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Vanishing residue characterization of the sine-Gordon hierarchy

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Abstract. The sine(hyperbolic)-Gordon hierarchy is shown to be the extension of the modified Korteweg–de Vries (MKdV) hierarchy in the integrodifferential algebra extending the standard differential algebra by means of one antiderivative. The characterization by vanishing residues of the MKdV hierarchy yields the same characterization of the sine(hyperbolic)-Gordon hierarchy in the integrodifferential algebra.

1. Introduction

This article is concerned with the hierarchy built upon the so-called *sine-Gordon* (SG) *equation*

$$\partial_t \partial_x u = \sin u, \tag{1.1}$$

one of the classical soliton equations. Another soliton equation intimately related to (1.1) is the modified Korteweg–de Vries (MKdV) equation which plays an important role in the present work:

$$\partial_t u = \partial_x^3 u - 6u^2 \partial_x u. \tag{1.2}$$

Each one of the equations (1.1) or (1.2) is the starting point of a sequence (called a *hierarchy*, see below) of evolution equations of increasing order m,

$$\partial_t u = P[elfu], \tag{1.3}$$

where

$$u \longmapsto P[u] = P(u, \partial u, \dots, \partial^m u)$$

is a *differential polynomial.* Here u is a smooth function of t valued in a differential algebra $\mathbf{A}$, i.e., a commutative algebra equipped with a derivation ∂ (also equipped with a topology compatible with its differential algebra structure). In many applications $\mathbf{A}=\mathcal{C}^\infty(\mathbf{S}^1)$ (the periodic case) or $\mathbf{A}=\mathcal{S}(\mathbf{R}^1)$, the Schwartz space (the rapidly decaying case). Here, when we do make use of a differential algebra, it will be the algebra $\mathbf{M}[[x]]$ of formal meromorphic series in one indeterminate x,

$$u(x)=\sum_{n=-N}^{\infty} a_n x^n, \tag{1.4}$$

with coefficients in $\mathbf{C}$ ($N\in\mathbf{Z}$ may vary with u), the derivation being the usual one, $\partial_x=d/dx$. But mostly we shall reason at the symbolic level in the spirit of [GD1], the j^{th} derivative $\partial^j u$ being replaced by the symbol ξ_j and $P(u,\partial u,\dots,\partial^m u)$ by the true polynomial $P(\xi)=P(\xi_0,\xi_1,\dots,\xi_m)$.

We shall be primarily interested in the "conserved quantities" of the evolution equation (1.3), by which we mean most often other polynomials $Q(\xi_0,\dots,\xi_n)$ endowed with the following property:

(1) *there is a polynomial* $\Phi(\xi_0,\dots,\xi_r)$ (called a *flux*) *such that*

$$\partial_t(Q[u])=\partial(\Phi[u]) \tag{1.5}$$

for every solution $u\in\mathcal{C}^1(\mathbf{R};\mathbf{A})$ *of* (1.3).

The effect of (1.5) is that, when $\mathbf{A}=\mathcal{C}^\infty(\mathbf{S}^1)$ or $\mathbf{A}=\mathcal{S}(\mathbf{R}^1)$, then

$$t\longmapsto\int Q[u(t,x)]\,dx$$

is a constant ("of motion": the integration is carried out over $\mathbf{S}^1$ or $\mathbf{R}^1$).

The evolution equations (1.3) under consideration in this article have infinitely many (independent) conserved polynomials. The most famous of these equations is the Korteweg–de Vries (abbreviated KdV) equation. The MKdV and SG equations also admit infinite sequences of conserved polynomials and so do a number of other equations. Much attention, on the part of analysts, geometers and algebraists, has focused on those relatively few that have *soliton* solutions. On this vast subject we refer to the texts [AC], [D] and [FT]. There are equations, such as the Airy equation $\partial_t u=\partial_x^3 u$, which admit infinitely many conserved polynomials [for Airy these are the solution u and the "energies" $\frac{1}{2}(\partial_x^k u)^2$, $k\in\mathbf{Z}_+$] but do not have soliton solutions.

Two differences between (1.1) and (1.3) jump to the eye: the right-hand side is a transcendental series, not a polynomial, and $\partial_t u$ is replaced by $\partial_t\partial_x u$. In the present work the first of those differences is dealt with by the routine extension from the algebra $\mathfrak{P}=\mathbf{C}[\xi_0,\xi_1,\dots]$ of polynomials in the (countable) infinity of indeterminates ξ_n to its natural completion $\widehat{\mathfrak{P}}$, the algebra of formal power series in the indeterminates ξ_n. The latter difference is handled by a subtler extension of $\widehat{\mathfrak{P}}$ to include the symbol ξ_{-1} of an *antiderivative* $\partial^{-1}u$. This allows us to reinterpret the SG equation (1.1) as the integrodifferential equation

$$\partial_t u = \sin \partial_x^{-1} u. \tag{1.6}$$

Once this is correctly done one notices that the conserved polynomials of equation (1.6) or rather, more accurately, of the *sine-hyperbolic-Gordon* (SHG) equation

$$\partial_t u = \tfrac{1}{2} \sinh 2\partial_x^{-1} u \tag{1.7}$$

are the same as those of the MKdV equation (passage from (1.6) to (1.7) is through the substitution $u \mapsto 2iu$; for us the scalar field is $\mathbf{C}$ and (1.6) and (1.7) are equivalent). Conversely $\frac{1}{4}(\cosh 2\xi_{-1} - 1) = \frac{1}{2}\int_0^{\xi_{-1}} \sinh 2\tau \, d\tau$ is a conserved series of the MKdV equation; it gives rise to what is called a *nonlocal* constant of motion.

The equations under consideration are all Hamiltonian, essentially in the sense of Lax ([L2] and [L3]). The Hamiltonian formalism associates to each *conserved* polynomial (or series) a *differential* polynomial (or an integrodifferential series). For instance to the conserved series $\frac{1}{4}(\cosh 2\xi_{-1} - 1)$ it associates the integrodifferential series $\frac{1}{2}\sinh 2\partial^{-1}u$ (see below). Thus "on top of" our initial equation, be it KdV, MKdV, SHG, etc., stands a tower of (evolution) differential equations of increasing order called a *hierarchy*. What was said earlier is that the SHG hierarchy is an extension (to a suitably augmented algebra akin to $\mathfrak{P}[[\xi_{-1}]]$) of the MKdV hierarchy.

The following result was proved in [T2].

Theorem 1. *For $Q \in \mathfrak{P}$ to be a conserved polynomial of the MKdV equation it is necessary and sufficient that*

$$\operatorname{Res} Q\left[\frac{1}{x} + \sum_{n=1}^{\infty} \xi_n \frac{x^n}{n!}\right] + \operatorname{Res} Q\left[-\frac{1}{x} + \sum_{n=1}^{\infty} \eta_n \frac{x^n}{n!}\right] = 0 \tag{1.8}$$

for all $\xi = (\xi_1, \xi_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$.

It is checked immediately (see end of Section 3) that (1.8) remains valid when we replace the polynomial Q by the series $\cosh 2\xi_{-1}$. We see thus that the SHG hierarchy is characterized by the same vanishing residue property as the MKdV hierarchy but in an enlarged algebra. At the outset one could have thought that the transcendental nature of the right-hand side in (1.1) did not lend itself to an algebraic characterization of the kind of Theorem 1; but it does. At the end of this article we indicate why a similar characterization is likely to be valid for a hierarchy discovered by Ablowitz, Kaup, Newell and Segur [AKNS]. In a separate article we prove (by very different methods but also relying on Theorem 1) that the nonlinear Schrödinger hierarchy admits a similar characterization. All these examples suggest that vanishing residue theorems might be common for soliton equations. An explanation for their recurrence remains to be found.

2. The algebraic framework

2.1. Basics of differential algebra

As we have said the starting framework is the algebra $\mathfrak{P}=\mathbf{C}[\xi_0,\xi_1,\dots]$ of polynomials in the (countable) infinity of indeterminates ξ_n (cf. [K]); we shall reason most often within the subalgebra $\mathfrak{P}_0$ of polynomials without constant term, i.e., vanishing at $\xi=0$.

In the algebra $\mathfrak{P}$ we define the *weight* of a monomial $\xi^p=\xi_0^{p_0}\dots\xi_\nu^{p_\nu}$ to be $w(p)=\sum_{j=0}^{\nu}(j+1)p_j$; the integer ν will be referred to as the *order* of ξ^p (in view of the differential connotation). We define the weight $w(P)$ of a polynomial $P\in\mathfrak{P}$ as the *minimum* weight of its (nonzero) monomials. A polynomial is said to be *weight-homogeneous of weight* k if all its monomials have weight k ($k\in\mathbf{Z}_+$; the zero polynomial is assigned any weight).

We denote by $\widehat{\mathfrak{P}}$ the completion of $\mathfrak{P}$ for a metric associated to the weight: a fundamental system of neighborhoods of zero consists of the ideals $\mathfrak{P}_k=\{P\in\mathfrak{P};\ w(P)\geq k+1\}$, $k\in\mathbf{Z}_+$. A generic element of $\widehat{\mathfrak{P}}$ is a formal series

$$f(\xi)=\sum_p c_p\xi^p$$

with coefficients $c_p\in\mathbf{C}$. The series f converges in $\widehat{\mathfrak{P}}$ since to each positive integer N there are only finitely many multiindices p such that $w(p)\leq N$. The weight $w(f)$ of the series f is the minimum of the weights $w(p)$ for p such that $c_p\neq 0$. The subalgebra $\widehat{\mathfrak{P}}_0$ of the series without a constant term is the maximal ideal of $\widehat{\mathfrak{P}}$ ($\widehat{\mathfrak{P}}_0$ is the closure of $\mathfrak{P}_0$ in $\widehat{\mathfrak{P}}$).

The most important operator in the algebra $\widehat{\mathfrak{P}}$ is the *chain rule derivation*

$$\mathfrak{d}=\sum_{j=0}^{\infty}\xi_{j+1}\frac{\partial}{\partial\xi_j}.$$

The justification for its name is that $\partial(P[u])=(\mathfrak{d}P)[u]$ for every $P\in\mathfrak{P}$ and every element u of the commutative algebra $\mathbf{A}$ equipped with the derivation ∂. Note that $\mathfrak{d}\widehat{\mathfrak{P}}\subset\widehat{\mathfrak{P}}_0$, the subalgebra of formal power series without constant term. Restricted to $\widehat{\mathfrak{P}}_0$ the chain rule derivation $\mathfrak{d}$ is *injective* (but not surjective, see below). The following evident property is important:

$$w(\mathfrak{d}f)=w(f)+1 \quad \text{for all } f\in\widehat{\mathfrak{P}}.$$

As the sequel will show it is advantageous to quotient out $\mathfrak{d}\widehat{\mathfrak{P}}$, i.e., to deal with $\widehat{\mathfrak{P}}_0/\mathfrak{d}\widehat{\mathfrak{P}}$ rather than with $\widehat{\mathfrak{P}}_0$ itself. Actually, it is preferable to deal with true series rather than with cosets in $\widehat{\mathfrak{P}}_0/\mathfrak{d}\widehat{\mathfrak{P}}$. We make use of the *reduced polynomials* (originally introduced and called *irreducible* in [KMGZ]) as per Definition 1.

Definition 1. A *reduced polynomial* (resp., *series*) is a polynomial (resp., series) in which each monomial is a constant multiple of one of the following monomials:

$$(2.1) \qquad \xi_0 \quad \text{and} \quad \xi^p=\xi_0^{p_0}\dots\xi_\nu^{p_\nu} \text{ with } p=(p_0,\dots,p_\nu)\in \mathbf{Z}_+^{\nu+1},\ p_\nu\geq 2.$$

The reduced series makes up a subring $\dot{\widehat{\mathfrak{P}}}_0$ of $\widehat{\mathfrak{P}}_0$. The next statement is easy to prove.

Proposition 1. ([KMGZ]) *The quotient map* $\widehat{\mathfrak{P}}_0\to\widehat{\mathfrak{P}}_0/\mathfrak{d}\widehat{\mathfrak{P}}$ *induces a bijection* $\dot{\widehat{\mathfrak{P}}}_0\to\widehat{\mathfrak{P}}_0/\mathfrak{d}\widehat{\mathfrak{P}}$.

