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Fundamental solutions of the acoustic and diffusion equations in nonhomogeneous medium

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Abstract. A fundamental solution of the acoustical equation with a variable refraction coefficient is constructed. The solution satisfies the limiting absorption and radiation conditions. The optimal high frequency estimate is proved for square means of the solution. The source function for the diffusion equation is a by-product of this construction.

1. Introduction

Consider the generalized Helmholtz equation

$$(\Delta+\omega^2\mathbf{n}^2+\mathbf{m})u=0 \tag{1}$$

in a Euclidean space X with time frequency ω and real variable refraction coefficients $\mathbf{n}$ and "mass" $\mathbf{m}$. The acoustical (wave) equation in frequency domain is the particular case ($\mathbf{m}=0$). A fundamental solution for (1) is a function $S(y,x;\omega)$ in $X\times X$ that satisfies the equation

$$(\Delta_x+\omega^2\mathbf{n}^2(x)+\mathbf{m}(x))S(y,x;\omega)=\delta(x-y).$$

We say that the fundamental solution satisfies the *limiting absorption condition*, if it admits an analytic continuation to the halfplane $\mathbf{C}_+(\omega_0)\doteq\{\omega:\operatorname{Im}\omega>0,\ |\omega|>\omega_0\}$ for some $\omega_0>0$ that tends to zero, as $|\omega|\to\infty$. We call such a fundamental solution a *source function* of (1). The uniqueness of the source function is easy to check (see below). The source function describes a divergent time harmonic wave with the phase function $\phi=-\omega t$. We here state the existence of a source function for a smooth medium in the plane X, which is homogeneous outside a compact set. We

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also give the estimate (Section 3) which shows the rate of decrease of the weighted square means of the source function, as $|\omega|\to\infty$. This information is applied to the analysis of near scattering fields, see [8].

The diffusion equation in the plane can be reduced to (1) and we show that the decreasing fundamental solution can be obtained from the source function by transformation of the frequency ω.

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2. Uniqueness

Proposition 2.1. *If S is a source function for* (1), *then the integral*

$$(2)\quad E(y,x,t) \doteq \frac{1}{2\pi}\int_\Gamma \exp(-\imath\omega t)S(y,x;\omega)\,d\omega, \quad \Gamma \doteq \{\omega : \operatorname{Im}\omega = \omega_0+\tau \text{ and } \tau>0\},$$

is the forward fundamental solution of the operator $\Box \doteq \Delta_x - \mathbf{n}^2\partial_t^2 + \mathbf{m}$.

Proof. The integral (2) converges in the space of tempered distributions to a fundamental solution of the wave operator. On the other hand, it does not change if we replace Γ by $\Gamma+\tau\imath$, $\tau>0$. This gives the factor $\exp(-\tau t)$ in an estimate of E, which implies $E=0$ for $t<0$, i.e. E is the forward fundamental solution of the wave operator. □

Corollary 2.2. *Any two source functions coincide.*

Proof. This follows from the uniqueness of the forward fundamental solution. To check the uniqueness, we note that the difference of any two forward fundamental solutions vanishes for $t<0$ and consequently for all t by compactness of the dependence domain. □

3. The main result

We assume the following conditions:

(*) $\mathbf{n}\in C^3(X)$, $\mathbf{n}>0$, and $\mathbf{n}=\mathbf{n}_0$ is constant in $X\backslash D$, where D is a compact set in the plane X;

(**) the metric $g=\mathbf{n}^2\,ds^2$ is *nontrapping* in the plane, i.e. any geodesic curve γ in D quits the set D in finite time;

(***) $\mathbf{m}\in C^0(X)$, and $\mathbf{m}=\mathbf{m}_0$ is constant in $X\backslash D$.

Denote by $t(\cdot,\cdot)$ the distance function in the space (X,g) and by $t_0(y,x)$ the distance for the metric $g_0=\min\{\mathbf{n}^2(x),\mathbf{n}_0^2\}\,ds^2$.

Theorem 3.1. *Under the conditions* (*), (**) *and* (***) *there exist a positive* ω_0, *and a source function* $S(y,x;\omega)$ *in* $X\times X\times \mathbf{C}_+(\omega_0)$ *for* (1) *that satisfies the inequality*

$$\int_{t_0(y,x)\leq\tau} |\exp(\operatorname{Im}\omega t_0(y,x))S(y,x;\omega)|^2\,dx \leq \frac{1}{|\omega|}(C_1\tau+C_2\tau^5) \tag{3}$$

uniformly for $y\in X$ *and* $\tau\geq 0$, *that is holomorphic and continuous with respect to* ω *in the closure of* $\mathbf{C}_+(\omega_0)$ *for* $x\neq y$ *and in the sense of* $L_{2.\mathrm{loc}}(X)$ *near the diagonal* $x=y$.

Note that a result of N. Burq [4] contains the estimates $\|S\|=O(\omega^{-1/2})$ for local operator L_2-norms in spaces X of arbitrary dimension for $\omega\to\infty$ near the real axis.

Given a forward fundamental solution E for the wave operator in a medium with velocity $\mathbf{n}^{-1}$, we could find the source function by means of the inverse Fourier transform of E, provided we can control the growth of E as $t\to\infty$. A local construction of the fundamental solution E for second order operators with analytic coefficients was done by Hadamard [7]. The case of smooth coefficients was considered by Sobolev [9]. Duistermaat and Hörmander [5] have constructed the forward propagator for arbitrary strictly hyperbolic operators by the method of Fourier integral operators. However, this approach is not easy to implement directly, even in the situation of Theorem 3.1, since there is not enough information on the growth of the fundamental solution as $t\to\infty$. We here apply a more involved method. First, we construct a forward parametrix P for the wave equation. For this we choose a consistent chain of singular functions starting from Hadamard's fundamental solution. Our construction is parallel to that of [5], but we use simple singular functions rather than Fourier integrals. The time Fourier transform of P will be a parametrix for (1). Finding the correction term, we complete the construction.

Remark. The condition $\dim X=2$ helps to keep the volume of this paper limited. No essential difficulty appears in the general case, except for that a more detailed analysis of the energy in the caustic area is necessary.

Corollary 3.2. *The forward fundamental solution for arbitrary* τ *satisfies the inequality*

$$\int_{t_0(y,x)\leq\tau}\int_{\mathbf{R}} |E_\varepsilon(y,x,t)|^2\,dt\,dx \leq C_\phi(C_1\tau+C_2\tau^5)\log\varepsilon,$$

where

$$E_\varepsilon(y,x,t)\doteq\frac{1}{\varepsilon}\int_{\mathbf{R}}\exp(-\omega_0 s)E(y,x,s)\varphi\left(\frac{t-s}{\varepsilon}\right)ds$$

and φ is an arbitrary function in $\mathbf{R}$ *with compact support such that*

$$\int_0^\infty |\widehat{\varphi}(\pm \exp\sigma)|^2 \, d\sigma < \infty.$$

Proof. By Parseval's theorem,

$$\int_{\mathbf{R}} |E_\varepsilon|^2 \, dt = \frac{1}{2\pi} \int_{\mathbf{R}} |\widehat{\varphi}(\varepsilon\omega) S(y, x; \omega + \imath\omega_0)|^2 \, d\omega$$

and we apply (3). □

Corollary 3.3. *The source function has a symmetric kernel (the reciprocity property), i.e.* $S(y,x;\omega) = S(x,y;\omega)$.

Proof. Take an arbitrary ω, such that $\operatorname{Im}\omega > 0$; then for arbitrary y and x

$$\begin{aligned} S(y,x;\omega) &= \int_X S(y,z;\omega)\delta(z-x)\,dz = \int_X S(y,z;\omega)\Box_\omega S(x,z;\omega)\,dz \\ &= \int_X S(x,z;\omega)\Box_\omega S(y,z;\omega)\,dz = \int_X S(x,z;\omega)\delta(z-y)\,dz = S(x,y;\omega). \end{aligned}$$

where $\Box_\omega \doteq \Delta_z + \omega^2\mathbf{n}^2 + \mathbf{m}$ and the function $S(y,z;\omega)$ is fast decreasing with respect to z according to (3). The equation $S(y,x;\omega) = S(x,y;\omega)$ is valid for real ω because of the uniqueness of the analytic continuation. □

Corollary 3.4. *The function* $S(y,x;\imath v)$ *is real for* $v > \omega_0$.

Proof. The function $\widetilde{S}(y,x;\omega) \doteq \overline{S}(y,x;-\bar{\omega})$ is also a source function for the real operator (1). By uniqueness it coincides with S. □

Proposition 3.5. *The source function satisfies the Sommerfeld radiation conditions*:

$$S(y,x;\omega) = O(r^{-1/2}) \text{ and } \left(\frac{\partial}{\partial r} - \imath\omega\mathbf{n}_0\right) S(y,x;\omega) = O(r^{-3/2}), \text{ as } r \doteq |x-y| \to \infty.$$

Proof. Apply the Helmholtz operator $\Box_{\omega 0} \doteq \Delta_x + \omega^2\mathbf{n}_0^2 + \mathbf{m}_0$ for the homogeneous medium

$$\Box_{\omega 0} S(y,x;\omega) = \delta_y(x) + T(y,x;\omega),$$

where $T(y,x;\omega) \doteq [\mathbf{m}_0 - \mathbf{m}(x) + \omega^2(\mathbf{n}_0^2 - \mathbf{n}^2(x))] S(y,x;\omega)$ and set

$$\widetilde{S}(y,x;\omega) = S_0(x-y;\omega) + \int_D S_0(x-z;\omega) T(y,z;\omega)\,dz,$$

where $S_0(x;\omega)$ is the source function in the homogeneous medium. We have

$$\square_{\omega 0}\widetilde{S}(y,x;\omega)=\delta_y+T(y,x;\omega).$$

It follows that $\square_{\omega 0}U(y,x;\omega)=0$, where $U=S-\widetilde{S}$. Note that the function $\widetilde{S}$ has analytic continuation to the halfplane $\mathbf{C}_+(0)$ since the function S_0 has such a continuation. Apply Proposition 16.1 for the coefficient $\mathbf{p}_0=\sqrt{\omega^2\mathbf{n}_0^2+\mathbf{m}_0}$, where $|\omega|>|\mathbf{m}_0|^{1/2}\mathbf{n}_0^{-1}$, and $\operatorname{Im}\mathbf{p}\geq 0$. This gives $\widetilde{S}=O(|\omega|^{3/2}\exp(-\operatorname{Im}\mathbf{p}_0|x|))$ for large ω; this, together with (3), implies

$$\int_{t_0(y,x)\leq\tau}|\exp(\operatorname{Im}\omega t_0(y,x))U(y,x;\omega)|^2\,dx\leq|\omega|^3(C_1\tau+C_2\tau^5). \tag{4}$$

The Fourier–Laplace transform

$$V(y,x,t)\doteq\int_\Gamma\exp(-\imath\omega t)U(y,x;\omega)\,d\omega$$

is well defined for the curve $\Gamma\doteq\{\omega:\operatorname{Im}\omega=\tau\}$. By (4) the integral converges for arbitrary $\tau>0$ in the distribution sense and satisfies $\square_0 V=0$. It decays to zero for $t<0$, as $\tau\to\infty$, but, on the other hand, does not depend on τ. Therefore $V=0$ for $t<0$; hence $V=0$, by the uniqueness theorem for the wave operator. It follows that $S=\widetilde{S}$. We check that $\widetilde{S}$ satisfies the radiation condition. It is true for S_0 and also for the term $\int_D S_0(x-z;\omega)T(y,z;\omega)\,dz$, since the integral is taken over the compact set D. □

Remark. For Schrödinger-type equations, the limiting absorption principle and the radiation conditions were studied by D. Eidus [6].

4. The diffusion equation

The diffusion equation in the optical tomography is

$$\frac{1}{\mathbf{c}}\frac{\partial\Phi}{\partial t}-\langle\nabla,\varkappa\nabla\rangle\Phi+\mu\Phi=q, \tag{5}$$

where $\Phi=\Phi(x,t)$ - the photon density, q the density of the source, $\mu=\mu(x)$ is the absorption coefficient measured in the unit metre^{-1}, $\varkappa(x)>0$ is the inverse diffusion coefficient measured in metres, and $\mathbf{c}$ is the light velocity measured in metre/second. This is the P_1-approximation to the transport equation, [1]. In frequency domain it appears as

$$-\langle\nabla,\varkappa\nabla\rangle\widehat{\Phi}+\mu\widehat{\Phi}+\imath\frac{\lambda}{\mathbf{c}}\widehat{\Phi}=\hat{q}.$$

For the unknown function $\Psi = \varkappa^{1/2}\Phi$ the equation

$$\Delta\Psi + \left(-\imath\frac{\lambda}{\mathbf{c}\varkappa} + \mathbf{m}\right)\Psi = -\varkappa^{-1/2} q$$

holds, where $\mathbf{m} \doteq -\varkappa^{-1/2}\Delta\varkappa^{1/2} - \varkappa^{-1}\mu$. It takes the form (1) for $\mathbf{n} = (\mathbf{c}\varkappa)^{-1/2}$, $\omega^2 = -\imath\lambda$, where λ is endowed by dimension second^{-2}. Define

$$S_D(y, x; \lambda) \doteq -\varkappa^{-1/2} S_H(y, x; \sqrt{-\imath\lambda}),$$

where S_H is the source function of (1) and $\operatorname{Im}\omega \doteq \operatorname{Im}\sqrt{-\imath\lambda} > 0$.

