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Rademacher chaos: tail estimates versus limit theorems

Ron Blei and Svante Janson

Abstract. We study Rademacher chaos indexed by a sparse set which has a fractional combinatorial dimension. We obtain tail estimates for finite sums and a normal limit theorem as the size tends to infinity. The tails for finite sums may be much larger than the tails of the limit.

1. Introduction and results

A (homogeneous) Rademacher chaos is a random variable of the type

$$S = \sum_{i_1<\dots<i_d} a_{i_1\dots i_d} r_{i_1} \dots r_{i_d}, \tag{1.1}$$

where $d\geq 1$, $a_{i_1\dots i_d}$ are real or complex numbers and $r_1, r_2, \dots$ is a sequence of independent random variables with the symmetric two-point distribution $\mathbf{P}(r_i=1)=\mathbf{P}(r_i=-1)=\frac{1}{2}$. (For example, r_i could be the classical Rademacher functions [19], defined on $[0,1]$ (with the usual Lebesgue measure) by $r_i(x)=1-2b_i$ when $x\in[0,1]$ has the binary expansion $0.b_1b_2\dots$, but it is often more convenient to let r_i be defined on the Cantor group $\mathbf{Z}_2^\infty$. For our purposes, the choice of r_i does not matter.) Equivalently, S is a linear combination of the Walsh functions of the type $r_{i_1} \dots r_{i_d}$.

We will consider only finite sums (1.1), so there is no problem of convergence, and all moments of S are finite.

We are interested in two related properties of the random variables S: the tail behaviour, i.e. the size of the probabilities $\mathbf{P}(|S|>x)$ for large x, and the size of the L^q norms $\|S\|_q=(\mathbf{E}|S|^q)^{1/q}$ for large q. For convenience, we define $\widetilde{S}=S/\|S\|_2$; thus $\mathbf{E}\widetilde{S}=0$ and $\operatorname{Var}\widetilde{S}=\mathbf{E}|\widetilde{S}^2|=1$.

Bonami's hypercontractive inequality [4] implies that every S in (1.1) satisfies

$$\|S\|_q \leq (q-1)^{d/2}\|S\|_2 = (q-1)^{d/2}\Bigl(\sum_{i_1<\dots<i_d} |a_{i_1\dots i_d}|^2\Bigr)^{1/2}, \quad q\geq 2, \tag{1.2}$$

or, equivalently, $\|\widetilde{S}\|_q\leq(q-1)^{d/2}$, $q\geq 2$. (See also [1], [2], [10] and [12].)

In general, this estimate is best possible, up to a constant depending on d but not on q. For example, it is easily seen that $a_{i_1\dots i_d}=1$ for $1\le i_1<\dots<i_d\le n$ (and 0 otherwise) yields an $S=S_n$ that after suitable normalization converges, as $n\to\infty$, to a (Hermite) d-degree polynomial in a Gaussian random variable; see Example 3.2. It thus follows that for some $c(d)>0$ and every $q\ge 2$, $\|S\|_q\ge c(d)(q-1)^{d/2}\|S\|_2$ provided n is large enough. (See e.g. [10, Chapter XIII].)

In this paper we study Rademacher chaos (1.1) where most coefficients $a_{i_1\dots i_d}=0$ so that we really only sum over an indexing set which is combinatorially sparse in the sense of [2, Chapters XII and XIII]. In this case, Bonami's hypercontractive inequality (1.2) can be improved, precisely reflecting the sparsity of the indexing set.

We first recall some definitions [2, Chapter XIII], which we modify and adapt to our purposes in this paper.

For $F\subseteq\mathbf{N}^d$ and $\alpha>0$, define

$$\Psi_F(s)=\max\{|F\cap(A_1\times\dots\times A_d)|: A_j\subset\mathbf{N},\ |A_j|\le s,\ j=1,\dots,m\},$$

and

$$d_F(\alpha)=\sup_{s\ge 1}\frac{\Psi_F(s)}{s^\alpha}=\sup_{A_1,\dots,A_d}\frac{|F\cap(A_1\times\dots\times A_d)|}{\left(\max_{1\le j\le d}|A_j|\right)^\alpha}.$$

In [2, Section XIII.4], the combinatorial dimension of a set $F\subseteq\mathbf{N}^d$ is defined to be

$$\dim F=\limsup_{s\to\infty}\frac{\log\Psi_F(s)}{\log s}=\sup\{\alpha: d_F(\alpha)=\infty\}=\inf\{\alpha: d_F(\alpha)<\infty\}. \tag{1.3}$$

In this paper, we consider sequences of index sets $F_N\subseteq[N]^d$, where $[N]=\{1,\dots,N\}$, and adopt the definition below. Because we want to consider only non-empty index sets, we consider sequences starting at some index $N_0\ge 1$; this allows for some empty F_N for smaller N that we ignore.

Definition. A sequence $F_N\subseteq[N]^d$, $N=N_0,N_0+1,\dots$, has combinatorial dimension α if there exist positive constants C_1 and C_2 such that for all $N\ge N_0$,

$$d_{F_N}(\alpha)\le C_1,$$

(i.e. $|F_N\cap(A_1\times\dots\times A_d)|\le C_1(\max_{1\le j\le d}|A_j|)^\alpha$), and

$$|F_N|\ge C_2N^\alpha.$$

We write $\dim\{F_N\}=\alpha$.

Given a set $F \subseteq \mathbf{N}^d$, we define $F_N = F \cap [N]^d$. In the present paper, we define $\dim F = \dim\{F_N\}$ when the latter exists (and leave the dimension undefined otherwise).

Remark 1.1. Note that this is a stricter definition than (1.3); there are sets F with no dimension in the present sense, but it is easily seen that when the dimension exists in the present sense, it coincides with (1.3).

If the cardinalities of F_N are uniformly bounded, then $\dim\{F_N\}=0$; otherwise $1 \le \dim\{F_N\} \le d$ (if $\dim\{F_N\}$ exists at all). We are mainly interested in the case $1 < \dim\{F_N\} < d$.