In other words, each coset in $\widehat{\mathfrak{P}}_0/\mathfrak{d}\widehat{\mathfrak{P}}$ contains a unique reduced series. A restatement of Proposition 1 is the direct sum decomposition

$$(2.2) \qquad \widehat{\mathfrak{P}}_0=\dot{\widehat{\mathfrak{P}}}_0\oplus\mathfrak{d}\widehat{\mathfrak{P}}.$$

A fact frequently used is that $\mathfrak{d}$ is skew-symmetric mod $\mathfrak{d}\widehat{\mathfrak{P}}$:

$$(2.3) \qquad f\mathfrak{d}^n g-(-1)^n g\mathfrak{d}^n f\in\mathfrak{d}\widehat{\mathfrak{P}} \quad \text{for all } f,g\in\widehat{\mathfrak{P}}.$$

The range $\mathfrak{d}\mathfrak{P}$ of $\mathfrak{d}$ can be characterized by means of the following linear differential operator (of infinite order) on $\mathfrak{P}$,

$$(2.4) \qquad Q\longmapsto\nabla Q=\sum_{j=0}^{\infty}(-\mathfrak{d})^j\left(\frac{\partial Q}{\partial\xi_j}\right).$$

The operator ∇ will play a crucial role in what follows; it is the same as the *variational derivative* of Gelfand and Dickey (see for instance [GD1], [GD2] and [GD3]). These authors denote it by $\delta/\delta u$; we are not using the natural notation $\delta/\delta\xi$ in order to avoid confusion with the partial derivatives $\partial/\partial\xi_j$. Since ∇ lowers the weight by one unit it extends straightforwardly to the completion $\widehat{\mathfrak{P}}$.

Proposition 2. *We have* $\mathfrak{d}\widehat{\mathfrak{P}}=\widehat{\mathfrak{P}}_0\cap\ker\nabla$.

To characterize the range of ∇ we introduce the following differential operators in $\mathfrak{P}$ (or in $\widehat{\mathfrak{P}}$)

$$\nabla^{(j)}F=\sum_{k=j}^{\infty}(-1)^k\binom{k}{j}\mathfrak{d}^{k-j}\left(\frac{\partial F}{\partial\xi_k}\right),\quad j=0,1,\dots,\tag{2.5}$$

with the understanding that $\nabla^{(0)}=\nabla$.

Proposition 3. *For a series* $f\in\widehat{\mathfrak{P}}$ *to belong to* $\nabla\widehat{\mathfrak{P}}$ *it is necessary and sufficient that*

$$\frac{\partial f}{\partial\xi_j}=\nabla^{(j)}f\quad \text{for all } j\in\mathbf{Z}_+.\tag{2.6}$$

We are also going to need the following proposition.

Proposition 4. *We have, for all* $f,g\in\widehat{\mathfrak{P}}$,

$$\nabla(fg)=\sum_{j=0}^{\infty}[(\mathfrak{d}^j f)\nabla^{(j)}g+(\mathfrak{d}^j g)\nabla^{(j)}f];\tag{2.7}$$

$$\nabla^{(j)}\mathfrak{d}f=-\nabla^{(j-1)}f,\quad j=1,2,\dots.\tag{2.8}$$

The proofs of Propositions 2, 3 and 4 are straightforward and will be omitted here.

We associate to any series $f\in\widehat{\mathfrak{P}}_0$ the formal vector field

$$\vartheta_f=\sum_{n=0}^{\infty}(\mathfrak{d}^n f)\frac{\partial}{\partial\xi_n}.\tag{2.9}$$

Such formal vector fields are characterized by the fact that they commute with $\mathfrak{d}=\vartheta_{\xi_1}$. The evolution equation

$$\partial_t u=f[u]\tag{2.10}$$

has the symbolic equivalent

$$\frac{d\xi}{dt}=\vartheta_f|_\xi$$

which is simply short-hand for the sequence of equations

$$\frac{d\xi_n}{dt}=\mathfrak{d}^n f(\xi),\quad n=0,1,\dots.\tag{2.11}$$

whose formal solution is $(\exp t\vartheta_f)(\xi)$. All this can be rigorously interpreted in terms of the infinite-dimensional Lie group of Bäcklund transformations and its Lie algebra. The latter is isomorphic to $\widehat{\mathfrak{P}}_0$ equipped with a Lie algebra structure by means of the *Poisson bracket*

$$\{f_1, f_2\} = \vartheta_{f_1} f_2 - \vartheta_{f_2} f_1. \tag{2.12}$$

The center of the Lie algebra $\widehat{\mathfrak{P}}_0$ is spanned (over $\mathbf{C}$) by the monomial ξ_1.

Observe that, given any $g \in \widehat{\mathfrak{P}}_0$ and any solution u of (2.10),

$$\partial_t(g[u]) = \sum_{j=0}^{\infty} (\partial_x^j \partial_t u) \frac{\partial g}{\partial \xi_j}[u] = \sum_{j=0}^{\infty} (\mathfrak{d}^j f[u]) \frac{\partial g}{\partial \xi_j}[u] = (\vartheta_f g)[u].$$

It is therefore natural to define the *conserved series* of the equation (2.10) as those series g such that $\vartheta_f g \in \mathfrak{d}\widehat{\mathfrak{P}}$. Thanks to the "integration by parts" formula (2.3) we see that this is equivalent to saying that $f \nabla g \in \mathfrak{d}\widehat{\mathfrak{P}}$. Since $g \in \mathfrak{d}\widehat{\mathfrak{P}}$ entails $\nabla g = 0$ we can state: *every series belonging to* $\mathfrak{d}\widehat{\mathfrak{P}}$ *is conserved for* (2.10) *whatever* $f \in \widehat{\mathfrak{P}}$, which is one reason for modding out $\mathfrak{d}\widehat{\mathfrak{P}}$.

Translating within this formalism the notion introduced in [L2] we say that the evolution equation (or, equivalently, the series f) is *Hamiltonian* if there is a series $g \in \widehat{\mathfrak{P}}$ such that $f = \mathfrak{d}\nabla g$. Then g is automatically conserved for (2.10) since $(\nabla g)\mathfrak{d}\nabla g = \mathfrak{d}\left(\frac{1}{2}(\nabla g)^2\right)$.

If $f_i = \mathfrak{d}\nabla g_i$, $i = 1, 2$, are two Hamiltonian series then

$$\{f_1, f_2\} = \mathfrak{d}\nabla((\nabla g_2)\mathfrak{d}\nabla g_1). \tag{2.13}$$

This shows that the Hamiltonian series form a Lie subalgebra $\widehat{\mathfrak{H}}$ of the Lie algebra $\widehat{\mathfrak{P}}_0$. We also see that the commutation relation $\{f_1, f_2\} = 0$ is equivalent to the property

$$f_1 \mathfrak{d}^{-1} f_2 = (\nabla g_2)\mathfrak{d}\nabla g_1 \in \mathfrak{d}\widehat{\mathfrak{P}}. \tag{2.14}$$

If (2.14) holds then g_1 and g_2 are conserved series for both f_1 and f_2.

2.2. Evolution equations of the sine-Gordon type

Henceforth we focus our attention on evolution equations quite different from (2.10), equations of the kind

$$u_{xt} = f(u) \tag{2.15}$$

in which f is a formal power series in a single indeterminate without constant term but with a nonzero term of order 1, i.e., such that $f(0)=0$ and $f'(0)\neq 0$.

The classical example of an equation (2.15) is the SG equation (1.1) or the SHG equation

$$\partial_t\partial_x u = \tfrac{1}{2}\sinh 2u. \tag{2.16}$$

We shall be interested in the conserved series $E[u]$ (with symbol $E\in\widehat{\mathfrak{P}}_0$) of (2.15). They are defined by the property that

$$\partial_t(E[u]) = \sum_{j=0}^{\infty} \frac{\partial E}{\partial \xi_j}[u]\partial_x^j u_t = \partial_x \Phi[u] \tag{2.17}$$

for some "flux" $\Phi\in\widehat{\mathfrak{P}}$ and all solutions u of (2.15). The equation (2.17) is equivalent to

$$\begin{aligned}\partial_t(E[u]) &= u_t\frac{\partial E}{\partial \xi_0}[u] + \sum_{j=0}^{\infty}\frac{\partial E}{\partial \xi_j}[u]\partial_x^{j-1}(f(u))\\ &= u_t\frac{\partial E}{\partial \xi_0}[u] - f(u)\sum_{j=0}^{\infty}(-1)^j\partial_x^j\left(\frac{\partial E}{\partial \xi_{j+1}}[u]\right) + \partial_x\Psi[u]\end{aligned}$$

for some $\Psi\in\widehat{\mathfrak{P}}$. In order to ensure that (2.17) be valid we require, first of all,

$$\frac{\partial E}{\partial \xi_0}[u] = \lambda f(u) \tag{2.18}$$

for some constant $\lambda\in\mathbf{C}$; by (2.15) this entails

$$u_t\frac{\partial E}{\partial \xi_0}[u] = \lambda u_t(\partial_x u_t) = \partial_x\left(\tfrac{1}{2}\lambda(u_t)^2\right).$$

Note that (2.18) entails

$$E[u] = \lambda\int_0^u f(\tau)\,d\tau + F(\partial_x u, \partial_x^2 u, \ldots) \tag{2.19}$$

with $F(\xi_1,\xi_2,\ldots)\in\widehat{\mathfrak{P}}_0$. We are therefore left with the requirement that

$$f(u)\sum_{j=0}^{\infty}(-1)^j\partial_x^j\left(\frac{\partial F}{\partial \xi_{j+1}}[u]\right) = \partial_x\Theta[u] \tag{2.20}$$

for some $\Theta\in\widehat{\mathfrak{P}}$.

Example 1. It is a simple exercise in the application of (2.19) and (2.20) to show that the following differential series and polynomials are conserved for the SHG equation:

$$E_0 = \tfrac{1}{2}(\cosh 2u - 1), \qquad E_1 = \tfrac{1}{2}(\partial_x u)^2,$$
$$E_2 = -\tfrac{1}{2}((\partial_x^2 u)^2 + (\partial_x u)^4), \quad E_3 = \tfrac{1}{2}(\partial_x^3 u)^2 + 5(\partial_x u)^2(\partial_x^2 u)^2 + (\partial_x u)^6. \quad \square$$

2.3. The integrodifferential algebra $\widehat{\mathfrak{Q}}_0$

Returning to equation (2.15) we note that $f(u) = \partial_x(\partial_t u)$. In order to make sense of this at the symbolic level we shall introduce a new indeterminate η_0 to represent $\partial_t u$. This is akin to re-interpreting equation (2.15) as the pair of equations

$$\partial_t u = v, \quad \partial_x v = f(u). \tag{2.21}$$

For a rigorous treatment we are going to quotient out a particular ideal in the differential algebra $\widehat{\mathfrak{P}}_0^{(2)}$ of formal power series, without constant term, in two sequences of indeterminates ξ_i, η_j, $i, j \in \mathbf{Z}_+$. The chain rule derivation in $\widehat{\mathfrak{P}}_0^{(2)}$ is given by $\mathfrak{d} = \mathfrak{d}_\xi + \mathfrak{d}_\eta$, where

$$\mathfrak{d}_\xi = \sum_{j=0}^{\infty} \xi_{j+1} \frac{\partial}{\partial \xi_j}, \quad \mathfrak{d}_\eta = \sum_{j=0}^{\infty} \eta_{j+1} \frac{\partial}{\partial \eta_j}.$$

As before $f \in \mathbf{C}[[\xi_0]]$ is a formal power series such that $f(0) = 0$ and $f'(0) \neq 0$.

Definition 2. We shall denote by $\widehat{\mathfrak{I}}_f$ the smallest closed ideal in $\widehat{\mathfrak{P}}_0^{(2)}$ stable under the chain rule derivation $\mathfrak{d}$ and containing $\eta_1 - f(\xi_0)$, and by $\widehat{\mathfrak{Q}}_f$ the quotient algebra $\widehat{\mathfrak{P}}_0^{(2)}/\widehat{\mathfrak{I}}_f$.

