(\overline{U})$ and $\mathcal{O}'(\partial U)$ are gentle FS spaces which tolerate the use of functional analysis.

We first recall the fact (this is Proposition 2 in [26]) that local and global solvability are the same on small open sets, when hyperfunction solutions are allowed, and as far as there are no compatibility conditions! Hence such phenomena as P-convexity play no role in hyperfunction solvability. Let us denote by $\mathcal{B}_0$ the space of germs of hyperfunctions at $0 \in M$.

Theorem 1.2. *The following properties are equivalent:*

(1) *The map $P: \mathcal{B}_0^d \to \mathcal{B}_0$ is onto.*

(2) *There exists an open neighborhood U of 0 in M such that $P: \mathcal{B}(U)^d \to \mathcal{B}(U)$ is onto, hence by the flabbiness of $\mathcal{B}$, $P: \mathcal{B}(V)^d \to \mathcal{B}(V)$ is onto for every open subset $V \subset U$.*

Proof. (2) $\Rightarrow$ (1) by the flabbiness of $\mathcal{B}$. Let us assume that property (1) holds. Let $V_k \subset\subset V \subset\subset M$ be open neighborhoods of 0 in M, with $\bigcap_k V_k = \{0\}$. The space

$$E_k = \{(\phi, \psi, \nu) \in \mathcal{O}'(\bar{V}) \times \mathcal{O}'(\bar{V}_k)^d \times \mathcal{O}'(\bar{V} \setminus V_k) : \phi = P\psi + \nu\}$$

is a closed subspace of a Fréchet space, hence a Fréchet space. If $\pi_k : E_k \to \mathcal{O}'(\bar{V})$ denotes the projection on the first factor, then $\bigcup_k \pi_k(E_k) = \mathcal{O}'(\bar{V})$ by property (1). By Baire's category theorem and the open mapping theorem, $\pi_k(E_k) = \mathcal{O}'(\bar{V})$ for some k. Finally, property (2) holds true with $U = V_k$, thanks again to the flabbiness of $\mathcal{B}$. □

In the preceding argument, let us replace property (1) by the weaker hypothesis that $P : \mathcal{B}_0^d \to \mathcal{B}_0$ has a range with finite codimension, so that for some $\theta_1, \dots, \theta_N \in \mathcal{O}'(\bar{V})$, $\mathcal{B}_0$ is spanned by the range of P and the germs defined by $\theta_1, \dots, \theta_N$. Considering the space

$$E_k = \left\{ (\phi, \psi, \nu, a) \in \mathcal{O}'(\bar{V}) \times \mathcal{O}'(\bar{V}_k)^d \times \mathcal{O}'(\bar{V} \setminus V_k) \times \mathbf{C}^N : \phi = P\psi + \nu + \sum_{i=1}^{N} a_i \theta_i \right\}$$

and repeating the previous proof, we obtain that, for some $V_k = U$, the map $\mathcal{O}'(\bar{U})^d \times \mathcal{O}'(\partial U) \times \mathbf{C}^N \to \mathcal{O}'(\bar{U})$, $(\psi, \nu, a) \mapsto P\psi + \nu + \sum_{i=1}^N a_i \theta_i$, is onto, hence a homomorphism. We deduce from this that the map $(\psi, \nu) \mapsto P\psi + \nu$ has closed range, hence is surjective, since its range is dense. This shows that local solvability in the space of hyperfunctions is insensitive to finite dimensional obstructions (see Section 2.1 for a simple application of this fact to holonomic systems).

Theorem 1.3. *If $P : \mathcal{B}_0^d \to \mathcal{B}_0$ has a range with finite codimension, then P is onto.*

We now show that the solvability of P in $\mathcal{B}(U)$ is equivalent to an *a priori* inequality for ${}^tP$. This improves a result of Schapira [26].

Theorem 1.4. *Let M, Ω be as in Notation* 1.1 *and $U \subset\subset M$ an open subset without compact connected component. The differential system* (1.1) *on M induces a surjective map $P : \mathcal{B}(U)^d \to \mathcal{B}(U)$ if and only if, for every small $\varepsilon > 0$ there exist $\eta > 0$ and C such that*

$$|h|_{\bar{U}_\eta} \le C(|{}^tPh|_{\bar{U}_\varepsilon} + |h|_{(\partial U)_\varepsilon}) \quad \textit{for all } h \in \mathcal{O}(\Omega). \tag{1.3}$$

Proof. The map $P : \mathcal{B}(U)^d \to \mathcal{B}(U)$ is onto if and only if the map of FS spaces

$$\mathcal{O}'(\bar{U})^d \times \mathcal{O}'(\partial U) \ni (\psi, \nu) \longmapsto P\psi + \nu \in \mathcal{O}'(\bar{U})$$

is onto. Since U has no compact component, this map has dense range, hence it is onto if and only if its range is closed. Consequently, this map is onto if and only if the range of the transpose map T

$$\mathcal{O}(\bar{U}) \ni h \longmapsto Th = ({}^tPh, h|_{\partial U}) \in \mathcal{O}(\bar{U})^d \times \mathcal{O}(\partial U)$$

of DFS spaces is closed, or (see [19, p. 18]) sequentially closed. This is the case if (1.3) holds: if Th_k converges, it converges in $\mathcal{O}(\bar{U}_\varepsilon)^d \times \mathcal{O}((\partial U)_\varepsilon)$ for some $\varepsilon>0$; since $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(\bar{U})$, (1.3) implies that the sequence h_k is bounded in $\bar{U}_\eta$ for some $\eta>0$, hence admits a subsequence converging in $\mathcal{O}(\bar{U})$. Conversely let us assume that the (injective) map T has closed range. It induces an isomorphism from $\mathcal{O}(\bar{U})$ onto its image, so $h_k \to 0$ if $Th_k \to 0$. If (1.3) did not hold for some small $\varepsilon>0$, we could select a sequence $h_k \in \mathcal{O}(\Omega)$ with $|h_k|_{\bar{U}_{1/k}}=1$ while $Th_k \to 0$, a contradiction. □

1.4. Solvability and non-confinement of singularities

We now come to the main result of this section, which is a hyperfunction version of Theorem 1.2.4 of Hörmander [11]. We shall deal with an open set $U \subset\subset M$ and a differential system (1.1) of order m. Let F be a subspace of $\mathcal{B}(U)$ with the following property.

Hypothesis 1.5. *The space F is a Fréchet space such that*

- *$\mathcal{O}(\bar{U}) \subset F$ with continuous injection, and*
- *if $\Delta \subset \mathbf{C}^n$ is open and Q is a differential operator on M of order $\leq m$, then the space*

$$\{(f,g) \in F \times \mathcal{O}(\Delta) : g|_{U\cap\Delta} = Qf|_{U\cap\Delta}\}$$

is closed in $F \times \mathcal{O}(\Delta)$.

It is not clear whether there always exists a Fréchet space with these properties. However, if P is of order m and M is of class C^m, we can take $F=C^m(U)$. If M is smooth, a stronger statement is obtained by taking a smaller F in the following theorem. For example, if M is real analytic, it may be interesting to take a Gevrey–Beurling space of functions as a space F.

Theorem 1.6. *Let M be as in Notation 1.1 and $U \subset\subset M$ an open set without compact connected component. If the differential system (1.1) on M is of order m and $F \subset \mathcal{B}(U)$ is a Fréchet space as in Hypothesis 1.5, the following condition is sufficient for the induced map $P: \mathcal{B}(U)^d \to \mathcal{B}(U)$ to be onto:*

If f is any hyperfunction in a neighborhood of $\overline{U}$ such that $f|_U \in F$, f is analytic in a neighborhood of ∂U and ${}^tPf$ is analytic in U, then f is analytic in U.

Proof. Given $\varepsilon > 0$ small, we shall show that the estimate

$$h \in \mathcal{O}(\Omega), \quad |h|_{\overline{U}_\eta} \le C(\|h|_U\|_F + |{}^tPh|_{\overline{U}_\varepsilon} + |h|_{(\partial U)_\varepsilon}) \tag{1.4}$$

holds for some $\eta > 0$, some C, and some continuous semi-norm $\|\cdot\|_F$ on F. The *a priori* inequality (1.3) must then hold for $\eta' < \eta$, since, if it did not, we could select a sequence $h_k \in \mathcal{O}(\Omega)$ with $|h_k|_{\overline{U}_{\eta'}} = 1$ while $|{}^tPh_k|_{\overline{U}_\varepsilon} + |h_k|_{(\partial U)_\varepsilon} \to 0$; since the inclusion $\mathcal{O}(\overline{U}) \to F$ is continuous, $\|h_k|_U\|_F$ would be bounded, hence also $|h_k|_{\overline{U}_\eta}$ by (1.4), and we would reach a contradiction: some subsequence would converge uniformly in $\overline{U}_{\eta'}$, to 0 close to ∂U, hence everywhere by the uniqueness of analytic continuation.

If $\Delta \subset \mathbf{C}^n$ is open, we denote by $\mathcal{O}^\infty(\Delta)$ the Banach space of bounded holomorphic functions on Δ, with the norm $|\cdot|_\Delta$. Let us consider the subspace

$$E = \{(f, g, h) : g|_U = {}^tPf|_U,\ h|_{U \cap (\partial U)_\varepsilon} = f|_{U \cap (\partial U)_\varepsilon}\}$$

of the Fréchet space $F \times \mathcal{O}^\infty(\overline{U}_\varepsilon)^d \times \mathcal{O}^\infty((\partial U)_\varepsilon)$. Due to Hypothesis 1.5, E is closed, hence a Fréchet space. By the assumption in the theorem, E is the union of the closed balanced convex sets $E(k)$ consisting of all $(f, g, h) \in E$ such that f is the restriction of a function $\hat{f} \in \mathcal{O}^\infty(\overline{U}_{1/k})$ with $|\hat{f}|_{\overline{U}_{1/k}} \le k$. By Baire's theorem, one of these sets is a neighborhood of 0 in E, which implies the estimate (1.4) and finishes the proof of the theorem. □

In general, there is no reason why the sufficient condition for solvability in Theorem 1.6 should be necessary. It is however *locally* the case when P has constant coefficients; recall that we do not assume $M = \mathbf{R}^n$, so this covers the case of a differential operator on $\mathbf{R}^n$ biholomorphically equivalent to a differential operator with constant coefficients.