Let $\Delta^d = \{(i_1, \dots, i_d) : 1 \le i_1 < \dots < i_d < \infty\}$ and $\Delta^d_N = \Delta^d \cap [N]^d$. We will in the sequel consider only $F \subseteq \Delta^d$ and $F_N \subseteq \Delta^d_N$; this is not essential, but restriction to ordered sets of indices is convenient when we study sums (1.6).

It is proved in [2, Chapter XIII] that for every $\alpha \in [1, d]$, there exist sets $F \subset \Delta^d$ of combinatorial dimension α (also in the stricter sense used here). Such sets can always be constructed by a random procedure; for rational $\alpha \ge 1$ and d such that $d\alpha$ is an integer, it is also possible to use the following deterministic construction.

Example 1.2. (Minimal fractional Cartesian products [2, Section XIII.1 and p. 493].) Fix arbitrary integers $d \ge 3$ and $1 \le m \le d$, and let $\{S_1, \dots, S_d\}$ be a cover of $[d]$ consisting of m-subsets of $[d]$, such that every $i \in [d]$ appears in exactly m elements of $S_1, \dots, S_d$; i.e., $\bigcup_{j=1}^d S_j = [d]$, $|S_j| = m$, and for every $i \in [d]$, $|\{j : i \in S_j\}| = m$.

We employ the following notation: if X is a set, $\mathbf{y} = (y_1, \dots, y_d) \in X^d$ and $S \subseteq [d]$, then

$$\pi_S \mathbf{y} = (y_i : i \in S).$$

For an integer $N \ge d^m$, let n be the greatest integer such that $n \le N^{1/m}$. Fix a one-to-one map φ from $[n]^m$ into $[N]$, and consider

$$F_N^* = \{(\varphi(\pi_{S_1}\mathbf{k}), \dots, \varphi(\pi_{S_d}\mathbf{k})) : \mathbf{k} \in [n]^d\}. \tag{1.4}$$

In order to obtain a subset of Δ^d_N, for the purposes of this paper, we modify this set to

$$F_N = \{(i_1, \dots, i_d) \in \Delta^d_N : (i_{\varrho 1}, \dots, i_{\varrho d}) \in F_N^* \text{ for some permutation } \varrho\}. \tag{1.5}$$

We call the sequence $\{F_N\}$ a *fractional Cartesian product*.

We further say the the fractional Cartesian product is *disconnected* if $[d]$ can be partitioned into two disjoint non-empty subsets T_1 and T_2 such that each S_j is a subset of either T_1 or T_2, and *connected* otherwise.

The archetypal case is $d=3$, $m=2$, $S_1=\{1,2\}$, $S_2=\{1,3\}$ and $S_3=\{2,3\}$. This gives a connected fractional Cartesian product.

Claim. $\dim\{F_N\}=\dim\{F_N^*\}=d/m$.

Proof. We verify the claim in the archetypal case $d=3$, $m=2$ only. The general case is similar; see [2, Corollary XIII.16].

Let $1\le s\le N$ be an integer, and let A, B and C be arbitrary subsets of $[N]$. Then,

$$|F_N^*\cap(A\times B\times C)|=\sum_{k_1,k_2,k_3\in[n]}\mathbf{1}_A(\varphi(k_1,k_2))\mathbf{1}_B(\varphi(k_1,k_3))\mathbf{1}_C(\varphi(k_2,k_3)).$$

A three-fold application of the Cauchy–Schwarz inequality yields

$$\begin{aligned}|F_N^*\cap(A\times B\times C)|&\le\Big(\sum_{k_1,k_2\in[n]}\mathbf{1}_A(\varphi(k_1,k_2))\Big)^{1/2}\Big(\sum_{k_1,k_3\in[n]}\mathbf{1}_B(\varphi(k_1,k_3))\Big)^{1/2}\\&\quad\times\Big(\sum_{k_2,k_3\in[n]}\mathbf{1}_C(\varphi(k_2,k_3))\Big)^{1/2}\\&\le|A|^{1/2}|B|^{1/2}|C|^{1/2},\end{aligned}$$

which implies $\Psi_{F_N^*}(s)\le s^{3/2}$ and thus $\Psi_{F_N}(s)\le 6s^{3/2}$. In the opposite direction,

$$|F_N|\ge|\Delta_n^3|=\binom{n}{3}\ge c_1n^3\ge c_2N^{3/2}.\quad\square$$

Remark 1.3. Again, the definition differs slightly from [2]; there the fractional Cartesian product is defined on an infinite set ($n=\infty$ in (1.4)).

Remark 1.4. Note that the function φ appears in the definition of a fractional Cartesian product only because we let the indices be integers in this paper. We might avoid φ by changing the notation slightly: for example, for the case $d=3$, $m=2$, we could equivalently write (1.9) below as $S_N=\sum_{i<j<k\le n}r_{ij}r_{ik}r_{jk}$, where r_{ij}, $i<j$, are independent Rademacher variables.

It is shown in [2] (e.g., Corollary XIII.29; see Remark 1.9 below) that if $F\subseteq\Delta^d$ (finite or infinite), and S is a Rademacher chaos

$$S=\sum_{(i_1,\dots,i_d)\in F}a_{i_1\dots i_d}r_{i_1}\cdots r_{i_d},\tag{1.6}$$

then

$$\|S\|_q\le Kd_F(\alpha)^{1/2}q^{\alpha/2}\|S\|_2,\quad q\ge1,\tag{1.7}$$

where $K<\infty$ depends only on the ambient dimension d. In particular, if $\dim\{F_N\}<d$, the exponent in (1.2) can be improved, with d replaced by the combinatorial dimension.

