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An explicit inversion formula for the exponential Radon transform using data from 180°

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Abstract. We derive a direct inversion formula for the exponential Radon transform. Our formula requires only the values of the transform over an 180° range of angles. It is an explicit formula, except that it involves a holomorphic function for which an explicit expression has not been found. In practice, this function can be approximated by an easily computed polynomial of rather low degree.

1. Introduction

The set of all oriented lines in the plane can be identified with the space $S^1\times\mathbf{R}$ by associating the pair $(\theta,s)\in S^1\times\mathbf{R}$ with the line $L(\theta,s)=\{x\in\mathbf{R}^2;x\cdot\theta=s\}$. A parameterization of this line is then given by the mapping $t\mapsto s\theta+t\theta^\perp$, where $\theta^\perp$ is obtained by rotating θ counterclockwise through a right angle.

Let f be a smooth, compactly supported function in the plane $\mathbf{R}^2$. The Radon transform of f is the function Rf on $S^1\times\mathbf{R}$ defined so that $Rf(\theta,s)$ is the integral of f along the line $L(\theta,s)$. If μ is a real number, the exponential Radon transform $R_\mu f$ is defined by the weighted integral

$$R_\mu f(\theta,s)=\int_{-\infty}^{\infty} f(s\theta+t\theta^\perp)e^{\mu t}\,dt. \tag{1}$$

Note that the ordinary Radon transform is obtained as a special case of the exponential Radon transform when $\mu=0$.

Both the Radon transform and the exponential Radon transform, as well as the still more general attenuated Radon transform, arise in applications to medical imaging, see [2]. It is then of interest to invert the transform, that is to determine the function f from measurements of its transform. For the ordinary Radon transform, an explicit inversion formula was found by J. Radon in 1917, [6], and a generalization to the exponential Radon transform was derived by O. Tretiak and C. Metz in 1980, [7]. A further generalization to the attenuated Radon transform was recently discovered by R. Novikov, see [5] and [3]. An iterative algorithm for inversion of the exponential Radon transform, which only requires $R_\mu f(\theta,s)$ to be known for θ on half of the unit circle has been published by F. Noo and J. M. Wagner, [4].

The goal of this paper is to find an explicit inversion formula, solving the same problem as the algorithm of Noo and Wagner. More precisely, we consider the following problem.

Problem. *Let $D\subset\mathbf{R}$ be a compact set, let μ be a real number, and let $S^1_+=\{\theta\in S^1;\theta_1\geq 0\}$ be the right half of the unit circle. Find an explicit formula for computing f, given the values of $R_\mu f$ on $S^1_+\times\mathbf{R}$, where f is any smooth function with* $\operatorname{supp} f\subset D$.

We will present an essentially explicit formula meeting the requirements of this problem. For the purpose of numerical computations it is not clear that this formula offers any advantages over the algorithm given in [4]. We hope that nevertheless an explicit inversion formula might be of some interest.

2. Statement of results

The adjoint of the exponential Radon transform is known as the dual Radon transform and is denoted $R^\sharp_\mu$. It takes functions on $S^1\times\mathbf{R}$ to functions on $\mathbf{R}^2$ and is given by the formula

$$R^\sharp_\mu g(x)=\int_{S^1} g(\theta,x\cdot\theta)e^{\mu x\cdot\theta^\perp}\,d\theta, \tag{2}$$

where $d\theta$ denotes arc length measure on the unit circle. Computation of $R^\sharp_\mu g$ is also known as backprojection. The first step in our inversion formula is to compute $R^\sharp_{-\mu}g$, where $g=\partial R_\mu f/\partial s$ on $S^1_+\times\mathbf{R}$ and 0 on the other half of $S^1\times\mathbf{R}$.

Theorem 1. *If $f\in C^1_0(\mathbf{R}^2)$, then*

$$\int_{S^1_+}\frac{\partial R_\mu f}{\partial s}(\theta,x\cdot\theta)e^{-\mu x\cdot\theta^\perp}\,d\theta=2\int_{-\infty}^{\infty} f(x_1+t,x_2)\frac{\cosh\mu t}{t}\,dt, \tag{3}$$

where the singularity at $t=0$ in the integral on the right-hand side is treated as a principal value.

Let ch_μ denote the distribution

$$\operatorname{ch}_\mu(t)=\frac{\cosh\mu t}{t}$$

with a principal value at the origin. (Later, we will also use ch_μ to denote the meromorphic function defined by the same expression.) The next step is to find a compactly supported distribution u such that the convolution $u*\mathrm{ch}_\mu$ restricted to a given compact set is a point mass δ_0 at the origin. This is transformed into a problem about functions of one complex variable by means of the following definitions.