Theorem 1.7. *If the open set $U \subset\subset M$ is small enough and P has constant coefficients, then $P: \mathcal{B}(U)^d \to \mathcal{B}(U)$ is onto if and only if every hyperfunction f in a neighborhood of $\overline{U}$, satisfying that f is analytic in a neighborhood of ∂U and that ${}^tPf$ is analytic in a neighborhood of $\overline{U}$, is actually analytic in U.*

Proof. Let us recall how local approximation by entire functions is obtained in Baouendi–Treves [1], using the Gaussian kernel. Performing a complex linear transformation, we may assume that the tangent space to M at 0 is $\mathbf{R}^n$. For $\varepsilon > 0$, let us define

$$K_\varepsilon(z) = \frac{1}{(\varepsilon\pi)^{n/2}} e^{-z^2/\varepsilon}, \quad z \in \mathbf{C}^n.$$

If $\phi \in \mathcal{O}'(\overline{U})$, $U \subset\subset M$, $\phi_\varepsilon(z) = \phi(K_\varepsilon(z - \cdot))$ is an entire function and

$$\frac{\partial \phi_\varepsilon}{\partial z_k} = \left(\frac{\partial \phi}{\partial z_k}\right)_\varepsilon, \quad k = 1, \dots, n.$$

The following property follows from the proof in [1], provided $U \ni 0$ is small enough (this condition can be dropped in the case $M = \mathbf{R}^n$).

The function ϕ is analytic close to $z \in U$ if and only if ϕ_ε converges uniformly in a complex neighborhood of z as $\varepsilon \to 0^+$. Moreover, ϕ_ε then converges to the holomorphic extension of ϕ.

Let us now assume that $P: \mathcal{B}(U)^d \to \mathcal{B}(U)$ is onto, so that (1.3) holds, and let us assume that U is so small that the previous property holds for a neighborhood V of $\overline{U}$ in M. Let f be as in the statement of the theorem and $\phi \in \mathcal{O}'(K)$, $\overline{U} \subset K \subset V$, such that $\phi = f$ in a neighborhood of $\overline{U}$. By the previous property, ϕ_ε converges in a complex neighborhood of ∂U while ${}^tP\phi_\varepsilon = ({}^tP\phi)_\varepsilon$ converges in a complex neighborhood of $\overline{U}$. We deduce from (1.3) that ϕ_ε converges in a complex neighborhood of $\overline{U}$, hence f is analytic, thanks again to the previous property. □

2. Examples, applications and remarks

In this section we give several examples of how the results of Section 1 apply to hyperfunction solvability.

2.1. Ordinary differential equations and holonomic systems

Let us first consider the case of an analytic operator

$$P = \sum_{i=0}^{m} a_i(z) \frac{d^i}{dz^i}$$

on an open interval $I \subset \mathbf{R}$, with $m \geq 0$ and $a_m \not\equiv 0$. It is a theorem of Sato, and in fact a simple (striking) application of Sato's theory, that $P: \mathcal{B}(I) \to \mathcal{B}(I)$ is onto. We note that hyperfunction solutions of ${}^tPf = 0$ may have confined analytic singularities: the Dirac measure δ satisfies $z\delta = 0$, the smooth function f which is 0 on $]-\infty, 0]$ and $e^{-1/z}$ on $]0, +\infty[$ is analytic outside 0 and satisfies $(z^2(d/dz) - 1)f = 0$. However the condition in Theorem 1.6 is locally satisfied, using as a Fréchet space F an *ad hoc* space of ultradifferentiable functions.

Lemma 2.1. *If $J \subset\subset I$ is an open interval, there exists $s>1$ such that, if u belongs to the Gevrey–Beurling space $G^{(s)}(J)$ and ${}^tPu \in \mathcal{A}(J)$, then $u \in \mathcal{A}(J)$.*

Proof. Ramis [24] has computed the index of P acting on any space of formal power series with coefficients satisfying a growth condition of Gevrey type. Similar results certainly hold for the usual spaces of Gevrey functions. The partial result of Komatsu [18] is however more than sufficient for our purpose. We sketch a proof using both references. The claim is of a local nature, so we may assume that $0 \in J$ and 0 is the only point of J at which a_m vanishes. Let $\sigma \geq 1$ be the irregularity of the operator ${}^tP$ at 0 (see [24] or [18] for a definition) and $\hat{u}$ the Taylor series of $u \in G^{(s)}(J)$ at 0. As ${}^tP\hat{u} = \widehat{{}^tPu} \in \mathcal{O}_0$, it follows from [24] or [18] that $\hat{u} \in \mathcal{O}_0$ provided $s < \sigma/(\sigma-1)$. Taking $1 < s < \sigma/(\sigma-1)$, we find that u is locally the sum of an analytic function and a $G^{(s)}$ function v with $\hat{v}=0$, hence ${}^tPv=0$. It follows from Lemma 4 in Komatsu [18] that $v=0$. □

Using the finiteness theorem of Kashiwara [15], we shall obtain a local analogue of the just mentioned theorem of Sato, for holonomic systems, as an obvious consequence of the functional analysis Theorem 1.3. Though the following results extend to the general case, we shall assume that $M=\mathbf{R}^n$ for the sake of simplicity. Let $\mathcal{D}$ denote the sheaf of (analytic) differential operators on $\mathbf{C}^n$, and let us consider a general system of differential equations, that is a coherent left $\mathcal{D}$-module $\mathcal{M}$ near $0 \in \mathbf{C}^n$. The module $\mathcal{M}$ admits a free resolution

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{D}^{d_0} \overset{P^1}{\longleftarrow} \mathcal{D}^{d_1} \overset{P^2}{\longleftarrow} \dots \overset{P^n}{\longleftarrow} \mathcal{D}^{d_n} \overset{P^{n+1}}{\longleftarrow} \dots,$$

where $P^1, P^2, \dots, P^n, \dots$ are matrices of differential operators, acting on the right. Applying the functor $\mathrm{Hom}(\,\cdot\,, \mathcal{B}_0)$ to it, we obtain the complex

$$0 \longrightarrow \mathcal{B}_0^{d_0} \overset{P^1}{\longrightarrow} \mathcal{B}_0^{d_1} \overset{P^2}{\longrightarrow} \dots \overset{P^n}{\longrightarrow} \mathcal{B}_0^{d_n} \overset{P^{n+1}}{\longrightarrow} \dots,$$

where P^k acts on the left. The k^{th} cohomology space $\mathrm{Ext}^k(\mathcal{M}, \mathcal{B}_0)$ of this complex does not depend, up to an isomorphism, on the choice of the above resolution of $\mathcal{M}$. Kashiwara proved in [14] vanishing theorems that imply the existence of a resolution of length $\leq n$,

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{D}^{d_0} \overset{P^1}{\longleftarrow} \mathcal{D}^{d_1} \overset{P^2}{\longleftarrow} \dots \overset{P^n}{\longleftarrow} \mathcal{D}^{d_n} \longleftarrow 0,$$

so $\mathrm{Ext}^k(\mathcal{M}, \mathcal{B}_0)=0$ for $k>n$. We refer to Kashiwara [15] for the notion of a holonomic system and the fundamental result that the spaces $\mathrm{Ext}^k(\mathcal{M}, \mathcal{B}_0)$ are finite-dimensional if $\mathcal{M}$ is holonomic. In particular $P^n : \mathcal{B}_0^{d_{n-1}} \to \mathcal{B}_0^{d_n}$ has a range of finite codimension and an obvious generalization of Theorem 1.3 gives the following vanishing theorem, which was obtained by Schapira in [28], using a very different method.

Theorem 2.2. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module defined near $0\in\mathbf{C}^n$ and $\mathcal{B}_0$ the space of germs of hyperfunctions at $0\in\mathbf{R}^n$. Then* $\mathrm{Ext}^n(\mathcal{M},\mathcal{B}_0)=0$.

Let us consider again the more concrete equations (1.1) and (1.2). It is quite clear that the surjectivity of $P\colon\mathcal{B}_0^d\to\mathcal{B}_0$ only depends on the right ideal $\mathcal{I}$ of $\mathcal{D}$ generated by $P_1,\dots,P_d$ near $0\in\mathbf{C}^n$. Let us consider the germ

$$V(P)=\{(z,\zeta)\in T^*\mathbf{C}^n:\sigma(Q)(z,\zeta)=0 \text{ for all } Q\in\mathcal{I}\}$$

of a complex variety in $T^*\mathbf{C}^n$ over 0, where $\sigma(Q)$ denotes the principal symbol of Q. It is a well-known theorem of Sato–Kawai–Kashiwara [25] that $V(P)$ is involutive.

Theorem 2.3. *If $V(P)$ is Lagrangian, then $P\colon\mathcal{B}_0^d\to\mathcal{B}_0$ is onto.*

Proof. Let $\mathcal{J}$ be the left ideal generated by ${}^tP_1,\dots,{}^tP_d$. The left $\mathcal{D}$-module $\mathcal{M}=\mathcal{D}/\mathcal{J}$ has $V(P)$ as its characteristic variety, hence it is holonomic. We could conclude invoking Theorem 1.6 and the results of Honda [10] which imply that an analogue of Lemma 2.1 holds in the general case of a holonomic system.

We shall instead present a different approach, identifying P with the last compatibility condition of a holonomic system; we owe the following proof to P. Schapira. We start with a free resolution of length n of the *right* $\mathcal{D}$-module $\mathcal{N}=\mathcal{D}/\mathcal{I}$

$$0\longleftarrow\mathcal{N}\longleftarrow\mathcal{D}^{d_0}\xleftarrow{\widehat{P}}\mathcal{D}^{d_1}\longleftarrow\dots\longleftarrow\mathcal{D}^{d_n}\longleftarrow 0, \tag{2.1}$$

where $d_0=1$, $d_1=d$ and $\widehat{P}(A_1,\dots,A_d)=\sum_{i=1}^d P_iA_i$. We recall the following results of Kashiwara, see [14, Theorem 3.1.2 and Proposition 3.1.7], which hold for any left (respectively right) holonomic module: $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{N},\mathcal{D})=0$ if $j\neq n$ while $\mathcal{N}^*:=\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{N},\mathcal{D})$ is a right (respectively left) holonomic $\mathcal{D}$-module. Thus, applying the functor $\mathcal{H}om_{\mathcal{D}}(\,\cdot\,,\mathcal{D})$ to the resolution (2.1), we obtain the resolution

$$0\longrightarrow\mathcal{D}^{d_0}\xrightarrow{Q}\mathcal{D}^{d_1}\longrightarrow\dots\longrightarrow\mathcal{D}^{d_n}\longrightarrow\mathcal{N}^*\longrightarrow 0,$$

of the holonomic left $\mathcal{D}$-module $\mathcal{N}^*$. Here we have used the canonical identification of $\mathcal{H}om_{\mathcal{D}}(\mathcal{D}^{d_k},\mathcal{D})$ (morphisms of right $\mathcal{D}$-modules!) with $\mathcal{D}^{d_k}$. It remains to identify the morphism Q; by the definitions, we have, using obvious notation

$$Q(A)=Q(B\mapsto AB)=((B_1,\dots,B_d)\mapsto A\widehat{P}(B_1,\dots,B_d))=(AP_1,\dots,AP_d).$$

This means precisely that equation (1.1) is the n^{th} "compatibility system" of the holonomic module $\mathcal{N}^*$. Theorem 2.2 applies. □

2.2. Differential equations with constant coefficients

We now consider the case of an operator $P\neq 0$ with constant coefficients. If $M=\mathbf{R}^n$, it is well known that the analogue of the Malgrange–Ehrenpreis Theorem for systems holds true in the context of hyperfunctions, see Komatsu [17] or Schapira [27]. The proofs in [17], [27] make use of the Malgrange–Ehrenpreis Theorem. In the case of a single operator, a simpler proof of the surjectivity of $P:\mathcal{B}(U)\to\mathcal{B}(U)$ is well known *when* $U\subset\mathbf{R}^n$ *is bounded*, using the existence of a fundamental solution E of P: if $f\in\mathcal{B}(U)$, $f=\phi|_U$ for some $\phi\in\mathcal{O}'(\overline{U})$ and $u=(E\star\phi)|_U$ solves $Pu=f$. We note that another proof of this fact is possible, which does not use the existence of a fundamental solution but the classical and easier fact that, if $U\subset\subset\mathbf{R}^n$, there is an estimate

$$\|u\|_{L^2}\leq C\|{}^tPu\|_{L^2} \quad \text{for all } u\in C_0^\infty(U).$$

In fact we can state, more generally, the following theorem.