These norm estimates lead to tail estimates by the customary procedure: If (1.7) holds and $d_F(\alpha)<\infty$, then for any $x>0$ and $q\geq 1$, by Markov's inequality,

$$\mathbf{P}(|\widetilde{S}|\geq x)\leq x^{-q}\mathbf{E}|\widetilde{S}|^q = x^{-q}\|\widetilde{S}\|_q^q \leq (x^{-1}Cq^{\alpha/2})^q,$$

where $C=Kd_F(\alpha)^{1/2}$. Taking $q=(x/C)^{2/\alpha}e^{-1}$ (if $x\geq Ce^{\alpha/2}$), we obtain

$$\mathbf{P}(|\widetilde{S}|\geq x)\leq e^{-\alpha q/2}=\exp(-cx^{2/\alpha}), \tag{1.8}$$

for a constant $c>0$ depending on d, α and $d_F(\alpha)$ only.

The norm and tail estimates above are in fact sharp, in a sense made precise below. (Cf. [2, Corollary XIII.29].) For simplicity, we will consider only the case where $a_{i_1\dots i_d}$ is 0 or 1. Specifically, we consider a sequence of non-empty sets $F_N\subseteq\Delta_N^d$ and Rademacher chaos

$$S_N=\sum_{(i_1,\dots,i_d)\in F_N} r_{i_1}\dots r_{i_d}. \tag{1.9}$$

Clearly, $\|S_N\|_2=|F_N|^{1/2}$, and thus $\widetilde{S}_N=|F_N|^{-1/2}S_N$.

Theorem 1.5. *Suppose* $\dim\{F_N\}=\alpha\geq 1$, *where* $F_N\subseteq\Delta_N^d$. *Let* S_N *be given by* (1.9). *Then there exist positive constants* c_1, c_2, c_3 *and* c_4 (*depending only on* d, α, C_1, C_2 *and* N_0 *above*) *such that for every* $q\geq 1$,

$$c_1q^{\alpha/2}\leq\sup_N\|\widetilde{S}_N\|_q\leq c_2q^{\alpha/2}, \tag{1.10}$$

and for all $x\geq 2$,

$$\exp(-c_3x^{2/\alpha})\leq\sup_N\mathbf{P}(|\widetilde{S}_N|>x)\leq\exp(-c_4x^{2/\alpha}). \tag{1.11}$$

A natural question arises: Is it possible to replace $\sup_N$ in (1.10) and (1.11) by $\lim_{N\to\infty}$? (See Remark (ii) in [2, p. 524].) In the standard integer-dimensional case $F_N=\Delta^d$, the answer is affirmative (by a d-fold application of the usual central limit theorem). But in many fractional-dimensional cases, the answer is negative: the precise relation between tail estimates and combinatorial dimension, as per (1.11), is completely wiped out in the limit. We illustrate this in two important cases.

Theorem 1.6. *Let $F_N \subseteq \Delta_N^d$, $N=1, \dots,$ and let S_N be given by* (1.9). *Suppose either* (i) $d=2$ *and* $1<\dim\{F_N\}<2$, *or* (ii) F_N *is a connected fractional Cartesian product as in Example* 1.2. *Then* $\widetilde{S}_N \xrightarrow{\mathrm{d}} N(0,1)$ *with convergence of all moments. In particular, if* $\xi \sim N(0,1)$, *then, for all* $q \geq 1$,

$$\lim_{N\to\infty} \|\widetilde{S}_N\|_q = \|\xi\|_q \leq q^{1/2}$$

and for all $x \geq 2$,

$$\lim_{N\to\infty} \mathbf{P}(|\widetilde{S}_N| > x) = \mathbf{P}(|\xi| > x) \leq \exp\left(-\tfrac{1}{2}x^2\right).$$

Case (ii) with $m=2$ can be translated (using Remark 1.4) into a result for random graphs, which is a special case (with $p=\frac{1}{2}$) of [8, Theorem 1]; see also [9] and [10, Chapter XI].

Theorems 1.5 and 1.6 complement one another in the following (heuristic) sense. Let us agree that tail probabilities of sums of uncorrelated symmetric variables provide a gauge of interdependence between the variables: larger tail probabilities (smaller likelihood of cancellations) convey higher degree of interdependence, and conversely. In this light, Theorem 1.5 provides a precise assessment of interdependence of the random variables $r_{i_1} \dots r_{i_d}$, $(i_1, \dots, i_d) \in F_N$. As a counterpoint, reflecting increasing sparsity of F_N relative to the full product set Δ_N^d, Theorem 1.6 asserts that F_N *in the limit*, as $N \to \infty$, is asymptotically independent.

Theorems 1.5 and 1.6 show that, for large q or x, the limits as $N \to \infty$ are much smaller than the largest values for finite N. If we fix a large q and study $\|\widetilde{S}_N\|_q$ as N grows, we begin with rather small values (at most $|F_N|^{1/2}$) that grow to a maximum of the order $q^{\alpha/2}$ (when N is about q, see Section 2), but then the norms decrease again towards a limit of the order $q^{1/2}$. (We do not know whether the increase and decrease are monotone; there might be several local maxima.) A similar story holds for $\mathbf{P}(|\widetilde{S}_N| > x)$ for a fixed large x. Consequently, the limit results in Theorem 1.6 are misleading when we consider $\widetilde{S}_N$ for finite N.

A central limit theorem in fact holds generally under a condition of sparsity in F_N that is milder than the sparsity implied by non-integer combinatorial dimension. The condition is in effect that F_N is not "too close" to a product set. To express this precisely we use the following terminology. For $j \in [N]$, define

$$F_{Nj}^* = \{(i_1, \dots, i_d) \in F_N : j \in \{i_1, \dots, i_d\}\}.$$

Further, let $F_N^\#$ be the subset of $F_N \times F_N$ defined as follows: a pair of d-tuples $((i_1, \dots, i_d), (j_1, \dots, j_d)) \in F_N^\#$ if $\{i_1, \dots, i_d\} \cap \{j_1, \dots, j_d\} = \emptyset$ and there are $(k_1, \dots, k_d) \in F_N$ and $(l_1, \dots, l_d) \in F_N$ such that $\{k_1, \dots, k_d, l_1, \dots, l_d\} = \{i_1, \dots, i_d, j_1, \dots, j_d\}$ but $(k_1, \dots, k_d)$ does not equal $(i_1, \dots, i_d)$ or $(j_1, \dots, j_d)$. (In other words, the $2d$ indices $i_1, \dots, i_d, j_1, \dots, j_d$ can be partitioned in at least two ways into elements of F_N.)