A generic element of $\widehat{\mathfrak{I}}_f$ is a series

$$g(\xi, \eta) = \sum_{j=0}^{\infty} a_j(\xi, \eta)(\eta_{j+1} - \mathfrak{d}^j(f(\xi_0))), \quad a_j \in \widehat{\mathfrak{P}}_0^{(2)}. \tag{2.22}$$

We denote by φ the quotient map $\widehat{\mathfrak{P}}_0^{(2)} \to \widehat{\mathfrak{P}}_0^{(2)}/\widehat{\mathfrak{I}}_f = \widehat{\mathfrak{Q}}_f$. Since $\mathfrak{d}\widehat{\mathfrak{I}}_f \subset \widehat{\mathfrak{I}}_f$, the chain rule derivation in $\widehat{\mathfrak{P}}_0^{(2)}$ induces a derivation in $\widehat{\mathfrak{Q}}_f$ which we shall also call the chain rule derivation and denote by $\mathfrak{d}$. If we continue to call $\widehat{\mathfrak{P}}_0$ the ring of formal power series without constant term in the indeterminates ξ_j, $j \in \mathbf{Z}_+$, then $\widehat{\mathfrak{P}}_0 \cap \widehat{\mathfrak{I}}_f = \{0\}$

and therefore φ induces an isomorphism of $\widehat{\mathfrak{P}}_0$ onto a subalgebra of $\widehat{\mathfrak{Q}}_f$; henceforth $\varphi(\widehat{\mathfrak{P}}_0)$ will be identified with $\widehat{\mathfrak{P}}_0$ itself and we shall talk of the *natural injection of* $\widehat{\mathfrak{P}}_0$ *into* $\widehat{\mathfrak{Q}}_f$. With this identification we can write $\varphi(RS)=R\varphi(S)$ if $R(\xi)\in\widehat{\mathfrak{P}}_0$ and $S(\xi,\eta)\in\widehat{\mathfrak{P}}_0^{(2)}$.

Let $\widehat{\mathfrak{P}}_0[[\eta_0]]$ denote the ring of formal series in the (nonnegative) powers of η_0 with coefficients in $\widehat{\mathfrak{P}}_0$ (where the indeterminates are the ξ_j): $\widehat{\mathfrak{P}}_0[[\eta_0]]$ can be viewed as a subalgebra of $\widehat{\mathfrak{P}}_0^{(2)}$, not stable, however, under the chain rule derivation in $\widehat{\mathfrak{P}}_0^{(2)}$. The map

$$(2.23)\quad \widehat{\mathfrak{P}}_0^{(2)}\ni S(\xi_0,\eta_0,\dots,\xi_j,\eta_j,\dots)\longmapsto S(\xi_0,\eta_0,\xi_1,f(\xi_0),\xi_2,\mathfrak{d}(f(\xi_0)),\dots)\in\widehat{\mathfrak{P}}_0[[\eta_0]]$$

induces a commutative algebra isomorphism of $\widehat{\mathfrak{Q}}_f$ onto $\widehat{\mathfrak{P}}_0[[\eta_0]]$ to which we shall refer as the *canonical isomorphism of* $\widehat{\mathfrak{Q}}_f$ *onto* $\widehat{\mathfrak{P}}_0[[\eta_0]]$. It transforms the chain rule derivation $\mathfrak{d}$ of $\widehat{\mathfrak{Q}}_f$ into the following derivation of $\widehat{\mathfrak{P}}_0[[\eta_0]]$:

$$\mathfrak{d}_f=\mathfrak{d}_\xi+f(\xi_0)\frac{\partial}{\partial\eta_0}.$$

So long as we take f to be the basic series, when we speak of the differential algebra $\widehat{\mathfrak{P}}_0[[\eta_0]]$ we assume that its canonical derivation is $\mathfrak{d}_f$. In a sense $\widehat{\mathfrak{P}}_0[[\eta_0]]$ is the concrete realization of $\widehat{\mathfrak{Q}}_f$. We can say that in $\widehat{\mathfrak{P}}_0[[\eta_0]]$ there are the *antiderivatives* $\xi_j=\mathfrak{d}_f^{-1}\xi_{j+1}$ and $\eta_0=\mathfrak{d}_f^{-1}f(\xi_0)$.

Composition of the natural injection of $\widehat{\mathfrak{P}}_0$ into $\widehat{\mathfrak{Q}}_f$ with the canonical isomorphism (2.23) identifies $\widehat{\mathfrak{P}}_0$ with the subalgebra of $\widehat{\mathfrak{P}}_0[[\eta_0]]$ consisting of the series that are independent of η_0.

2.4. The operator ∇_f and the range of $\mathfrak{d}_f$

Next we propose to characterize the subspaces $\mathfrak{d}\widehat{\mathfrak{Q}}_f$ and $\mathfrak{d}_f\widehat{\mathfrak{P}}[[\eta_0]]$. For this we need the sequence of formal differential operators acting on $\mathfrak{P}_0^{(2)}$ [cf. (2.5)]

$$\nabla_1^{(k)}F=\sum_{k=j}^{\infty}(-1)^k\binom{k}{j}\mathfrak{d}^{k-j}\left(\frac{\partial F}{\partial\xi_k}\right)\quad\text{and}\quad\nabla_2^{(k)}F=\sum_{k=j}^{\infty}(-1)^k\binom{k}{j}\mathfrak{d}^{k-j}\left(\frac{\partial F}{\partial\eta_k}\right).$$

Lemma 1. *We have*

$$(2.24)\qquad \nabla_i(PQ)=\sum_{l=0}^{\infty}((\mathfrak{d}^lP)\nabla_i^{(l)}Q+(\mathfrak{d}^lQ)\nabla_i^{(l)}P).$$

for all pairs of polynomials $P,Q\in\mathfrak{P}_0^{(2)}$, $i=1,2$. *We also have*

$$(2.25)\qquad \nabla_i^{(j)}\mathfrak{d}=-\nabla_i^{(j-1)},\quad j=1,2,\dots.$$

The proof of formulas (2.24) and (2.25) is straightforward and we leave it to the reader.

We shall now exploit the property that $f'(0)\neq 0$: this means that the series $f'(\xi_0)$ is invertible in the commutative algebra $\mathbf{C}[[\xi_0]]$: we denote its reciprocal by $f'(\xi_0)^{-1}$.

Lemma 2. *The differential operator in* $\widehat{\mathfrak{P}}_0^{(2)}$,

$$g \longmapsto D_f g = \mathfrak{d}(f'(\xi_0)^{-1}\nabla_1 g) - \nabla_2 g.$$

maps the ideal $\widehat{\mathfrak{I}}_f$ *into itself.*

Proof. According to formulas (2.24) and (2.25) we see that, for each $j\in\mathbf{Z}_+$ and $a(\xi,\eta)\in\widehat{\mathfrak{P}}_0^{(2)}$,

$$\nabla_1((\eta_{j+1}-\mathfrak{d}^j(f(\xi_0)))a) = (-1)^{j+1}f'(\xi_0)(\mathfrak{d}^j a) + \sum_{l=0}^{\infty}(\mathfrak{d}^l(\eta_{j+1}-\mathfrak{d}^j(f(\xi_0))))\nabla_1^{(l)}a,$$

$$\nabla_2((\eta_{j+1}-\mathfrak{d}^j(f(\xi_0)))a) = (-1)^{j+1}(\mathfrak{d}^{j+1}a) + \sum_{l=0}^{\infty}(\mathfrak{d}^l(\eta_{j+1}-\mathfrak{d}^j(f(\xi_0))))\nabla_2^{(l)}a.$$

Therefore, if g is the series (2.22) then

$$\varphi(\nabla_1 g) = f'(\xi_0)\sum_{j=0}^{\infty}(-1)^{j+1}\mathfrak{d}^j a_j \quad\text{and}\quad \varphi(\nabla_2 g) = \sum_{j=0}^{\infty}(-1)^{j+1}\mathfrak{d}^{j+1}a_j,$$

whence

$$\mathfrak{d}(f'(\xi_0)^{-1}\varphi(\nabla_1 g)) - \varphi(\nabla_2 g) = 0.$$

The claim then follows from the commutation relations $\varphi\mathfrak{d}=\mathfrak{d}\varphi$ and $f'(\xi_0)^{-1}\varphi = \varphi f'(\xi_0)^{-1}$. $\square$

Lemma 2 tells us that D_f induces, via φ, an operator $\mathbf{D}_f$ of $\widehat{\mathfrak{Q}}_f$ into itself. The canonical isomorphism $\widehat{\mathfrak{Q}}_f\cong\widehat{\mathfrak{P}}_0[[\eta_0]]$ transforms $\mathbf{D}_f$ into an operator ∇_f on $\widehat{\mathfrak{P}}_0[[\eta_0]]$. To find out the expression for ∇_f we make use of the surjection (2.23). We regard a given series $S\in\widehat{\mathfrak{P}}_0[[\eta_0]]$ also as an element of $\widehat{\mathfrak{P}}_0^{(2)}$: we form the series $\mathfrak{d}(f'(\xi_0)^{-1}\nabla_1 S)-\nabla_2 S$ which we send to $\widehat{\mathfrak{P}}_0[[\eta_0]]$ by the map (2.23). Keeping in mind that S is independent of η_j, $j\geq 1$, we see that

$$\nabla_f S = \mathfrak{d}_f\left(f'(\xi_0)^{-1}\sum_{m=0}^{\infty}(-1)^m\mathfrak{d}_f^m\left(\frac{\partial S}{\partial\xi_m}\right)\right) - \frac{\partial S}{\partial\eta_0}. \tag{2.26}$$

We see directly that $\mathbf{D}_f f=0$; since $f(\xi_0)=-\mathfrak{d}_f\eta_0$ this is consistent with the next statement, which generalizes part of Proposition 2.

Proposition 5. *We have* $\mathfrak{d}\widehat{\mathfrak{Q}}_f \subset \ker \mathbf{D}_f$ *and* $\mathfrak{d}_f \widehat{\mathfrak{P}}_0[[\eta_0]] \subset \ker \mathbf{D}_f^\flat$.

Proof. Since

$$\begin{aligned}
&\frac{\partial}{\partial \xi_0}\mathfrak{d}_f S = \frac{\partial}{\partial \xi_0}\left(\mathfrak{d}_\xi S + f(\xi_0)\frac{\partial S}{\partial \eta_0}\right) = \mathfrak{d}_f\left(\frac{\partial S}{\partial \xi_0}\right) + f'(\xi_0)\frac{\partial S}{\partial \eta_0},\\
&\frac{\partial}{\partial \xi_m}\mathfrak{d}_f S = \mathfrak{d}_f\left(\frac{\partial S}{\partial \xi_m}\right) + \frac{\partial S}{\partial \xi_{m-1}} \quad \text{if } m \geq 1,\\
&\frac{\partial}{\partial \eta_0}\mathfrak{d}_f S = \mathfrak{d}_f\left(\frac{\partial S}{\partial \eta_0}\right),
\end{aligned}$$

we get

$$\begin{aligned}
\mathbf{D}_f^\flat \mathfrak{d}_f S &= \mathfrak{d}_f\left(f'(\xi_0)^{-1}\mathfrak{d}_f\left(\frac{\partial S}{\partial \xi_0}\right) + \frac{\partial S}{\partial \eta_0}\right)\\
&\quad + \mathfrak{d}_f\left(f'(\xi_0)^{-1}\left(\sum_{m=1}^{\infty}(-1)^m \mathfrak{d}_f^{m+1}\left(\frac{\partial S}{\partial \xi_m}\right) + \sum_{m=1}^{\infty}(-1)^m \mathfrak{d}_f^m\left(\frac{\partial S}{\partial \xi_{m-1}}\right)\right)\right)\\
&\quad - \mathfrak{d}_f\left(\frac{\partial S}{\partial \eta_0}\right)\\
&= 0.
\end{aligned}$$

Then the claim about $\widehat{\mathfrak{Q}}_f$ and $\mathbf{D}_f$ follows from the canonical differential isomorphism $\widehat{\mathfrak{Q}}_f \cong \widehat{\mathfrak{P}}_0[[\eta_0]]$. □

The following consequence of Proposition 5 will be of use below.

Proposition 6. *If a series* $S \in \widehat{\mathfrak{P}}_0$ *belongs to* $\mathfrak{d}_f(\widehat{\mathfrak{P}}_0[[\eta_0]])$ *then there is a series* $\Phi \in \widehat{\mathfrak{P}}_0$ *and a constant* λ *such that* $S(\xi) = \lambda f(\xi_0) + \mathfrak{d}\Phi(\xi)$.