Theorem 2.4. *Let M, Ω be as in Notation* 1.1 *and assume that M is smooth and that $P_1, \dots, P_d$ have constant coefficients. If the inequality*

$$\|u\|_{L^2}\leq C\|{}^tPu\|_{H^N} \quad \textit{for all } u\in C_0^\infty(M), \tag{2.2}$$

holds for some C and some Sobolev norm $\|\cdot\|_{H^N}$, then $P:\mathcal{B}(U)^d\to\mathcal{B}(U)$ is onto for every open set $U\subset\subset M$ without compact connected component.

Proof. Let $\varepsilon>0$ and $\phi\in C_0^\infty(\overline{U}_\varepsilon)$ a function which is 1 in a neighborhood of $\overline{U}$. If $h\in\mathcal{O}(\Omega)$, applying (2.2) to the (holomorphic) partial derivatives of sufficiently high order of ϕh, Sobolev's inequality on the left-hand side and Cauchy's inequalities on the right-hand side, we obtain $|h|_{\overline{U}}\leq C(|{}^tPh|_{\overline{U}_\varepsilon}+|h|_{(\partial U)_\varepsilon})$. Replacing h by $h(\cdot+\zeta)$, $|\zeta|$ small, and taking into account the fact that P commutes with translations, we obtain the *a priori* inequality (1.3). Theorem 1.4 applies. □

2.3. Differential operators of principal type

The result in this section will be generalized in Section 3, so we shall be brief, referring the reader to Section 3 for any notation which might be used here without having been introduced. We first note that if $U\subset\subset\mathbf{R}^n$ is open and P is an elliptic differential operator in a neighborhood of $\overline{U}$, ${}^tPf\in\mathcal{A}(U)\Rightarrow f\in\mathcal{A}(U)$ by a theorem of Sato. Theorem 1.6 applies: $P:\mathcal{B}(U)\to\mathcal{B}(U)$ is onto, hence $P:\mathcal{A}(U)\to\mathcal{A}(U)$ is onto, again by Sato's theorem. However this is a weaker result than the classical existence theorem of Malgrange, since in Malgrange's theorem, P is defined merely in U, not necessarily in a neighborhood of $\overline{U}$.

We now consider a differential operator P of principal type satisfying the condition (P) of Nirenberg–Treves. It would be tempting to obtain the hyperfunction solvability of P directly from a known L^2 estimate, that of Nirenberg–Treves [23] if M is real analytic, that of Beals–Fefferman [2] if M is merely smooth, using a substitute of Theorem 2.4. We have not been able to find this substitute. Of course, this can be done when P has order one.

Theorem 2.5. *If M is smooth and P is a differential operator of order one whose principal part is a non-vanishing vector field, then $P: \mathcal{B}_0 \to \mathcal{B}_0$ is onto if and only if P satisfies condition* (P) *in a neighborhood of* 0 *in M.*

Proof. We only discuss the sufficiency of the condition. By performing standard reductions, we may assume that $P=\partial/\partial z_1$, so P induces a complex vector field on M. In this simpler situation, it is an earlier result of Nirenberg and Treves that condition (P) implies the estimate (2.2) with $N=1$; later on Treves improved it to $N=0$, see Treves [34]. □

The case when $M=\mathbf{R}^n$ can also be settled without much effort. The proof follows the strategy introduced by Hörmander in [12], but the needed microlocal results are already available. The case of a maximally real manifold with low regularity will be treated in Section 3.

Theorem 2.6. *Let $U \subset\subset \mathbf{R}^n$ be open and let P be an* (*analytic*!) *differential operator on a neighborhood of $\bar{U}$, of principal type on U and satisfying condition* (P) *there. If no complete bicharacteristic of P over U lies over a compact subset of U, then $P: \mathcal{B}(U) \to \mathcal{B}(U)$ is onto.*

Sketch of a proof. By a *complete bicharacteristic of P*, we mean a Nagano leaf B in $\dot{T}^*_U\mathbf{C}^n$ of the vector distribution spanned by the radial vector field and the real and imaginary parts of the Hamilton field of P, with the property that B is contained in the characteristic variety of P. Let $f \in \mathcal{B}(U)$ be analytic close to ∂U, with ${}^tPf \in \mathcal{A}(U)$. If $\theta \in \dot{T}^*_U\mathbf{C}^n$, either θ is a non-characteristic point of ${}^tP$, or θ is a characteristic point of "finite type", or θ is a characteristic point of "infinite type" and belongs to a complete bicharacteristic of ${}^tP$; θ cannot belong to the microsupport (or analytic wave front set), in the first case by Sato's theorem, in the second case by a theorem of Trépreau [30], in the third case, since we assume that the complete bicharacteristics of P escape every compact subset of U, by a theorem of Hanges–Sjöstrand [7]: if B is a complete bicharacteristic of ${}^tP$ over U, either B is contained in the microsupport of f, or B does not meet it. Note that [7] is concerned with classical solutions of ${}^tP$, which is sufficient for our purpose, due to the formulation of Theorem 1.6, but the result is actually true for hyperfunction solutions as shown in an unpublished manuscript of the second author. □

2.4. Hypo-analytic structures

The notion of a hypo-analytic structure is defined in Treves [34]. It is locally equivalent to the data of a maximally real manifold $M\subset\mathbf{C}^n$ and a partial de Rham system $\partial/\partial z_1, \dots, \partial/\partial z_d$. For the sake of briefness, we shall only consider this local model, keeping in mind that the next results can be given an invariant meaning in terms of the underlying hypo-analytic structure; we refer to Cordaro–Treves [4] for details. So, we consider a maximally real manifold $M\subset\mathbf{C}^n$ and the equations

$$P(u_1, \dots, u_d)=\sum_{i=1}^{d}\frac{\partial u_i}{\partial z_i}=f, \tag{2.3}$$

$$^tPf=-\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_d}\right)=(u_1, \dots, u_d). \tag{2.4}$$

From Theorem 1.4, we deduce that, if P is solvable, the solutions of $^tPu=0$ satisfy a weak maximum principle.

Theorem 2.7. *Let $U\subset\subset M$ be open and assume that $P\colon\mathcal{B}(U)^d\to\mathcal{B}(U)$ is onto. Then, if $u\in C^0(U)$ satisfies $^tPu=0$, $|u|$ has no strict local maximum.*

Proof. We may assume U as small as we wish and take $\Omega=\mathbf{C}^n$ in Notation 1.1. By the flabbiness of $\mathcal{B}$, $P\colon\mathcal{B}(V)^d\to\mathcal{B}(V)$ is onto for every open set $V\subset U$. By Theorem 1.4, we have

$$|h|_{\overline{V}}\le C|h|_{(\partial V)_\varepsilon}$$

for all $h\in\mathcal{O}(\mathbf{C}^n)$ satisfying the equation $^tPh=0$. Applying this inequality to h^k, taking k^{th} roots and letting $k\to+\infty$, we obtain that the inequality holds with $C=1$. Letting $\varepsilon\to0^+$, we get

$$|h|_{\overline{V}}\le|h|_{\partial V}.$$

To finish the proof, we note that if u is a continuous solution and $z\in U$, by the Baouendi–Treves approximation theorem, there exists a sequence $u_k\in\mathcal{O}(\mathbf{C}^n)$ such that $u_k\to u$ uniformly in a neighborhood of z in M, and $^tPu_k\to0$ in a complex neighborhood of z. We can solve $^tPv_k={}^tPu_k$ with $v_k\in\mathcal{O}(\mathbf{C}^n)$ and $v_k\to0$ in a complex neighborhood of z. Defining $h_k=u_k-v_k$, we have, if W is a small open neighborhood of z in M, $|h_k|_{\overline{W}}\le|h_k|_{\partial W}$, hence $|u|_{\overline{W}}\le|u|_{\partial W}$. □

It is tempting to make the conjecture that the strong maximum principle (that is, $|u|_K\le|u|_{\partial K}$ for every continuous solution of $^tPu=0$ and every compact set $K\subset U$) is a necessary and sufficient condition for $P\colon\mathcal{B}(U)^d\to\mathcal{B}(U)$ to be onto, if U is small enough. We hope to return to this question in the future. Here we shall only illustrate this conjecture by evoking a few known results. First, in the case of a CR structure (in our local model this corresponds to the case when the system ${}^tP$ and the Cauchy–Riemann system induce on M a system of d vector fields $Z_1, \dots, Z_d$ such that $(\operatorname{Re} Z_1, \dots, \operatorname{Re} Z_d, \operatorname{Im} Z_1, \dots, \operatorname{Im} Z_d)$ has rank $2d$), the strong and the weak maximum principles are equivalent and are equivalent to the fact that the Levi form of the structure is definite at no point. This is a consequence of a result of Berhanu [3]. In the special but important case of a CR structure associated to the induced Cauchy–Riemann system $\bar\partial_N$ on a real hypersurface $N \subset \mathbf{C}^m$, this is equivalent to the hyperfunction solvability of P. This is a consequence of more general results of Michel [21]. It is an interesting situation, since the solvability in the smooth category is not known in that case (see however Michel [22], where solvability is obtained in some spaces of Gevrey functions, including the space of analytic functions). Another important case where the result is known to be true is the case, in some sense opposite to the CR case, when $d = n-1$: it is then a special case of the result of Cordaro–Treves [5]. We shall not pursue this question further here.