Corollary 4.1. *If the coefficients $\mu \in C^0(X)$ and $\varkappa \in C^3(X)$ are constant in $X \backslash D$ for a compact set D and $\mathbf{n} = (\mathbf{c}\varkappa)^{-1/2}$ satisfies the condition (**), then there exists a function $S_D(y, x; \lambda)$ defined in $X \times X \times (\mathbf{C} \backslash \imath\mathbf{R}_+)$ that possesses the following properties:*

(i) *it satisfies*

$$\left(-\langle\nabla, \varkappa\nabla\rangle + \mu + \imath\frac{\lambda}{\mathbf{c}}\right) S_D(y, \cdot\,; \lambda) = \delta(\cdot - y);$$

(ii) *it is holomorphic in λ, and*

(iii) *it decreases fast as $t_0(y, x) \to \infty$ and the inequality*

$$\int_{t_0(y,x) \le \tau} \left|\exp\left(\sqrt{|\lambda| - \operatorname{Im}\lambda}\, t_0(y, x)\right) S_D(y, x; \lambda)\right|^2 dx \le \frac{1}{|\lambda|^{1/2}}(C_1\tau + C_2\tau^5)$$

holds for $\tau > 0$, where t_0 is the distance function as in Theorem 3.1.

Corollary 4.2. *The function*

$$E(y, x, t) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = \tau < 0} S_D(y, x; \lambda)\exp(\imath\lambda t)\, d\lambda$$

is a fundamental solution of the diffusion equation (5).

5. The parametrix of the wave equation

First, we construct a forward parametrix $P = P(y, x, t)$ for the wave operator $\Box \doteq \Delta_x - \mathbf{n}^2\partial_t^2 + \mathbf{m}$ in space-time $X \times \mathbf{R}$. We shall use the coordinates $x, t; \xi, \tau$ in the phase space $\Phi \doteq (X \times \mathbf{R}) \times (X \times \mathbf{R})^*$, where ξ and τ are conjugate to the space-time coordinates x and t. This means that $\alpha \doteq \xi\, dx + \tau\, dt$ is the canonical contact form in Φ, where $\xi\,dx=\xi_1\,dx_1+\xi_2\,dx_2$. The function $\mathbf{n}^2(x)\tau^2-|\xi|^2$ is the symbol of this operator. Consider the Hamiltonian flow generated by $h(x;\xi,\tau)\doteq\frac{1}{2}(\mathbf{n}^2(x)\tau^2-|\xi|^2)$:

$$\frac{dx}{dr}=-\xi,\quad \frac{dt}{dr}=\mathbf{n}^2(x),\quad \frac{d\xi}{dr}=-\mathbf{n}(x)\nabla\mathbf{n}(x),\quad r\geq 0;\ \tau=1. \tag{6}$$

Fix a point $y\in X$ and take all the solutions with initial data

$$x(0)=y,\quad t(0)=0,\quad \xi(0)\in T_y^*(X)\quad\text{and}\quad |\xi(0)|=\mathbf{n}(y).$$

Let Λ_y be the union of all trajectories of the flow. This is a smooth surface in Φ. We have $|\xi|=\mathbf{n}(x)$ and the contact form α vanishes in Λ_y, by Jacobi's theorem. This means that the conic set in Φ generated by Λ_y is a Lagrange manifold.

Let $\pi\colon\Phi\to X$ be the natural projection; the image of a solution of (6) is called a *ray*. It is a geodesic for the metric g. The set $L_y\doteq\pi(\Lambda_y)$ is the union of rays starting at y; it is called the front of the wave which starts at y. The set $K_y\doteq L_y+\{(0,t):t\geq 0\}$ is called the future conoid of y; the boundary ∂K_y is the *first* wave front. Consider the union $K\doteq\bigcup_{y\in X}(\{y\}\times K_y)=\{(y,x,t):t\geq t(y,x)\}$.

Lemma 5.1. *There exists a kernel P in $X\times X\times\mathbf{R}_+$ supported by K such that its Fourier transform in $\mathbf{C}_+(0)$ satisfies the inequalities*

$$\int_{t(y,x)\leq\tau}\exp(2\,\mathrm{Im}\,\omega t(y,x))|\widehat{P}(y,x;\omega)|^2\,dx\leq\frac{1}{|\omega|}(C_1\tau+C_2\tau^5), \tag{7}$$

$$\int_{t(y,x)\leq\tau}\exp(2\,\mathrm{Im}\,\omega t(y,x))|\widehat{Q}(y,x;\omega)|^2\,dx\leq\frac{1}{|\omega|^3}(C_1\tau+C_2\tau^5), \tag{8}$$

where $Q\doteq\Box P-\delta_{y,0}$.

The notation $\hat{A}$ is used for the time Fourier transform:

$$\hat{A}(\omega)\doteq F(A)=\frac{1}{2\pi}\int_{\mathbf{R}}\exp(-\imath\omega t)A(t)\,dt.$$

Proof. We construct the kernel P in several steps. We will let e be a real smooth function with compact support in $\mathbf{R}$ that is equal to 1 in a neighbourhood of the origin; it need not be the same in all steps.

Step 1. Choose $\tau>0$ so small that the geodesic coordinates are defined in the neighbourhood $U_0\doteq\{(y,x):t(y,x)\leq\tau_0\}$ of the diagonal in $X\times X$. We define the parametrix in U_0 by Hadamard's method:

$$\begin{aligned} A_0(y,x,t)&\doteq\theta(t-t(y,x))e(t-t(y,x))(t^2-t^2(y,x))^{-1/2}a(y,x,t),\\ a(y,x,t)&=a_0(y,x)+(t^2-t^2(y,x))a_1(y,x). \end{aligned} \tag{9}$$

Here $t(y,x)$ denotes the distance from y to x in the metric space (X,g); it is a smooth function in U_0. The amplitude $a(y,x)$ is a smooth function with respect to the polar coordinates centered in y such that

$$\Box A_0(y,x,t)=\delta(x-y)+\theta(t-t(y,x))(t^2-t^2(y,x))^{1/2}b(y,x,t)+B(y,x,t) \tag{10}$$

for some smooth functions b and B; where $\theta(t)=1$ for $t\geq 0$ and $\theta(t)=0$ otherwise.

Step 2. We extend Hadamard's construction to $D_0\times D_0$, where D_0 is a compact neighbourhood of D, by choosing a consistent chain of singular functions A_α starting with A_0. The union $\Lambda\doteq\bigcup_X(\{y\}\times\Lambda_y)$ is a smooth manifold in $X\times\Phi$. Consider the mapping $\pi:\Lambda\to X\times\Phi\to X\times X$, that equals the composition of projections. For an arbitrary $y\in X$ we denote by $\pi_y:\Lambda_y\to X$ the restriction of π. The rank of π_y is equal to 1 or 2 in each point, since the projection $\pi_y:\gamma\to X$ is an immersion for an arbitrary trajectory γ of the flow (6).

Take a point $(y,\lambda)\in\Lambda$ such that rank $d\pi_y(\lambda)=2$ and choose a generating function for the germ of Λ in the form $\phi(y,x,t)=t-\varphi(y,x)$, where φ is an eikonal function. We take the singular function

$$A^{(2)}(y,x,t)\doteq\phi(y,x,t)_\pm^{-1/2}a(y,x,t), \tag{11}$$

that depends on the phase ϕ and the amplitude a. Here and later we use the notation $s_\pm^\lambda\doteq|s|^\lambda$ for $\pm s\geq 0$ and $s_\pm^\lambda=0$ otherwise. The amplitude has the form $a(y,x,t)=e(\phi(y,x,t))[a_0(y,x)+2\phi(y,x,t)a_1(y,x)]$, where a_0 and a_1 are some smooth functions and $e(\tau)\in\mathcal{D}(\mathbf{R})$ is a function that is equal to 1 in a neighbourhood of the origin. They are subjected to the system of transport equations

$$\begin{gathered}2\langle\nabla\varphi,\nabla a_0\rangle+\Delta(\varphi)a_0=0,\\ 2\langle\nabla\varphi,\nabla a_1\rangle+[\Delta(\varphi)+\mathbf{m}]a_1=\Delta a_0.\end{gathered} \tag{12}$$

Note that the function A_0 (see (9)) is of type $A^{(2)}$ everywhere in the set $\{(y,x,t):x\neq y\}$. We take the kernel $\phi(y,x,t)_+^{-1/2}$ in (11), if the point $(x,t=\varphi(y,x))$ belongs to the first front, i.e. to the boundary of K_y and also to any regular point of L_y such that the ray $\gamma(y,x,t)$ from $(y,0)$ to (x,t) has even Morse index. The kernel $\phi(y,x,t)_-^{-1/2}$ is used for any regular point of L_y with odd Morse index of the corresponding ray $\gamma(y,x,t)$. We have

$$\Box A^{(2)}=\phi_\pm^{1/2}e(\phi)\Delta a_1+B, \tag{13}$$

where B is a smooth function.

Lemma 5.2. *The function* e *can be chosen in such a way that* $\operatorname{supp} A^{(2)} \subset K$.

Proof. The choice of e is not important in the case of the kernel $\phi_+^{-1/2}$, since $\operatorname{supp} \phi_+^{-1/2} \subset K$ for an arbitrary eikonal function φ. The kernel $\phi_-^{-1/2}$ appears only if the Morse index of the corresponding geodesic $\gamma(y, x, t)$ is odd. By Jacobi's theorem (see e.g. [3]) this implies that this geodesic is not the shortest path from y to x. Therefore the point $(y, x, \varphi(y, x)) \in L_y$ does not belong to the boundary of K_y. We can choose the function e in such a way that the function $e(\phi(y, x, t))$ vanishes on ∂K. $\square$

Step 3. Take an arbitrary point $(y, \lambda) \in \Lambda$ such that rank $d\pi_y(\lambda) = 1$. We have $dt \neq 0$ in Λ and there exists a Euclidean coordinate system $x = (u, v), t$ in a neighbourhood of $\pi(\lambda)$ such that $\xi\, dx \doteq \eta\, du + \theta\, dv$ for the dual coordinates $\xi = (\eta, \theta)$ and $(y, u; \theta)$ is a local coordinate system in Λ. We have $t = t(u, \theta)$ and $v = v(u, \theta)$ in Λ for some smooth functions. The phase function

$$\phi(y, x, t; \theta) = t - \varphi(y, x; \theta), \quad \varphi(y, x; \theta) = t(u, \theta) + \theta(v(u, \theta) - v),$$

generates a neighbourhood Λ' of λ in Λ, i.e.

$$\Lambda' = \{(y, x, t; \xi, \tau) : \eta = \phi'_u(y, x, t; \theta),\ \phi(y, x, t; \theta) = 0 \text{ and } \phi'_\theta(y, x, t; \theta) = 0\}$$

is a neighbourhood of λ in Λ. Indeed, the equations $t'_\theta + \theta v'_\theta = 0$, $t'_u + \theta v'_u + \eta = 0$ hold in Λ, since $\alpha = 0$.

We shall use the notation $p_2(s) = s \log(s + 0\imath) - s$, $p_1(s) = \log(s + 0\imath)$ and $p_{k-1} = p'_k$ for $k = -1, 0, 1, 2$. Take a singular function of the form

$$A^{(1)}(y, x, t) = \int_{\mathbf{R}} e(\phi) \operatorname{Re}[p_0(\phi) a_0 + p_1(\phi) a_1]\, d\theta, \tag{14}$$

where the phase function $\phi = t - \varphi$ satisfies the *generalized* eikonal equation $\mathbf{n}^2 - |\nabla\varphi|^2 = \varphi'_\theta \psi$, where $\psi = \psi(x, \theta)$ is a smooth function. It exists and is unique, since the left-hand side vanishes as $\varphi'_\theta = v(u, \theta) - v = 0$. The amplitude functions $a_k = a_k(y, x; \theta)$, $k = 0, 1$, are smooth and have proper supports (that is the projections $\operatorname{supp} a_k \to X \times X$, $(y, x; \theta) \mapsto (y, x)$ are proper). They are subjected to the transport equations

$$\begin{aligned} 2\langle \nabla\varphi, \nabla a_0 \rangle + (\psi a_0)'_\theta + \Delta(\varphi) a_0 &= \varphi'_\theta b_0, \\ 2\langle \nabla\varphi, \nabla a_1 \rangle + (\psi a_1)'_\theta + [\Delta(\varphi) + \mathbf{m}] a_1 &= 2\Delta a_0 + \varphi'_\theta b_1 - b'_{0\theta} \end{aligned} \tag{15}$$

for some smooth functions b_0 and b_1. Note that (12) and (15) are the only points where the coefficient $\mathbf{m}$ contributes to our construction.

Lemma 5.3. *If the functions* a_0 *and* a_1 *are supported by* $X\times X\times\Theta$, *where* Θ *is a sufficiently small neighbourhood of* $\theta_0=\theta(\lambda)$, *we can choose a function* e *in such a way that* $\operatorname{supp} A^{(1)}\subset K$.