Let φ be a function, holomorphic in the whole complex plane except on some subset of the real line. Define $B_+\varphi$ and $B_-\varphi$ to be the boundary values of φ on the real line from above and from below

$$B_\pm\varphi(t) = \lim_{\varepsilon\searrow 0} \varphi(t\pm i\varepsilon) \tag{4}$$

provided that these limits exist as distributions, and let $B_\Sigma\varphi = B_+\varphi + B_-\varphi$ and $B_\Delta\varphi = B_+\varphi - B_-\varphi$.

Furthermore, if φ and ψ are holomorphic outside some compact set in the complex plane, define an entire function $[\varphi,\psi]$ by the formula

$$[\varphi,\psi](z) = \frac{1}{2\pi i}\int_\Gamma \varphi(\zeta)\psi(z-\zeta)\,d\zeta, \tag{5}$$

where Γ is a closed curve, depending on z, so large that the integrand is holomorphic in the unbounded component of $\mathbf{C}\setminus\Gamma$.

Theorem 2. *Let $r<R$ be positive numbers, and let F be a function holomorphic in $\mathbf{C}\setminus[-R,R]$. If*

$$B_\Sigma F(t) = -\frac{1}{2\pi i}\delta_0(t) \quad \textit{for } |t|<r \tag{6}$$

and

$$[\mathrm{ch}_\mu, F] \equiv 0, \tag{7}$$

then $u=B_\Delta F$ satisfies $2(u*\mathrm{ch}_\mu)(t)=\delta_0(t)$ for $|t|<r$.

Finally, we show that it is possible to find functions F satisfying the hypothesis of Theorem 2. See Section 4 for some remarks on the computation of F.

Theorem 3. *Let $w(\alpha)$ be a positive function on the interval $[0,1]$, and let $G(z)$ be defined for $z\in\mathbf{C}\setminus[-R,R]$ by*

$$G(z) = \int_0^1 \frac{w(\alpha)}{\sqrt{r^2+(R^2-r^2)\alpha-z^2}}\,d\alpha, \tag{8}$$

where for any positive real number c, we have chosen the branch of $1/\sqrt{c-z^2}$ which is holomorphic outside the interval $[-\sqrt{c},\sqrt{c}]$ and satisfies $z/\sqrt{c-z^2}\to i$, as $z\to\infty$. For generic choice of r and R there exist a number a and an odd, entire function h such that

$$F(z)=\Big(\frac{a}{z}+h(z)\Big)G(z) \tag{9}$$

satisfies the hypothesis (6) *and* (7) *in Theorem* 2.

Remark. It is likely that the conclusion of the theorem holds for all r, R and w, but we do not have a proof.

Combining these results, we obtain an inversion formula for the exponential Radon transform.

Corollary 1. *Let $D\subset\mathbf{R}^2$ be a compact set, and let $r\geq\sup\{|x_1-y_1|;x,y\in D\}$. Choose a number $R>r$ and a positive function w on the interval $[0,1]$, let F be the function constructed in Theorem* 3 *which satisfies the hypothesis of Theorem* 2, *and let $u=B_\Delta F$. If f is any function of class C^2 with* $\operatorname{supp} f\subset D$, *then*

$$f(x)=\int_{-R}^{R}u(t)\int_{S^1_+}\frac{\partial R_\mu f}{\partial s}(\theta,\theta_1(x_1+t)+\theta_2x_2)e^{-\mu(\theta_1(x_1+t)+\theta_2x_2)}\,d\theta\,dt \tag{10}$$

for all $x\in D$.

3. Proofs

Proof of Theorem 1. Let ∂_θ denote the directional derivative in direction θ. Then it follows from the definition of the exponential Radon transform, that

$$\frac{\partial R_\mu f}{\partial s}(\theta,s)=\int_{-\infty}^{\infty}\partial_\theta f(s\theta+t\theta^\perp)e^{\mu t}\,dt$$

and hence

$$\begin{aligned}\int_{S^1_+}\frac{\partial R_\mu f}{\partial s}(\theta,x\cdot\theta)e^{-\mu x\cdot\theta^\perp}\,d\theta&=\int_{S^1_+}\int_{-\infty}^{\infty}\partial_\theta f((x\cdot\theta)\theta+t\theta^\perp)e^{\mu t-\mu x\cdot\theta^\perp}\,dt\,d\theta\\&=\int_{S^1_+}\int_{-\infty}^{\infty}\partial_\theta f(x+\tau\theta^\perp)e^{\mu\tau}\,d\tau\,d\theta\\&=\int_{-\infty}^{\infty}\int_{S^1_+}\partial_\theta f(x+\tau\theta^\perp)e^{\mu\tau}\,d\theta\,d\tau,\end{aligned}$$