Let us set $z = (z', z'')$, with $z' = (z_1, \dots, z_d)$, $z'' = (z_{d+1}, \dots, z_n)$ and let $\pi: z \mapsto z''$ denote the projection. An analytic solution of ${}^tPu = 0$ on M extends to a holomorphic function which does not depend on z'. The maximum property in Theorem 2.7 depends, roughly speaking, on the topological geometry of the fibers of $\pi|_M$ on the one side, on the holomorphic geometry of the space of the fibers on the other. In the case of a CR structure, the fibers are points and the structure coincides with its space of fibers; the opposite case is when $d = n-1$, see Cordaro–Treves [5]; in that case the projection takes its values in $\mathbf{C}$ so the holomorphic geometry is trivial, and everything depends on the topology of the fibers. We shall end this section with a sufficient condition for solvability in terms of the topology of the fibers only.

Theorem 2.8. *Suppose that the open set $U \subset M$ is small enough and that, for all $z_0 \in U$, the fiber $\{z \in U : z'' = z_0''\}$ has no compact connected component. Then $P: \mathcal{B}(U)^d \to \mathcal{B}(U)$ is onto.*

Proof. We first recall the content of Lemma 2.2 in Treves [33]. If U is small enough and the assumption in Theorem 2.8 is verified, for every $\varepsilon > 0$, there exists C, such that, for every $z \in U$, there exists a piecewise smooth curve $\gamma: [0,1] \to \overline{U}_\varepsilon$ with the following properties: $\gamma(0) = z$, γ takes its values in the complex fiber $\mathbf{C}^d \times \{z''\}$ of z, $\gamma(1) \in \partial U$ and γ has length $\le C$. If $h \in \mathcal{O}(\mathbf{C}^n)$ we may write

$$h(z) = h(\gamma(1)) - \int_\gamma \partial h = h(\gamma(1)) - \int_\gamma \partial' h,$$

where ∂' stands for the partial holomorphic differential with respect to z'. Thus

$$|h|_{\overline{U}} \le C |\partial' h|_{\overline{U}_\varepsilon} + |h|_{\partial U}.$$

Arguing as at the end of the proof of Theorem 2.4, we obtain the *a priori* inequality (1.3). □

3. Condition (P) on a maximally real manifold

3.1. Statement of the main result

Let $M\subset\mathbf{C}^n$ be an oriented maximally real manifold of class C^3 and

$$P=\sum_{|\alpha|\le m} a_\alpha(z)\frac{\partial^\alpha}{\partial z^\alpha} \tag{3.1}$$

a differential operator of order m, defined and holomorphic in a complex neighborhood Ω of M. The main result of this section is the following result.

Theorem 3.1. *Assume that M is of class $C^{\sup(3,m)}$ and let $U\subset\subset M$ be an open subset without compact connected component. If P is of principal type on U, satisfies condition* (P) *on U, and if no complete bicharacteristic of P over U lies over a compact set in U, then $P\colon\mathcal{B}(U)\to\mathcal{B}(U)$ is onto.*

The meaning of the hypothesis in this statement will be made precise in Section 3.2. Theorem 3.1 is a hyperfunction version of Theorem 7.3 in Hörmander [12]. Actually we follow the strategy introduced in [12] to obtain it, that is we prove the non-confinement of analytic singularities for the adjoint equation. M being of class C^m, we may then apply Theorem 1.6. However we only need that M be C^3 in the following statement, from which Theorem 3.1 follows.

Theorem 3.2. *Let M be of class C^3 and P a differential operator of principal type on M, satisfying condition* (P), *with the property that no complete bicharacteristic of P lies over a compact subset of M. If $u\in\mathcal{B}(M)$ is analytic outside a compact subset of M and Pu is analytic, then u is analytic.*

Proof of Theorem 3.1. If the hypothesis in Theorem 3.1 is satisfied, Theorem 3.2, applied to the operator ${}^tP$ on U, shows that the condition in Theorem 1.6 is verified, taking $F=C^m(U)$ as a Fréchet space. □

We saw in the proof of Theorem 2.6 how the property of non-confinement of singularities in Theorem 3.2 follows from known results when M is real analytic. In the general case the non-confinement property depends on microlocal results on the singularities of u when Pu is analytic, which may be of independent interest. They are announced in Section 3.3. The proof of these statements is reduced in Section 3.4, using the microlocal transformation theory, to the proof of similar statements for the operator $\partial/\partial z_1$ acting at the boundary of a strictly pseudoconvex domain in $\mathbf{C}^n$. This simpler situation is dealt with in Section 3.5.

3.2. Geometry of condition (P)

Let $T^*\mathbf{C}^n$ be the holomorphic vector bundle of $(1,0)$ forms $\lambda=\sum_{i=1}^n \zeta_i dz_i$, with coordinates $(z,\zeta)=(z_1,\dots,z_n,\zeta_1,\dots,\zeta_n)$. It is endowed with the canonical one form $i\mu=\sum_{i=1}^n \zeta_i dz_i$. The holomorphic symplectic form $i\sigma=id\mu$ gives rise to the two real symplectic forms $\mathrm{Re}\, i\sigma$ and $\mathrm{Im}\, i\sigma$. The conormal bundle of M is the real vector bundle $T_M^*\mathbf{C}^n$ over M with fiber

$$(3.2)\qquad (T_M^*\mathbf{C}^n)_z=\{\lambda\in T_z^*\mathbf{C}^n : \mathrm{Re}\,\lambda|_{T_zM}=0\},\quad z\in M.$$

It is a submanifold of $T^*\mathbf{C}^n$, actually a maximally real manifold in $T^*\mathbf{C}^n$, with the important properties that it is *R-Lagrangian* (i.e. $\mathrm{Re}\, i\sigma$ vanishes on it, this is obvious) and *I-symplectic* (i.e. $\mathrm{Im}\, i\sigma$ is non-degenerate on it, this is easy). Hence μ induces a real one form μ^M and σ a real symplectic form σ^M on $T_M^*\mathbf{C}^n$. We shall denote by $\{\cdot,\cdot\}$ and H the Poisson bracket and the Hamilton map on $T^*\mathbf{C}^n$, by $\{\cdot,\cdot\}^M$ and H^M the Poisson bracket and the Hamilton map on $T_M^*\mathbf{C}^n$ associated with the symplectic form σ^M. We note that the radial field $i\varrho=\sum_{i=1}^n \zeta_i(\partial/\partial\zeta_i)$ is related to the canonical one form by the formula $i\varrho=-H(i\mu)$ and we define the radial vector field $\varrho^M=-H^M(\mu^M)$ on $T_M^*\mathbf{C}^n$ by analogy. A basic fact to compute brackets on $T_M^*\mathbf{C}^n$ is the following formula which holds *when a and b are holomorphic*,

$$(3.3)\qquad \{a,b\}^M=i\{a,b\}\quad \text{on } T_M^*\mathbf{C}^n.$$

Finally, we shall denote by $\dot{T}_M^*\mathbf{C}^n$ the manifold obtained from $T_M^*\mathbf{C}^n$ by removing the zero section and by $S_M^*\mathbf{C}^n$ its quotient space under the natural action of $\mathbf{R}^{+*}$. We have the natural maps

$$\dot{T}_M^*\mathbf{C}^n \xrightarrow{\pi} S_M^*\mathbf{C}^n \xrightarrow{\omega} M.$$

The principal symbol p of the operator P is the homogeneous holomorphic function on $T^*\Omega$ defined by

$$p(z,\zeta)=\sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha.$$

Its zero set is the complex characteristic variety of P, the points of which are the characteristic points of P. We shall also denote by p the restriction of this function to $\dot{T}_M^*\mathbf{C}^n$. A characteristic point $\theta\in\dot{T}_M^*\mathbf{C}^n$, or its image $\pi(\theta)=\vartheta\in S_M^*\mathbf{C}^n$ will be called a characteristic point of P over M, or simply a characteristic point. M being of class C^3, $\dot{T}_M^*\mathbf{C}^n$ is of class C^2 and, if $q\in C^2(\dot{T}_M^*\mathbf{C}^n)$ is a real function, H_q^M is a C^1 vector field with well-defined integral curves. An integral curve of H_q^M on which $q=0$ is called a *bicharacteristic* of q; since $H_q^M q=0$, $q=0$ on an integral curve of H_q^M if $q=0$ at some of its points. Here an integral curve is a C^1 map $\gamma\colon I\to\dot{T}_M^*\mathbf{C}^n$ defined on a non-empty interval $I\subset\mathbf{R}$, such that $\gamma'(t)=H_q^M(\gamma(t))$ for all $t\in I$.

We shall always assume that P is of principal type.

Definition 3.3. The operator P is of *principal type* if $dp\wedge\mu\neq 0$ at every characteristic point $\theta\in\dot{T}^*_M\mathbf{C}^n$.

Using the Hamilton isomorphism, an equivalent condition is that $H_p\wedge\varrho\neq 0$ at θ. Since $T^*_M\mathbf{C}^n$ is maximally real, other equivalent conditions are $dp\wedge\mu^M\neq 0$ or $H^M_p\wedge\varrho^M\neq 0$ on $T^*_M\mathbf{C}^n$ at θ. We recall the formulation of condition (P).

Definition 3.4. The operator of principal type P satisfies condition (P), if there is no C^2 complex-valued non-vanishing homogeneous function q in $\dot{T}^*_M\mathbf{C}^n$ such that $\operatorname{Im} qp$ takes both positive and negative values on a bicharacteristic of $\operatorname{Re} qp$.

Condition (P) is necessary for the hyperfunction solvability of P, if P is of principal type; this is a consequence of the stronger result that a weaker condition, the so called condition (Ψ), is necessary for the microlocal solvability of P. Actually condition (Ψ) is also sufficient for the *microlocal* solvability of P (see Trépreau [31] or the updated and more easily available version in Hörmander [13, Chapter VII]), which in some sense is a much stronger result than the one obtained below, but unfortunately it is not clear how to get local from microlocal solvability.

Let $\theta_0\in\dot{T}^*_M\mathbf{C}^n$ be a characteristic point and let $O(\theta_0)$ be the Sussmann orbit of θ_0 (see [29]) for the vector distribution F on $\dot{T}^*_M\mathbf{C}^n$ with fiber

$$F_\theta=\mathbf{R}H^M_{\operatorname{Re} p}(\theta)+\mathbf{R}H^M_{\operatorname{Im} p}(\theta)+\mathbf{R}\varrho^M(\theta),\quad \theta\in\dot{T}^*_M\mathbf{C}^n, \tag{3.4}$$

(the rank of F_θ may depend on θ). More precisely, a point θ belongs to $O(\theta_0)$ if there exists a continuous curve $\gamma\colon[0,1]\to\dot{T}^*_M\mathbf{C}^n$ and real numbers $0=t_0<t_1<\ldots<t_N=1$, such that $\gamma(t_0)=\theta_0$, $\gamma|_{[t_i,t_{i+1}]}$ is an integral curve of a vector field

$$X_i=a_iH^M_{\operatorname{Re} p}+b_iH^M_{\operatorname{Im} p}+c_i\varrho^M$$

with C^1 coefficients a_i, b_i, c_i, $i=0,\ldots,N-1$, and $\gamma(1)=\theta$. The orbit $O(\theta_0)$ has a natural structure of a C^1 manifold (its topology may be finer than the one induced by $\dot{T}^*_M\mathbf{C}^n$), such that the injection $O(\theta_0)\to\dot{T}^*_M\mathbf{C}^n$ is an immersion. Note that the notion is global, even the dimension of the orbit may shrink as M is shrunk.