Proof. By Proposition 5, $S \in \mathfrak{d}_f \widehat{\mathfrak{P}}_0[[\eta_0]]$ implies that $\mathbf{D}_f^\flat S \equiv 0$. If $\partial S/\partial \eta_0 \equiv 0$ this means that $\mathfrak{d}(f'(\xi_0)^{-1}\sum_{m=0}^{\infty}(-1)^m \mathfrak{d}^m(\partial S/\partial \xi_m)) = 0$ and therefore that there is a constant $\lambda \in \mathbf{C}$ such that

$$\sum_{m=0}^{\infty}(-1)^m \mathfrak{d}^m\left(\frac{\partial S}{\partial \xi_m}\right) = \lambda f'(\xi_0),$$

i.e., $\nabla(S(\xi) - \lambda f(\xi_0)) = 0$ where ∇ has the meaning (2.4). It suffices then to apply Proposition 2. □

2.5. Index shift

Actually it is convenient to make the change of variables $\xi_{j+1} \mapsto \xi_j$, $j \in \mathbf{Z}_+$; ξ_0 becomes a new indeterminate ξ_{-1}, standing for an "antiderivative". This means that the symbolic equivalent of equation (2.15) is now

$$\frac{d\xi_0}{dt} = f(\xi_{-1}). \tag{2.27}$$

Henceforth we deal with the ring $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ of formal power series in ξ_{-1} with coefficients in the ring $\widehat{\mathfrak{P}}$ in which the indeterminates are $\xi_0, \xi_1, \dots$. A generic element of $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ is a series of the form

$$S(\xi) = \sum_p \sum_{k=0}^{\infty} c_{k,p} \xi_{-1}^k \xi^p, \tag{2.28}$$

where $\xi^p = \xi_0^{p_0} \dots \xi_\mu^{p_\mu}$. The role played by the algebra $\widehat{\mathfrak{P}}_0$ in the preceding subsections will now be played by the ideal $\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$ in $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ consisting of the series without constant terms (to be distinguished from the strictly smaller ideal $\widehat{\mathfrak{P}}_0[[\xi_{-1}]]$ of the formal power series in ξ_{-1} with coefficients in $\widehat{\mathfrak{P}}_0$). The chain rule derivation in $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ is given by

$$\mathfrak{d} S = \sum_{j=-1}^{\infty} \xi_{j+1} \frac{\partial S}{\partial \xi_j}. \tag{2.29}$$

The derivation $\mathfrak{d}\colon \widehat{\mathfrak{P}}[[\xi_{-1}]]_0 \to \widehat{\mathfrak{P}}[[\xi_{-1}]]_0$ is injective; its restriction to $\widehat{\mathfrak{P}}$ (the subalgebra of series independent of ξ_{-1}) is equal to the usual derivation $\mathfrak{d}$. The range of $\mathfrak{d}$ is equal to the kernel of the formal differential operator

$$S \longmapsto \sum_{j=-1}^{\infty} (-1)^j \mathfrak{d}^{j+1} \left(\frac{\partial S}{\partial \xi_j} \right). \tag{2.30}$$

The restriction of the operator (2.30) to $\widehat{\mathfrak{P}}$ is equal to $\mathfrak{d}\nabla$, where ∇ is given by (2.4).

The shift of indices leads us to identify the integrodifferential algebra $\widehat{\mathfrak{Q}}_0$ with the differential algebra $(\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]]$ of formal power series in η_0 with coefficients in $\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$. The derivation in $(\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]]$ is given by

$$\mathfrak{d}_f = \mathfrak{d} + f(\xi_{-1}) \frac{\partial}{\partial \eta_0}. \tag{2.31}$$

We get the following expression for the operator $\mathbf{D}_f^\flat$:

$$\mathbf{D}_f^\flat S = \mathfrak{d}_f \left(f'(\xi_{-1})^{-1} \sum_{j=-1}^{\infty} (-1)^j \mathfrak{d}^{j+1} \left(\frac{\partial S}{\partial \xi_j} \right) \right) - \frac{\partial S}{\partial \eta_0}. \tag{2.32}$$

2.6. Hamiltonian series in $\widehat{\mathfrak{P}}[[\xi_{-1}]]$

We now describe the Hamiltonian approach to the conserved series of the equation (2.27). Let $\Phi=\mathfrak{d}\nabla\Psi$ be a Hamiltonian series in $\widehat{\mathfrak{P}}$. We propose to give a meaning to the product $\Phi\nabla S$ for an arbitrary series $S\in\widehat{\mathfrak{P}}[[\xi_{-1}]]$. For this, we first extend the operator ∇ defined in $\widehat{\mathfrak{P}}$ to the linear subspace of $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ consisting of the series S such that $\partial S/\partial\xi_{-1}\in\mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]])$ by setting

$$\nabla S=-\mathfrak{d}_f^{-1}\left(\frac{\partial S}{\partial\xi_{-1}}\right)+\sum_{j=0}^{\infty}(-1)^j\mathfrak{d}^j\left(\frac{\partial S}{\partial\xi_j}\right). \tag{2.33}$$

Then $\nabla S\in(\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]]$. The condition on S is very restrictive as shown by the following lemma.

Lemma 3. *If a series $S\in\widehat{\mathfrak{P}}_0[[\xi_{-1}]]$ is such that $\partial S/\partial\xi_{-1}\in\mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]])$ then*

$$S(\xi)=\lambda\int_0^{\xi_{-1}}f(\tau)\,d\tau+W(\xi_0,\xi_1,\ldots)+\mathfrak{d}\Phi(\xi) \tag{2.34}$$

with $W\in\widehat{\mathfrak{P}}_0$, $\Phi\in\widehat{\mathfrak{P}}[[\xi_{-1}]]$ and $\lambda\in\mathbf{C}$.

Proof. If $S\in\widehat{\mathfrak{P}}_0[[\xi_{-1}]]$ then $\partial S/\partial\xi_{-1}\in\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$. By Proposition 6, if $\partial S/\partial\xi_{-1}\in\mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]])$ then $\partial S/\partial\xi_{-1}=\lambda f(\xi_{-1})+\mathfrak{d}\Psi(\xi)$ for some $\Psi\in\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$. It follows that

$$S(\xi)-\lambda\int_0^{\xi_{-1}}f(\tau)\,d\tau-\mathfrak{d}\int_0^{\xi_{-1}}\Psi(\tau,\xi_0,\xi_1,\ldots)\,d\tau$$

is independent of ξ_{-1}. $\square$

The operator $\mathfrak{d}_f\nabla$, however, is defined in the whole of $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ and maps it into itself; as a matter of fact it coincides with the operator (2.30).

Definition 3. We shall say that a series $\Phi\in\widehat{\mathfrak{P}}[[\xi_{-1}]]$ is *f-Hamiltonian* if there is a series $\Psi=\lambda\int_0^{\xi_{-1}}f(\tau)\,d\tau+W(\xi)$ with $W\in\widehat{\mathfrak{P}}_0$ and $\lambda\in\mathbf{C}$, such that $\Phi=\mathfrak{d}_f\nabla\Psi$.

The series f itself is f-Hamiltonian since $f(\xi_{-1})=-\mathfrak{d}_f\nabla\int_0^{\xi_{-1}}f(\tau)\,d\tau$. In passing, note that $\nabla(\int_0^{\xi_{-1}}f(\tau)\,d\tau)=-\eta_0$ and therefore does not belong to $\widehat{\mathfrak{P}}[[\xi_{-1}]]$. Generally, if $S(\xi)$ is given as in (2.34) then

$$\mathfrak{d}_f\nabla S=-\lambda f(\xi_{-1})+\mathfrak{d}\nabla W. \tag{2.35}$$

To any f-Hamiltonian series Φ we associate the formal vector field

$$\vartheta_\Phi=(\mathfrak{d}_f^{-1}\Phi)\frac{\partial}{\partial\xi_{-1}}+\sum_{j=0}^{\infty}(\mathfrak{d}^j\Phi)\frac{\partial}{\partial\xi_j}. \tag{2.36}$$

When $\Phi \in \widehat{\mathfrak{P}}$ this definition coincides with (2.9). In general, if $\Phi = \mathfrak{d}_f \nabla \Psi$ with $\Psi(\xi) = \lambda \int_0^{\xi_{-1}} f(\tau)\, d\tau + W(\xi)$, $W \in \widehat{\mathfrak{P}}_0$, $\lambda \in \mathbf{C}$, then

$$\vartheta_\Phi = -(\nabla \Psi) \frac{\partial}{\partial \xi_{-1}} + \sum_{j=0}^{\infty} (\mathfrak{d}^j \Phi) \frac{\partial}{\partial \xi_j}. \tag{2.37}$$

Since $\nabla \Psi = -\lambda \eta_0 + \nabla W \in (\widehat{\mathfrak{P}}[[\xi_{-1}]])[[\eta_0]]$, (2.37) implies that ϑ_Φ is a derivation from $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ into $(\widehat{\mathfrak{P}}[[\xi_{-1}]])[[\eta_0]]$.

If $\Phi = f(\xi_{-1})$ we get

$$\vartheta_f = \mathfrak{d}_f^{-1}(f(\xi_{-1})) \frac{\partial}{\partial \xi_{-1}} + \sum_{j=0}^{\infty} (\mathfrak{d}^j (f(\xi_{-1}))) \frac{\partial}{\partial \xi_j} \tag{2.38}$$

(keep in mind that $\mathfrak{d}_f^{-1}(f(\xi_{-1})) = \eta_0$). If $\Phi = \mathfrak{d}_f \nabla \big(\lambda \int_0^{\xi_{-1}} f(\tau)\, d\tau + W(\xi)\big)$ we get, by (2.35),

$$\vartheta_\Phi = -\lambda \vartheta_f + \vartheta_{\mathfrak{d}\nabla W}. \tag{2.39}$$

Note that

$$\vartheta_{\mathfrak{d}\nabla W} = \sum_{j=-1}^{\infty} (\mathfrak{d}^{j+1} \nabla W) \frac{\partial}{\partial \xi_j}. \tag{2.40}$$

This means that the change of variables $\xi_{j-1} \mapsto \xi_j$, $j \in \mathbf{Z}_+$, transforms the vector field $\vartheta_{\mathfrak{d}\nabla W}$ in $\widehat{\mathfrak{P}}[[\xi_{-1}]]$ into the vector field $\vartheta_{\nabla W}$ in $\widehat{\mathfrak{P}}$. It follows right away that $\vartheta_{\mathfrak{d}\nabla W} \mathfrak{d} = \mathfrak{d} \vartheta_{\mathfrak{d}\nabla W}$. This property extends routinely.

Proposition 7. *If $\Phi \in \widehat{\mathfrak{P}}[[\xi_{-1}]]$ is f-Hamiltonian and $S \in \widehat{\mathfrak{P}}[[\xi_{-1}]]$ then $\vartheta_\Phi \mathfrak{d} S = \mathfrak{d}_f \vartheta_\Phi S$.*

Next we define the *Poisson bracket* between two f-Hamiltonian series $\Phi_1 = \mathfrak{d}_f \nabla \Psi_1$ and $\Phi_2 = \mathfrak{d}_f \nabla \Psi_2$. If Ψ_1 and Ψ_2 belong to $\widehat{\mathfrak{P}}_0$ the Poisson bracket $\{\Phi_1, \Phi_2\}$ has been defined in (2.12). We shall adopt the same definition here but we must keep in mind that when an f-Hamiltonian series Φ depends effectively on ξ_{-1} the associated vector field ϑ_Φ depends on the choice of f, as is made clear by formula (2.39). Thus we set

$$\{\Phi_1, \Phi_2\} = \vartheta_{\Phi_1} \Phi_2 - \vartheta_{\Phi_2} \Phi_1. \tag{2.41}$$

The bracket (2.41) induces the bracket (2.12) on $\widehat{\mathfrak{P}}_0$. There is no claim here that $\{\Phi_1, \Phi_2\}$ is also f-Hamiltonian (see the proof of Proposition 8 below). We can only claim that $\{\Phi_1, \Phi_2\} \in (\widehat{\mathfrak{P}}[[\xi_{-1}]])[[\eta_0]]$.

We need to prove the analogue of formula (2.13). This would suggest that we prove the formula

$$\{\Phi_1, \Phi_2\} = \eth\nabla(\Phi_1 \eth_f^{-1}\Phi_2)$$

with the meaning (2.33) for ∇. The trouble is that $\eth_f^{-1}\Phi_2 = \nabla\Psi_2$ might well depend on η_0 and ∇ does not act on such a series. But we may avail ourselves of formula (2.32) which shows that $\eth_f\nabla = f'(\xi_{-1})\eth_f^{-1}\mathbf{D}_f^?$ when acting on series that belong to $\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$. In view of this we can extend formula (2.13) as follows.