Theorem 1.7. *Suppose*

$$\lim_{N\to\infty} \max_j \frac{|F^*_{Nj}|}{|F_N|} = 0 \tag{1.12}$$

and

$$\lim_{N\to\infty} \frac{|F^\#_N|}{|F_N|^2} = 0. \tag{1.13}$$

Then $\widetilde{S}_N \xrightarrow{\mathrm{d}} N(0,1)$, *with convergence of all moments.*

We have the following partial converse. (The trivial example $F=\{(1,j):j\geq 2\}$ shows that (1.12) is not necessary; we do not know whether it is needed at all in Theorem 1.7.)

Theorem 1.8. *Suppose that* $\widetilde{S}_N \xrightarrow{\mathrm{d}} N(\mu,\sigma^2)$ *for some* μ *and* $\sigma^2>0$. *Then* $\mu=0$, $\sigma^2=1$ *and* (1.13) *holds.*

The proof of Theorem 1.5 is given in Section 2, and the proofs of Theorems 1.6, 1.7 and 1.8 are given in Section 3. Some simple examples of non-normal limits when (1.13) is not satisfied are given also in Section 3. Further remarks and open problems are presented in Section 4.

Remark 1.9. The results in the present paper use Corollary XIII.29 in [2]. A correction to an argument in the proof of that theorem is included in the preprint version of the present paper [3]. The referee has pointed out that (1.7) also follows from [2, Corollary XIII.28] together with the decoupling inequality, see e.g. [18, Theorem 3.1.1].

2. The proof of Theorem 1.5

The proof of Theorem 1.5. The upper bounds follow by (1.7) and (1.8) (for $x\geq x_0$, say; the case $2\leq x\leq x_0$ follows by Chebyshev's inequality if c_4 is small enough).

To verify the lower bounds, let $\mathcal{E}_N$ be the event $r_1=...=r_N=1$; thus $\mathbf{P}(\mathcal{E}_N)=2^{-N}$. On $\mathcal{E}_N$, we have $S_N=|F_N|$ and thus $\widetilde{S}_N=|F_N|^{1/2}$. Hence, for every $q\geq 1$,

$$\|\widetilde{S}_N\|_q \geq |F_N|^{1/2}\mathbf{P}(\mathcal{E}_N)^{1/q} \geq cN^{\alpha/2}2^{-N/q}.$$

To verify the left inequality in (1.10), in the line above choose $N=\max\{N_0,\lfloor q\rfloor\}$.

Similarly, given x (large enough), let $N=\lceil Cx^{2/\alpha}\rceil$ for a constant $C>C_2^{-1/\alpha}$, where C_2 is as in the definition above. Then, on $\mathcal{E}_N$,

$$\widetilde{S}_N = |F_N|^{1/2} \geq C_2^{1/2}N^{\alpha/2} > x,$$

and thus

$$\mathbf{P}(\widetilde{S}_N > x) \geq \mathbf{P}(\mathcal{E}_N) = 2^{-N} \geq \exp(-c_3x^{2/\alpha}). \quad \square$$

3. Asymptotic normality

Lemma 3.1. *If $d=2$, then for any α and any finite subsets A and B of* $\mathbf{N}$ *such that $|A|\le|B|$,*

$$|F\cap(A\times B)|\le 2d_F(\alpha)|A|^{\alpha-1}|B|,$$
$$|F\cap(B\times A)|\le 2d_F(\alpha)|A|^{\alpha-1}|B|.$$

Proof. We may assume $|A|\ge1$. Partition B into $\lceil |B|/|A|\rceil\le 2|B|/|A|$ subsets B_j with $|B_j|\le|A|$. For each j, $|F\cap(A\times B_j)|\le\Psi_F(|A|)\le d_F(\alpha)|A|^\alpha$, and similarly for $|F\cap(B_j\times A)|$. The result follows by summing over j. $\square$

Proof of Theorem 1.6. We verify the conditions of Theorem 1.7. Let $\alpha=\dim\{F_N\}$.

First consider case (i), i.e., suppose that $d=2$ and $1<\alpha<2$. Then, $|F^*_{Nj}|\le N=o(|F_N|)$, which verifies (1.12).

Next, choose $\varepsilon_N>0$ such that $\varepsilon_N\to0$ and $N\varepsilon_N^{1/(2-\alpha)}\to\infty$. (For example, $\varepsilon_N=N^{-\delta}$ for $0<\delta<2-\alpha$.) Let $A=A_N=\{i:|F^*_{Ni}|\ge\varepsilon_N N\}$. Then,

$$|F_N\cap(A\times[N])|+|F_N\cap([N]\times A)|=\sum_{i\in A}|F^*_{Ni}|\ge\varepsilon_N N|A|,$$

and thus, by Lemma 3.1,

$$\varepsilon_N N|A|\le 4d_{F_N}(\alpha)|A|^{\alpha-1}N,$$

which implies

$$|A|\le\left(\frac{4d_{F_N}(\alpha)}{\varepsilon_N}\right)^{1/(2-\alpha)}=o(N). \tag{3.1}$$

By definition, $F_N^\#$ is the set of all $((i,j),(k,l))\in F_N\times F_N$, all of which entries are distinct, such that either $(\{i,k\},\{j,l\})\in F_N\times F_N$ or $(\{i,l\},\{j,k\})\in F_N\times F_N$ (or both), where $\{i,k\}=(i,k)$ when $i<k$ and (k,i) when $i>k$. We let $F^\#_{N1}$ be the subset of $F^\#_N$ where $i\in A$, and $F^\#_{N2}$ the subset where $i\notin A$.