The vector space (3.4) and the definition of the orbit $O(\theta_0)$ are not invariant under multiplication by an elliptic operator, since $H^M_{ap}=aH^M_p+pH^M_a$. However, this formula shows that the fact that the orbit is contained in the characteristic variety of P is invariant and that the orbit is also invariant in that case, since H_{ap} is proportional to H_p when $p=0$. Using the fact that the orbit of θ is homogeneous by the definition of the vector distribution F, we define the orbit $o(\vartheta)$ of a point $\vartheta\in S^*_M\mathbf{C}^n$ to be the projection in $S^*_M\mathbf{C}^n$ of the orbit of any point $\theta\in\dot{T}^*_M\mathbf{C}^n$ with $\pi(\theta)=\vartheta$.

Definition 3.5. A non-empty set $B \subset S^*_M \mathbf{C}^n$ is called a *complete bicharacteristic of* P (over M) if B is contained in the characteristic variety of P and is the orbit of one, hence of any, of its points.

If $\vartheta \in S^*_M \mathbf{C}^n$ and $\theta \in \pi^{-1}(\vartheta)$, then $\pi_* H^M_{\operatorname{Re} p}(\theta)$ and $\pi_* H^M_{\operatorname{Im} p}(\theta)$ depend on θ, but $\mathbf{R}\pi_* H^M_{\operatorname{Re} p}(\theta)$ and $\mathbf{R}\pi_* H^M_{\operatorname{Im} p}(\theta)$ do not depend on θ, due to the homogeneity of p: F_θ has a well-defined image $\pi_*(F_\theta) \subset T_\vartheta S^*_M \mathbf{C}^n$, which depends only on ϑ, and which we denote by E_ϑ,

$$E_\vartheta = \mathbf{R}\pi_* H^M_{\operatorname{Re} p}(\theta) + \mathbf{R}\pi_* H^M_{\operatorname{Im} p}(\theta), \quad \theta \in \pi^{-1}(\vartheta).$$

Since P is of principal type, E_ϑ is a one or two dimensional vector space.

Definition 3.6. A *bicharacteristic interval* of P is a C^1 curve $\gamma: I \to S^*_M \mathbf{C}^n$, $I \subset \mathbf{R}$ a non-empty interval, such that $E_{\gamma(t)} = \mathbf{R}\gamma'(t)$ for every $t \in I$, and $p(\gamma(t)) = 0$ for some, hence for all, $t \in I$.

If we identify two bicharacteristic intervals which coincide up to reparametrization, there is an obvious notion of *a maximal bicharacteristic interval.* For the classification, it is convenient to endow $S^*_M \mathbf{C}^n$ with a complete Riemannian metric (by the completeness assumption, a curve of finite length is relatively compact in $S^*_M \mathbf{C}^n$), so that we may assume that a maximal bicharacteristic interval is parametrized by arc length; then the parametrization is unique up to the orientation and a translation in $\mathbf{R}$. Let $\gamma: I \to S^*_M \mathbf{C}^n$ be a maximal bicharacteristic interval parametrized by arc length. If $t_0 \in \mathbf{R}$ is an endpoint of I, clearly $\gamma(t)$ has a limit as $t \to t_0$, $t \in I$. Looking at a bicharacteristic of $\operatorname{Re} p$ or $\operatorname{Im} p$ through the limit point, it is clear that $t_0 \in I$. So, either $I = \mathbf{R}$ and γ is a complete bicharacteristic over M, or $I =]-\infty, a]$ (or $I = [a, +\infty[$ according to the orientation), or $I = [a, b]$. In the last two cases we shall refer to $\gamma(a)$, $\gamma(b)$ as the endpoint(s) of γ.

Condition (P) has strong consequences on the geometry of the complete bicharacteristics of P. Let us recall the following theorem of Hörmander [12].

Theorem 3.7. *We assume that condition* (P) *is satisfied. If the orbit* $o(\vartheta)$ *of* $\vartheta = \pi(\theta) \in S^*_M \mathbf{C}^n$ *contains a characteristic point* ϑ' *with* $\dim E_{\vartheta'} = 2$, *then* $o(\vartheta)$ *is a complete two dimensional bicharacteristic. Moreover, if one identifies any two points of the orbit which belong to the same bicharacteristic interval, the resulting* C^0*-manifold has a natural structure of a Riemann surface. A function* u *in* $o(\vartheta)$ *is "holomorphic" if* $H^M_p(u \circ \pi) = 0$ *on* $O(\theta)$ (*then* u *induces a well-defined function on the reduced orbit*).

The first part of this statement is contained in [12, Proposition 2.1 and Proposition 2.4], the second part in [12, Section 4]. Presumably, the description of the complex structure of the reduced bicharacteristics should be simpler in our setup than in [12], because P is analytic, if M is not.

Let $\mathcal{V}\subset S^*_M\mathbf{C}^n$ be (the projection of) the characteristic variety of P over M and $\vartheta\in\mathcal{V}$. If $\dim E_\vartheta=2$, then $o(\vartheta)\subset\mathcal{V}$ by Theorem 3.7: starting from ϑ and following successively (the projection of) integral curves of $H^M_{\mathrm{Re}\,p}$ or $H^M_{\mathrm{Im}\,p}$, one stays in $\mathcal{V}$ and travels through a two dimensional complete bicharacteristic of P. If $\dim E_\vartheta=1$, but ϑ is the limit of a sequence of points $\vartheta'\in\mathcal{V}$ with $\dim E_{\vartheta'}=2$, then clearly $o(\vartheta)\subset\mathcal{V}$ by continuity, hence ϑ belongs to a one or to a two dimensional complete bicharacteristic of P. Thus we have the following lemma.

Lemma 3.8. *If condition* (P) *is satisfied, the characteristic variety* $\mathcal{V}\subset S^*_M\mathbf{C}^n$ *of* P *over* M *is the* (*not necessarily disjoint*) *union of* $\mathcal{V}^b$, *the union of all one or two dimensional complete bicharacteristics of* P *over* M, *and the set* $\mathcal{V}^0\subset\mathcal{V}$ *defined by*

$$\mathcal{V}^0=\{\vartheta\in\mathcal{V}:\dim E_{\vartheta'}=1 \text{ for all } \vartheta'\in\mathcal{V} \text{ close to } \vartheta\}. \tag{3.5}$$

Roughly speaking, Theorem 3.2 will follow in Section 3.3 from the propagation of singularities along the bicharacteristic intervals and the complete two dimensional bicharacteristics of P and the hypoellipticity of P at certain points in $\mathcal{V}^0$. For these points, the following lemma will be used.

Lemma 3.9. *We assume that condition* (P) *is satisfied. Let* $\vartheta=\pi(\theta)\in\mathcal{V}^0$ *and let* a *be a complex positively homogeneous function of class* C^2 *near* θ, *with* $H^M_{\mathrm{Re}\,ap}\wedge\varrho^M\neq0$. *Let* γ_a *be the germ of the bicharacteristic of* $\mathrm{Re}\,ap$ *through* θ, *with* $\gamma_a(0)=\theta$. *The property that*

- $\mathrm{Im}\,ap$ *takes a negative value on* $\gamma_a(\,]-\varepsilon,0[\,)$ *or a positive value on* $\gamma_a(\,]0,+\varepsilon[\,)$, *for every small* $\varepsilon>0$,

does not depend on a. *If it is satisfied,* ϑ *is called a point of* positive type.

Proof. We may assume that

$$H^M_{\mathrm{Re}\,p}\wedge\varrho^M\neq0,\quad H^M_{\mathrm{Im}\,p}\wedge\varrho^M=0 \text{ at } \theta.$$

As $H^M_{\mathrm{Re}\,ap}=(\mathrm{Re}\,a)H^M_{\mathrm{Re}\,p}-(\mathrm{Im}\,a)H^M_{\mathrm{Im}\,p}$ at θ, $\mathrm{Re}\,a(\theta)\neq0$ if the condition in Lemma 3.9 is satisfied, so we may write $a=(1+i\beta)\,\mathrm{Re}\,a$. If $\beta=0$, that is p is multiplied by a non-vanishing real function, the bicharacteristic of $\mathrm{Re}\,p$ is preserved, with the preserved or reversed orientation depending on the sign of a, and the same happens to the sign of $\mathrm{Im}\,p$; the invariance is clear in that case. It remains to consider the case of $a=1+i\beta$. We have to look at the sign of $\beta\,\mathrm{Re}\,p+\mathrm{Im}\,p$ along the bicharacteristic of $\mathrm{Re}\,p-\beta\,\mathrm{Im}\,p$. As $\mathrm{Re}\,p=\beta\,\mathrm{Im}\,p$ along this bicharacteristic, this is the same as the sign of $(1+\beta^2)\operatorname{Im} p$, hence as the sign of $\operatorname{Im} p$. We use a homotopy argument. Let γ_t be the bicharacteristic of $\operatorname{Re} p - t\beta \operatorname{Im} p$ through θ, $0 \le t \le 1$. If (for example) $\operatorname{Im} p$ takes a positive value on $\gamma_0(\,]0,+\varepsilon[\,)$ for all $\varepsilon>0$ and is ≤ 0 on $\gamma_1(\,]0,+\varepsilon[\,)$, by the condition (P) and continuity, $\operatorname{Im} p$ must be zero on $\gamma_t(\,]0,+\varepsilon[\,)$ for some $t\in[0,1]$. Then $\gamma_t(\,]0,+\varepsilon[\,)$ is contained in the characteristic set of P, hence its image in $S^*_M\mathbf{C}^n$ is a bicharacteristic interval, since it is contained in $\mathcal{V}^0$. This implies that $\gamma_0(\,]0,+\varepsilon[\,)$ and $\gamma_1(\,]0,+\varepsilon[\,)$ have the same germ of image near ϑ, which is a contradiction. □

3.3. Propagation and non-confinement of singularities

In this section we reduce the proof of Theorem 3.2 to microlocal results concerning the singularities of u, when Pu is analytic, and their propagation. We first recall a little of Sato's theory for maximally real manifolds with low regularity. Let $\mathcal{E}$ denote the sheaf on $T^*\mathbf{C}^n$ of microdifferential operators; we shall also consider $\mathcal{E}$ as a sheaf on $S^*_M\mathbf{C}^n$, using the homogeneity. The sheaf $\mathcal{C}_M$ of microfunctions is a flabby sheaf of $\mathcal{E}$-modules on $S^*_M\mathbf{C}^n$ with the other main property: there is a sheaf morphism $\mathcal{B}_M \to \omega_*\mathcal{C}_M$ compatible with the natural action of differential operators, such that the support of the image of a section u of $\mathcal{B}_M$ is the microsupport of u. Concerning the notion of the microsupport, for the time being, we need only mention that a hyperfunction is analytic near a point $z\in M$ if and only if its microsupport does not meet the fiber of z in $S^*_M\mathbf{C}^n$. The proof of Theorem 3.2 depends on the next three microlocal results.