Proposition 8. *If* $\Phi_i \in \widehat{\mathfrak{P}}[[\xi_{-1}]]$, $i=1,2$, *are two f-Hamiltonian series then*

$$\mathbf{D}_f^?(\Phi_1\eth_f^{-1}\Phi_2) = \eth_f(f'(\xi_{-1})^{-1}\{\Phi_1, \Phi_2\}). \tag{2.42}$$

If moreover $\Phi_2 = \eth\nabla W_2$ *with* $W_2 \in \widehat{\mathfrak{P}}_0$ *then*

$$\{\Phi_1, \Phi_2\} = \eth_f\nabla(\Phi_1\eth_f^{-1}\Phi_2). \tag{2.43}$$

Proof. Let us write $\Psi_i(\xi) = \lambda_i \int_0^{\xi_{-1}} f(\tau)\,d\tau + W_i(\xi)$, $W_i \in \widehat{\mathfrak{P}}_0$, $\lambda_i \in \mathbf{C}$, i.e., $\Phi_i = -\lambda_i f(\xi_{-1}) + \eth\nabla W_i$. Making use of (2.39) shows that

$$\begin{aligned}\{\Phi_1, \Phi_2\} = &-\lambda_1\vartheta_f(\eth\nabla W_2) + \lambda_2\vartheta_f(\eth\nabla W_1) + \lambda_2\vartheta_{\eth\nabla W_1} f(\xi_{-1}) \\ &-\lambda_1\vartheta_{\eth\nabla W_2} f(\xi_{-1}) + \{\eth\nabla W_1, \eth\nabla W_2\}.\end{aligned}$$

Since the series $\eth\nabla W_i$ is independent of ξ_{-1} we have, according to (2.13),

$$\{\eth\nabla W_1, \eth\nabla W_2\} = \eth\nabla((\eth\nabla W_1)\nabla W_2),$$

where ∇W_i has its usual meaning. Going back to (2.37) we get

$$\vartheta_{\eth\nabla W_i} f(\xi_{-1}) = -f'(\xi_{-1})\nabla W_i$$

as well as

$$\vartheta_f(\eth\nabla W_i) = \eth\vartheta_f(\nabla W_i) = \eth\sum_{j=0}^{\infty}(\eth^j(f(\xi_{-1})))\frac{\partial}{\partial\xi_j}(\nabla W_i).$$

Concerning this last expression we apply Proposition 3,

$$\sum_{j=0}^{\infty}(\eth^j(f(\xi_{-1})))\frac{\partial}{\partial\xi_j}(\nabla W_i) = \sum_{j=0}^{\infty}(\eth^j(f(\xi_{-1})))\nabla^{(j)}(\nabla W_i).$$

Comparing the right-hand side with formula (2.7) and taking into account the fact that $\nabla^{(j)}(f(\xi_{-1}))\equiv 0$ for all $j\in\mathbf{Z}_+$ we can write

$$\sum_{j=0}^{\infty}(\mathfrak{d}^j(f(\xi_{-1})))\nabla^{(j)}(\nabla W_i)=\sum_{j=0}^{\infty}(-1)^j\mathfrak{d}^j\Big(\frac{\partial}{\partial\xi_j}(f(\xi_{-1})\nabla W_i)\Big)\nabla(f(\xi_{-1})\nabla W_i)$$
$$+\mathfrak{d}_f^{-1}(f'(\xi_{-1})\nabla W_i).$$

At this stage we have gotten

$$\begin{aligned}\{\Phi_1,\Phi_2\}=\;&\mathfrak{d}\nabla((\mathfrak{d}\nabla W_1)\nabla W_2)\\&+\lambda_2\Big(-f'(\xi_{-1})\nabla W_1+\sum_{j=0}^{\infty}(-1)^j\mathfrak{d}^{j+1}\Big(\frac{\partial}{\partial\xi_j}(f(\xi_{-1})\nabla W_1)\Big)\Big)\\&-\lambda_1\Big(-f'(\xi_{-1})\nabla W_2+\sum_{j=0}^{\infty}(-1)^j\mathfrak{d}^{j+1}\Big(\frac{\partial}{\partial\xi_j}(f(\xi_{-1})\nabla W_2)\Big)\Big).\end{aligned}$$

Returning to (2.33) we see that

$$\mathfrak{d}_f\nabla(f(\xi_{-1})\nabla W_i)=-f'(\xi_{-1})\nabla W_i+\sum_{j=0}^{\infty}(-1)^j\mathfrak{d}^{j+1}\Big(\frac{\partial}{\partial\xi_j}(f(\xi_{-1})\nabla W_i)\Big),$$

whence

$$\{\Phi_1,\Phi_2\}=\mathfrak{d}\nabla((\mathfrak{d}\nabla W_1)\nabla W_2)+\lambda_2\mathfrak{d}_f\nabla(f(\xi_{-1})\nabla W_1)-\lambda_1\mathfrak{d}_f\nabla(f(\xi_{-1})\nabla W_2).$$

It is here that we use the fact that $\mathfrak{d}_f\nabla=f'(\xi_{-1})\mathfrak{d}_f^{-1}\mathbf{D}_f^{\flat}$ when acting on series belonging to $\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$. It enables us to derive from the preceding equation

$$\mathfrak{d}_f(f'(\xi_{-1})^{-1}\{\Phi_1,\Phi_2\})=\mathbf{D}_f^{\flat}((-\lambda_1 f+\mathfrak{d}\nabla W_1)(-\lambda_2\mathfrak{d}_f^{-1}f+\nabla W_2))-\lambda_1\lambda_2\mathbf{D}_f^{\flat}(f\mathfrak{d}_f^{-1}f).$$

But Proposition 5 entails $\mathbf{D}_f^{\flat}(f\mathfrak{d}_f^{-1}f)=\mathbf{D}_f^{\flat}(\mathfrak{d}_f(\frac{1}{2}(\mathfrak{d}_f^{-1}f)^2))=0$. □

Corollary 1. *If $\Phi_i\in\widehat{\mathfrak{P}}[[\xi_{-1}]]$, $i=1,2$, are two f-Hamiltonian series then the property $\Phi_1\mathfrak{d}_f^{-1}\Phi_2\in\mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]])$ entails $\{\Phi_1,\Phi_2\}=0$.*

Proof. This follows immediately from (2.42) and Proposition 5. □

2.7. Conserved series in $\widehat{\mathfrak{P}}[[\xi_{-1}]]$

In accordance with the discussion in Subsection 1.1 we are interested solely in the conserved series E of equation (2.27) such that $\partial E/\partial\eta_0\equiv 0$, i.e., $E\in\widehat{\mathfrak{P}}[\xi_{-1}]_0$.

We shall also take E to be reduced, which simply means that the image of E under the map $\xi_j \mapsto \xi_{j+1}$ is reduced (see Definition 1). To say that E is *conserved* for f is to say that

$$\vartheta_f E \in \mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]]). \tag{2.44}$$

Note that $\vartheta_f \xi_0 = f(\xi_{-1}) = -\mathfrak{d}_f \eta_0$, i.e., ξ_0 is conserved for every f.

To ensure (2.44) we require first that (2.18) be satisfied, that is to say, after the index shift,

$$\frac{\partial E}{\partial \xi_{-1}} = \lambda f(\xi_{-1}) = \lambda \mathfrak{d}_f(\mathfrak{d}_f^{-1} f(\xi_{-1})), \tag{2.45}$$

hence

$$f(\xi_{-1})\mathfrak{d}_f^{-1}\left(\frac{\partial E}{\partial \xi_{-1}}\right) = -\lambda \mathfrak{d}_f\left(\frac{1}{2}\eta_0^2\right);$$

and (cf. Lemma 3)

$$E(\xi) = \lambda \int_0^{\xi_{-1}} f(\tau)\, d\tau + F(\xi_0, \xi_1, \dots). \tag{2.46}$$

The series $F \in \widehat{\mathfrak{P}}_0$ must satisfy (cf. (2.19))

$$\vartheta_f F = \sum_{j=0}^{\infty} \mathfrak{d}^j (f(\xi_{-1})) \frac{\partial F}{\partial \xi_j} \in \mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]])$$

or equivalently by (2.3),

$$f(\xi_{-1}) \sum_{j=0}^{\infty} (-1)^j \mathfrak{d}^j \left(\frac{\partial F}{\partial \xi_j}\right) \in \mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]]).$$

By Proposition 6 this last property amounts to the fact that $f(\xi_{-1})(\nabla F - \mu) \in \mathfrak{d}(\widehat{\mathfrak{P}}[[\xi_{-1}]])$ with ∇ given by (2.4) and $\mu \in \mathbf{C}$. But since $\mu f(\xi_{-1}) = -\mu \mathfrak{d}_f \eta_0$ we conclude that

$$f(\xi_{-1}) \nabla F = -\mu \mathfrak{d}_f \eta_0 + \mathfrak{d}\Psi \tag{2.47}$$

with $\Psi \in \widehat{\mathfrak{P}}[[\xi_{-1}]]$.

We can rephrase all this in the Hamiltonian terminology of the preceding subsection: by (2.46) and Definition 3 the series $\mathfrak{d}_f \nabla E$ is f-Hamiltonian. Combining (2.45) and (2.47) shows that

$$\begin{aligned} f(\xi_{-1}) \nabla E &= -(\lambda - \mu) f(\xi_{-1}) \mathfrak{d}_f^{-1}(f(\xi_{-1})) + f(\xi_{-1})(\nabla F - \mu) \\ &= \tfrac{1}{2}(\lambda - \mu)\mathfrak{d}_f(\eta_0^2) + f(\xi_{-1})(\nabla F - \mu) \in \mathfrak{d}_f((\widehat{\mathfrak{P}}[[\xi_{-1}]]_0)[[\eta_0]]). \end{aligned}$$

Corollary 1 allows us to conclude that $\{f, \mathfrak{d}_f \nabla E\} = 0$.

The following result will be used in the next section.

Proposition 9. *Let $F\in\widehat{\mathfrak{P}}_0$ be a reduced polynomial verifying* (2.47). *If $F-\mu\xi_0$ is homogeneous of weight $\varkappa\geq 2$ then $\varkappa$ must be even.*

Proof. We may as well assume $\mu=0$. Let m_1 be the order of F and let us write

$$F=P_0(\xi')\xi_{m_1}^{r_1}+P_1(\xi')\xi_{m_1}^{r_1-1}+\text{ terms of degree less than } r_1-1 \text{ with respect to } \xi_{m_1},$$

where $\xi'=(\xi_0,\dots,\xi_{m_1-1})$, $r_1\geq 2$ and the polynomial P_0 does not vanish identically unless $F\equiv 0$. Then

$$\begin{aligned}\vartheta_f F&=\sum_{j=-1}^{m_1}\mathfrak{d}^j(f(\xi_{-1}))\frac{\partial F}{\partial\xi_j}\\ &=\left(\sum_{j=-1}^{m_1-1}\frac{\partial P_0}{\partial\xi_j}\mathfrak{d}^j(f(\xi_{-1}))\right)\xi_{m_1}^{r_1}\\ &\quad+\left(r_1P_0\mathfrak{d}^{r-1}(f(\xi_{-1}))+\sum_{j=-1}^{m_1-1}\frac{\partial P_1}{\partial\xi_j}\mathfrak{d}^j(f(\xi_{-1}))\right)\xi_{m_1}^{r_1-1}\\ &\quad+\text{ terms of degree less than } r_1-1 \text{ with respect to } \xi_{m_1}.\end{aligned}$$

Since $r_1\geq 2$ and the order of $\mathfrak{d}^j(f(\xi_{-1}))$ does not exceed m_1-2 if $j\leq m_1-1$ the series

$$\left(\sum_{j=-1}^{m_1-1}\mathfrak{d}^j(f(\xi_{-1}))\frac{\partial P_0}{\partial\xi_j}\right)\xi_{m_1}^{r_1}$$

is reduced. If $\vartheta_f F\in\mathfrak{d}(\widehat{\mathfrak{P}}[[\xi_{-1}]])$ then necessarily

$$\sum_{j=-1}^{m_1-1}\mathfrak{d}^j(f(\xi_{-1}))\frac{\partial P_0}{\partial\xi_j}\equiv 0. \tag{2.48}$$