The number of possible $(i,j)\in F_N$ with $i\in A$ is $|F_N\cap(A\times[N])|$, and thus, by Lemma 3.1 and (3.1),

$$|F^\#_{N1}|\le|F_N\cap(A\times[N])|\,|F_N|\le 2d_{F_N}(\alpha)|A|^{\alpha-1}N|F_N|=o(N^\alpha|F_N|)=o(|F_N|^2). \tag{3.2}$$

On the other hand, let $F_{Ni}^{**}=\{k:(i,k)\in F_{Ni}^{*}$ or $(k,i)\in F_{Ni}^{*}\}$. Thus $|F_{Ni}^{**}|=|F_{Ni}^{*}|$. If $((i,j),(k,l))\in F_N^{\#}$, then either k or l is in F_{Ni}^{**}, and thus the number of possible $(k,l)\in F_N$ for a given $i\notin A$ is at most, again by Lemma 3.1,

$$|F_N\cap(F_{Ni}^{**}\times[N])|+|F_N\cap([N]\times F_{Ni}^{**})|\leq 4d_F(\alpha)|F_{Ni}^{**}|^{\alpha-1}N\leq 4d_F(\alpha)\varepsilon_N^{\alpha-1}N^{\alpha}.$$

Summing over all possible (i,j) we find

$$|F_{N2}^{\#}|\leq 4d_F(\alpha)\varepsilon_N^{\alpha-1}N^{\alpha}|F_N|=o(|F_N|^2). \tag{3.3}$$

Combining (3.2) and (3.3), we obtain (1.13) and the result follows in this case.

In case (ii), we first observe that fixing an index j in F_N means that the corresponding $\mathbf{k}$ in (1.4) is such that $\pi_{S_i}\mathbf{k}=\varphi^{-1}(j)$ for some i; for each i this means that m of the d coordinates of $\mathbf{k}$ have given values, so the number of choices of $\mathbf{k}$ is at most dn^{d-m}. Consequently, $|F_{Nj}^{*}|\leq dn^{d-m}=o(|F_N|)$, proving (1.12).

Next, suppose that

$$((i_1,\dots,i_d),(j_1,\dots,j_d))\in F_N^{\#},$$

and that the d-tuples $(i_1,\dots,i_d)$ and $(j_1,\dots,j_d)$ are generated by (1.4) and (1.5) by some vectors $\mathbf{i}$ and $\mathbf{j}$ in $[n]^d$, respectively. By the definition of $F_N^{\#}$, there exists also $(k_1,\dots,k_d)\in F_N$, generated in the same way by, say, $\mathbf{k}\in[n]^d$, such that $\{k_1,\dots,k_d\}\subseteq\{i_1,\dots,i_d,j_1,\dots,j_d\}$ but $(k_1,\dots,k_d)$ does not equal $(i_1,\dots,i_d)$ or $(j_1,\dots,j_d)$.

Hence, each $\pi_{S_\nu}\mathbf{k}$, $1\leq\nu\leq d$, coincides with some $\pi_{S_\mu}\mathbf{i}$ or $\pi_{S_\mu}\mathbf{j}$, $1\leq\mu\leq d$. Define

$$\begin{aligned} J_1&=\{\nu\in[d]:\pi_{S_\nu}\mathbf{k}=\pi_{S_\mu}\mathbf{i}\text{ for some }\mu\};\\ J_2&=\{\nu\in[d]:\pi_{S_\nu}\mathbf{k}=\pi_{S_\mu}\mathbf{j}\text{ for some }\mu\};\\ T_s&=\bigcup_{\nu\in J_s}S_\nu,\qquad s=1,2.\end{aligned}$$

Then $J_1\cup J_2=[d]$ and $T_1\cap T_2\neq\emptyset$, because otherwise the fractional Cartesian product would be disconnected.

If $q\in T_1$, then $q\in S_\nu$ for some $\nu\in J_1$, and thus $\pi_{S_\nu}\mathbf{k}=\pi_{S_\mu}\mathbf{i}$ for some μ. In particular, the qth coordinate of $\mathbf{k}$ is one of the coordinates of $\mathbf{i}$. Similarly, if $q\in T_2$, then the qth coordinate of $\mathbf{k}$ is one of the coordinates of $\mathbf{j}$.

Because $T_1\cap T_2\neq\emptyset$, it follows that $\mathbf{i}$ and $\mathbf{j}$ have at least one coordinate in common (not necessarily in the same position). Consequently, the number of possible pairs $(\mathbf{i},\mathbf{j})$ is $O(n^{2d-1})$, and

$$|F_N^{\#}|=O(n^{2d-1})=O(N^{2(d/m)-1})=o(N^{2\alpha})=o(|F_N|^2),$$

verifying (1.13). $\square$

Proof of Theorem 1.7. All limits in the proof are as $N\to\infty$. We begin by observing that the assumption (1.12) implies

$$\sum_{j=1}^{N}|F^*_{Nj}|^2\le\max_j|F^*_{Nj}|\sum_{j=1}^{N}|F^*_{Nj}|\le\max_j|F^*_{Nj}|\cdot d|F_N|=o(|F_N|^2). \tag{3.4}$$

We use the martingale central limit theorem, as stated in [15, Corollary (2.13)]. We let

$$F_{Nj}=\{(i_1,\dots,i_d)\in F_N:i_d=j\}\subseteq F^*_{Nj},$$

and let

$$X_{Nj}=\sum_{(i_1,\dots,i_d)\in F_{Nj}}r_{i_1}\dots r_{i_d}=r_j\sum_{(i_1,\dots,i_d)\in F_{Nj}}r_{i_1}\dots r_{i_{d-1}}.$$

Then

$$S_N=\sum_{j=1}^{N}X_{Nj},$$

and with $\widetilde{X}_{Nj}=|F_N|^{-1/2}X_{Nj}$,

$$\widetilde{S}_N=\sum_{j=1}^{N}\widetilde{X}_{Nj}.$$

Evidently, $(\widetilde{X}_{Nj})_{j=1}^N$ is a martingale difference sequence for the filtration $\mathcal{F}_j=\mathcal{F}(r_1,\dots,r_j)$, and we have $\mathbf{E}\widetilde{S}_N^2=\sum_{j=1}^N\mathbf{E}\widetilde{X}_{Nj}^2=1$.