Proof. We argue as in Lemma 5.2. The Morse index of the ray $\gamma(y,x,t)$ changes just when the curve passes a singular point $\pi(\lambda)\in L_y$ such that rank $d\pi_y(\lambda)=1$. This point does not belong to ∂K_y, due to Jacobi's theorem. Therefore there is a space for choosing an appropriate function e. □

Lemma 5.4. *We have*

$$\Box A^{(1)}(x,t)=\int_{\mathbf{R}} e(\phi)\operatorname{Re}(p_1(\phi)(\Delta a_1(x,\theta)-b'_{1\theta}(x,\theta)))\,d\theta+B(x,t), \tag{16}$$

where B *is a smooth function.*

Proof. We calculate

$$\begin{aligned}\Box A^{(1)}(x,t)=&\int_{\mathbf{R}}\operatorname{Re}p_{-2}(\phi)((\nabla\varphi(x,\theta))^2-\mathbf{n}^2(x))a_0\,d\theta\\&+\int_{\mathbf{R}}\operatorname{Re}p_{-1}(\phi)[((\nabla\varphi)^2-\mathbf{n}^2)a_1-2\langle\nabla\varphi,\nabla a_0\rangle-\Delta\varphi a_0]\,d\theta\\&+\int_{\mathbf{R}}\operatorname{Re}p_0(\phi)[-2\langle\nabla\varphi,\nabla a_1\rangle-\Delta\varphi a_1+(\Delta+\mathbf{m})a_0]\,d\theta\\&+\int_{\mathbf{R}}\operatorname{Re}p_1(\phi)\Delta a_1\,d\theta+B.\end{aligned}$$

The term B contains derivatives of $e(\phi)$ and is a smooth function. We integrate the first term by parts using the eikonal equation,

$$\int_{\mathbf{R}}p_{-2}(\phi)((\nabla\varphi)^2-\mathbf{n}^2)a_0\,d\theta=-\int_{\mathbf{R}}p_{-2}(\phi)\varphi'_\theta\psi a_0\,d\theta=-\int_{\mathbf{R}}p_{-1}(\phi)(\psi a_0)'_\theta\,d\theta.$$

Combining this with the second term we obtain

$$\begin{aligned}&\int_{\mathbf{R}}p_{-1}(\phi)(((\nabla\varphi)^2-\mathbf{n}^2)a_1-2\langle\nabla\varphi,\nabla a_0\rangle-\Delta\varphi a_0-(\psi a_0)'_\theta)\,d\theta\\&\qquad=-\int_{\mathbf{R}}p_{-1}(\phi)\varphi'_\theta(\psi a_1+b_0)\,d\theta=-\int_{\mathbf{R}}p_0(\phi)(\psi a_1+b_0)'_\theta\,d\theta.\end{aligned}$$

This together with the third term gives

$$\begin{aligned}\int_{\mathbf{R}}p_0(\phi)[-2\langle\nabla\varphi,\nabla a_1\rangle-\Delta\varphi a_1+(\Delta+\mathbf{m})a_0-(\psi a_1+b_0)'_\theta]\,d\theta&=-\int_{\mathbf{R}}p_0(\phi)\varphi'_\theta b_1\,d\theta\\&=-\int_{\mathbf{R}}p_1(\phi)b'_{1\theta}\,d\theta\end{aligned}$$

and so on. □

Step 4. Take a bounded convex set D_0 that contains a neighbourhood of D; we shall specify it later.

Lemma 5.5. *There exist a finite system of open sets* $\{U_i \subset X \times X : 0 \le i \le i_*\}$ *that covers* $D \times D_0$, *a system of open sets* $\{\Lambda_\alpha : 0 \le \alpha \le \alpha_*\}$ *in* Λ, *a mapping* $i = i(\alpha)$ *for* $0 \le \alpha \le \alpha_*$ *such that* $\pi(\Lambda_\alpha) \subset U_{i(\alpha)}$ *for any* $\alpha \ge 0$, *and for any* α *a function* A_α *in* $U_{i(\alpha)}$ *such that*

(i) U_0 *is a neighbourhood of the diagonal in* $D \times D_0$ *and* A_0 *is defined by* (9);

(ii) *for any* $\alpha > 0$, A_α *is a function of form* (11) *or* (14), *where the phase function* $\phi = \phi_\alpha(y, x, t)$ *and* $\phi = \phi_\alpha(y, x, t; \theta)$, *respectively, generates* Λ_α;

(iii) *for arbitrary* j *and* k *the function*

$$B_{jk} \doteq \sum_{i(\alpha)=j} A_\alpha - \sum_{i(\beta)=k} A_\beta$$

can be represented in $U_j \cap U_k$ *as*

$$\begin{aligned} B_{jk}(y,x,t) = &\sum (\phi_\gamma(y,x,t))_\pm^{1/2} b_\gamma(y,x,t) \\ &+ \sum \int_{\mathbf{R}} \operatorname{Re}(p_2(\phi_\gamma(y,x,t;\theta)) b_\gamma(y,x,t;\theta))\, d\theta, \end{aligned} \tag{17}$$

where the sums are taken over all γ *such that* $i(\gamma) = j, k$; *and* b_γ *are smooth functions with proper supports.*

Proof. The function (9) is well defined in U_0. The projection $\pi : \Lambda_0 \to U_0$ is bijective for $\Lambda_0 = \pi^{-1}(U_0) \cap \Lambda$. Next, we construct the functions A_α for all $\alpha = 1, \dots, \alpha_1$ such that Λ_α has a nonempty intersection with Λ_0. Then we find the functions A_β for all $\beta = \alpha_1 + 1, \dots, \alpha_2$ such that $\Lambda_\beta \cap \Lambda_\alpha \ne \emptyset$ at least for one $\alpha \le \alpha_1$ and so on. The initial conditions for the amplitudes are defined by means of (iii). Consider one step of the continuation in detail. If the functions A_α and A_β are of type $A^{(2)}$, and the phase functions φ_α and φ_β coincide in $\pi(\Lambda_\alpha) \cap \pi(\Lambda_\beta)$. We can take a smooth continuation of amplitudes $a_{\alpha k}$ to $a_{\beta k}$ for $k = 0, 1$ preserving the equation (12).

Assume that A_α and A_β are of type $A^{(1)}$. Suppose that there exists a point $(y, \lambda) \in \Lambda_\alpha \cap \Lambda_\beta$ such that rank $d\pi_y(\lambda) = 1$. Then there exists a coordinate system (v, η) in a neighbourhood of λ such that $\theta = \theta(\eta)$, $x = x(v, \eta)$,

$$\varphi_\alpha(x(v,\eta); \theta(\eta)) = \pm \varphi_\beta(v; \eta) \quad \text{and} \quad \frac{a_\alpha(x(v,\eta); \theta(\eta))}{|\eta'_\theta|} = a_\beta(v; \eta).$$

The integral A_α is transformed to A_β by the coordinate change. This transformation also does not contribute to the sum $B_{U,V}$. If there is no such point λ, we can transform A_α to A_β through a function of type $A^{(2)}$.

The most complicated case is the transformation of a function A_α of type $A^{(1)}$ to a sum of functions A_β of type $A^{(2)}$. We assume for simplicity that $a_1=0$. Take an arbitrary point $(y,\lambda)\in\Lambda_\alpha\cap\Lambda_\beta$ such that rank $d\pi_y(\lambda)=2$ and set $x=\pi_y(\lambda)$. We have $\theta=\theta_\beta(x)$ in a neighbourhood of x, due to the rank condition and $\partial_\theta^2\varphi_\alpha(x;\theta_\beta(x))\neq 0$ since ϕ_α is a nondegenerate phase function. Therefore we have

$$\varphi_\alpha(x;\theta)=\varphi_\beta(x)\pm\eta^2, \quad \text{where } \varphi_\beta(x)=\varphi_\alpha(x;\theta_\beta(x)), \tag{18}$$

for a smooth function $\eta=\eta(x,\theta)$ defined in a neighbourhood of $(y,\theta_\beta(x))$ such that $\eta'_\theta\neq 0$, where the phase function $\phi_\beta(x,t)=t-\varphi_\beta(x)$ generates Λ_β. By means of a partition of unity we write the above integral as a sum of integrals of the form

$$B_\beta(x,t)=\int_{\mathbf{R}}\operatorname{Re}p_0(\phi_\beta\pm\eta^2)b(x,t;\eta)\,d\eta,$$

where $b=a|\theta'_\eta|$ is a smooth function with proper support. Write $b(x,t;\eta)=b(x,t;0)+c(x,t;\eta)\eta$ for a smooth function c and get

$$\begin{aligned} B_\beta(x,t)&=\operatorname{Re}\left[b(x,t;0)\int_{\mathbf{R}}p_0(\phi_\beta\pm\eta^2)\,d\eta\right]+\int_{\mathbf{R}}\operatorname{Re}p_0(\phi_\beta\pm\eta^2)c(x,t;\eta)\eta\,d\eta\\ &=\pi\operatorname{Re}[b(x,t;0)((\phi_\beta)_+^{-1/2}\mp\imath(\phi_\beta)_-^{-1/2})]-\frac{1}{2}\int_{\mathbf{R}}\operatorname{Re}p_{-1}(\phi_\alpha)c'_\eta(x,t;\eta)\,d\eta. \end{aligned} \tag{19}$$

The first term is the sum of functions A_β of type $A^{(2)}$. The second term is of type $A^{(1)}$; we denote it $B_{\alpha\beta}$. We have $A_\alpha^{(1)}=\sum(A_\beta^{(2)}+B_{\alpha\beta})$, where the sum is taken over all β such that $\Lambda_\alpha\cap\Lambda_\beta\neq\emptyset$ and

$$B_{\alpha\beta}(x,t)=\int_{\mathbf{R}}\operatorname{Re}p_2(\phi_\beta(x,t)\pm\eta^2)b_\beta(x,t;\eta)\,d\eta$$

for a smooth function b_β with proper support.

We check that the system (15) for φ_α is consistent with (12) for φ_β in the hypersurface $H\doteq\{(x;\eta):(\varphi_\alpha)'_\theta(x;\eta)=0\}$. By (18), $(\varphi_\alpha)'_\theta=\pm 2\eta\eta'_\theta$, where $\eta'_\theta\neq 0$, and we have $\nabla\varphi_\alpha=\nabla\varphi_\beta$ in $H=\{(x,\theta):\eta(x,\theta)=0\}$. From the generalized eikonal equation and (18) we also have

$$\begin{aligned}(\varphi_\alpha)'_\theta\psi&=\mathbf{n}^2-|\nabla\varphi_\alpha|^2=\mathbf{n}^2-|\nabla\varphi_\beta|^2\mp 4\eta\langle\nabla\varphi_\beta,\nabla\eta\rangle-4\eta^2|\nabla\eta|^2\\ &=-4\eta[\pm\langle\nabla\varphi_\beta,\nabla\eta\rangle+\eta|\nabla\eta|^2]\end{aligned}$$

which yields $\psi=-2(\eta'_\theta)^{-1}\langle\nabla\varphi_\beta,\nabla\eta\rangle=-2(\eta'_\theta)^{-1}\langle\nabla\varphi_\alpha,\nabla\eta\rangle$ in H. Therefore the field $2\langle\nabla\varphi_\alpha,\nabla\rangle+\psi\partial_\theta=2\langle\nabla\varphi_\alpha,\nabla\rangle-2(\eta'_\theta)^{-1}\langle\nabla\varphi_\alpha,\nabla\eta\rangle\partial_\theta$ annihilates the function η in H;

hence this field is defined in H. It coincides with $2\langle\nabla\varphi_\beta,\nabla\rangle$ on functions which do not depend on θ, q.e.d. This implies that we can continue the amplitude functions from $\pi(\Lambda_\alpha)$ to $\pi(\Lambda_\beta)$ and vice versa. Finally we conclude that

$$B_{jk}=\sum_{i(\alpha)=j}A_\alpha^{(1)}-\sum_{i(\beta)=k}A_\beta^{(2)}=\sum B_{\alpha\beta}$$

and Lemma 5.5 follows. $\square$

Step 5. Choose a family of functions $\{h_i\in\mathcal{D}(U_i)\}$ such that $\sum_i h_i(y,x)=1$ in $D\times D_0$. We set $P_D\doteq\sum_\alpha h_{i(\alpha)}A_\alpha$; the sum is finite and A_α are functions as in Lemma 5.5. We have

$$Q_D\doteq\square P_D=\sum_j\sum_{\alpha:i(\alpha)=j}[h_j\square A_\alpha+2\langle\nabla h_j,\nabla A_\alpha\rangle+\Delta h_jA_\alpha]. \tag{20}$$

The term $\square A_\alpha$ is of the form (13) or (16). We show that the other terms cancel, up to a function of the form (17). Take an arbitrary point $(y,x)\in D\times D_0$ and fix an arbitrary index j. We have at this point

$$\sum_{i(\alpha)=j}\langle\nabla h_j,\nabla A_\alpha\rangle=\sum_{i(\beta)=k}\langle\nabla h_j,\nabla A_\beta\rangle+\sum_k\langle\nabla h_j,\nabla B_{jk}\rangle.$$

Take the sum with respect to j; the first sum on the right-hand side vanishes, since $\sum_j\nabla h_j=0$. The second sum is of the form (17) according to Lemma 5.5. Therefore the third sum in (20) is equal to the sum of Δh_jB_{jk}, which yields

$$\square P_D=\sum_j\sum_{i(\alpha)=j}h_j\square A_\alpha+2\sum_{j,k}\langle\nabla h_j,\nabla B_{jk}\rangle+\sum_{j,k}\Delta h_jB_{jk}. \tag{21}$$

6. The energy near the source point

Lemma 6.1. *The following inequality holds for* $\tau\le\tau_0$,

$$\int_{t_0(y,x)\le\tau}\exp(2\operatorname{Im}\omega t(y,x))(|\hat A_0(y,x;\omega)|^2+|\omega\square_\omega\hat A_0(y,x;\omega)|^2)\,dx\le C\frac{\tau}{|\omega|},$$

where $\square_\omega\doteq\Delta+\omega^2\mathbf{n}^2+\mathbf{m}$.