Suppose the order m_2 of P_0 is ≥ 1 and let us write

$$P_0(\xi')=P_{0,1}(\xi'')\xi_{m_2}^{r_2}+\text{ terms of degree less than } r_1 \text{ with respect to } \xi_{m_1},$$

where $\xi'=(\xi_0,\dots,\xi_{m_1-1})$, $r_2\geq 1$ and the polynomial $P_{0.1}$ does not vanish identically unless $P_0\equiv 0$. We see that

$$\begin{aligned}\sum_{j=-1}^{m_1-1}\mathfrak{d}^j(f(\xi_{-1}))\frac{\partial P_0(\xi')}{\partial\xi_j}&=\xi_{m_2}^{r_2}\sum_{j=-1}^{m_2-1}\mathfrak{d}^j(f(\xi_{-1}))\frac{\partial P_{0.1}(\xi')}{\partial\xi_j}\\ &\quad+\text{ terms of degree less than } r_2 \text{ with respect to } \xi_{m_2}.\end{aligned}$$

Therefore (2.48) implies

$$\sum_{j=-1}^{m_2-1} \mathfrak{d}^j(f(\xi_{-1}))\frac{\partial P_{0,1}(\xi')}{\partial \xi_j} \equiv 0.$$

We see that

$$F = P_{0,1}(\xi'')\xi_{m_2}^{r_2}\xi_{m_1}^{r_1} + \text{ terms of degree less than } r_1 \text{ with respect to } \xi_{m_1}$$
$$+\text{terms of degree less than } r_2 \text{ with respect to } \xi_{m_2}.$$

Repeating this argument we come to the conclusion that

$$F(\xi) = c\xi_{m_\nu}^{r_\nu} \dots \xi_{m_2}^{r_2}\xi_{m_1}^{r_1} + R(\xi),$$

where

$$r_\nu < \dots < r_2 < r_1, \quad (m_1+1)r_1 + (m_2+1)r_2 + (m_\nu+1)r_\nu = \varkappa$$

and the polynomial $R(\xi)$ is a sum of polynomials

$$\varrho(\xi_0, \dots, \xi_{m_\lambda - 1})\xi_{m_\lambda}^{r'_\lambda} \dots \xi_{m_2}^{r'_2}\xi_{m_1}^{r'_1}, \tag{2.49}$$

where $1 \le \lambda \le \nu$, $r'_1 \le r_1$, $r'_2 \le r_2, \dots, r'_{\lambda-1} \le r_{\lambda-1}$ and $r'_\lambda < r_\lambda$. Since

$$\vartheta_f F = c r_\nu \mathfrak{d}^{m_\nu}(f(\xi_{-1}))\xi_{m_\nu}^{r_\nu - 1} \dots \xi_{m_2}^{r_2}\xi_{m_1}^{r_1} + R_1(\xi) \tag{2.50}$$

with $R_1(\xi)$ being a sum of polynomials (2.49) where $1 \le \lambda \le \nu$, $r'_1 \le r_1$, $r'_2 \le r_2, \dots$, $r'_{\lambda-1} \le r_{\lambda-1}$ and $r'_\lambda < r_\lambda$, with the proviso, now, that if $\lambda = \nu$ then $r'_\nu \le r_\nu - 2$. It follows from (2.50) that $\vartheta_f F \in \mathfrak{d}(\widehat{\mathfrak{P}}[[\xi_{-1}]])$ implies that $r_\alpha = 0$ for all $\alpha < \nu$ and $r_\nu = 2$; or else, that $c=0$ in which case $F \equiv 0$. If $F \not\equiv 0$ the weight of F is equal to $2(m_\nu + 1)$. □

3. The sinh-Gordon hierarchy as an extension of the MKdV hierarchy

Henceforth we use the notation introduced in the preceding subsection. We refer the reader to Example 1. The SHG equation corresponds to the formal differential equation (2.27) in which $f(\xi_{-1}) = \frac{1}{2}\sinh 2\xi_{-1}$. The conserved series for the SHG equation in Example 1 have now the expressions

$$H_0 = \tfrac{1}{2}(\cosh 2\xi_{-1} - 1), \quad H_1 = \tfrac{1}{2}\xi_0^2, \quad H_2 = -\tfrac{1}{2}(\xi_1^2 + \xi_0^4), \quad H_3 = \tfrac{1}{2}\xi_2^2 + 5\xi_0^2\xi_1^2 + \xi_0^6.$$

Inspection of the standard lists of low weight conserved polynomials of the MKdV equation (e.g. in [T2]) shows that H_1, H_2 and H_3 are the (normalized, weight-homogeneous of weight 2, 4 and 6, respectively) conserved polynomials of the MKdV equation. This observation suggests the following statement.

Theorem 2. *Every conserved series of the sinh-Gordon equation, i.e., of the series* $\frac{1}{2}\sinh 2\xi_{-1}$, *has the form*

$$E(\xi)=\tfrac{1}{2}\lambda(\cosh 2\xi_{-1}-1)+F(\xi_0,\xi_1,\dots), \tag{3.1}$$

where F is a conserved series of the MKdV equation and $\lambda\in\mathbf{C}$.

Proof. We begin by proving that every conserved series of the MKdV equation is conserved for the sinh-Gordon equation. We know that the conserved polynomials of the MKdV equation of weight 2, 4 and 6 are conserved for the sinh-Gordon equation. We shall apply the recursion formula for the MKdV polynomials $M_m=(-1)^m\mathfrak{d}\nabla H_{m+1}$:

$$M_{m+1}=T\mathfrak{d}^{-1}M_m,$$

i.e.,

$$-\mathfrak{d}\nabla H_{m+1}=T\nabla H_m, \tag{3.2}$$

where

$$T\nabla Q=\mathfrak{d}^3\nabla Q-4\mathfrak{d}(\xi_0^2\nabla Q)+4\xi_0\xi_1\nabla Q+4\xi_1\mathfrak{d}^{-1}(\xi_1\nabla Q).$$

[We are availing ourselves of the fact that

$$\xi_1\nabla Q-\mathfrak{d}Q\in\mathfrak{d}\mathfrak{P}, \tag{3.3}$$

a direct consequence of (2.3) and (2.4).]

Using the notation $\cong$ to mean congruent modulo $\mathfrak{d}_f\widehat{\mathfrak{P}}_0[[\eta_0]]$ and keeping in mind that $\mathfrak{d}_f=\mathfrak{d}$ on $\widehat{\mathfrak{P}}_0$ we have

$$\begin{aligned}
f(\xi_{-1})\nabla H_{m+1}&\cong-(\mathfrak{d}_f^{-1}f(\xi_{-1}))\mathfrak{d}\nabla H_{m+1}\\
&=(\mathfrak{d}_f^{-1}f(\xi_{-1}))(\mathfrak{d}^3\nabla H_m-4\mathfrak{d}(\xi_0^2\nabla H_m)\\
&\quad+4\xi_0\xi_1\nabla H_m+4\xi_1\mathfrak{d}^{-1}(\xi_1\nabla H_m))\\
&\cong-f(\xi_{-1})\mathfrak{d}^2\nabla H_m+4\xi_0^2f(\xi_{-1})\nabla H_m\\
&\quad+4\xi_0\xi_1(\mathfrak{d}_f^{-1}f(\xi_{-1}))\nabla H_m-4\xi_0\mathfrak{d}_f((\mathfrak{d}_f^{-1}f(\xi_{-1}))\mathfrak{d}^{-1}(\xi_1\nabla H_m))\\
&\cong\xi_0f'(\xi_{-1})\mathfrak{d}\nabla H_m+4\xi_0f(\xi_{-1})(\xi_0\nabla H_m-\mathfrak{d}^{-1}(\xi_1\nabla H_m))\\
&=\xi_0f'(\xi_{-1})\mathfrak{d}\nabla H_m+4(\xi_0\nabla H_m-\mathfrak{d}^{-1}(\xi_1\nabla H_m))\mathfrak{d}\left(\int_0^{\xi_{-1}}f(\tau)\,d\tau\right)\\
&\cong\left(f'(\xi_{-1})-4\int_0^{\xi_{-1}}f(\tau)\,d\tau\right)\xi_0\mathfrak{d}\nabla H_m.
\end{aligned}$$

When $f(\xi_{-1})=\frac{1}{2}\sinh 2\xi_{-1}$ we have $f'(\xi_{-1})-4\int_0^{\xi_{-1}} f(\tau)\,d\tau=1$, whence

$$\tfrac{1}{2}\sinh(2\xi_{-1})\nabla H_{m+1}\cong -\xi_1\nabla H_m.$$

It suffices to apply (3.3) to conclude that $\frac{1}{2}\sinh(2\xi_{-1})\nabla H_{m+1}\in\mathfrak{d}\widehat{\mathfrak{P}}$.

Next we show that if a reduced series $F\in\widehat{\mathfrak{P}}_0$ satisfies (2.47) with $f(\xi_{-1})=\frac{1}{2}\sinh 2\xi_{-1}$, then F is a conserved series for the MKdV equation. It suffices to deal with a weight-homogeneous polynomial $F\in\dot{\mathfrak{P}}_0$ of weight $\varkappa\geq 2$, ξ_0 being obviously conserved. According to Proposition 9 we need only consider the case of an *even* weight. We begin by proving the following claim.

Claim. *Let F be a reduced series belonging to $\widehat{\mathfrak{P}}_0$. For a series $G\in\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$ to exist such that*

$$\sinh(2\xi_{-1})\nabla F=\mathfrak{d}G \tag{3.4}$$

it is necessary and sufficient that there be two series $A, B\in\widehat{\mathfrak{P}}_0$ such that

$$\nabla F=\mathfrak{d}A+2\xi_0 B \quad \textit{and} \quad \mathfrak{d}B+2\xi_0 A\equiv 0. \tag{3.5}$$

Proof of Claim. Suppose (3.4) holds. Letting $\partial^2/\partial\xi_{-1}^2$ act on both sides of the equation (3.4) leads to

$$\sinh(2\xi_{-1})\nabla F=\frac{1}{4}\mathfrak{d}\frac{\partial^2 G}{\partial\xi_{-1}^2},$$

whence $\partial^2 G/\partial\xi_{-1}^2=4G$ by comparing the last equation to (3.4). This means that

$$G(\xi_{-1},\xi_0,\xi_1,\ldots)=A(\xi)\sinh 2\xi_{-1}+B(\xi)\cosh 2\xi_{-1}$$

with $A\in\widehat{\mathfrak{P}}$ and $B\in\widehat{\mathfrak{P}}_0$. Putting this into (3.4) yields

$$\sinh(2\xi_{-1})\nabla F=(\mathfrak{d}A+2\xi_0 B)\sinh 2\xi_{-1}+(\mathfrak{d}B+2\xi_0 A)\cosh 2\xi_{-1}$$

and (3.5) follows. The implication (3.5) $\Rightarrow$ (3.4) is proved by tracing back the preceding argument. $\square$

From (3.5) we derive, for all $\nu\in\mathbf{Z}_+$,

$$\nabla\frac{\partial^\nu F}{\partial\xi_0^\nu}=\mathfrak{d}\frac{\partial^\nu A}{\partial\xi_0^\nu}+2\xi_0\frac{\partial^\nu B}{\partial\xi_0^\nu}+2\nu\frac{\partial^{\nu-1}B}{\partial\xi_0^{\nu-1}} \quad \text{and} \quad \mathfrak{d}\frac{\partial^\nu B}{\partial\xi_0^\nu}+2\xi_0\frac{\partial^\nu A}{\partial\xi_0^\nu}+2\nu\frac{\partial^{\nu-1}A}{\partial\xi_0^{\nu-1}}\equiv 0. \tag{3.6}$$

The second equation requires the degree of A with respect to ξ_0 to be less than $m=\deg_{\xi_0} B$ (if $m=0$ this will mean $A\equiv 0$). Indeed, for any $\nu\geq m$ we have

$$\frac{\partial}{\partial\xi_0}\left(\xi_0\frac{\partial^\nu A}{\partial\xi_0^\nu}+\nu\frac{\partial^{\nu-1}A}{\partial\xi_0^{\nu-1}}\right)\equiv 0$$

which in turn implies $\partial^\nu A/\partial\xi_0^\nu\equiv 0$. Now suppose $m\geq\mu_0=\deg_{\xi_0} F$. Taking $\nu=m+1$ in the first equation of (3.6) yields $\partial^m B/\partial\xi_0^m\equiv 0$ which contradicts the definition of m. We must therefore have $m<\mu_0$. Note that if $\mu_0=0$ this implies $B\equiv A\equiv 0$ and therefore $\nabla F\equiv 0$, i.e., $F\equiv 0$ since F is reduced.