Theorem 3.10. *Let P be an operator of principal type and let b be a bicharacteristic interval of P. If $\hat{u}$ is a microfunction defined in a neighborhood of b and $P\hat{u}=0$, then either the support of $\hat{u}$ contains b, or it does not meet b.*

As a special case, complete one dimensional bicharacteristics propagate singularities. This is also the case for complete two dimensional bicharacteristics, at least when condition (P) is satisfied.

Theorem 3.11. *Let P satisfy condition* (P) *and let B be a complete one or two dimensional bicharacteristic of P. If $\hat{u}$ is a microfunction defined in a neighborhood of B and $P\hat{u}=0$, then either the support of $\hat{u}$ contains B, or it does not meet B.*

Finally, for points in $\mathcal{V}^0$ (see (3.5) and Lemma 3.9), we have the following result.

Theorem 3.12. *Let P satisfy condition* (P) *and let $\vartheta\in\mathcal{V}^0$ be a point of positive type. If $\hat{u}$ is a microfunction defined in a neighborhood of ϑ and $P\hat{u}=0$, then ϑ does not belong to the support of $\hat{u}$.*

Proof of Theorem 3.2. Let $u \in \mathcal{B}(M)$ be analytic outside a compact set $K \subset M$ and such that Pu is analytic. We must prove that u is analytic. Assuming it is not and that ϑ belongs to the support of the microfunction $\hat{u} \in \mathcal{C}_M(S^*_M\mathbf{C}^n)$ associated to u, we shall reach a contradiction.

The point ϑ must be a characteristic point of P since by Sato's theorem, P is invertible at every non-characteristic point. The point ϑ does not belong to a complete bicharacteristic of P, since by Theorem 3.11, this bicharacteristic would be contained in the support of $\hat{u}$, hence lie over K, which is forbidden by the assumption in Theorem 3.2. As a consequence, $\vartheta \in \mathcal{V}^0 \setminus \mathcal{V}^b$. By Theorem 3.12, $\vartheta \in \mathcal{V}^0$ cannot be a point of positive type, hence with the notation in Lemma 3.9, $\operatorname{Im} ap$ must be $\equiv 0$ on $\gamma(\,]-\varepsilon, 0])$ or on $\gamma([0, +\varepsilon[\,)$, which by the definition of $\mathcal{V}^0$ implies that ϑ belongs to a maximal bicharacteristic interval $\gamma: I \to S^*_M\mathbf{C}^n$. Since $\vartheta \notin \mathcal{V}^b$ either $I = [t^-, t^+]$ or $I = [0, +\infty[$. In the first case, we may select a non-vanishing homogeneous complex-valued function a such that $H^M_{\operatorname{Re} ap} \wedge \varrho^M \neq 0$ along $\pi^{-1}(\gamma)$, hence γ is the projection of a bicharacteristic of $\operatorname{Re} ap$. Now, as γ is maximal and contained in $\mathcal{V}^0$, $\operatorname{Im} ap$ must be $\neq 0$ somewhere on the left (respectively on the right) of $\gamma(t^-)$ (respectively $\gamma(t^+)$) on the bicharacteristic of $\operatorname{Re} ap$. Using condition (P), it is then clear that at least one of the endpoints of γ is of positive type, which is forbidden by Theorem 3.12 and the propagation along γ. So $I = [0, +\infty[$. Since the support of $\hat{u}$ lies over K, so does γ by Theorem 3.10, hence it has a cluster point ϑ_0 as $t \to +\infty$. Then ϑ_0 belongs to a complete one dimensional bicharacteristic lying over K (see the proof of Theorem 7.1 in [12]), which is not allowed by the hypothesis in Theorem 3.2. □

It may be worth mentioning that we have obtained the non-confinement of *analytic* singularities as a by-product of simple statements about the existence of analytic singularities and their propagation. The case of C^∞ singularities is much more complicated, see the analysis of Hörmander in [12].

3.4. Complex canonical reduction

We shall now reduce the problem to a model problem. This is similar to what was done in [31] apart from the fact that we shall use the transformation theory of Kashiwara and Schapira [16] in place of the earlier but less general transformation theory of Kashiwara and Kawai.

Let $\Omega \subset \mathbf{C}^n$ be a strictly pseudoconvex domain of class C^2, near its boundary point $0 \in N := \partial\Omega$, and f a defining function of Ω:

$$\Omega := \{z : f(z) < 0\}. \tag{3.6}$$

We still define the conormal bundle $T^*_N\mathbf{C}^n$ by (3.2) and it is still the case, due to the assumption that the Levi form is non-degenerate, that $T^*_N\mathbf{C}^n \subset T^*\mathbf{C}^n$ is maximally real, R-Lagrangian and I-symplectic. We shall use the same notation as in Section 3.2, with superscript N, for the objects associated with the induced symplectic structure. The manifold $\dot{T}^*_N\mathbf{C}^n$ is defined by the equation

$$(3.7)\qquad \dot{T}^*_N\mathbf{C}^n=\left\{(z,\zeta):f(z)=0,\ \zeta_j=k\frac{\partial f}{\partial z_j}(z)\text{ for }1\le j\le n\text{ and some }k\in\mathbf{R}^*\right\}.$$

We shall work on the *outer* component $T^{*+}_N\mathbf{C}^n$ of $\dot{T}^*_N\mathbf{C}^n$, defined by taking $k>0$ in (3.7), and the outer component

$$(3.8)\qquad S^{*+}_N\mathbf{C}^n=T^{*+}_N\mathbf{C}^n/\mathbf{R}^{+*}$$

of $S^*_N\mathbf{C}^n$. Since its fiber at $z\in N$ consists of just one point, we shall often identify $S^{*+}_N\mathbf{C}^n$ with N, using $\omega: S^{*+}_N\mathbf{C}^n\to N$. The role of the sheaf of microfunctions will be played by the sheaf on N of "holomorphic functions on the Ω side of N modulo holomorphic functions on N". More precisely, we introduce the sheaf $\mathcal{C}_N$ on N or $S^{*+}_N\mathbf{C}^n$ with stalk at $z\in N$:

$$(3.9)\qquad \mathcal{C}_{N,z}=\varinjlim_{\Delta\ni z}\mathcal{O}(\Delta\cap\Omega)/\mathcal{O}(\Delta),$$

Δ running through the set of all open complex neighborhoods of z. The sheaf $\mathcal{C}_N$ is a sheaf of $\mathcal{E}$-modules.

Let us return to the situation considered in Section 3.3. Let P be of principal type and $\pi(\theta)=\vartheta\in S^*_M\mathbf{C}^n$ be a characteristic point of P. We can find a homogeneous holomorphic canonical transformation χ of $T^*\mathbf{C}^n$, defined in a complex conic neighborhood of θ, such that the complex characteristic variety of P is transformed into the complex hypersurface $\{\zeta_1=0\}$, while $\dot{T}^*_M\mathbf{C}^n$ is locally transformed into the outer conormal bundle to the boundary $N\ni 0$ of a strictly pseudoconvex domain of class C^3 in $\mathbf{C}^n$ (we refer to Section 5 in [31] or Section 7.4 in [13], where this statement is proved when $M=\mathbf{R}^n$, but the proof does not use this fact). By the theory of Sato–Kawai–Kashiwara [25], χ can be quantized as an isomorphism $\widehat{\chi}: \chi_*\mathcal{E}_\theta\to\mathcal{E}_{\chi(\theta)}$, in such a way that the principal symbol $\sigma(Q)$ of a microdifferential operator Q is transformed according to the formula $\sigma(\widehat{\chi}(Q))=\sigma(Q)o\chi^{-1}$. In particular, $\sigma(\widehat{\chi}(P))$ vanishes at order one on $\{\zeta_1=0\}$ and by another classical result of [25], there exists an elliptic, hence invertible, microdifferential operator A defined in a conic complex neighborhood of θ such that $\widehat{\chi}(AP)=\partial/\partial z_1$.

Finally, by the transformation theory of Kashiwara and Schapira (see [16, Theorem 11.4.9]), χ can be locally quantized as a sheaf isomorphism $\widehat{\chi}: \chi_*\mathcal{C}_M\to\mathcal{C}_N$, which is compatible with the transformation of microdifferential operators, that is $\widehat{\chi}(Qu)=\widehat{\chi}(Q)\widehat{\chi}(u)$.

So any microlocal statement about a differential operator of principal type on a maximally real manifold M of class C^3, whose formulation is invariant under canonical transformations and multiplication by an elliptic operator, has only to be tested in the case of the simple operator $\partial/\partial z_1$ acting at the boundary of a strictly pseudoconvex domain of class C^3. This is, almost, the case for the statements in the previous section. So, from now on, we shall consider the operator

$$\frac{\partial}{\partial z_1} : \mathcal{C}_{N,0} \longrightarrow \mathcal{C}_{N,0}. \tag{3.10}$$

3.5. The study of the model

We consider now the operator (3.10). We use the notation

$$z' = (z_2, \dots, z_n), \quad z'' = (z_2, \dots, z_{n-1}), \quad z_k = x_k + iy_k \text{ for } k = 1, \dots, n.$$

Since the non-characteristic case is trivial and our problem is invariant under a local biholomorphism that preserves the equation $\partial u/\partial z_1 = 0$, we may perform an elementary reduction (we refer to Section 2 in [31] or Section 7.1 in [13] for the details of this reduction), so that Ω has a local equation

$$\Omega := \{z : f(z) := -x_n + h(z_1, z'', y_n)^2 + g(y_1, z'', y_n) < 0\}, \tag{3.11}$$

with $h \in C^2$, $g \in C^3$,

$$h(0) = g(0) = 0, \quad dg(0) = 0, \quad \partial h(0)/\partial x_1 \neq 0. \tag{3.12}$$

Let $\hat{u} \in \mathcal{C}_{N,0}$, $\partial \hat{u}/\partial z_1 = 0$, and let u be a holomorphic representative of $\hat{u}$. As $\partial u/\partial z_1 \in \mathcal{O}_0$, solving $\partial v/\partial z_1 = \partial u/\partial z_1$ with $v \in \mathcal{O}_0$, replacing u by $u - v$, we see that we may assume that $\partial u/\partial z_1 = 0$. In this situation, it is easy to show, see [31], [13], that $u = v \circ \delta$ for some holomorphic function v in a "local projection" of Ω under the map

$$\delta : \mathbf{C}^n \longrightarrow \mathbf{C}^{n-1}, \quad \delta((z_1, z')) = z'. \tag{3.13}$$