Proof. We choose geodesic normal coordinates z in the set $\{(y,x):t_0(y,x)\le\tau_0\}$ such that $t^2(y,x)=|z|^2\doteq z_1^2+z_2^2$. We have by (9),

$$\begin{aligned}\hat A_0(y,x;\omega)&=a_0(y,z)\int_{t\ge|z|}\exp(\imath\omega t)(t^2-|z|^2)^{-1/2}e(t-|z|)\,dt\\&+a_1(y,z)\int_{t\ge|z|}\exp(\imath\omega t)(t^2-|z|^2)^{1/2}e(t-|z|)\,dt.\end{aligned} \tag{22}$$

By changing $s=t-|z|$, we can write $\hat{A}_0(y,x;\omega)=\exp(\imath\omega|z|)[a_0(y,z)I_-+a_1(y,z)I_+]$, where

$$I_\pm \doteq \int_0^\infty \exp(\imath\omega s)(s^2+2|z|s)^{\pm 1/2}e(s)\,ds.$$

We estimate I_- for large ω; the only singular point in the integrand is the origin of the ray $\{s\geq 0\}$. We assume that $e(\tau)=1$ for $|\tau|\leq 1$ and continue e as a smooth function in $\mathbf{C}$, which equals 1 in the unit disc. Following the Laplace method, we replace the ray in (22) by the chain

$$\begin{aligned}
&\gamma=\gamma_1\cup\gamma_2\cup\gamma_3,\\
&\gamma_1=\left\{s=\tau\exp\left(\tfrac{1}{2}\pi-\arg\omega\right):0\leq\tau\leq 1\right\},\\
&\gamma_2=\left\{s=\exp(\imath\psi):0\leq|\psi|\leq\tfrac{1}{2}\pi-\arg\omega\right\},\\
&\gamma_3=\{s\geq 1\}
\end{aligned}$$

and get $I_-=I_1+I_2+I_3$, where I_1 gives the main contribution:

$$\begin{aligned}
|I_1(z)|\doteq\left|\int_0^1 \exp(-\omega\tau)(-\tau^2+2\imath|z|\tau)^{-1/2}\,d\tau\right| &\leq \int_0^\infty \exp(-\omega\tau)(2\tau|z|)^{-1/2}\,d\tau\\
&=\left(\frac{\pi}{2|\omega z|}\right)^{1/2}.
\end{aligned}$$

We have $|s+2|z|s|\geq C(|z|+1)$ in $\gamma_2\cup\gamma_3$ and $\operatorname{Re}\imath\omega s\leq 0$. Integration by parts in I_2+I_3 yields $|I_2+I_3|\leq C|\omega|^{-1}(|z|+1)^{-1/2}$. For the integral I_+ the same arguments yield the estimate $\leq C|z/\omega|^{1/2}$, which implies

$$\int_{t_0\leq\tau}|\exp(\operatorname{Im}\omega t(y,x))\hat{A}_0(y,x;\omega)|^2\,dx\leq\frac{C}{|\omega|}\int_{|z|\leq\tau}\frac{dx}{|z|}\leq C'\frac{\tau}{|\omega|}.$$

We can estimate the kernel $\square_\omega\hat{A}_0$ in a similar way by means of (10). $\square$

7. The energy in the perturbed domain

We estimate the Fourier transforms of P_D and of $Q_D\doteq\square P_D-\delta_{y,0}$. For any $\lambda\in\mathbf{C}$, $\operatorname{Re}\lambda>0$ and arbitrary $b\in\mathcal{D}(\mathbf{R})$,

$$\int_{\mathbf{R}}\exp(\imath\omega\tau)\tau_\pm^{\lambda-1}b(\tau)\,d\tau=\Gamma(\lambda)(\imath\omega\mp 0)^{-\lambda}\left[b(0)+O\left(\frac{1}{\omega}\right)\right],$$

consequently the Fourier transform of a function of type (11) equals

$$\hat{A}^{(2)}(x;\omega)=\int_{\mathbf{R}}\exp(\imath\omega t)A^{(2)}(x,t)\,dt=\int_{\mathbf{R}}\exp(\imath\omega t)(t-\varphi)_{\pm}^{-1/2}e(t-\varphi)a(x)\,dt$$
$$=\pm(\imath\pi\omega\mp 0)^{-1/2}\exp(\imath\omega\varphi(x))\left[a(x)+O\left(\frac{1}{\omega}\right)\right].$$

Note that for the function $A_\alpha=A^{(2)}$ the phase $\varphi(x)$ coincides with the length of a geodesic from $(y,0)$ to (x,t). Therefore $\varphi(x)\geq t(y,x)$ and the exponential factor does not surpass $\exp(-\operatorname{Im}\omega t(y,x))$. By (13) the function $\Box_\omega\hat{A}^{(2)}=F(\Box A^{(2)})$ has a similar structure; hence

$$|\hat{A}^{(2)}(x;\omega)|+|\omega\Box_\omega\hat{A}^{(2)}(x;\omega)|\leq\frac{C}{|\omega|^{3/2}}\exp(-\operatorname{Im}\omega t(y,x)). \tag{23}$$

This inequality agrees with (7) and (8). For a singular function of type (14) defined in an open set $Y\subset X$ the estimate

$$\int_X h\exp(2\operatorname{Im}\omega t(y,x))(|\hat{A}^{(1)}(x;\omega)|^2+|\omega\Box_\omega A^{(1)}(x;\omega)|^2)\,dx\leq\frac{C}{|\omega|}$$

holds for an arbitrary continuous function h compactly supported in Y. It can be checked by the method of Lemma 11.1 below. In the same way we can estimate the integrals of $|\nabla B_{jk}|^2$ in (21), since each kernel ∇B_{jk} is of type (14) or of type (11). Summing these inequalities over α and taking into account (21), we get

$$\int_X h\exp(2\operatorname{Im}\omega t(y,x))(|\widehat{P}_D(y,x;\omega)|^2+|\omega\widehat{Q}_D(y,x;\omega)|^2)\,dx\leq\frac{C}{|\omega|} \tag{24}$$

for any continuous h compactly supported in D_0.

8. Phase functions for constant velocity

Step 6. We extend the above construction for the domain $D\times X$. First, we choose a special generating function for $\Lambda_\infty\doteq\Lambda\cap\pi^{-1}(D\times X\setminus D)$. Denote by Γ_s the solution of (6) with the initial data $x_s=y$ and $\xi_s=(\mathbf{n}(y)\cos s,\mathbf{n}(y)\sin s)$, $0\leq s<2\pi$. By $(*)$ $\mathbf{n}=\mathbf{n}_0$ is constant in $X\setminus D$; the Hamiltonian system has the form

$$\frac{dx}{dr}=-\xi,\quad\frac{dt}{dr}=\mathbf{n}^2,\quad\frac{d\xi}{dr}=0,\quad\tau=\pm1,$$

and the ray $\gamma(s)\doteq\pi_y(\Gamma_s)$ is a straight line in $X\setminus D$. We assume that D_0 is a convex neighbourhood of D with smooth boundary ∂D_0. Denote by $(x_0(s),\xi_0(s))$ the point in Γ_s, where the ray $\gamma(s)$ reaches the boundary. The vector $\xi(s)=-dx/dr$ cannot be tangent to the convex curve ∂D_0, because otherwise the ray $\gamma(s)$ could not enter D. The conditions $(*)$ and $(**)$ imply that x_0 and ξ_0 are C^2-functions in $D\times S^1$. The ray $\gamma(s)\backslash D_0$ is parameterized by

$$x(s,r)=-\xi_0(s)r+x_0(s) \text{ and } t(s,r)=\mathbf{n}^2r+t_0(s) \quad r\geq 0; \tag{25}$$

hence the parameter r vanishes just at ∂D_0. The ray $\gamma(s)$ and its direction $\xi_0(s)$ will be called *critical* if $\xi_0'(s)=0$. The union Ξ of all critical directions has zero angular measure. The caustic set, i.e. the image of the critical set of π_y, is contained in $V\cup G$, where V is an arbitrary open conic neighbourhood of Ξ and G is a compact set.

Consider the phase function $\phi(x,t;s)=t-\varphi(x;s)$, where

$$\varphi(x;s)\doteq t_0(s)+\langle\xi_0(s),x_0(s)-x\rangle=t(s,r)+\langle\xi_0(s),x(s,r)-x\rangle \tag{26}$$

and $\langle\cdot,\cdot\rangle$ stands for the inner product in X. This function satisfies the eikonal equation $|\nabla_x\varphi|^2=|\xi_0|^2=\mathbf{n}^2$. The function ϕ vanishes in Λ_∞ for $t=\tau(s,r)$ and

$$d\varphi(x;s)=[dt_0(s)+\langle\xi_0(s),dx_0(s)\rangle]-\langle\xi_0(s),dx\rangle+\langle d\xi_0(s),x_0(s)-x\rangle.$$

The first bracket vanishes in the Lagrange manifold Λ_∞. We have $\varphi_s'(x;s)=\langle\xi_0'(s),x_0(s)-x\rangle$ and by the eikonal equation, $\langle\xi_0',\xi_0\rangle=0$. Consequently the equation

$$-\phi_s'=\varphi_s'=\langle\xi_0'(s),x_0(s)-x(s,r)\rangle=r\langle\xi_0'(s),\xi_0(s)\rangle=0$$

holds in Λ_∞. Vice versa, for any noncritical direction, the equation $\varphi_s'=0$ implies that $x_0(s)-x=r\xi_0(s)$ for some $r\geq 0$ and $\phi(x,t;s)=t-\tau(s,r)$. Moreover, $d_x\phi=-d_x\varphi=\langle\xi_0,dx\rangle$; hence ϕ generates the noncritical part of Λ_∞.

Set $\sigma\doteq\xi_0\times\xi_0'$; then $|\sigma|\doteq\mathbf{n}|\xi_0'|$. For a vector $v=(v_1,v_2)$ we let $v^*=(-v_2,v_1)$; then $\langle v^*,u\rangle=v\times u$, where u is an arbitrary vector u and $\times$ denotes the cross product.

Lemma 8.1. *The quantity $j(s,r)=r\sigma+x_0'\times\xi_0$ is equal to the divergence of the family of rays and*

$$s_x'=\frac{\xi_0^*}{j},\quad r_x'=\frac{1}{j}(-r\xi_0'+x_0')^*. \tag{27}$$

Proof. The divergence equals the Jacobian

$$j(s,r)=\det\frac{\partial x}{\partial(r,s)}=x_r'\times x_s'=-\xi_0\times(-r\xi_0'+x_0')=r\sigma+x_0'\times\xi_0. \quad\square \tag{28}$$

It follows that there is focusing near any critical ray. Calculate the transport system in (s, r)-coordinates for the phase function (26). We have $\nabla_x \varphi = -\xi_0(s)$ and

$$\Delta\varphi = \langle \nabla, \nabla_x \varphi \rangle = -\langle \xi_0', s_x' \rangle = -\frac{1}{j}\langle \xi_0', \xi_0^* \rangle = \frac{\sigma}{j}.$$

Hence (12) appears as

$$\partial_r (a|j|^{1/2}) = 0. \tag{29}$$

It follows that $L_r(|a|^2\, dx) = 0$, where L_r is the Lie derivative with respect to the field $\partial_r = \partial/\partial r$.

Lemma 8.2. *The equations $x_0' \times \xi_0 = 0$ and $\xi_0' = 0$ have no common solution.*

Proof. The vector x_0' is tangent to ∂D_0 and the vector ξ_0 is not. Therefore the first equation is equivalent to $x_0' = 0$. This together with the second equation implies that s is no more a local coordinate in Λ for a point (s_0, r_0), which contradicts Liouville's theorem. □

9. The parametrix in the nonperturbed domain

Step 7. Choose small numbers $\varepsilon > 0$ and $\delta > 0$ such that

$$|\sigma| \le 3\varepsilon \quad \text{implies} \quad |\xi_0 \times x_0'| \ge \delta \tag{30}$$

and set $\mu \doteq \max_{0 \le s \le 2\pi} |\xi_0(s) \times x_0'(s)| > 0$. We can assume that the set D_0 chosen in Step 4 contains a $2\mu\mathbf{n}\varepsilon^{-1}$-neighbourhood of D where $\mathbf{n} = \mathbf{n}_0$. Take the covering of Λ_∞ by the sets

$$\Lambda_> \doteq \{(s, r) : |\sigma(s)| \ge \varepsilon\},$$
$$\Lambda_< \doteq \left\{(s, r) : |\sigma(s)| \le 3\varepsilon \text{ and either } r|\sigma(s)| \le \tfrac{1}{2}\delta \text{ or } r|\sigma(s)| \ge 2\mu\right\},$$
$$\Lambda_\flat \doteq \left\{(s, r) : |\sigma(s)| \le 3\varepsilon \text{ and } \tfrac{1}{3}\delta \le r|\sigma(s)| \le 3\mu\right\}.$$

By (25), $\mathbf{n}r = |x(s, r) - x_0(s)| \ge 2\mu\mathbf{n}\varepsilon^{-1}$ in $X \setminus D_0$, which yields $r \ge 2\mu\varepsilon^{-1}$. Therefore $r|\sigma| \ge 2\mu \ge |\xi_0 \times x_0'| + \mu$ and $|j| \ge r|\sigma| - |x_0' \times \xi_0| \ge \mu$ in $\Lambda_>$. Hence there is no focusing in $\Lambda_>$ and we can use x as a local coordinate system. We have by (25),

$$\mathbf{n}r - d_0 \le |x - y| \le \mathbf{n}_0 r + d_0, \quad d_0 \doteq \max_{x \in \partial D_0} |x - y|. \tag{31}$$

since $|\xi_0|=\mathbf{n}$. Take a smooth function $h_>$ in $\mathbf{R}$ such that $h_>(\tau)=1$, if $|\tau|\geq 3\varepsilon$, and $h_>(\tau)=0$, if $|\tau|\leq 2\varepsilon$. Set

(32)
$$A_>(x,t) \doteq \sum_{x(s,r)=x} (t-\tau(s,r))_\pm^{-1/2} e(t-\tau(s,r))a(s,r)h_>(\sigma(s)), \quad \pm 1 = \operatorname{sgn} j(s,r),$$

where the sum is taken over all points $(s,r)\in\Lambda_>$ such that $x(s,r)\doteq -\xi_0(s)r+x_0(s)=x$, and the amplitude function a_0 is smooth and satisfies (29). We take the initial data for a from the consistency condition with the kernel P_D. This means that the difference $A_> - P_D$ equals the sum of terms of type (17) in a neighbourhood of ∂D_0.