By [15, Corollary (2.13)], to prove $\widetilde{S}_N\xrightarrow{\mathrm{d}}N(0,1)$ it suffices to verify the Lindeberg condition

$$\sum_{j=1}^{N}\mathbf{E}(\widetilde{X}_{Nj}^2\mathbf{1}[|\widetilde{X}_{Nj}|>\varepsilon])\to0\quad\text{for every }\varepsilon>0, \tag{3.5}$$

together with

$$\limsup_{N\to\infty}\sum_{i\ne j}\mathbf{E}(\widetilde{X}_{Ni}^2\widetilde{X}_{Nj}^2)\le1. \tag{3.6}$$

Because every moment of $\widetilde{S}_N$ stays bounded by (1.2), moment convergence will follow as well.

To prove (3.5) it suffices to show that

$$\sum_{j=1}^{N} \mathbf{E}\widetilde{X}_{Nj}^4 \to 0. \tag{3.7}$$

In our case, by (1.2) we note that $\|\widetilde{X}_{Nj}\|_4 \le 3^{d/2}\|\widetilde{X}_{Nj}\|_2$, and therefore

$$\sum_{j=1}^{N} \mathbf{E}\widetilde{X}_{Nj}^4 \le 3^{2d}\sum_{j=1}^{N} \|\widetilde{X}_{Nj}\|_2^4 = 3^{2d}\sum_{j=1}^{N} \frac{|F_{Nj}|^2}{|F_N|^2} \le \frac{3^{2d}}{|F_N|^2}\sum_{j=1}^{N} |F_{Nj}^*|^2,$$

which by (3.4) implies (3.7).

It remains to verify (3.6). For simplicity we first treat the case $d=2$, and will later describe the modifications needed in the general case. If $d=2$, then

$$\mathbf{E}(X_{Ni}^2 X_{Nj}^2) = \sum_{k,l,m,n} \mathbf{E}\mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(l,i)\mathbf{1}_{F_N}(m,j)\mathbf{1}_{F_N}(n,j) r_k r_l r_m r_n.$$

We have, $\mathbf{E}r_k r_l r_m r_n = 0$ unless the indices k, l, m, n coincide in pairs, and obtain (overcounting the case when all four indices coincide)

$$\begin{aligned}\mathbf{E}(X_{Ni}^2 X_{Nj}^2) \le &\sum_{k,m} \mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(m,j)\mathbf{1}_{F_N}(m,j)\\ &+2\sum_{k,l} \mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(l,i)\mathbf{1}_{F_N}(k,j)\mathbf{1}_{F_N}(l,j).\end{aligned}$$

Summing the first term on the right over all i and j, we obtain $|F_N|^2$. Therefore, to show (3.6), it suffices to verify that

$$\sum_{i\neq j}\sum_{k,l} \mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(l,i)\mathbf{1}_{F_N}(k,j)\mathbf{1}_{F_N}(l,j) = o(|F_N|^2). \tag{3.8}$$

The sum above equals the number of pairs $((k,i),(l,j)) \in F_N \times F_N$ such that also $((l,i),(k,j)) \in F_N \times F_N$. The number of such pairs with distinct i, j, k, l is at most $|F_N^{\#}|$. Further, the number of pairs $((k,i),(l,j)) \in F_N \times F_N$ where two indices are equal to some r is at most $|F_{Nr}^*|^2$. Consequently, the sum in (3.8) is at most

$$|F_N^{\#}| + \sum_{r=1}^{N} |F_{Nr}^*|^2,$$

and (3.8) follows by (1.13) and (3.4).

In the case $d \geq 2$, we similarly find that $\mathbf{E}X_{Ni}^2 X_{Nj}^2$ equals the number of quadruples I_1, I_2, I_3, I_4 of d-tuples in F_N wherein the $4d$ indices coincide in pairs, and the last index is i in I_1 and I_2, and j in I_3 and I_4. We group such quadruples according to the positions of the pairs of coinciding elements (again overcounting in the cases with less than $2d$ distinct indices, when there are several possibilities of pairing).

To do this precisely, let $\hat{I}_k = \{1, \dots, d\} \times \{k\}$, $k=1,2,3,4$; thus, $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$, $\hat{I}_4$ are four disjoint copies of $\{1, \dots, d\}$. We define a *pattern* to be a complete matching in $\hat{I}_1 \cup \hat{I}_2 \cup \hat{I}_3 \cup \hat{I}_4$, i.e. a partition of the $4d$ points into $2d$ pairs, which are regarded as the edges of a graph.

For a pattern π, any assignment of indices in $\{1, \dots, N\}$ to the $2d$ edges defines 4 d-tuples I_1, I_2, I_3, I_4 in the obvious way. Let $T_N(\pi)$ be the number of quadruples $(I_1, I_2, I_3, I_4) \in F_N^4$ generated in this way, i.e., the number of all assignments such that $I_1 \in F_N$, $I_2 \in F_N$, $I_3 \in F_N$ and $I_4 \in F_N$. Finally, let Π' denote the set of all patterns that contain the two edges $\{(d,1),(d,2)\}$ and $\{(d,3),(d,4)\}$.