Localizing everything near $0 \in \mathbf{C}^n$, we define

$$\Omega := \{z : |x_1| \le \varepsilon,\ |y_1| \le \varepsilon,\ |x_n| < \varepsilon,\ |z''| + |y_n| < \eta,\ f(x_1 + iy_1, z') < 0\},$$
$$N := \{z : |x_1| < \varepsilon,\ |y_1| < \varepsilon,\ |x_n| < \varepsilon,\ |z''| + |y_n| < \eta,\ f(x_1 + iy_1, z') = 0\}.$$

In the sequel, we shall assume that $0<\eta\ll\varepsilon\ll 1$ are chosen small enough. Then $\hat{u}$ is represented by a function $u=v\circ\delta$ where $v\in\mathcal{O}(\Delta)$ and $\Delta\subset\mathbf{C}^{n-1}$ is the open set defined by

$$\Delta:=\{z:|x_n|<\varepsilon,\ |z''|+|y_n|<\eta,\ -x_n+G(z'',y_n)<0\},$$

where $G(z'',y_n)=\min_{|y_1|\le\varepsilon} g(y_1,z'',y_n)$. We note that Δ is a supergraph of a Lipschitz function since, with $\xi=(z'',y_n)$ and an obvious notation, we may write

$$G(\xi')-G(\xi)=g(y_1(\xi'),\xi')-g(y_1(\xi),\xi)\le g(y_1(\xi),\xi')-g(y_1(\xi),\xi)\le C|\xi'-\xi|.$$

Whether 0 belongs to the support of $\hat{u}$ or not depends, on one hand, on the position of the fiber $\delta^{-1}(\delta(0))$ of 0 with respect to Ω, on the other, on the holomorphic convexity of Δ at $\delta(0)=0\in\mathbf{C}^{n-1}$. We have the following lemma.

Lemma 3.13. *The following properties hold for* $0<\eta\ll\varepsilon\ll 1$.

(1) *If* $\delta(0)\in\Delta$, *or if* $\delta(0)\in\partial\Delta$ *and* $\delta(0)$ *belongs to the envelope of holomorphy of* Δ, *then* 0 *does not belong to the support of* $\hat{u}$.

(2) *If* $\delta(0)\in\partial\Delta$, *then either the support of* $\hat{u}$ *contains the set* $N\cap\delta^{-1}(\delta(0))$, *or it does not meet it.*

(3) *If* $\delta(0)\in\partial\Delta$ *and* $A:\bar{D}\to\bar{\Delta}$ ($D\subset\mathbf{C}$ *denotes the open unit disc*) *is an analytic disc with* $A(0)=0$, *the following property holds*: 0 *does not belong to the support of* $\hat{u}$ *if* $A(D)$ *intersects* Δ, *while if* $A(D)\subset\partial\Delta$, *then the projection of the support of* $\hat{u}$ *either contains or does not meet* $A(D)$.

Proof. Properties (1) and (2) are self-evident. To prove property (3), we argue as in [32, Lemma 3.2]. Let us assume that v is holomorphic in the union of Δ and a small ball around $A(\tau_0)$ ($\tau_0\in D\setminus\{0\}$, $A(\tau_0)$ may belong to Δ or not) but cannot be holomorphically extended near $0\in\mathbf{C}^{n-1}$. Introducing (as in [32, Lemma 1.2]) the envelope of holomorphy $\hat{\Delta}$ of the union of Δ and this small ball, this means that $0\notin\hat{\Delta}$. We shall reach a contradiction. Set $e=(0,\dots,0,1)\in\mathbf{C}^{n-1}$ and let $\hat{d}(z)$ denote the euclidean distance of $z\in\hat{\Delta}$ to the boundary of $\hat{\Delta}$. Since $A(\bar{D})\subset\bar{\Delta}$ and Δ is a supergraph of a Lipschitz function, $A(\tau)+ite\in\Delta$, for small $t>0$ and every $\tau\in\bar{D}$, and $\hat{d}(A(\tau)+te)\ge t/C$. On the other hand, $\hat{d}(te)\le t$, since $0\in\partial\hat{\Delta}$. The function $u_t(\tau)=-\log\hat{d}(A(\tau)+te)$, $t>0$, is subharmonic in D, uniformly bounded in some disc of center τ_0, radius $\varrho>0$. Using the mean value property we obtain

$$-\log t\le u_t(0)\le\iint_D u_t(\tau)\frac{d\bar{\tau}\wedge d\tau}{2i\pi}\le\frac{1}{\pi}((\pi-\pi\varrho^2)(-\log t/C)+\pi\varrho^2 M).$$

Letting $t\to 0^+$, we reach a contradiction. $\square$

The differential operator $\partial/\partial z_1$ is of principal type, with principal symbol ζ_1. To emphasize the link with Section 3.3, we shall use the notation

$$p=\zeta_1|_{T_N^*\mathbf{C}^n}.$$

Taking (3.7), (3.11), (3.12) into account, we have

$$p=k\frac{\partial f}{\partial z_1},\quad \operatorname{Re}p=\frac{k}{2}\left(2h\frac{\partial h}{\partial x_1}\right),\quad \operatorname{Im}p=-\frac{k}{2}\left(2h\frac{\partial h}{\partial y_1}+\frac{\partial g}{\partial y_1}\right).$$

The characteristic points over N are thus defined by $h=0$, $\partial g/\partial y_1=0$. Our task is now to establish a link between the properties of the symbol p and its Hamiltonian field, which occur in the statements of Theorems 3.10–3.12, and the properties in Lemma 3.13, which are stated in terms of the fibers and the image of the map $\delta\colon(z_1,z')\mapsto z'$. For example, we would like to recognize a bicharacteristic interval, or a characteristic point of positive type, by looking at the behaviour of the function $y_1\mapsto\partial g(y_1,z'',y_n)/\partial y_1$. This will be possible, thanks to the following lemma.

Lemma 3.14. *Let $\theta\in\dot{T}_N^*\mathbf{C}^n$ be a characteristic point with $dp\wedge d\bar{p}\wedge\mu^N=0$ at θ. In the coordinate system (x_1,y_1,x'',y',k), we have*

$$H_p^N=i\frac{\partial}{\partial z_1}-i\bar{a}\frac{\partial}{\partial\bar{z}_1}+\beta\frac{\partial}{\partial k}\quad \text{at } \theta,$$

with $a=(\partial h/\partial\bar{z}_1)(\partial h/\partial z_1)^{-1}$, $\beta\in\mathbf{C}$.

Hence, $\pi_*H_p^N$ is tangent to the zero set $\Sigma=\{h=0\}$ of Re p *at $\pi(\theta)$, and if we use (y_1,x'',y') as a coordinate system on Σ, we have*

$$\pi_*H_{\operatorname{Re}p}^N=\alpha\frac{\partial}{\partial y_1}\quad \text{at } \pi(\theta),\ \text{with } \alpha=\frac{1}{4}\left(\frac{\partial h}{\partial x_1}\right)^2\left|\frac{\partial h}{\partial z_1}\right|^{-2}>0.$$

Proof. By our assumption we have the relation

$$d\bar{\zeta}_1=a\,d\zeta_1+ib\sum_{\alpha=1}^{n}\zeta_\alpha\,dz_\alpha\quad \text{at } \theta, \tag{3.14}$$

on $\dot{T}_N^*\mathbf{C}^n$, for some $a,b\in\mathbf{C}$. Using only formula (3.3) we get

$$\langle H_{\zeta_1}^N,dz_j\rangle=\{\zeta_1,z_j\}^N=i\{\zeta_1,z_j\}=i\delta_{1j},\quad j=1,\dots,n.$$

Using (3.14) and formula (3.3) we compute

$$\langle H^N_{\zeta_1}, d\bar z_j\rangle = \{\zeta_1, \bar z_j\}^N = \overline{\{\bar\zeta_1, z_j\}^N} = \bar a\overline{\{\zeta_1, z_j\}^N} - i\bar b\sum_{\alpha=1}^n \bar\zeta_\alpha\overline{\{z_\alpha, z_j\}^N}$$
$$= -i\bar a\delta_{1j} \quad \text{at } \theta,\ j=1,\dots,n.$$

To compute a, we rewrite (3.14),

$$d\left(k\frac{\partial f}{\partial \bar z_1}\right) = ad\left(k\frac{\partial f}{\partial z_1}\right) + ibk\partial f,$$

and chase the coefficients of dz_1 and $d\bar z_1$ in this equality, in our coordinate system. For the sake of simplicity, we shall write f_{z_1} for $\partial f/\partial z_1$, etc. We note that the term $ibk\partial f$ gives no contribution, since at θ, $f_{z_1}=0$ and $dx_n=dg$ with $g_{z_1}=0$, $g_{\bar z_1}=0$. As f_{z_1}, $f_{\bar z_1}$ do not depend on x_n, we obtain

$$f_{\bar z_1 z_1} = af_{z_1 z_1}, \quad f_{\bar z_1\bar z_1} = af_{\bar z_1 z_1}.$$

By addition we get $f_{\bar z_1 x_1}=af_{z_1 x_1}$. As g does not depend on x_1 and $h=0$ at θ, we obtain $a=h_{\bar z_1}/h_{z_1}$ and the first part of the lemma.

The second part follows, since the formula for H^N_p gives $H^N_p h=0$ at θ and $H^N_p y_1=\frac{1}{2}(1+\bar a)$ with real part $\alpha=h^2_{x_1}/4|h_{z_1}|^2$. □

We are now in a position to prove the statements of Section 3.3.

Proof of Theorem 3.10. If $b\ni 0$ is a bicharacteristic interval of $\partial/\partial z_1$, then b is the projection of an integral curve of $H^N_{\operatorname{Re} p}$ along which $p=0$, and the condition in Lemma 3.14 is satisfied. Hence $h=0$, $\partial g/\partial y_1=0$ along b, and by Lemma 3.14, z'', y_n are constant $=0$ along b; hence also $g(y_1, z'', y_n)=g(y_1, 0, 0)$ is constant $=0$ along b. So $z'=0$ on b and $b=N\cap\delta^{-1}(0)$, and Lemma 3.13, property (2) applies. □

From now on we assume that $\partial/\partial z_1$ satifies condition (P) on $S^{*+}_N\mathbf{C}^n$, though this is perhaps not necessary for the next result (we shall avoid proving an analogue of the Hanges–Sjöstrand Theorem [7] in our context, condition (P) simplifies things).