where we have made the change of variables $\tau=t-x\cdot\theta^\perp$. For fixed x and τ, θ is the tangent vector to the semicircle $\{x+\tau\theta^\perp;\theta\in S^1_+\}$, so that

$$\int_{S^1_+}\partial_\theta f(x+\tau\theta^\perp)\,d\theta=\frac{f(x_1+\tau,x_2)-f(x_1-\tau,x_2)}{\tau}.$$

Combining these computations, it follows that

$$\begin{aligned}\int_{S^1_+}\frac{\partial R_\mu f}{\partial s}(\theta,x\cdot\theta)e^{-\mu x\cdot\theta^\perp}\,d\theta&=\int_{-\infty}^{\infty}\frac{(f(x_1+\tau,x_2)-f(x_1-\tau,x_2)}{\tau}e^{\mu\tau}\,d\tau\\&=\lim_{\varepsilon\to0}\int_{|\tau|>\varepsilon}\frac{e^{\mu\tau}+e^{-\mu\tau}}{\tau}f(x_1+\tau,x_2)\,d\tau.\quad\square\end{aligned}$$

Theorem 2 is a direct consequence of the following identity.

Lemma 1. *If φ and ψ are holomorphic outside a compact subset of the real line, then*

$$[\varphi,\psi]|_{\mathbf{R}}=\frac{1}{4\pi i}(B_\Sigma\varphi*B_\Delta\psi-B_\Delta\varphi*B_\Sigma\psi).\tag{11}$$

Proof. Let $z\in\mathbf{R}$ and suppose that $\varphi(\zeta)$ and $\psi(z-\zeta)$ are holomorphic for ζ outside the interval $[a,b]$. Then

$$\begin{aligned}[\varphi,\psi](z)&=\lim_{\varepsilon\to0}\frac{1}{2\pi i}\left(\int_a^b\varphi(t-i\varepsilon)\psi(z-(t-i\varepsilon))\,dt+\int_b^a\varphi(t+i\varepsilon)\psi(z-(t+i\varepsilon))\,dt\right)\\&=\frac{1}{2\pi i}\int_a^b(B_-\varphi(t)B_+\psi(z-t)-B_+\varphi(t)B_-\psi(z-t))\,dt\\&=\frac{1}{4\pi i}\int_a^b(B_\Sigma\varphi(t)B_\Delta\psi(z-t)-B_\Delta\varphi(t)B_\Sigma\psi(z-t))\,dt\\&=\frac{1}{4\pi i}((B_\Sigma\varphi*B_\Delta\psi)(z)-(B_\Delta\varphi*B_\Sigma\psi)(z)).\quad\square\end{aligned}$$

Proof of Theorem 2. To prove the theorem, take $\varphi=\mathrm{ch}_\mu$ and $\psi=F$. Then $B_\Sigma\varphi=2\,\mathrm{ch}_\mu$ and $B_\Delta\varphi=-2\pi i\delta_0$. From the assumption that $[\mathrm{ch}_\mu,F]=0$ it follows that

$$2\,\mathrm{ch}_\mu*u=B_\Sigma\,\mathrm{ch}_\mu*B_\Delta F=B_\Delta\,\mathrm{ch}_\mu*B_\Sigma F=-2\pi i\delta_0*B_\Sigma F.$$

The conclusion follows from the assumption on $B_\Sigma F$. $\square$

Proof of Theorem 3. We must show that there exist a number a and an entire function h such that

$$F(z)=\left(\frac{a}{z}+h(z)\right)G(z)$$

satisfies (6) and (7). Note that for φ and ψ holomorphic outside the real line,

$$B_\Sigma(\varphi\psi)=\tfrac{1}{2}(B_\Sigma\varphi B_\Sigma\psi+B_\Delta\varphi B_\Delta\psi) \tag{12}$$

provided that the products on the right make sense. Since $B_\Sigma G=B_\Delta h=0$ in $(-r,r)$, it follows that in the interval $(-r,r)$,

$$\begin{aligned} B_\Sigma F&=\frac{1}{2}B_\Delta\left(\frac{a}{z}\right)B_\Delta G=-\pi i a B_\Delta G(0)\delta_0\\ &=-2\pi i a\left(\int_0^1 \frac{w(\alpha)}{\sqrt{r^2+(R^2-r^2)\alpha}}\,d\alpha\right)\delta_0. \end{aligned} \tag{13}$$

Hence, the condition (6) will be satisfied precisely if

$$a=-\frac{1}{4\pi^2\displaystyle\int_0^1 \frac{w(\alpha)}{\sqrt{r^2+(R^2-r^2)\alpha}}\,d\alpha}. \tag{14}$$

It remains to determine h so that (7) is satisfied. To prove the existence of h it is useful to reformulate the condition by means of the following lemma.