In this framework, we then observe that

$$\sum_{i,j} \mathbf{E}X_{Ni}^2 X_{Nj}^2 \leq \sum_{\pi \in \Pi'} T_N(\pi).$$

We classify the patterns in Π' into three types: a pattern is of type I if all its edges are inside $\hat{I}_1 \cup \hat{I}_2$ or $\hat{I}_3 \cup \hat{I}_4$; it is of type II if it is not of type I and there are no edges connecting $\hat{I}_2$ and $\hat{I}_3$, and type III otherwise.

First, consider a pattern π of type I. Since the d-tuples in F_N are ordered, it follows that $T_N(\pi)=0$ unless π is a pattern with the edges $\{(i,1),(i,2)\}$ and $\{(i,3),(i,4)\}$, $i=1, \dots, d$. In this case, $I_1=I_2$ and $I_3=I_4$, which are arbitrary elements of F_N, and thus $T_N(\pi)=|F_N|^2$.

Because the set of patterns is finite, it suffices to show that $T_N(\pi)=o(|F_N|^2)$ for every pattern π of type II or III.

If π is of type II, then I_1 and I_4 together determine I_2 and I_3. As in the case $d=2$, the number of allowed pairs (I_1, I_4) with distinct indices is at most $|F_N^{\#}|$, and the number of pairs (I_1, I_4) with at least one common index is at most $\sum_{r=1}^{N} |F_{Nr}^*|^2$. Therefore $T_N(\pi)=o(|F_N|^2)$ by (1.13) and (3.4).

Finally, suppose that π is of type III. Let $\hat{I}_L = \hat{I}_1 \cup \hat{I}_2$ and $\hat{I}_R = \hat{I}_3 \cup \hat{I}_4$, and call these the left and right sides of the pattern. We further say that the points $(i,k) \in \hat{I}_L$ and $(i,k+2) \in \hat{I}_R$ are the *mirror images* of one another. Suppose that there are r edges between $\hat{I}_L$ and $\hat{I}_R$; call these r edges *crossing*, and order them (in some way). Let $t_N^L(k_1, \dots, k_r)$ be the number of ways to assign indices to the edges inside $\hat{I}_L$ such that, with $k_1, \dots, k_r$ assigned to the crossing edges, $I_1, I_2 \in F_N$. Similarly, let $t_N^R(k_1, \dots, k_r)$ be the corresponding number of ways to assign indices in $\hat{I}_R$ such that $I_3, I_4 \in F_N$. Then,

$$T_N(\pi) = \sum_{k_1,\dots,k_r=1}^{N} t_N^L(k_1,\dots,k_r) t_N^R(k_1,\dots,k_r).$$

Further, let π' be the pattern obtained by taking the edges inside $\hat{I}_L$ in π together with their mirror images in $\hat{I}_R$, and the edges connecting each remaining point to its mirror image. Define π'' similarly, starting with the edges inside $\hat{I}_R$ in π. Note that both π' and π'' are patterns of type II. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned} T_N(\pi) &= \sum_{k_1,\dots,k_r=1}^{N} t_N^L(k_1,\dots,k_r) t_N^R(k_1,\dots,k_r) \\ &\le \Big(\sum_{k_1,\dots,k_r=1}^{N} t_N^L(k_1,\dots,k_r)^2 \Big)^{1/2} \Big(\sum_{k_1,\dots,k_r=1}^{N} t_N^R(k_1,\dots,k_r)^2 \Big)^{1/2} \\ &= T_N(\pi')^{1/2} T_N(\pi'')^{1/2} \\ &= o(|F_N|)^2, \end{aligned}$$

where the final estimate holds because π' and π'' are of type II.

This completes the proof of (3.6) and thus of the theorem. □

Proof of Theorem 1.8. If $\widetilde{S}_N$ converges in distribution, then (1.2) implies that all moments converge (as remarked in the proof of Theorem 1.7). In particular, $\mu = \lim_{N\to\infty} \mathbf{E}\widetilde{S}_N = 0$ and $\sigma^2 = \lim_{N\to\infty} \mathbf{E}\widetilde{S}_N^2 = 1$; further,

$$\mathbf{E}\widetilde{S}_N^4 \to \mathbf{E}\xi^4 = 3. \tag{3.9}$$

Similarly, as in the proof above, ES_N^4 equals the number of quadruples (I_1, I_2, I_3, I_4) of d-tuples in F_N such that the $4d$ indices in them coincide in pairs. To estimate this number from above, we note that the number of possibilities that I_1, I_2, I_3, I_4 can coincide in two different pairs is $3|F_N|(|F_N|-1)$, and that each element in $F_N^{\#}$ contributes (at least) one more to the count. Hence,

$$|F_N|^2 \mathbf{E}\widetilde{S}_N^4 = \mathbf{E}S_N^4 \ge 3|F_N|^2 - 3|F_N| + |F_N^{\#}|. \tag{3.10}$$

Obviously, $|F_N| \to \infty$ if $\widetilde{S}_N \xrightarrow{\mathrm{d}} N(0,1)$. Hence, (3.9) and (3.10) imply (1.13). □

We end this section with some counterexamples where the set F_N is close to a product set and asymptotic normality does not hold.

Example 3.2. Take $F_N=\Delta_N^d$ with $d\geq 2$. It is easily seen that (1.13) does not hold, so asymptotic normality fails by Theorem 1.8. Actually, it is easy to see that in this case, $\widetilde{S}_N$ converges to a Hermite polynomial of degree d in a standard normal variable [20]; see also [10, Section XI.1] and [2, Theorem X.26].

In particular, with $d=2$, this example shows that Theorem 1.6(i) does not extend to $\dim\{F_N\}=2$.

Example 3.3. Fix an integer $l\geq 1$, and for $N>l$ let F_N be the product set $\{1,\dots,l\}\times\{l+1,\dots,N\}$.

Clearly, $S_N=\sum_{i=1}^{l} r_i\sum_{j=l+1}^{N} r_j$ and it follows from the central limit theorem that

$$\widetilde{S}_N \xrightarrow{\mathrm{d}} Y\xi,$$

where Y and ξ are independent, $\xi\sim N(0,1)$ and $Y=l^{-1/2}\sum_{i=1}^{l} r_i$.