Proof of Theorem 3.11. Theorem 3.10 applies to one dimensional bicharacteristics, so we consider the case of a complete two dimensional bicharacteristic. We must be careful since its definition is global while our reduction is local. However, this is not important for the following reason: if ϑ belongs to a two dimensional bicharacteristic, and also to a bicharacteristic interval, as propagation along a bicharacteristic interval has already been established, we may localize our study close to an endpoint of it. Hence we may assume without loss of generality, that

ϑ does not belong to, or is an endpoint of, a bicharacteristic interval. Then it is easily seen that the complex structure of the reduced two dimensional bicharacteristic through ϑ is locally determined and the following argument is meaningful. By Theorem 3.7, we may assume that there exists a germ of a two dimensional manifold $0 \in B \subset N$ such that the space obtained by shrinking any bicharacteristic interval in B to a point has a complex structure, the holomorphic functions of which are induced by solutions of $H_p^N u=0$. We note that the map δ is well defined on the reduced space, since a bicharacteristic interval projects in a fiber of δ by the proof of Theorem 3.10. It is holomorphic since $H_p^N z_j = i\{\zeta_1, z_j\}=0$ if $j>1$. Hence $\delta\colon B \to \mathbf{C}^{n-1}$ may be locally considered as an analytic disc, as in Lemma 3.13, property (3). The lemma applies (actually Δ is pseudoconvex [31] and $\delta(B) \subset \partial\Delta$ but we do not need these facts.) □

Proof of Theorem 3.12. It is an immediate consequence of Lemma 3.13 and the following result.

Lemma 3.15. *With the notation in Lemma* 3.13, $\delta(0) \in \Delta$ *if* $0 \in \mathcal{V}^0$ *is a point of positive type.*

It was proved in [31] that, if $\operatorname{Im} p = -k\partial g/\partial y_1$ does not change sign from $-$ to $+$ along any bicharacteristic of $\operatorname{Re} p$, it does not change sign from $-$ to $+$ along any integral curve of $\partial/\partial y_1$. No such result holds for sign changes from $+$ to $-$. The main ingredient in the proof of Lemma 3.15 is that this is however the case on the part $\mathcal{V}^0$ of the characteristic variety, when condition (P) is satisfied.

Proof of Lemma 3.15. We identify $S_N^{*+}\mathbf{C}^n$ with N and we parametrize the zero set $\Sigma = \{z \in N : h(z)=0\}$ of $\operatorname{Re} p$ by y_1, x'', y'. The vector field $H_{\operatorname{Re} p}^N$ induces a vector field $\pi_* H_{\operatorname{Re} p}^N$ on Σ (because p is homogeneous of degree 1) and by condition (P), $\partial g/\partial y_1$ does not change sign along its integral curves. We shall use the homotopy

$$X_t = (1-t)\pi_* H_{\operatorname{Re} p}^N + t\frac{\partial}{\partial y_1}, \quad 0 \le t \le 1,$$

between the vector fields $\pi_* H_{\operatorname{Re} p}^N$ and $\partial/\partial y_1$ on Σ. By the definition (3.5) of $\mathcal{V}^0$ and Lemma 3.14, we have near 0,

$$X_t = \alpha_t \frac{\partial}{\partial y_1} \text{ with } \alpha_t > 0, \quad \text{if } \frac{\partial g}{\partial y_1} = 0.$$

It is then a consequence of the Bony–Brézis lemma, see [13, Lemma 7.3.4], applied to $\pm\partial g/\partial y_1$, that $\partial g/\partial y_1$ does not change sign along the integral curves of X_t, $0 \le t \le 1$. We may assign a sign $s(t)$ to t: $s(t)=-1, 0, +1$, depending on if $\partial g/\partial y_1$

takes a negative value in every neighborhood of 0, or is $\equiv 0$, or takes a positive value in every neighborhood of 0, on the half integral curve of X_t ending at 0. If $s(0)>0$, then $s(t)>0$ for all t, since otherwise, by continuity, $s(t)$ would vanish for some t, which would mean that the half integral curve of X_0 ending at 0 is a bicharacteristic interval. This is a contradiction. So $s(1)>0$ and $g(y_1, 0)$ is a nonconstant nondecreasing function on $[-\varepsilon, 0]$, hence takes negative values, which implies that $\delta(0) \in \Delta$, with the notation in Lemma 3.13. The other case in the definition of a point of positive type can be treated similarly. □

References

1. BAOUENDI, M. S. and TREVES, F., A property of the functions and distributions annihilated by a locally integrable system of complex vector fields, *Ann. of Math.* **113** (1981), 387–421.
2. BEALS, R. and FEFFERMAN, C., On local solvability of linear partial differential equations, *Ann. of Math.* **97** (1973), 482–498.
3. BERHANU, S., Liouville's theorem and the maximum modulus principle for a system of complex vector fields, *Comm. Partial Differential Equations* **19** (1994), 1805–1827.
4. CORDARO, P. D. and TREVES, F., *Hyperfunctions on Hypo-analytic Manifolds,* Annals of Math. Studies **136**, Princeton Univ. Press, Princeton, N. J., 1994.
5. CORDARO, P. D. and TREVES, F., Necessary and sufficient conditions for the local solvability in hyperfunctions of a class of systems of complex vector fields, *Invent. Math.* **120** (1995), 339–360.
6. GROTHENDIECK, A., *Espaces vectoriels topologiques,* 3rd ed., Publicação da Sociedade de Matemática de São Paulo, São Paulo, 1964.
7. HANGES, N. and SJÖSTRAND, J., Propagation of analyticity for a class of non-microcharacteristic operators, *Ann. of Math.* **116** (1982), 559–577.
8. HARVEY, R., The theory of hyperfunctions on totally real subsets of a complex manifold with applications to extension problems, *Amer. J. Math.* **91** (1969), 853–873.
9. HARVEY, R. and WELLS JR, R. O., Holomorphic approximation and hyperfunction theory on a C^1 totally real submanifold of a complex manifold, *Math. Ann.* **197** (1972), 287–318.
10. HONDA, N., On the reconstruction theorem of holonomic modules in the Gevrey classes, *Publ. Res. Inst. Math. Sci.* **27** (1991), 923–943.
11. HÖRMANDER, L., On the existence and the regularity of solutions of linear pseudo-differential equations, *Enseign. Math.* **17** (1971), 99–163.
12. HÖRMANDER, L., Propagation of singularities and semi-global existence theorems for (pseudo-)differential operators of principal type, *Ann. of Math.* **108** (1978), 569–609.
13. HÖRMANDER, L., *Notions of Convexity,* Progr. Math. **127**, Birkhäuser, Boston, Mass., 1994.

14. KASHIWARA, M., Algebraic study of systems of differential equations, Master's Thesis, Tokyo University, 1970 (Japanese). English transl.: *Mém. Soc. Math. France* **63** (1995).
15. KASHIWARA, M., On the maximally overdetermined systems of linear differential equations I, *Publ. Res. Inst. Math. Sci.* **10** (1975), 563–579.
16. KASHIWARA, M. and SCHAPIRA, P., *Sheaves on Manifolds,* Grundlehren Math. Wiss. **292**, Springer-Verlag, Berlin–Heidelberg, 1990.
17. KOMATSU, H., Resolution by hyperfunctions of sheaves of solutions of differential equations with constant coefficients, *Math. Ann.* **176** (1968), 77–86.
18. KOMATSU, H., Linear ordinary differential equations with Gevrey coefficients, *J. Differential Equations* **45** (1982), 272–306.
19. KÖTHE, G., *Topological Vector Spaces II,* Springer-Verlag, Berlin–Heidelberg, 1979.
20. MARTINEAU, A., Les hyperfonctions de M. Sato, *Séminaire Bourbaki, Exp. No.* **214,** 1960/61.
21. MICHEL, V., Sur la régularité C^∞ du $\bar\partial$ au bord d'un domaine de $\mathbf{C}^n$ dont la forme de Levi a exactement s valeurs propres strictement négatives, *Math. Ann.* **295** (1993), 135–161.
22. MICHEL, V., Résolution locale du $\bar\partial$ avec régularité Gevrey au bord d'un domaine r-convexe, *Math. Z.* **218** (1995), 305–317.
23. NIRENBERG, L. and TREVES, F., On local solvability of partial differential equations, Part I, Necessary conditions, *Comm. Pure Appl. Math.* **23** (1970), 1–38; Part II, Sufficient conditions, *Comm. Pure Appl. Math.* **23** (1970), 459–509; correction: *Comm. Pure Appl. Math.* **24** (1971), 279–288.
24. RAMIS, J.-P., Théorèmes d'indices Gevrey pour les équations différentielles ordinaires, *Mem. Amer. Math. Soc.* **296** (1984).
25. SATO, M., KAWAI, T. and KASHIWARA, M., Microfunctions and pseudo-differential equations, in *Hyperfunctions and Pseudodifferential Operators* (Komatsu, H., ed.), Lecture Notes in Math. **287**, pp. 265–529, Springer-Verlag, Berlin–Heidelberg, 1973.
26. SCHAPIRA, P., Solutions hyperfonctions des équations aux dérivées partielles du premier ordre, *Bull. Soc. Math. France* **97** (1969), 243–255.
27. SCHAPIRA, P., *Théorie des hyperfonctions,* Lecture Notes in Math. **126**, Springer-Verlag, Berlin–Heidelberg, 1970.
28. SCHAPIRA, P., Vanishing in highest degree for solutions of $\mathcal{D}$-modules and perverse sheaves, *Publ. Res. Inst. Math. Sci.* **26** (1990), 535–538.
29. SUSSMANN, H. J., Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* **180** (1973), 171–188.
30. TRÉPREAU, J.-M., Sur l'hypoellipticité analytique microlocale des opérateurs de type principal, *Comm. Partial Differential Equations* **9** (1984), 1119–1146.
31. TRÉPREAU, J.-M., Sur la résolubilité analytique microlocale des opérateurs de type principal, Thesis, Université de Reims, 1984.
32. TRÉPREAU, J.-M., Sur le prolongement holomorphe des fonctions CR définies sur une hypersurface réelle de classe C^2 dans $\mathbf{C}^n$, *Invent. Math.* **83** (1986), 583–592.
33. TREVES, F., On the local solvability for top-degree forms in hypo-analytic structures, *Amer. J. Math.* **112** (1990), 403–421.

34. Treves, F., *Hypo-analytic Structures. Local Theory*, Princeton Univ. Press, Princeton, N. J., 1992.

Received January 7, 1997

Paulo D. Cordaro
Instituto de Matemática e Estatistica
Universidade de São Paulo
Caixa Postal 66281, CEP 05315-970
São Paulo, SP.
Brazil
email: cordaro@ime.usp.br

Jean-Marie Trépreau
Institut de Mathématiques
Université Paris VI
Case 247, 4 Place Jussieu
F-75252 Paris
France
email: jmt@ccr.jussieu.fr