Lemma 2. *Let h and ψ be entire holomorphic functions, and let φ be holomorphic outside a compact set, with a zero of order $k\geq 0$ at infinity. Then $[z^{-1},h\varphi]=\psi$ if and only if $h=[z^{-1},\psi/\varphi]+P$ for some polynomial P of degree at most $k-1$.*

Proof. Note first that for any φ holomorphic outside a compact set, $[z^{-1},\varphi]$ is the unique entire function with the property that $\varphi-[z^{-1},\varphi]=O(|z|^{-1})$, as $z\to\infty$. From this it follows that

$$[z^{-1},h\varphi]=\psi \iff h\varphi-\psi=O(|z|^{-1}) \iff h-\psi/\varphi=O(|z|^{k-1}) \iff h-[z^{-1},\psi/\varphi]$$

is a polynomial of degree at most $k-1$. $\square$

Rewrite the condition (7) as

$$\left[\frac{1}{z},hG\right]=-\left[\frac{\cosh(\mu z)-1}{z},hG\right]-\left[\mathrm{ch}_\mu,\frac{aG(z)}{z}\right]. \tag{15}$$

Since G has a zero of order 1 at ∞, it follows from Lemma 2 that this condition will be satisfied if

$$h=-\left[\frac{1}{z},\frac{1}{G}\left[\frac{\cosh(\mu z)-1}{z},hG\right]\right]-\left[\frac{1}{z},\frac{1}{G}\left[\mathrm{ch}_\mu,\frac{aG(z)}{z}\right]\right]. \tag{16}$$

Write the right-hand side of (16) as $-\Phi(h)-H$, where H is an entire function and Φ is a linear operator on the space of entire functions. Let K be a compact set containing the interval $[-R,R]$ in its interior, and let $A(K)$ be the Banach space of functions continuous in K and holomorphic in the interior of K. Since $(\cosh(\mu z)-1)/z$ is an entire function, the contour of integration in the definition of

$$\left[\frac{\cosh(\mu z)-1}{z},hG\right]=\frac{1}{2\pi i}\int_\Gamma\frac{\cosh(\mu\zeta)-1}{\zeta}h(z-\zeta)G(z-\zeta)\,d\zeta$$

can be chosen so that the argument of h always is in the interior of K. From this it is clear that Φ can be extended to a bounded operator from $A(K)$ to $A(K')$ for any compact set $K'\subset\mathbf{C}$. If K is contained in the interior of K', the restriction $A(K')\to A(K)$ is compact, and it follows that Φ is a compact operator on $A(K)$. So unless -1 happens to be an eigenvalue of Φ, the equation $h+\Phi(h)=-H$ has a unique solution $h\in A(K)$. Since both H and $\Phi(h)$ are entire functions, it follows that h is also entire. Since Φ takes odd functions to odd functions, even functions to even functions and H is odd, it follows that h is odd.

Finally, note that Φ depends analytically on r and R, and the norm of Φ converges to 0, as r and R approach 0. Hence, -1 is not an eigenvalue of Φ for generic choices of r and R. In fact, numerical experiments seem to suggest that the eigenvalues of Φ are always positive. □

4. Numerical tests

To use the inversion formula on numerical data, it is necessary to choose a weight w, compute approximations for the corresponding a and h to find a function $F=(a/z+h)G$ satisfying the hypothesis of Theorem 2, and then compute a list of values of $u=B_\Delta F$.

Choice of weight. Choosing w to be a piecewise linear function makes the computation of G straightforward. In order to make G fairly smooth, it is advisable to make $w(0)=w(1)=0$.