Suppose $\mu_0>0$. Then taking $\nu=\mu_0$ in the first equation of (3.6) and $\nu=\mu_0-1$ in the second one implies

$$\nabla F_0=2\mu_0 B_0 \quad \text{and} \quad \mathfrak{d}B_0+2(\mu_0-1)A_0\equiv 0.$$

where we have used the notation

$$F_0=\frac{\partial^{\mu_0}F}{\partial\xi_0^{\mu_0}},\quad B_0=\frac{\partial^{\mu_0-1}B}{\partial\xi_0^{\mu_0-1}}\quad\text{and}\quad A_0=\frac{\partial^{\mu_0-2}A}{\partial\xi_0^{\mu_0-2}}.$$

These series are independent of ξ_0; this means that $\mathfrak{d}$ can be equated to the operator

$$\mathfrak{d}_1=\sum_{j=1}^{\infty}\xi_{j+1}\frac{\partial}{\partial\xi_j}$$

and

$$\nabla F_0=\sum_{j=1}^{\infty}(-1)^j\mathfrak{d}_1^j\left(\frac{\partial F}{\partial\xi_j}\right).$$

We can repeat the same argument with F_0, A_0 and B_0 in the place of F, A and B, respectively, and with ξ_1 in the place of ξ_0.

$$\nabla\frac{\partial^\nu F_0}{\partial\xi_1^\nu}=2\mu_0\frac{\partial^\nu B_0}{\partial\xi_1^\nu}\quad\text{and}\quad\mathfrak{d}\frac{\partial^\nu B_0}{\partial\xi_1^\nu}+2(\mu_0-1)\frac{\partial^\nu A_0}{\partial\xi_1^\nu}\equiv 0.$$

But if we put $\nu>\mu_1=\deg_{\xi_1} F_0$ we get $\partial^\nu B_0/\partial\xi_1^\nu\equiv 0$ and therefore the degree of B_0 with respect to ξ_1 must be $\leq\mu_1$. Likewise the degree of A_0 with respect to ξ_1 cannot exceed that of B_0 by the second equation. Repeating this argument iteratively for

$$F_k=\frac{\partial^{\mu_k}F_{k-1}}{\partial\xi_k^{\mu_k}},\quad B_k=\frac{\partial^{\mu_k}B_{k-1}}{\partial\xi_k^{\mu_k}},\quad A_k=\frac{\partial^{\mu_k}A_{k-1}}{\partial\xi_k^{\mu_k}},\qquad k=2,3,\ldots,$$

we obtain equations

$$\nabla \frac{\partial^{\mu_k} F_{k-1}}{\partial \xi_k^{\mu_k}} = 2\mu_k \frac{\partial^{\mu_k} B_{k-1}}{\partial \xi_k^{\mu_k}} \quad \text{and} \quad \eth \frac{\partial^{\mu_k} B_{k-1}}{\partial \xi_k^{\mu_k}} + 2(\mu_k - 1) \frac{\partial^{\mu_k} A_{k-1}}{\partial \xi_k^{\mu_k}} \equiv 0.$$

This can be pursued until k reaches its maximum value, which we call ω, such that $\partial^{\mu_k} F_{k-1}/\partial \xi_k^{\mu_k} \not\equiv 0$; we have $\partial^{\mu_\omega} F_{\omega-1}/\partial \xi_\omega^{\mu_\omega} = C$ and therefore

$$F_{\omega-1} = \frac{\partial^{\mu_{\omega-1}} F_{\omega-2}}{\partial \xi_{\omega-1}^{\mu_{\omega-1}}} = \frac{C}{\mu_\omega !} \xi_\omega^{\mu_\omega} + \text{ terms of degree less than } \mu_\omega$$

with respect to ω. But since $F_{\omega-1}$ is weight-homogeneous and only depends on ξ_ω we must have

$$\frac{\partial^{\mu_{\omega-1}} F_{\omega-2}}{\partial \xi_{\omega-1}^{\mu_{\omega-1}}} = \frac{C}{\mu_\omega !} \xi_\omega^{\mu_\omega}.$$

Recalling that F is reduced we must have $\mu_\omega \geq 2$. Then, if $\omega \geq 1$ the left-hand side in the equation

$$\nabla \frac{\partial^{\mu_{\omega-1}} F_{\omega-2}}{\partial \xi_{\omega-1}^{\mu_{\omega-1}}} = 2\mu_\omega \frac{\partial^{\mu_{\omega-1}} B_{\omega-2}}{\partial \xi_{\omega-1}^{\mu_{\omega-1}}}$$

has the form

$$\frac{C}{(\mu_\omega - 2)!} \xi_\omega^{\mu_\omega - 2} \xi_{2\omega} + \text{ terms of order less than } 2\omega.$$

But the right-hand side is of order $\leq \omega$. We conclude that F_0 is a constant $c \neq 0$.

Thus, considering what are the *reduced* monomials of the kind $\xi^p \xi_0^{\varkappa - l}$, $1 \leq l \leq 8$ and of weight $\varkappa$ we must have

$$F(\xi) = c\xi_0^{\varkappa} + \text{ terms of degree } \leq \varkappa - 4 \text{ with respect to } \xi_0.$$

Take $\varkappa = 2n$; it follows easily from the recurrence formula (3.2) that

$$H_n(\xi) = \frac{(2m-3)!}{m!(m-2)!} \xi_0^{2n} + \text{ terms of degree } \leq \varkappa - 4$$

with respect to ξ_0. We conclude that the polynomial

$$F_1 = F - \frac{m!(m-2)!}{(2m-3)!} c H_n$$

also satisfies (2.47) with $f(\xi_{-1}) = \frac{1}{2} \sinh 2\xi_{-1}$ and therefore the same argument as above can be applied to it. Since F_1 cannot have any monomial of the form $c\xi_0^{\varkappa}$ it must vanish identically. The proof of Theorem 2 is complete. □

We can apply Corollary 1 to make the following observation.

When $f(\xi_{-1})=\frac{1}{2}\sinh 2\xi_{-1}$, *for an* f*-Hamiltonian series* $\mathfrak{d}\nabla E\in\widehat{\mathfrak{P}}[[\xi_{-1}]]_0$ *to satisfy the commutation relation* $\{f,\mathfrak{d}\nabla E\}=0$ *it is necessary and sufficient that* E *be of the form* (3.1) *with* F *being a conserved series of the MKdV equation and* $\lambda\in\mathbf{C}$; *and therefore that* $M(\xi)=\mathfrak{d}\nabla E(\xi)-\frac{1}{2}\lambda\sinh 2\xi_{-1}$ *belong to the closure in* $\widehat{\mathfrak{P}}_0$ *of the span of the MKdV polynomials.*

The set of all the f-Hamiltonian series $\mathfrak{d}\nabla E$ above is a kind of maximal abelian Lie algebra, since the MKdV series M commute among themselves. But the f-Hamiltonian series do not form a Lie algebra for the bracket (2.41). At any rate the inclusion among them of the series $\frac{1}{2}\lambda\sinh 2\xi_{-1}$ determines that abelian Lie algebra uniquely, precisely by Theorem 2.

The commutation relation

$$\left\{\tfrac{1}{2}\sinh 2\xi_{-1},\xi_3-6\xi_0^2\right\}\equiv 0$$

means that $\sinh 2\xi_{-1}$ is a conserved series of the MKdV equation. Viewed as a functional $u\mapsto\sinh 2\int^x u$ it is what is called a *nonlocal* functional. On the other hand, any conserved polynomial $P(\xi_0,\xi_1,\ldots)$ of the MKdV equation defines a *local* functional of the SHG equation, namely

$$u\longmapsto P(\partial_x u,\partial_x^2 u,\ldots)$$

(cf. Example 1).

As announced in the introduction, the characterization of the conserved series of the MKdV equation provided by Theorem 1 extends to $\sinh 2\xi_{-1}$. Indeed, consider a meromorphic series

$$u(x)=\pm\frac{1}{x}+\sum_{n=1}^{\infty}\xi_n\frac{x^n}{n!}.$$

We can define

$$\partial_x^{-1}u=\int^x u(x)\,dx=\pm\log x+\sum_{n=2}^{\infty}\xi_{n-1}\frac{x^n}{n!}.$$

The residues of the formal meromorphic series

$$\exp\left(\pm 2\int^x u(x)\,dx\right)=x^{\pm 2}\exp\left(2\sum_{n=2}^{\infty}\xi_{n-1}\frac{x^n}{n!}\right)$$

are equal to zero.

4. Other equations of sine-Gordon type

4.1. Pairs of series and polynomials of sine-Gordon type

In this section we discuss briefly the existence of pairs consisting of a series f and of a reduced polynomial $E\in\mathfrak{P}_0$ satisfying (2.44), which here means

$$f(\xi_{-1})\nabla E\in\mathfrak{d}(\widehat{\mathfrak{P}}[[\xi_{-1}]]). \tag{4.1}$$

where ∇ has its usual meaning. We assume that $f(0)=0$, $f'(0)\neq 0$ and that E is weight-homogeneous for a given weight. Thus E will be a conserved polynomial of equation (2.27) and $\partial E/\partial\xi_{-1}\equiv\partial E/\partial\eta_0\equiv 0$ (cf. (2.46); we limit ourselves to low weights).

As before, $\cong$ will stand for *congruent* mod $\mathfrak{d}(\widehat{\mathfrak{P}}[[\xi_{-1}]])$. We shall not distinguish between pairs (f, E) that can be transformed into one another by a transformation

$$(f(\xi), E(\xi))\longmapsto(c_1f(\varrho\xi), c_2E(\varrho\xi)) \tag{4.2}$$

with $c_1, c_2, \varrho\in\mathbf{C}$ and $c_1c_2\varrho\neq 0$.

4.1.1. *Weight* 4. Every polynomials homogeneous of weight 4 is a constant multiple of $E_2(\xi)=\frac{1}{2}\xi_1^2+\frac{1}{4}A\xi_0^4$. For $E=E_2$ condition (4.1) reads

$$Af(\xi_{-1})\xi_0^3-f(\xi_{-1})\xi_2\cong\big(Af(\xi_{-1})-\tfrac{1}{2}f''(\xi_{-1})\big)\xi_0^3\cong 0,$$

and requires $f''=2Af$. Assuming $A\neq 0$ and $f(0)=0$, $f'(0)\neq 0$ leads to

$$f(\xi_{-1})=\frac{1}{2A}\sinh 2A\xi_{-1}.$$

All these pairs are equivalent to the SG hierarchy.

4.1.2. *Weight* 6. Reduced polynomials that are weight-homogeneous of weight 6 are constant multiples of $E_3(\xi)=\frac{1}{2}\xi_2^2+A\xi_1^3+B\xi_0^2\xi_1^2+C\xi_0^6$.

A brief calculation shows that condition (4.1), in which $E=E_3$, reads

$$\begin{aligned}&\big(6Cf(\xi_{-1})-\tfrac{1}{2}Bf''(\xi_{-1})+\tfrac{1}{4}f^{(iv)}(\xi_{-1})\big)\xi_0^5\\&+\big(2Bf(\xi_{-1})+3Af'(\xi_{-1})-\tfrac{5}{2}f''(\xi_{-1})\big)\xi_0\xi_1^2\cong 0.\end{aligned}$$

This requires

$$f^{(iv)}-2Bf''+24Cf=0\quad\text{and}\quad\tfrac{5}{2}f''-3Af'-2Bf=0.$$

We seek solutions of the kind $f(z)=\varkappa(\exp\lambda z-\exp\mu z)$ with $\lambda, \mu, \varkappa\in C$, $\lambda\mu\neq 0$, and $\lambda\neq\mu$. This demands that both λ and μ be solutions of the pair of equations

$$r^4-2Br^2+24C=0 \quad\text{and}\quad \tfrac{5}{2}r^2-3Ar-2B=0.$$

Note that $\lambda\mu\neq 0$ implies $B\neq 0$, and $\lambda\neq\mu$ implies $9A^2+20B\neq 0$.