Lemma 9.1. *The number $N(x)$ of terms in* (32) *that are nonvanishing for $x\in X\backslash D_0$ is uniformly bounded.*

Proof. We call an interval $I\subset S^1$ *univalent* if $|\sigma(s)|\geq\varepsilon$ for $s\in I$. The inequality $|j(s,r)|>0$ holds for $r\geq\mu\varepsilon^{-1}$ in an arbitrary univalent interval. Consider the mapping

$$\pi_y\colon I\times[\mu_0,\infty)\longrightarrow X, \quad (s,r)\longmapsto x(s,r), \tag{33}$$

which is affine with respect to r. It is an imbedding, since the Jacobian does not vanish and I is connected. Hence for any $x\in X\backslash D_0$ there is no more than one solution to the equation $x=x(s,r)$. If the whole circle S^1 is univalent, the number of terms is equal to 1. Suppose the opposite, i.e. there is a point s' such that $|\sigma(s')|<\varepsilon$. We call an interval $I=(s_0,s_1)$ *good*, if it is univalent and $|\sigma(s_0)|=|\sigma(s_1)|=2\varepsilon$. We have $|\sigma(s)|<2\varepsilon$, if a point s does not belong to a good interval, and the corresponding term in (32) vanishes since $h_>(\sigma)=0$. If two good intervals have a common point, then their union is again a good interval and the number of nonvanishing terms in (32) is bounded by the number N of maximal good intervals. We show that $N<\infty$. Indeed, for arbitrary consecutive maximal good intervals $(s_0,s_1),(s_2,s_3)\subset S^1$ such that $s_1<s_2$, there is a point $\tau\in(s_1,s_2)$ such that $|\sigma(\tau)|<\varepsilon$, whereas $|\sigma(s_1)|=|\sigma(s_2)|=2\varepsilon$. These relations imply that $s_2-s_1>2\varepsilon/\max|\sigma'(s)|$. The same inequality holds for $s_0+2\pi-s_3$. Therefore $N<\pi\varepsilon^{-1}\max|\sigma'|$. □

We have the inequality

$$\int_{\Gamma(\tau)} \sum_{x(s,r)=x} |a(s,r)h_>(\sigma)|^2\, dl \leq C, \quad \Gamma(\tau)\doteq\{x : t(y,x)=\tau\},$$

where dl denotes the arc element of $\Gamma(\tau)$ and C does not depend on $\tau\geq\tau_0$. This follows from the equation $dl=(1+O(r^{-1}))\,dx/dr$, where the form dr and the density

$|a|^2\,dx$ are constant along any ray by virtue of (29). The number of terms in the sum is uniformly bounded by Lemma 9.1. By (23) and (32) we find

$$\int_{\Gamma(\tau)} |\hat{A}_>(x;\omega)|^2\,dl \le \frac{C}{|\omega|}\int_{\Gamma(\tau)} |a(s,r)h_>(\sigma)|^2\,dl \le \frac{C}{|\omega|} \tag{34}$$

and

$$\int_{\Gamma(\tau)} |\hat{A}_>(x;\omega)|^2\,dl \le \frac{C}{|\omega|}. \tag{35}$$

where the constant C does not depend on τ and ω.

Step 8. Choose a smooth even function $h_<$ in $\mathbf{R}$ such that $h_<=1$ in $\left[0,\frac{1}{3}\delta\right]\cup[\mu+2,\infty)$ and $h_<=0$ in $\left[\frac{1}{2}\delta,\mu+1\right]$. Define the singular function which is similar to (32):

$$A_<(x,t)=\sum_{x(s,r)=x}(t-t(s,r))_\pm^{-1/2}e(t-\tau(s,r))a(s,r)(1-h_>(\sigma(s)))h_<(r\sigma(s)), \tag{36}$$

where $\pm1=\operatorname{sgn} j(s,r)$ and the amplitude a is defined as in Step 7. Note that $j\neq0$ in $\operatorname{supp}(1-h_>)h_<$; hence the equation $x(s,r)=x$ has a locally smooth solution $s=s(x)$, $r=r(x)$. The function $A_<$ is continued to $D\times(X\backslash D_0\times\mathbf{R})$ by setting $A_<=0$ for $x\in X\backslash(D_0\cup\pi_y(\Lambda_<))$. It is smooth in $D\times X$.

Lemma 9.2. *The number of nonzero terms in* (36) *is uniformly bounded for* $x\in X\backslash D_0$.

Proof. We argue similarly to the proof of Lemma 9.1. We call an interval $J\subset S^1$ *critical* if $|\sigma|\le3\varepsilon$ in J. Let J be a critical interval. If $s\in J$, the inequality $j(s,r)\ge\frac{1}{2}\delta$ holds for $r\le\delta/2|\sigma|$, because of (30), whereas $j(s,r)\ge1$ for $r\ge2\mu|\sigma|^{-1}$, $\sigma\neq0$. Therefore the mapping $J\times[0,\delta(2|\sigma|)^{-1}]\cup[2\mu|\sigma|^{-1},\infty)\to X$ like (33) is an imbedding and for any $x\in X$ there is no more than two solutions to $x(s,r)=x$, if $s\in J$, $(s,r)\in\Lambda_<$. If $|\sigma|\ge2\varepsilon$ everywhere in S^1, the sum (36) vanishes since $h_>(\sigma)=1$. Suppose the opposite. We call a critical interval $J=(s_0,s_1)$ *good*, if $|\sigma(s_0)|=|\sigma(s_1)|=2\varepsilon$. The number of nonzero terms is bounded by the number N of maximal good intervals. We show that this number is finite. Two different maximal good intervals are disjoint. If a point s does not belong to a good interval, we have $|\sigma(s)|>3\varepsilon$ and the corresponding term in (36) vanishes since $h_>(\sigma)=1$. For arbitrary maximal good intervals (s_0,s_1) and (s_2,s_3), there is a point $\tau\in(s_1,s_2)$ such that $|\sigma(\tau)|>3\varepsilon$, whereas $|\sigma(x_1)|=|\sigma(x_2)|=2\varepsilon$. This yields $s_2-s_1\ge2\varepsilon(\max|\sigma'(s)|)^{-1}$ and $N\le\pi\varepsilon^{-1}\max|\sigma'|$. $\square$

10. The parametrix near caustics

Step 9. The integral

$$A_\flat(x,t) \doteq \int_{\mathbf{R}} e(\phi(x,t;s)) \operatorname{Re}(p_0(\phi_\flat(x,t;s))a_\flat(x,s))h_\flat(s,r)\,ds \tag{37}$$

represents the parametrix in the focusing area $\Lambda_\flat$. Here e is a function as in Lemma 5.2,

$$\begin{aligned} \phi_\flat(x,t;s) &= t-\varphi_\flat(x,s),\\ \varphi_\flat(x,s) &= t_0(s)+\langle\xi_0(s),x_0(s)-x\rangle,\\ h_\flat(s,r) &\doteq (1-h_>(\sigma(s)))(1-h_<(\sigma(s)r)),\\ r &\doteq \frac{1}{\mathbf{n}^2}\langle\xi_0(s),x_0(s)-x\rangle, \end{aligned}$$

and the amplitude $a_\flat$ is defined below. We have $(\mathbf{n}_0)'_v=0$ and $\varphi'_s=0$ for $x=x(s,r)$. Therefore, since $\Delta\varphi_\flat=0$, (15) looks as follows:

$$2\partial_r a_\flat = 2\langle\xi_0,\nabla a_\flat\rangle = (\varphi_\flat)'_s b. \tag{38}$$

Introduce the variable $u=\mathbf{n}^{-2}\xi_0\times(x-x_0)$; we have $(\varphi_\flat)'_s=\langle\xi'_0,x-x_0\rangle=\sigma u$. The function $a_\flat$ should satisfy (38) in the surface $u=0$. We extend this function by setting $a_\flat(x,s,r)\doteq a_\flat(x(s,r),s)g(u)$, $x(s,r)=x_0(s)-r\xi_0(s)$, where $g\in\mathcal{D}((-1,1))$ and $g=1$ in a neighbourhood of the origin. It is easy to see that (38) now is satisfied with $b=0$ for any u.

Next we find the consistency condition for amplitudes in the set $\pi(\Lambda_\sharp)$, where $\Lambda_\sharp\doteq(\Lambda_>\cup\Lambda_<)\cap\Lambda_\flat$. First we change the integration variable s in (37) to $u=u(x,s)$. From $u'_s=\mathbf{n}^{-2}(\xi'_0\times(x-x_0)-\xi_0\times x'_0)$, by substituting $x=x_0-r\xi_0+u\xi^*_0$, we obtain $u'_s=\mathbf{n}^{-2}(r\sigma-\xi_0\times x'_0)=\mathbf{n}^{-2}j(s,r)$. We have $\frac{1}{2}\delta\le r|\sigma|\le\frac{2}{3}\delta$ or $1\le r|\sigma|\le 2\mu+2$ in $\Lambda_\sharp$. Therefore the derivative v'_s is bounded from above and below, the function $s=s(x,v)$ is smooth, and we can write

$$A_\flat(x,t) \doteq \int_{\mathbf{R}} e(\phi_\flat(x,t;s)) \operatorname{Re}(p_0(\phi_\flat(x,t;s))a_\flat(x,s))\frac{h_\flat(s,r)}{|j(s,r)|}\mathbf{n}^2\,du.$$

Next we calculate this integral as in Step 4. We have $\partial_u\varphi_\flat=\partial_s\varphi_\flat(u'_s)^{-1}=\mathbf{n}^2\sigma uj^{-1}$; hence $\varphi_\flat(x,s)=\varphi(x)+\chi(s,r)u^2$, $\chi\doteq\mathbf{n}^2\sigma(2j)^{-1}$, where $\varphi(x)=\tau(s,r)$ is the phase function of $A_>$. The function $|\chi|$ is smooth and bounded from below since $|j|\ge\frac{1}{2}\delta$ and $|\sigma|\ge\delta/2r$. As $\phi_\flat(x,t;s)\equiv t-\varphi_\flat(x;s)$, we can integrate by method (19), getting

$$\begin{aligned} A_\flat(x,t) &= \int_{\mathbf{R}} e(\phi+\chi v^2)\operatorname{Re}(p_0(\phi+\chi v^2)a_\flat(x,s))\frac{h_\flat(x)}{|j(s,r)|}\mathbf{n}^2\,du\\ &= \operatorname{Re}((\phi_+^{-1/2}+\varepsilon\imath\phi_-^{-1/2})b(x,0)h_\flat(x)) - \int_{\mathbf{R}}\operatorname{Re}p_1(\phi_\flat)c'_v(x,v)h_\flat(x)\,du. \end{aligned} \tag{39}$$

where $\phi = t - \varphi(x)$, $\varepsilon = -\operatorname{sgn}\chi$ and

$$b(x,v) \doteq 2^{1/2}\pi \mathbf{n} a_\flat(x,s)|j\sigma|^{-1/2} = b(x,0) + 2vc(x,v).$$

Now we can write the consistency condition

$$a(s,r) = b(x,0) = 2^{1/2}\pi \mathbf{n} a_\flat(x,s(x,0))|j\sigma|^{-1/2}, \tag{40}$$

where $x = x(s,r)$, $u = 0$ and a is the amplitude of $A_<$. We see that the product $a(s,r)|j|^{1/2} = 2^{1/2}\pi \mathbf{n} a_\flat(x,s(x,0))|\sigma|^{-1/2}$ does not depend on r, since $a_\flat$ satisfies (38).

11. The energy in the caustic area

Now we estimate the integral of $\hat{A}_\flat$. First calculate the Fourier transform of the functions $p_k(\phi)$, where $\phi = t - \varphi$. We have for $\omega > 0$,

$$F(p_0(\phi)) = -2\pi i \exp(-i\omega\varphi),$$
$$F(p_1(\phi)) = -\frac{2\pi i}{\omega}\exp(-i\omega\varphi),$$
$$F(\bar{p}_0(\phi)) = F(\bar{p}_1(\phi)) = 0;$$

for $\omega < 0$ we need only to replace p_k by $\bar{p}_k$. Therefore

$$\hat{A}_\flat(x;\omega) = \int_{\mathbf{R}} \exp(-i\omega\varphi_\flat(x,s))\tilde{a}(x,s)\,ds + R_\flat(x;\omega),$$

where $\tilde{a}(x,s) = 2\pi i a_\flat(x,s)h_\flat(s,r)$ and $|R_\flat(x;\omega)| \le C|\omega|^{-1}\exp(-\operatorname{Im}\omega t(y,x))$. We get

$$|\hat{A}_\flat(x;\omega)|^2 \le \int_{\mathbf{R}}\int_{\mathbf{R}} \exp(-i\operatorname{Re}\omega\Phi(x;s,s') - \operatorname{Im}\omega\Psi(x;s,s'))a(x;s,s')\,ds\,ds'$$
$$+ |R_\flat(x,\omega)|^2,$$

where

$$\Phi(x;s,s') \doteq \varphi_\flat(x,s) - \varphi_\flat(x,s') = t_0(s) - t_0(s') + \xi_0(s)(x_0(s) - x) - \xi_0(s')(x_0(s') - x),$$
$$\Psi(x;s,s') \doteq \varphi_\flat(x,s) + \varphi_\flat(x,s'),$$
$$a(x;s,s') \doteq \tilde{a}(x,s)\bar{\tilde{a}}(x,s').$$

Integrating over $\Gamma(\tau)$ for $\tau \ge \tau_0$, yields

$$\int_{\Gamma(\tau)} |\hat{A}_\flat(x;\omega)|^2\,dl \le I(\tau,\omega) + R(\tau,\omega),$$

where

$$I(\tau,\omega) \doteq \int_{\Gamma(\tau)}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(-i\operatorname{Re}\omega\Phi(x;s,s') - \operatorname{Im}\omega\Psi(x;s,s'))a(x;s,s')\,ds\,ds'\,dl,$$
$$R(\tau,\omega) \doteq \int_{\Gamma(\tau)} |R_\flat(x,\omega)|^2\,dl.$$