Computation of a and h. The constant a is found directly from (14). The function h is computed directly from the equation (7) rather than (16). More precisely, h is approximated by an odd polynomial. Choose a positive integer n, and let $\widetilde{\mathrm{ch}}_\mu$ and $\widetilde{G}$ be the Laurent series expansions of ch_μ and G up to terms of some finite degree, say $4n$. Use the relations

$$(17)\qquad [z^j,z^k]=\begin{cases} (-1)^{j+1}\binom{k}{-1-j}z^{j+k+1}, & \text{if } j<0 \text{ and } j+k+1\geq 0,\\ (-1)^k\binom{j}{-1-k}z^{j+k+1}, & \text{if } k<0 \text{ and } j+k+1\geq 0,\\ 0, & \text{otherwise,}\end{cases}$$

and the bilinearity of $[\,\cdot\,,\cdot\,]$ to compute an odd polynomial h_n of degree $2n+1$ such that the Taylor series of

$$\Big[\widetilde{\mathrm{ch}}_\mu, \Big(\frac{a}{z}+h_n\Big)\widetilde{G}\Big]$$

vanishes up to terms of degree $2n$. Note that this expression is an even function, so we have $n+1$ linear equations in the $n+1$ unknown coefficients of h_n. Then it is easy to show that if a solution h of (16) exists, h_n converges to h, as $n\to\infty$.

Computation of u. The distribution u is readily computed by the formula

$$(18)\qquad u(t)=2\Big(\frac{a}{t}+h_n(t)\Big)\int_{\max\{0,(t^2-r^2)/(R^2-r^2)\}}^1 \frac{w(\alpha)}{\sqrt{r^2+(R^2-r^2)\alpha-t^2}}\,d\alpha.$$

Here u is treated as a function rather than a distribution. When computing the integral in (10) numerically, it is necessary to deal with the singularity of u at the origin. One simple-minded approach is to use the trapezoid rule on a set of nodes symmetric with respect to the origin to approximate the principal value integral. More accurate results can be obtained by using the methods described in [2, Chapter III].

A reconstruction was made with the values $r=1$, $R=1.5$ and ten nonzero terms in the polynomial h_n, see Figures 1 and 2. The test object consists of circular discs, and the Radon transform was sampled at 200 values of θ equally spaced over S^1_+, and 101 values of s equally spaced between -0.5 and 0.5. The width and height of the image are 1 and the attenuation $\mu=3$.

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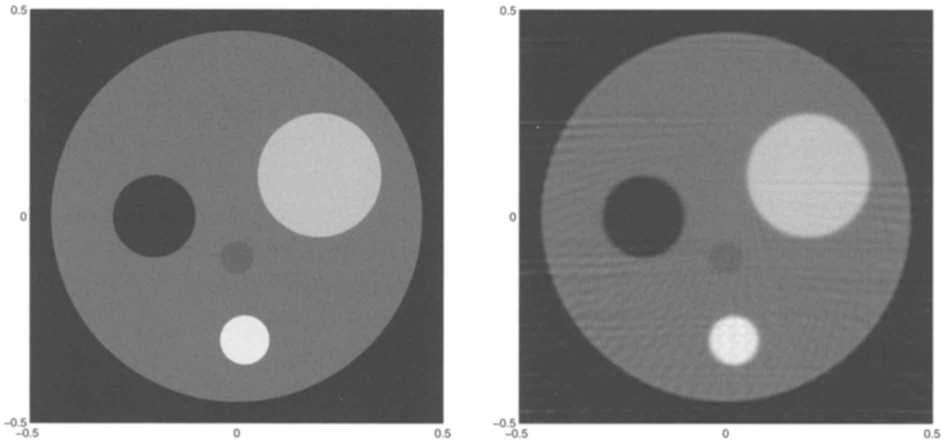

Figure 1. Exact image and reconstruction obtained using the inversion formula.

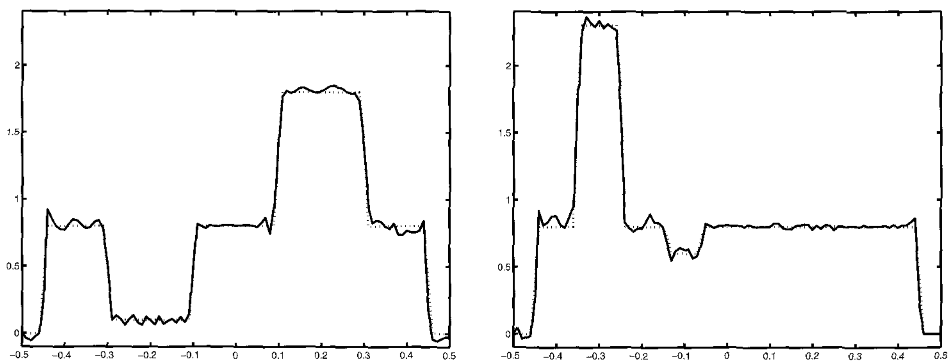

Figure 2. Cross section of exact image (dotted) and reconstruction (solid) along the horizontal (left) and vertical (right) axes through the center of the image.

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