The case $A=0$ is essentially that of the SG equation: $\lambda=-\mu=2\sqrt{B/5}$.

The cases $A\neq 0$ are very different. For the sake of simplicity we carry out a transformation (4.2) allowing us to take $B=\frac{5}{2}$. The previous equations become

$$r^4-5r^2+24C=0 \quad\text{and}\quad r^2-\tfrac{6}{5}Ar-2=0.$$

These two equations have two common roots if and only if $C=\frac{1}{6}$ and $A=\pm\frac{5}{6}$. If $A=-\frac{5}{6}$ we get the pair of roots $(1,-2)$; if $A=\frac{5}{6}$ we get the pair of roots $(-1,2)$; the corresponding pairs are obviously equivalent. We shall take

$$f(\xi_{-1})=\exp\xi_{-1}-\exp(-2\xi_{-1}) \quad\text{and}\quad E_3(\xi)=\tfrac{1}{2}\xi_2^2-\tfrac{5}{6}\xi_1^3+\tfrac{5}{2}\xi_0^2\xi_1^2+\tfrac{1}{6}\xi_0^6. \tag{4.3}$$

With the choice (4.3) of f, the equation (2.15) becomes the Ablowitz–Kaup–Newell–Segur (abbreviated AKNS) equation (see [AKNS]).

4.1.3. *Weight* 8. It suffices to look at the reduced polynomials homogeneous of degree 8,

$$E_4=\tfrac{1}{2}\xi_3^2+(a\xi_0^2+a'\xi_1)\xi_2^2+b\xi_1^4+c\xi_0^2\xi_1^3+d\xi_0^4\xi_1^2+\tfrac{1}{8}e\xi_0^8.$$

As before we take $f(z)=\varkappa(\exp\lambda z-\exp\mu z)$ with $\lambda\mu\neq 0$ and $\lambda\neq\mu$. A straightforward calculation yields

$$\begin{aligned}e^{r\xi_{-1}}\nabla E_4\cong{}&\big(2a+a'r-\tfrac{7}{2}r^2\big)\xi_0\xi_2^2e^{r\xi_{-1}}-\big(-2c+2r(a-2b)+\tfrac{8}{3}a'r^2-7r^3\big)\xi_0\xi_1^3e^{r\xi_{-1}}\\&+(4d+3cr-9ar^2-a'r^3+7r^4)\xi_0^3\xi_1^2e^{r\xi_{-1}}+\tfrac{1}{3}\big(ar^4-dr^2+3e-\tfrac{1}{2}r^6\big)\xi_0^7e^{r\xi_{-1}}\\\cong{}&0.\end{aligned}$$

We equate the coefficients to zero:

$$\begin{aligned}2a+a'r-\tfrac{7}{2}r^2&=0,\\-2c+2r(a-2b)+\tfrac{8}{3}a'r^2-7r^3&=0,\\4d+3cr-9ar^2-a'r^3+7r^4&=0,\\3e-dr^2+ar^4-\tfrac{1}{2}r^6&=0.\end{aligned}\tag{4.4}$$

Note that $\lambda\mu\neq 0$ implies $a\neq 0$, and $\lambda\neq\mu$ implies $28a+(a')^2\neq 0$.

The case $a'=0$ gives the roots $\pm 2\sqrt{a/7}$ corresponding to cases equivalent to the SG equation. In this case we may take $r=\pm 2$, i.e., $a=7$; the remaining coefficients are then determined,

$$E_4 = \tfrac{1}{2}\xi_3^2 + 7\xi_0^2\xi_2^2 - \tfrac{7}{2}\xi_1^4 + 35\xi_0^4\xi_1^2 + \tfrac{5}{2}\xi_0^8.$$

This is the fourth (normalized) conserved quantity of the MKdV equation, H_4 in the notation of the preceding subsection.

When $a'\neq 0$ we get pairs of roots $\varrho, -2\varrho$, with ϱ depending on the choice of the coefficient e. In other words we get pairs equivalent to those of the AKNS equation. We may as well select $\varrho=1$; then necessarily

$$E_4 = \tfrac{1}{2}\xi_3^2 + \tfrac{7}{2}(\xi_0^2 - \xi_1)\xi_2^2 - \tfrac{7}{6}\xi_1^4 - \tfrac{7}{3}\xi_0^2\xi_1^3 + 7\xi_0^4\xi_1^2 + \tfrac{1}{6}\xi_0^8. \tag{4.5}$$

4.1.4. *Weight* 10. The generic weight-homogeneous polynomial of weight 10 is a constant multiple of the polynomial

$$\begin{aligned} E_5 = {} & \tfrac{1}{2}\xi_4^2 + (a_1\xi_0^2 + a_2\xi_1)\xi_3^2 + b\xi_0\xi_2^3 + (c_1\xi_0^4 + c_2\xi_0^2\xi_1 + c_3\xi_1^2)\xi_2^2 \\ & + \gamma_1\xi_1^5 + \gamma_2\xi_0^2\xi_1^4 + \gamma_3\xi_0^4\xi_1^3 + \gamma_4\xi_0^6\xi_1^2 + \tfrac{1}{10}e\xi_0^{10}. \end{aligned}$$

Direct computation shows that

$$\begin{aligned} e^{-r\xi_{-1}} \sum_{j=0}^{\infty} \partial^j (e^{r\xi_{-1}}) \frac{\partial E_5}{\partial \xi_j} \cong {} & \tfrac{1}{2}(4a_1 + 2a_2 r - 9r^2)\xi_0\xi_3^2 + (b - a_2 r + 5r^2)\xi_2^3 \\ & + \text{ terms of order } \leq 2. \end{aligned}$$

Since we want E_5 to be a conserved polynomial of $e^{\lambda\xi_{-1}} - e^{-\mu\xi_{-1}}$ with $\lambda\neq\mu$ and $\lambda\mu\neq 0$ it follows that the two quadratic equations

$$9r^2 - 2a_2 r - 4a_1 = 5r^2 - a_2 r + b = 0$$

must have the same roots. This demands $a_2=0$ and therefore $\lambda=-\mu$. It means that through a transformation (4.2) brings us back to the SG situation.

4.2. The AKNS hierarchy

The polynomials E_3 and E_4 given by (4.3) and (4.5) provide us with the conserved polynomials of lowest weight (6 and 8, respectively) of the AKNS equation

$$\partial_t \partial_x u = e^u - e^{-2u}. \tag{4.6}$$

Keep in mind that the variable ξ_j corresponds to the derivative $\partial_x^{j+1}u$; to u itself there corresponds the variable ξ_{-1}. We have just seen that there is no conserved polynomial of (4.6) that is weight-homogeneous of weight 10.

The Hamiltonian polynomials

$$\begin{aligned}P_1&=\tfrac16\mathfrak{d}\nabla E_3=\xi_5-5(\xi_0^2-\xi_1)\xi_3+5\xi_2^2-20\xi_0\xi_1\xi_2-5\xi_1^3+5\xi_0^4\xi_1,\\ P_2&=-\tfrac16\mathfrak{d}\nabla E_4\\ &=\xi_7-7(\xi_0^2-\xi_1)\xi_5-21(2\xi_0\xi_1-\xi_2)\xi_4+14\xi_3^2-70\xi_0\xi_2\xi_3-14(4\xi_1^2-\xi_0^4+\xi_1\xi_0^2)\xi_3\\ &\quad-\tfrac{28}{3}\xi_0^6\xi_1+84\xi_0^2\xi_1^3-\tfrac{28}{3}\xi_1^4+56\xi_0(2\xi_0^2-\xi_1)\xi_1\xi_2-7(2\xi_0^2+11\xi_1)\xi_2^2,\end{aligned}$$

constitute the start of the AKNS hierarchy, after the series $e^{\xi_{-1}}-e^{-2\xi_{-1}}$. They commute: $\{P_1,P_2\}\equiv 0$, which is to say that $\nabla E_4(\mathfrak{d}\nabla E_3)\in\mathfrak{d}\mathfrak{P}$. This can be proved by a straightforward (although lengthy) calculation. The AKNS hierarchy spans an abelian subalgebra of the Lie algebra $\widehat{\mathfrak{P}}[[\xi_{-1}]]$.

The congruences mod $\mathfrak{d}\mathfrak{P}$,

$$\begin{aligned}E_3&\cong\tfrac16(\xi_0^2+\xi_1)^3+\tfrac12(2\xi_0\xi_1+\xi_2)^2,\\ E_4&\cong\tfrac16(\xi_0^2+\xi_1)^4+\tfrac32(\xi_0^2+\xi_1)(2\xi_0\xi_1+\xi_2)^2+\tfrac12(2\xi_1^2+2\xi_0\xi_2+\xi_3)^2\end{aligned}$$

point to an interesting feature of the polynomials E_3 and E_4: after division by 4 they are the pullbacks under the Miura transformation $\eta_0=\frac12(\xi_0^2+\xi_1)$ (see [M]) of the two polynomials

$$Q_3'(\eta)=\tfrac13\eta_0^3+\tfrac12\eta_1^2\quad\text{and}\quad Q_4'(\eta)=\tfrac23\eta_0^4+3\eta_0\eta_1^2+\tfrac12\eta_2^2.$$

The polynomials Q_3' and Q_4' can be compared to the normalized conserved polynomials of the KdV equation of the same order, Q_3 and Q_4:

$$\begin{aligned}&\tfrac1{36}Q_3'(6\eta)=2\eta_0^3+\tfrac12\eta_1^2=Q_3,\\ &\tfrac1{36}Q_4'(6\eta)=24\eta^4+18\eta_0\eta_1^2+\tfrac12\eta_2^2\neq Q_4=10\eta_0^4+10\eta_0\eta_1^2+\tfrac12\eta_2^2.\end{aligned}$$

After the change of variables $\eta\mapsto 6\eta$ and multiplication by a constant, $\mathfrak{d}\nabla Q_3'$ becomes the KdV polynomial $R_1=\eta_3-12\eta_0\eta_1$. However, the same change of variables does not transform $\mathfrak{d}\nabla Q_4'$ into a constant multiple of the second KdV polynomial $R_2=\xi_5-20\xi_0\xi_3-40\xi_1\xi_2+120\xi_0^2\xi_1$. It follows that $\mathfrak{d}\nabla Q_3'$ and $\mathfrak{d}\nabla Q_4'$ do not commute, otherwise the change of variables $\eta\mapsto 6\eta$ would bring $\mathfrak{d}\nabla Q_4'$ into the centralizer of R_1 which it does not. And indeed,

$$-\nabla Q_4'(\mathfrak{d}\nabla Q_3')=\big(\tfrac43\eta_0^3-3\eta_1^2-6\eta_0\eta_2+\eta_4\big)(\eta_3-\eta_0\eta_1)\cong\eta_0\eta_1^3+\tfrac{13}{2}\eta_1\eta_2^2.$$

Another interesting feature of the AKNS hierarchy is the following residue vanishing result, which is proved by direct calculation.

Proposition 10. *Let E_3 and E_4 be given by* (4.3) *and* (4.5), *and let* $F(\xi_{-1})=\int_0^{\xi_{-1}}(e^{\tau}-e^{-2\tau})\,d\tau=e^{\xi_{-1}}+\frac{1}{2}e^{-2\xi_{-1}}-\frac{3}{2}$. *If*

$$u(x)=\frac{k}{x}+\sum_{n=1}^{\infty}\frac{1}{n!}\gamma_n x^n \tag{4.7}$$

with $k=1$ *or* $k=-2$ *and arbitrary coefficients* $\gamma_n\in\mathbf{C}$, *then*

$$\operatorname{Res}F(\partial^{-1}u)=\operatorname{Res}E_3[u]=\operatorname{Res}E_4[u]=0,$$

where

$$\partial^{-1}u=k\log x+\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\gamma_n x^{n+1}.$$

Moreover, any weight-homogeneous polynomial P of weight $2j$ such that $\operatorname{Res}P[u]=0$ *for the series* (4.7) *is a constant multiple of* E_j, $j=3,4$.

This leads naturally to the following conjecture.

Conjecture 1. *For a series* $g\in\widehat{\mathfrak{P}}[[\xi_{-1}]]$ *to be a conserved series of the AKNS equation* (4.6) *it is necessary and sufficient that* $\operatorname{Res}g[u]=0$ *for the formal series* (4.7) *with* $k=1$ *or* $k=-2$.

For further information on the AKNS hierarchy we refer to [AKNS].

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