Lemma 11.1. *The following inequalities hold*

(41) $$|I(\tau,\omega)|\le \frac{C}{|\omega|}\tau\exp(-\operatorname{Im}\omega\tau),\ \ R(\tau,\omega)\le \frac{C}{|\omega|^2}\tau\exp(-\operatorname{Im}\omega\tau),\quad \tau\ge 0.$$

Proof. We will apply the stationary phase evaluation. The function $\varphi_\flat$ belongs to C^2 since ξ_0, x_0 are of this class (see Step 9). Therefore the phase function Φ is contained in C^2 too. Find the critical set of $\{\Phi_l=\Phi_s=\Phi_{s'}=0\}$ in the manifold $\Gamma(\tau)\times S^1\times S^1$. The l-derivative can be written as $\partial_l\doteq|x-y|^{-1}(x-y)\times\partial_x$. Therefore the first equation is equivalent to $(\xi_0(s)-\xi_0(s'))\times(x-y)=0$ and implies $\xi_0(s)=\xi_0(s')$. The second and third equations give $\langle\xi_0'(s),x-x_0(s)\rangle=\langle\xi_0'(s),x-x_0(s')\rangle=0$ which imply $x=x(s,r)$ for some $r\ge 0$ and $x_0(s)=x_0(s')$. It follows that $\Gamma(s)=\Gamma(s')$, hence $s'=s$. Calculating the second derivatives in s and l on the diagonal $\Delta(\tau)$ in $\Gamma(\tau)\times S^1\times S^1$ defined by $s=s'$, we get

(42)
$$\Phi_{ls}''=\xi_0'(s)\times|x-y|^{-1}(x-y)=\left(-\frac{1}{\mathbf{n}}+O\left(\frac{1}{\tau}\right)\right)\xi_0'(s)\times\xi_0(s)=\left(\frac{1}{\mathbf{n}}+O\left(\frac{1}{\tau}\right)\right)\sigma,$$

$$\Phi_{ss}''=r\langle\xi_0'',\xi_0\rangle+\langle\xi_0',x_0'\rangle=-r|\xi_0'|^2+\langle\xi_0',x_0'\rangle=\frac{-r\sigma^2+\sigma\xi_0\times x_0'}{\mathbf{n}^2}=\frac{\sigma j}{\mathbf{n}^2},$$

$$\Phi_{ll}''=0,$$

$$\det\nabla_{ls}^2\Phi=\left(-1+O\left(\frac{1}{\tau}\right)\right)\frac{\sigma^2}{\mathbf{n}^2},$$

since $r^{-1}x\to\xi_0$, as $\tau\to\infty$, and $r/\tau=1+O(\tau^{-1})$. We have $|\Phi_{ss}''|\le 4\mu|\sigma|\mathbf{n}^{-2}$, since $|j|\le 4\mu$ in $\Lambda_\flat$; hence all second derivatives in (42) are equal to $O(|\sigma|)$. We apply Proposition 17.1 to the function Φ for fixed s and $x\in\Gamma(\tau)$. There is only one critical point $x_*=x_0(s)-r\xi_0(s)$. We have $\|b^{-1}\|=O(|\sigma|^{-1})$ for $b=\frac{1}{2}d^2\Phi(x;s,s)$. From (42) we see that $|\nabla_{ls}^3\Phi|=O(1)$. Therefore the set $\{(s,x):|s-s'|+|x-x_*|\le c|\sigma(s')|\}$ is contained in U_0. Then we apply Proposition 17.2. All derivatives of first and second order of a are bounded in $\Lambda_\flat$. We have $\varphi_\flat(x,s)>t(y,x)$, since the caustic is always behind the first front of a wave. This yields $\Psi(x;s,s')>2t(y,x)$. Therefore by Proposition 17.3 we obtain $|I(\tau,\omega)|\le C|\omega\sigma|^{-1}\exp(-\operatorname{Im}\omega\tau)$. Taking into account that $\frac{1}{3}\delta\le r|\sigma|$, we estimate $|\sigma|^{-1}\le Cr\le C\tau$. This proves the estimate (41) for $I(\tau,\omega)$. The estimate for the remainder R is more simple. □

12. The parametrix in the homogeneous domain

Step 10. We set $P_\infty=A_>+A_<+A_\flat$, where we choose the amplitude functions in (32), (36) and (37) according to the rule (40). It follows from (34), (35) and (41) that

$$\int_{\Gamma(\tau)} |\widehat{P}_\infty(x;\omega)|^2\,dl \le \frac{C}{|\omega|}\tau\exp(-\operatorname{Im}\omega\tau). \tag{43}$$

We estimate $Q_\infty \doteq \Box P_\infty$ in $D\times(X\setminus D_0\times\mathbf{R})$. We have $Q_\infty = \Box(A_> + A_<) + \Box A_\flat$ and

$$\begin{aligned}\Box(A_>(x,t)+A_<(x,t)) \doteq{} & \sum_{x(s,r)=x} e(\phi)\phi_\pm^{1/2}\Delta a(1-h_\flat)\\ & -2\sum_{x(s,r)=x}\langle\nabla(e(\phi)\phi_\pm^{1/2}a),\nabla h_\flat\rangle - \sum_{x(s,r)=x} e(\phi)\phi_\pm^{1/2}a\Delta h_\flat,\end{aligned} \tag{44}$$

where a is defined in (32) and

$$\begin{aligned}\Box A_\flat(x,t) ={} & \int_{\mathbf{R}} e(\phi)\operatorname{Re}(p_1(\phi)(\Delta+\mathbf{m})a_\flat)h_\flat\,ds\\ & +2\int_{\mathbf{R}}\langle\nabla(\operatorname{Re}p_1(\phi)a_\flat),\nabla h_\flat\rangle\,ds + \int_{\mathbf{R}}\operatorname{Re}p_1(\phi)a_\flat\Delta h_\flat\,ds,\end{aligned} \tag{45}$$

where $a_\flat$ is defined in (38). We want to estimate the Fourier transform of the six terms in (44) and (45). Write the sum as $Q_\infty = Q^0 + Q^1 + Q^2$, where

$$Q^0 \doteq \sum_{x(s,r)=x} e(\phi)\phi_\pm^{1/2}(1-h_\flat)(\Delta+\mathbf{m})a + \int_{\mathbf{R}} e(\phi)\operatorname{Re}(p_1(\phi)(\Delta+\mathbf{m})a_\flat)h_\flat\,ds,$$

$$Q^1 \doteq -2\sum_{x(s,r)=x}\langle\nabla(e(\phi)\phi_\pm^{1/2}a),\nabla h_\flat\rangle + 2\int_{\mathbf{R}}\langle\nabla e(\phi)\operatorname{Re}(p_1(\phi)a_\flat),\nabla h_\flat\rangle\,ds,$$

$$Q^2 \doteq -\sum_{x(s,r)=x} e(\phi)\phi_\pm^{1/2}a\Delta h_\flat + \int_{\mathbf{R}}\operatorname{Re}p_1(\phi)a_\flat\Delta h_\flat\,ds.$$

First, check the inequality

$$\int_{\Gamma(\tau)} |\widehat{Q}^0(x;\omega)|^2\,dl \le \frac{C}{|\omega|^3}\tau\exp(-\operatorname{Im}\omega\tau). \tag{46}$$

For the first term in Q^0 this follows from (23). We estimate the second term similarly to (41), taking into account that derivatives of the amplitude $a_\flat$ are bounded.

The terms Q^1 and Q^2 contain derivatives of the cutting functions $h_>$ and $h_<$. The derivatives $\nabla h_>(\sigma)$ and $\Delta h_>(\sigma)$ are bounded by (27) and (31) since the dominator j has a positive lower bound. By similar arguments we show that $\nabla h_<(\sigma r)=O(|x|)$ and $\Delta h_<(\sigma r)=O(|x|^2)$, as $|x|\to\infty$; hence also

(47) $$h_\flat = O(1), \quad \nabla h_\flat = O(|x|) \quad \text{and} \quad \Delta h_\flat = O(|x|^2).$$

Therefore

(48) $$\int_{\Gamma(\tau)} |\widehat{Q}^2(x;\omega)|^2\,dl \le \frac{C}{|\omega|^3}\exp(-\tau\,\mathrm{Im}\,\omega)(\tau^5+\tau).$$

The function Q^1 is supported in $\pi(\Lambda_\ge)$, since $\nabla h_\flat=0$ in the complement. We write this function in a different form where no kernel $\nabla\phi_\pm^{-1/2}=c\nabla(\phi)\phi_\pm^{-3/2}$ appears. For this we apply (39):

$$\int_{\mathbf{R}} e(\phi)\,\mathrm{Re}(p_0(\phi_\flat)a_\flat)\,ds = \mathrm{Re}((\phi_+^{-1/2}\pm\imath\phi_-^{-1/2})b(x;0)) - \int_{\mathbf{R}} \mathrm{Re}\,p_1(\phi_\flat)c'_u(x;v)\,du,$$

where the first term in the right-hand side contributes to the main term of Q^1. These terms cancel and the rest is equal to

$$Q^1 = -2\int_{\mathbf{R}} \langle\nabla e(\phi)\,\mathrm{Re}(p_1(\phi)c'_u(x;u)), \nabla h_\flat\rangle\,dv.$$

We estimate the Fourier transform of this integral as in (46). For this we need an estimate for the amplitude. By Step 9, we have $c(x;u)=(2u)^{-1}(b(x;u)-b(x;0))$, where $b(x;u)\doteq 2^{1/2}\pi\mathbf{n}a_\flat(x,s)|j\sigma|^{-1/2}$, in the second term. Therefore

$$\max_u |c'_u(x;u)| \le C\max_u |b''_{uu}(x;u)| \le C|a_\flat(x,s)|\,|j\sigma|^{-1/2} \le C|x-y|^{1/2}|a_\flat(x,s)|$$

since the factor $|j|^{-1/2}$ is uniformly bounded and $|\sigma|\le C/r$ in $\Lambda_\ge$, otherwise $\nabla h_\flat=0$. Finally

(49) $$\int_{\Gamma(\tau)} |\widehat{Q}^1(x;\omega)|^2\,dl \le \frac{C}{|\omega|^3}\tau\exp(-\mathrm{Im}\,\omega\tau).$$

The inequalities (46), (49) and (48) imply (8).

Step 11. Set $U\doteq\bigcup_i U_i\subset X\times X$ and take a smooth function $h_D\in\mathcal{D}(U)$ that is equal to 1 in $D\times D_0$; define $P=h_D P_D+(1-h_D)P_\infty$. An estimate for P follows from (24) and (43). By applying Lemma 5.5 we obtain (7). Next we find

(50) $$Q \doteq \Box P - \delta_{y,0} = (h_D Q_D + R_D) + (1-h_D)Q_\infty,$$

where the term $R_D=[\Box, h_D]P_D+[\Box, 1-h_D]P_\infty=[\Box, h_D](P_D-P_\infty)$ is smooth and supported by $K\cap U$. The operator P_D-P_∞ can be represented in the form (17). By Lemma 5.5 this implies an estimate for the kernel $\widehat{R}_D$ similar to (8). This completes the construction for Lemma 5.1. $\Box$

13. The Helmholtz equation with constant velocity

Step 12. Now we are going to find a kernel W such that the sum $S\doteq\widehat{P}+W$ is a fundamental function. This equation is equivalent to $\Box_\omega W=-\widehat{Q}$. First we solve the Helmholtz equation with constant velocity

$$\Box_{\omega 0} W_0=-\widehat{Q}. \tag{51}$$

Lemma 13.1. *There exists a solution* W_0 *of* (51) *that is analytic in the set* $\{\omega:\operatorname{Im}\omega\geq 0$ *and* $\omega\neq 0\}$ *and satisfies*

$$\int_{t(y,x)\leq\tau}|\exp(\operatorname{Im}\omega t_0(y,x))W_0(y,x;\omega)|^2\,dx\leq\frac{1}{|\omega|^4}(C_1\tau+C_2\tau^5),\quad \tau\geq\tau_0. \tag{52}$$

Proof. We solve this equation by means of the convolution

$$W_0(y,x;\omega)\stackrel{?}{=}-\int_X S_0(x-z;\omega)\widehat{Q}(y,z;\omega)\,dz, \tag{53}$$

where S_0 is the source function for (51). This integral does not, however, converge absolutely. We regularize it, using the special structure (50) and set $\widehat{Q}=\widehat{U}_D+\widehat{U}_\infty$, where $\widehat{U}_\infty\doteq(1-h_D)\widehat{Q}_\infty$. The term $\widehat{U}_D\doteq h_D\widehat{Q}_D+\widehat{R}_D$ is supported by D_0, and hence the convolution $W_D\doteq S_0*\widehat{U}_D$ is well defined. To estimate this convolution we use the inequality

$$\int_{D_0}|\exp(\operatorname{Im}\omega t(y,x))\widehat{U}_D(y,x;\omega)|^2\,dx\leq\frac{C}{|\omega|^3} \tag{54}$$

which follows from (8) for $\widehat{Q}_D$ and $\widehat{R}_D$. We have

$$\begin{aligned}|\exp(\operatorname{Im}\omega t_0(y,x))W_D(y,x;\omega)|&=\left|\int_{D_0}\exp(\operatorname{Im}\omega t_0(y,x))S_0(x-z;\omega)\widehat{U}_D(y,z;\omega)\,dz\right|\\&\leq\int_{D_0}|\exp(\operatorname{Im}\omega t_0(z,x))S_0(x-z;\omega)\exp(\operatorname{Im}\omega t_0(y,z))\widehat{U}_D(y,z;\omega)|\,dz\end{aligned}$$

since $t_0(y,x)\leq t_0(y,z)+t_0(z,x)$. Therefore by Lemma 13.2,

$$\begin{aligned}\int_{t_0(y,x)\leq\tau}&|\exp(\operatorname{Im}\omega t_0(y,x))W_D(y,x;\omega)|^2\,dx\\&\leq\left(\int_{t_0(x,z)\leq\tau+\tau_0}|\exp(\operatorname{Im}\omega t_0(x,z))S_0(x-z;\omega)|\,dz\right)^2\\&\quad\times\int_{D_0}|\exp(\operatorname{Im}\omega t_0(y,z))\widehat{U}_D(y,z;\omega)|^2\,dx,\end{aligned}$$

where τ_0 is the diameter of D_0 in the metric g_0. The second factor is bounded by $C|\omega|^{-3}$, due to (54) and the inequality $t_0(y,z)\le t(y,z)$. The first factor is bounded by

$$\frac{C}{|\omega|}\left(\int_{t_0(x,z)\le\tau+\tau_0}|x-z|^{-1/2}\,dz\right)^2\le\frac{C}{|\omega|}(\tau+\tau_0)^3,$$

because of (64) and $t_0(x,z)\le\mathbf{n}_0|x-z|$. This implies

$$\int_{t_0(y,x)\le\tau}|\exp(\operatorname{Im}\omega t_0(y,x))W_D(y,x;\omega)|^2\,dx\le\frac{C}{|\omega|^4}(\tau+\tau_0+\tau_1)^3$$

since $t(y,x)\le t_0(y,x)+\tau_1$, where τ_1 is the g-diameter of D_0.