Hence, if $l=1$, the limit is normal, but not if $l\geq 2$. For example, if $l=2$, the limit variable is 0 with probability $\frac{1}{2}$. (The limit can be regarded as a mixture of normal distributions with different variances.)

In particular, this example shows that Theorem 1.6(i) does not extend to $\dim\{F_N\}=1$.

Example 3.4. Consider a *disconnected* fractional Cartesian product. For example, take $d=6$, $m=2$ and let $S_1,\dots,S_6$ be the sets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{4,5\}$, $\{4,6\}$ and $\{5,6\}$. It is easily seen that (1.13) does not hold, so asymptotic normality fails by Theorem 1.8.

This case is related to the case of disconnected G or H, respectively, in [8, Theorem 1] or [9, Theorem 1]. We expect that, as in those results, $\widetilde{S}_N$ converges to a polynomial in normal variables, but we have not checked the details.

4. Further remarks and open problems

Remark 4.1. It would be interesting to know more about $\|\widetilde{S}_N\|_q$ and of $\mathbf{P}(|\widetilde{S}_N|>x)$ as functions of N. For example, how fast is the transition from the maxima in Theorem 1.5 to the limits in Theorem 1.6 as N grows?

Remark 4.2. We considered for simplicity only $a_{i_1\dots i_d}=1$ in Theorems 1.5 and 1.6. The upper bounds in (1.7) and (1.8) are given for arbitrary $a_{i_1\dots i_d}$, and, in particular, $a_{i_1\dots i_d}=\pm 1$, but the proof of the lower bounds uses the fact that all coefficients have the same sign. In general, there will be cancellations among the terms in (1.6), for any values of $r_1,\dots,r_N$, and it seems likely that the lower bounds in Theorem 1.5 do not extend to general $a_{i_1\dots i_d}$. What is the correct result? Give an extension of Theorem 1.5 to arbitrary $a_{i_1\dots i_d}$.

Certainly, the central limit theorems 1.6 and 1.7 extend to sums (1.6) with suitable conditions on $a_{i_1\ldots i_d}$, but we have not worked out the details of such extensions.

See also [7] and [16] for some related bounds.

Remark 4.3. Note that if S is given by (1.6) and $d_F(\alpha)<\infty$, then

$$\|S\|_\infty \geq c\|\{a_{i_1\ldots i_d}\}\|_{l^{2\alpha/(\alpha+1)}},$$

where the exponent $2\alpha/(\alpha+1)$ is the best possible; see [2, Section XIII.7]. This generalizes a result for $F=\Delta^d$ (i.e. sums (1.1), with $\alpha=d$) proved by Littlewood [14] for $d=2$ and for general d by [5] and [11]. It would be interesting to obtain lower bounds for the probability $\mathbf{P}(S\geq c\|\{a_{i_1\ldots i_d}\}\|_{l^{2\alpha/(\alpha+1)}})$.

Remark 4.4. The proofs above show that the tail estimates in Theorem 1.5 hold for the upper tails $\mathbf{P}(\widetilde{S}_N>x)$ as well. If d is odd, we obtain the same results for $\mathbf{P}(\widetilde{S}_N<-x)$ by symmetry, but if d is even this fails. It seems likely that $\sup_N \mathbf{P}(\widetilde{S}_N<-x)$ is smaller than $\exp(-cx^{2/\alpha})$ for even d, for example for $d=2$. How small is it?

We can also replace the Rademacher system by other orthogonal systems. (See e.g. [12, Chapter 6] for a general background.)

Remark 4.5. If we replace the Rademacher variables r_i by Steinhaus functions χ_i, i.e. independent complex random variables that are uniformly distributed on the unit circle, then Theorem 1.5 still holds.

Indeed, (1.7) is still valid [2, Corollary XIII.29], and thus (1.8) holds by the same proof, so the upper bounds in Theorem 1.5 hold. For the lower bounds, we use the same proof as above, now taking $\mathcal{E}_N=\{\operatorname{Re}\chi_k\geq\frac{1}{2},\ k=1,\ldots,N\}$.

For the upper bound in (1.7), we can alternatively introduce a Rademacher system $\{r_i\}$ independent of $\{\chi_i\}$, replace χ_i by $\chi_i r_i$, which has the same distribution, and use the Rademacher version above conditioning on $\{\chi_i\}$. This standard trick works for all independent identically distributed sequences of bounded symmetric random variables.

Are the central limit theorems 1.6 and 1.7 true for the Steinhaus system too, now with complex Gaussian limits? (We believe so, but we have not checked the details.)

Remark 4.6. Let us instead consider a Gaussian chaos, obtained by replacing r_i by independent Gaussian variables $\xi_i\sim N(0,1)$.

The hypercontractive inequality (1.2) holds in this case too [17] (see also [1], [10] and [12]) but the combinatorial dimension version (1.7) fails in the Gaussian case, as is seen by taking F to be a set with a single element.

Hence Theorem 1.5 is not true in the Gaussian case. What is true? There is no problem with the lower bounds in Theorem 1.5; the proof in Section 2 works if we take $\mathcal{E}_N=\{\xi_i>1,\ i=1,\dots,N\}$.

We believe that Theorems 1.6 and 1.7 hold for the Gaussian case too, but we have not checked the details.

The estimates in [6] and [13] for $d=2$ might be useful.

Remark 4.7. Are the results true if we replace r_k by a lacunary sequence $\exp(2\pi i n_k t)$, where $\inf n_{k+1}/n_k>1$?

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Ron Blei
Department of Mathematics
University of Connecticut
U-9 Storrs, CT 06269
U.S.A.
email: blei@math.uconn.edu

Svante Janson
Department of Mathematics
Uppsala University
P. O. Box 480
SE-751 06 Uppsala
Sweden
email: svante.janson@math.uu.se