Lemma 13.2. *The inequality* $\|a*b\|_2\le\|a\|_2\|b\|_1$ *holds for arbitrary* $a\in L_2(X)$ *and* $b\in L_1(X)$.

Proof. A proof can be done by the Fourier transform. □

Next, we define the convolution of the functions S_0 and $\widehat{U}_\infty\doteq(1-h_D)\widehat{Q}_\infty$. For this we will use analytic continuation in the variable r. We have

$$\widehat{Q}_\infty=F(\Box A_>)+F(\Box A_<)+F(\Box A_\flat). \tag{55}$$

We consider the three terms separately. According to (32) and Lemma 9.1 the term $(1-h_D)F(\Box A_>(y,x,t))$ equals the sum of the functions

$$\begin{aligned}U^n_>(y,x;\omega)&\doteq F[(t-\tau(s,r))_\pm^{-1/2}e(t-\tau(s,r))(\Delta+\mathbf{m})a(s,r)h_>(\sigma)]\\&=\exp(\imath\omega\tau(s,r))q(\omega)(\Delta+\mathbf{m})a(s,r)h_>(\sigma)(1-h_D)\end{aligned} \tag{56}$$

for $n=1,\dots,N$, where $q(\omega)\doteq F(t_\pm^{-1/2}e(t))$ and the right-hand side is a univalent (and smooth) function of $x=x(s,r)$. The sign $\pm1=\operatorname{sgn}j(s,r)$ is constant for each n and the number N of terms depends on y, t, s and r. We wish to regularize the integral

$$W^n_>(y,x;\omega)\overset{?}{=}-\int_{X\setminus D_0}S_0(x-z;\omega)U^n_>(y,z;\omega)\,dz. \tag{57}$$

First, note that the integral converges absolutely for $\operatorname{Im}\omega>0$, since $S_0(x-z;\omega)=O(\exp(-\operatorname{Im}\omega|x-z|))$ and $U^n_>(y,z;\omega)=O(\exp(-\operatorname{Im}\omega t(y,z)))$ by (56). According to (25) the function $\tau(s,r)$ is linear with respect to r, and $a(s,r)$ is of the form $a_0(s)r^{-1/2}$ by (28) and (29). The function $(\Delta+\mathbf{m})a$ is also analytic in r, as can be seen from (27). Therefore the right-hand side has analytic continuation to the domain $\{\varrho=r+v\imath : r\geq 0 \text{ and } v\in\mathbf{R}\}$ and this continuation satisfies

$$
\begin{aligned}
|U^n_>(y,x;\omega)| &\leq C|q(\omega)|\exp(-\operatorname{Im}\omega\tau(s,\varrho)) \\
&\leq \frac{C}{|\omega|^{3/2}}\exp(-\operatorname{Im}\omega\tau(s,r)-\operatorname{Re}\omega\mathbf{n}_0^2 v(r)),
\end{aligned} \tag{58}
$$

since $\tau(s,\varrho)=\tau(s,r)+\mathbf{n}_0^2 v\imath$, $\tau(s,r)=t(y,x)$. Suppose that $\operatorname{Re}\omega\geq 0$. Then the right-hand side is bounded in the halfplane $v\geq 0$. First, we make the change of variables $z=z(r,s)$, where $dz=\jmath\, dr\, ds$, $\jmath=\jmath(s,r)$. Take into account that the integrand $S_0(x-z;\omega)U^n_>(y,z;\omega)\jmath(s,r)$ admits a holomorphic continuation in r.

Shift the domain $X\backslash D_0$ to the chain X_v defined by the mapping $(s,r)\mapsto z=z(s,r)\doteq-\xi_0(s)(r+v(r)\imath)+x_0(s)$ from $S'\times\mathbf{R}$ to the complex space $X_{\mathbf{C}}=X+\imath X$, where $v(r)$ is a suitable continuous function, provided the function $S_0(x-z;\omega)$ has a bounded holomorphic continuation to a 3-chain Z_v in $X_{\mathbf{C}}$ such that $\partial Z_v=X\backslash D_0-X_v$. The integrals over $X\backslash D_0$ and X_v will coincide by the Cauchy theorem. According to Proposition 16.1 this condition is satisfied if $\operatorname{Im}\omega\sqrt{(x-z)^2}\geq 0$ for $z\in Z_v$, where $w^2=w_1^2+w_2^2$. This follows from the inequality $|\mathbf{n}_0\operatorname{Im}\omega-\operatorname{Im}\mathbf{m}|\leq C$. Take a positive number v_0, to be specified later, and a continuous function $v=v(r)$ that vanishes for $0\leq r\leq r_0(x)\doteq 4\mathbf{n}_0^{-1}(|x|+\max|x_0(s)|)$, and equals $\frac{1}{4}(r-r_0)$ for $r\geq r_0$. Set $Z_v\doteq\{z=-\xi_0(s)(r+\tau v(r)\imath)+x_0(s) : 0\leq\tau\leq 1\}$ and show that

$$
\operatorname{Re}(x-z)^2\geq\operatorname{Im}(x-z)^2\geq 0 \quad \text{for } z\in Z_v. \tag{59}
$$

This is obvious for $r\leq r_0(x)$, since $v(r)=0$. Otherwise, it follows from the equations

$$
\begin{aligned}
\operatorname{Re}(x-z)^2 &= (\xi_0 r+x-x_0)^2-(\mathbf{n}_0\tau v(r))^2, \\
\operatorname{Im}(x-z)^2 &= 2\langle\operatorname{Re}z-x,\operatorname{Im}z\rangle = 2\tau v(r)(\mathbf{n}_0^2 r+\langle x-x_0,\xi_0\rangle).
\end{aligned}
$$

This property implies that Z_v is contained in the domain $\{\omega : \operatorname{Im}\omega\sqrt{(x-z)^2}\geq 0\}$ for $\operatorname{Re}\omega\geq 0$; the function $S_0(x-z;\omega)$ decreases exponentially in Z_v, as $r\to\infty$. Then we apply a version of Lemma 13.1 to the integral (57) taken over X_v. The last factor in (58) is bounded by $\exp(-c\operatorname{Re}\omega r)$ for some $c>0$. We have $\operatorname{Re}\omega>0$ and $r\geq\mathbf{n}_0^{-1}|y-z|-r_1$ for some r_1, which ensures convergence of an integral like (57) taken over X_v.

14. Quasianalytic continuation

Next we consider the second term in (55). It is a finite sum of the terms $U^n_<$ that are similar to (56) with the extra factor $h_<(r\sigma)$, $\nabla h_<(r\sigma)$ or $\Delta h_<(r\sigma)$. This factor has analytic continuation for complex $\varrho = r + v\imath \in \mathbf{C}$, except for the interval $K(s) \doteq \{r : \frac{1}{3}\delta \le r|\sigma(s)| \le 3\mu\}$. This set might be unbounded near critical values of the variable s. We continue h to the function in the complex plane:

$$\tilde{h}(\varrho) \doteq \sum_{q=0}^{3} \frac{(\imath v)^q}{q!} h^{(q)}(r), \quad \varrho = r + v\imath,$$

such that $\bar{\partial}\tilde{h}(\varrho) = \frac{1}{2} v^3 h^{(4)}(r) = O(v^3)$ and also

$$\partial^l \bar{\partial}^{k+1} \tilde{h}(\varrho) = O(v^{3-l-k}), \quad l+k \le 2. \tag{60}$$

Now we regularize the integral

$$W^n_<(x;\omega) \overset{?}{=} -\int_{X \setminus D_0} S_0(x-z;\omega) U^n_<(y,z;\omega)\, dz,$$

by means of the Cauchy–Green theorem as follows:

$$\begin{aligned} W^n_<(x;\omega) \doteq &-\int_{X_v} S_0(x-z;\omega) U^n_<(y,z;\omega) j\, dr\, ds \\ &-\int_{Z_v} S_0(x-z;\omega) \bar{\partial}_\varrho U^n_<(y,z;\omega) j\, d\bar{\varrho}\, d\varrho\, ds, \end{aligned}$$

where $Z_v = \{z = z(r+v\imath, s) : 0 \le v \le v(r)\} \subset X_{\mathbf{C}}$. The first integral is similar to that of the function $W^n_>$. The inequality $|x - \operatorname{Re} z| \le \operatorname{Re}\sqrt{(x-z)^2}$ holds for $z \in Z_v$, by (59). On the other hand, $\operatorname{Re}\omega \operatorname{Im}\sqrt{(x-z)^2} \ge 0$. This yields

$$\operatorname{Im}\omega|x-y| \le \operatorname{Im}\omega|x - \operatorname{Re} z| + \operatorname{Im}\omega|y - \operatorname{Re} z| \le \operatorname{Im}\omega\sqrt{(x-z)^2} + \operatorname{Im}\omega|y - \operatorname{Re} z|.$$

By this inequality

$$\begin{aligned} \exp(\operatorname{Im}\omega \mathbf{n}_0|x-y|) &\left| \int_{Z_v} S_0(x-z;\omega) \bar{\partial}_\varrho U^n_<(y,z;\omega) j\, d\bar{\varrho}\, d\varrho\, ds \right| \\ &\le \sup_{z \in Z_v} \left| \exp\left(\mathbf{n}_0 \operatorname{Im}\omega\sqrt{(x-z)^2}\right) S_0(x-z;\omega) \right| \\ &\quad \times \int_{Z_v} \exp(\operatorname{Im}\omega \mathbf{n}_0|y - \operatorname{Re} z|) |\bar{\partial}_\varrho U^n_<(y,z;\omega) j|\, dv\, dr\, ds \\ &\le \frac{C}{|\omega|^{1/2}} \int_{Z_v} \exp(\operatorname{Im}\omega \mathbf{n}_0|y - \operatorname{Re} z|) |\bar{\partial}_\varrho U^n_<(y,z;\omega) j|\, dv\, dr\, ds. \end{aligned}$$

The supremum is finite by Proposition 16.1. Applying an inequality like (58) to $U^n_<$ and taking into account that $\operatorname{Re}\omega\mathbf{n}_0^2 v\geq 0$, we get

$$|\omega|^{3/2}\int_{Z_v}\exp(\operatorname{Im}\omega\mathbf{n}_0|y-\operatorname{Re}z|)|\bar\partial_\varrho U^n_<(y,z;\omega)j|\,dv\,dr\,ds \tag{61}$$
$$\leq C\int_{Z_v}(|j\bar\partial_\varrho\tilde h|+|j\bar\partial_\varrho\nabla\tilde h|+|j\bar\partial_\varrho\Delta\tilde h|)\,dv\,dr\,ds,$$

where $\tilde h=\tilde h(\sigma\varrho)$. To estimate the first term in the right-hand side, we use the relations $\bar\partial_\varrho\tilde h(\sigma\varrho)=O(\sigma^4|v|^3)$, $j=\varrho\sigma+O(1)$. Then

$$\int_{Z_v}|j\bar\partial_\varrho\tilde h|\,dv\,dr\,ds\leq C_1\int_{S^1}|\sigma|^5\int_K\int_0^{v(r)}|v|^4|j|\,dv\,dr\,ds$$
$$\leq C_2\int_{S^1}|\sigma|^5\int_{K(s)}((\log r/r_0)^6+C)\,dr\,ds\leq C_3,$$

since $\sigma^3\log(r/r_0)$ is uniformly bounded, $r\in K(s)$, and $s\in S^1$. For the third term in (61) we use (27) and (60) which gives $|j\bar\partial_\varrho\Delta\tilde h(\sigma\varrho)|\leq C|\sigma|^4\varrho^2 v$. This estimate is sufficient for the finiteness of the integral of $|j\bar\partial_\varrho\Delta\tilde h|\,dv\,dr\,ds$ over Z_v. The second and third terms in (61) are estimated similarly.

Now we have only to consider the third term in (55). We have

$$\Box_\omega\hat A_\flat(x;\omega)=\int_{\mathbf{R}}\exp(i\omega\varphi_\flat(x,s))\operatorname{Re}q_1(\omega)(\Delta+\mathbf{m})(a_\flat(x,s)h_\flat(r,s))\,ds,$$

where $q_1(\omega)=F(p_1(t))$ and the omitted terms contain derivatives of $h_\flat$. The phase function $\varphi_\flat(x,s)=t_0(s)+\mathbf{n}_0^2 r$ for $r\doteq\mathbf{n}_0^{-2}\langle\xi_0(s),x_0(s)-x\rangle$ admits analytic continuation with respect to r and $\operatorname{Im}\varphi_\flat(x,s)=\mathbf{n}_0^2 v$. This function is nonnegative if $v\geq 0$. According to the choice (38) the function $a_\flat(x,s)$ does not depend on r at all. Therefore only the $\bar\partial_r$-derivative of the factors (47) appear in $\bar\partial_r\Box_\omega\hat A_\flat$. We estimate the corresponding integrals as in the previous case. In this way we define the regularized convolution $W_\flat=S_0*\Box_\omega\hat A_\flat$. Taking the sum $W_0=\sum W^n_>+\sum W^n_<+W_\flat$, we get a solution of (51) that admits an analytic continuation at the domain $\{\omega:\operatorname{Im}\omega\geq 0\text{ and }\operatorname{Re}\omega>0\}$, which satisfies (52). To get an analytic continuation to the domain $\operatorname{Re}\omega<0$, we replace v by $-v$ in the above arguments. This completes the proof of Lemma 13.1. □

15. The completion of the construction

Step 13. We have $\Box_\omega((1-h_D)W_0)+\widehat Q=-L$, where $h_D\in\mathcal D(X)$ is as in Step 11. By (51) the kernel $L=h_D\widehat Q-\langle\nabla h_D,\nabla W_0\rangle-\Delta h_D W_0$ is supported in $D\times D_0$. We take into account (52) and estimate the derivatives of W_0 by the standard energy integral method

$$\int_{D_0} |\exp(\operatorname{Im}\omega t_0(y,x))\nabla W_0(y,x;\omega)|^2\,dx \le \frac{C}{|\omega|^2}.$$

This and (8) imply

$$\int_X |\exp(\operatorname{Im}\omega t_0(y,x))L(y,x;\omega)|^2\,dx \le \frac{C_L}{|\omega|^2}. \tag{62}$$

Now we look for a kernel R such that $S=\widehat{P}+W_0+R$ is a fundamental solution. It is now equivalent to $\Box_\omega R=L$. The kernel $T\doteq\widehat{P}+W_0$ satisfies $\Box_\omega T=\delta_y-L$. By (7),

$$\int_{t(y,x)\le\tau} |\exp(\operatorname{Im}\omega t_0(y,x))T(y,x;\omega)|^2\,dx \le \frac{C_T}{|\omega|}(\tau^5+\tau).$$

The sum $S=T+R$ is a fundamental function, if R is a solution of the integral equation

$$R(y,x;\omega)-\int_X L(y,z;\omega)R(z,x;\omega)\,dz = \int_X L(y,z;\omega)T(z,x;\omega)\,dz.$$

Write the solution by means of the Neumann series $R\doteq R^{(1)}+R^{(2)}+\dots$, where

$$R^{(k)}(y,x;\omega)\doteq\int_X\dots\int_X L(y,z_k;\omega)\dots L(z_3,z_2;\omega)L(z_2,z_1;\omega)T(z_1,x;\omega)\,dz_k\dots dz_1.$$

For an arbitrary $y\in D$, we have by (62),

$$\begin{aligned}
&\int_X |\exp(\operatorname{Im}\omega t_0(y,x))R^{(k)}(y,x;\omega)|^2\,dx\\
&\qquad\le\int_X\Big|\exp(\operatorname{Im}\omega t_0(y,x))\\
&\qquad\quad\times\int_X\dots\int_X L(y,z_k;\omega)\dots L(z_2,z_1;\omega)T(z_1,x;\omega)\,dz_k\dots dz_1\Big|^2\,dx\\
&\qquad\le\int_X |\exp(\operatorname{Im}\omega t_0(y,z))L(y,z;\omega)|^2\,dz \qquad (63)\\
&\qquad\quad\times\left(\int_D\int_D |\exp(\operatorname{Im}\omega t_0(z,x))L(z,x;\omega)|^2\,dz\,dx\right)^{k-1}\\
&\qquad\quad\times\int_X |\exp(\operatorname{Im}\omega t_0(z,x))T(z,x;\omega)|^2\,dz\,dx
\end{aligned}$$

$$\leq \frac{C_T}{|\omega|}\left(V(D)\int_X |\exp(\operatorname{Im}\omega t_0(y,z))L(y,z;\omega)|^2\,dz\right)^k$$
$$\leq \frac{C_T(C_L V(D))^k}{|\omega|^{2k+1}},$$

where $V(D)$ stands for the volume of V, since $t_0(y,x) \leq t_0(y,z_k)+...+t_0(z_1,x)$. The series converges if $|\omega|^2 > C_L V(D)$, and we obtain the estimate for $|\omega| \geq \omega_0 \doteq (2C_L V(D))^{1/2}$:

$$\int_X |\exp(\operatorname{Im}\omega t_0(y,x))R(y,x;\omega)|^2\,dx \leq \frac{C_T}{V(D)|\omega|^3}.$$

The inequality (3) follows from (7) and (63). This completes the construction of the source function S that satisfies (3) for all $y \in D$. The compact D can be magnified, consequently the above arguments are valid for arbitrary compact subsets D of the plane. The sequence of the corresponding source functions gives the global function in $X \times X$ because of the uniqueness theorem. To complete the proof of Theorem 3.1 we only need to check that the inequality (3) is satisfied with some constants C_1 and C_2 that do not depend on $y \in X$. Starting from a point $y \in X \setminus D$, in the first step we use the source function S_0 for the homogeneous space. It satisfies (3) with $C_2 = 0$ according to Proposition 16.1 of the next section. Moreover, the arc integrals of the density $|\exp(\operatorname{Im}\omega t_0(y,x))S_0(y,x;\omega)|^2\,dl$ over $\Gamma(\tau)$ are uniformly bounded. Indeed, if y is at the distance r from D, only the portion $O(r^{-1})$ of the energy is scattered on the inhomogeneity of the medium. This, together with the arguments of Steps 2–13 shows that the estimate (3) is uniform for $y \in X$. This completes the proof of Theorem 3.1. □

16. The source function in homogeneous medium

Denote by $\log\zeta$ the univalent branch that is defined for $|\arg\zeta| < \pi$ and is positive for $\zeta > 1$. Set $\sqrt{\zeta} = \exp(\frac{1}{2}\log\zeta)$.

Proposition 16.1. *If* $\mathbf{p}$ *is constant and* $\operatorname{Im}\mathbf{p} \geq 0$, *then the source function* S_0 *of the operator* $\Delta + \mathbf{p}^2$ *has the form*

$$S_0(x;\omega) = \frac{1}{4\pi} J_0\left(\sqrt{(\mathbf{p}x)^2}\right)\log(\mathbf{p}x)^2 + A((\mathbf{p}x)^2),$$

where J_0 *is the Bessel function and* A *is an entire analytic function of order* $\frac{1}{2}$. *Moreover, it satisfies the inequality*

$$|S_0(x;\omega)| \leq 8\pi\left|\sqrt{(\mathbf{p}x)^2}\right|^{-1/2}\exp\left(-\operatorname{Im}\mathbf{p}\sqrt{x^2}\right) \tag{64}$$

in the domain $|\arg(\mathbf{p}x)^2| < \pi$, *of the complex 2-space* $X_{\mathbf{C}}$.

Proof. For real $\mathbf{p}$ the distribution $E_0(x,t)=-(2\pi)^{-1}\theta(t-\mathbf{p}|x|)(t^2-\mathbf{p}^2|x|^2)^{-1/2}$ defined in $X\times\mathbf{R}$ satisfies the equation $(\Delta-\mathbf{p}^2\partial_t^2)E_0=\delta_0$. Set $\varrho=\sqrt{(\mathbf{p}x)^2}$ and calculate

$$-2\pi S_0(x)=\int_{t\geq r}\exp(t\imath)(t^2-\varrho^2)^{-1/2}\,dt\overset{t=\varrho s}{=}\int_{s\geq 1}\exp(\varrho s\imath)(s^2-1)^{-1/2}\,ds$$
$$\overset{s=1+\sigma\imath}{=}-\imath\exp(\varrho\imath)\int_0^\infty\exp(-\varrho\sigma)(-\sigma^2+2\sigma\imath)^{-1/2}\,d\sigma.$$

The right-hand side admits an analytic continuation to the halfplane $\operatorname{Re}\varrho>0$. The second factor is estimated by means of the Cauchy theorem as follows:

$$\left|\int_0^\infty\exp(-\varrho\sigma)(-\sigma^2+2\sigma\imath)^{-1/2}\,d\sigma\right|=\left|\int_0^\infty\exp(-\varrho\sigma)\sigma^{-1/2}(2\imath-\sigma)^{-1/2}\,d\sigma\right|$$
$$\overset{\sigma=\bar\varrho\tau}{=}\bar\varrho\left|\int_0^\infty\exp(-|\varrho|^2\tau)(\bar\varrho\tau)^{-1/2}(2\imath-\bar\varrho\tau)^{-1/2}\,d\tau\right|$$
$$\leq\max_\tau|2\imath-\bar\varrho\tau|^{-1/2}|\varrho|^{1/2}\left|\int_0^\infty\frac{\exp(-|\varrho|^2\tau)}{\tau^{1/2}}\,d\tau\right|$$
$$=\left(\tfrac{1}{2}\pi\right)^{1/2}|\varrho|^{-1/2}.\quad\square$$

Remark. We apply this proposition to the coefficient $\mathbf{p}=\sqrt{\omega^2\mathbf{n}_0^2+\mathbf{m}_0}$ for positive $\mathbf{n}_0$. Therefore if $\operatorname{Im}\omega\geq 0$ and $|\omega|\geq|\mathbf{m}_0|^{1/2}\mathbf{n}_0^{-1}$, the coefficient $\mathbf{p}$ is also in the upper halfplane.

17. The quantitative stationary phase method

Introduce a norm in $\mathbf{R}^n$ and denote by $\|a\|$ the norm of an operator a in $\mathbf{R}^n$.

Proposition 17.1. *Let $x_0\in\mathbf{R}^n$ be a critical point of a smooth phase function φ and U be a starlike neighbourhood of x_0 such that*

$$\|\{\varphi''_{ij}(x)-\varphi''_{ij}(x_0)\}\|\,\|\{\varphi''_{ij}(x_0)\}^{-1}\|<\tfrac{1}{2}\tag{65}$$

for $x\in U$. Then there exists a smooth coordinate system $y=y(x)$ in U such that $y(x_0)=0$ and $\beta(y(x))=\varphi(x)$, where $\beta(y)=\frac{1}{2}\varphi''_{ij}(x_0)y^iy^j$.

Proof. Assume that $x_0=0$ and write $\varphi(x)-\beta(x)=\sum_{i,j=1}^n a_{ij}(x)x^ix^j$, where

$$a_{ij}(x)=\int_0^1\int_0^1[\varphi''_{ij}(stx)-\varphi''_{ij}(0)]\,ds\,dt.\tag{66}$$

Look for a solution of form $y(x)=x+zx$, where $z=z(x)$ is an unknown matrix function vanishing at the origin, and x is written as a column. The equation for z is $\beta(x,zx)+\beta(zx,x)+\beta(zx,zx)=x'ax$, where x' means the row of coordinate functions and $a=\{a_{ij}\}$. Therefore it is sufficient to solve the matrix equation $z+b^{-1}z'b+b^{-1}z'bz=c\doteq b^{-1}a$, where $b=\{\frac{1}{2}\varphi''_{ij}(x_0)\}$. Since the matrices b and a are symmetric, we have $b^{-1}(c')^k b=c^k$ for any natural k. Write z as a power series in c such that $z(0)=0$. The system takes the form $2z+z^2=c$ and we find $z=z(c)=(1+c)^{1/2}-1$. The series converges if $\|c(x)\|<1$. We have by (66) and (65) that

$$\|c(x)\|\le\|b^{-1}\|\,\|a(x)\|\le 2\|\{\varphi''_{ij}(x_0)\}^{-1}\|\,\|\{\varphi''_{ij}(x)-\varphi''_{ij}(x_0)\}\|<1. \quad \square$$

Proposition 17.2. *Let $\beta(x)=\frac{1}{2}b_{ij}x^ix^j$ be a nondegenerated form of signature s and f be a continuous function such that $f,g\in L_1(\mathbf{R}^n)$. The integral*

$$I(\omega)=\int_{\mathbf{R}^n}\exp(2\pi\imath\omega\beta(x))f(x)\,dx, \quad \omega>0,$$

can be written in the form

$$I(\omega)=\frac{\exp\frac{1}{4}\imath\pi s}{\omega^{n/2}\sqrt{\det b+0\imath}}f(0)+\frac{1}{\pi\omega\imath}\int_{\mathbf{R}^n}\exp(2\pi\imath\omega\beta(x))g(x)\,dx,$$

where

(67)

$$g\doteq-\sum_{j=1}^n\alpha^j(D)f_j,\quad \sum_{j=1}^n x^jf_j(x)=f(x)-f(0),\quad \alpha^j=a^{jk}\partial_k,\quad \{a^{jk}\}=\{b_{jk}\}^{-1}.$$

Proof. A proof can be done by the regularization $\beta\mapsto\beta+\imath\varepsilon x^2$ and partial integration. $\square$

An estimate can be done for the function g:

Proposition 17.3. *If $U\subset\mathbf{R}^n$ is an arbitrary starlike neighbourhood of the origin, then there exists a function g, that satisfies* (67) *and* $\max_U|g|\le\max_U|\alpha(D)f|$, *where $\alpha(D)=a^{jk}\partial_j\partial_k$.*

Proof. A proof follows from the equations

$$f_j(x)=\int_0^1\partial_jf(tx)\,dt,$$

$$g(x)=-\sum_{j=1}^n\int_0^1\alpha^j(D)\partial_jf(tx)t\,dt=-\int_0^1\alpha(D)f(tx)t\,dt. \quad \square$$

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