Gaussian measures and polar sets in locally convex spaces

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In this paper we apply infinite dimensional Gaussian measures to prove results about plurisubharmonic and holomorphic functions on locally convex spaces. Specifically we are interested in the "size" of polar subsets (i.e. the set of points where a plurisubharmonic functions takes the value $-\infty$) in a locally convex space.

Since every polar subset of a finite dimensional space has Lebesgue measure zero we were led to consider the following two problems:

Let A denote a polar subset of a locally convex space;

(a) *Is $\mu(A)=0$ for every non-degenerate centered complex Gaussian measure μ on E?*

(b) *Can one find a complex Gaussian measure μ on E such that $\mu(A-x)=0$ for every x in E?*

We show that the answer to (a) in general is no and modulo certain restrictions on a locally convex space E we show $\mu(A)=0$ for every non-degenerate centered complex Gaussian measure μ if an only if A does not contain a dense subspace of E. We show that (b) is true for a reasonably large class of polar subsets of E. We then apply the above results to show that the holomorphic completion of "most" dense hyperplanes of E is in fact equal to E.

§ 1.

If E is a real quasi-complete locally convex space and μ is a Radon Gaussian measure on E then (see [1]) (E, μ) can be considered as an abstract Wiener space (i.e. there exists a separable Hilbert space H_μ and an injection i from H_μ into E such that $\mu=i(\nu)$ where ν is the canonical Gaussian cylindrical measure on H_μ). In that case $i(H_\mu)$ is the set of admissible translates of μ i.e. $x\in i(H_\mu)$ if and only if $\mu\sim\mu_x$ where $\mu_x(A)=\mu(A-x)$ for every μ-measurable subset A of E.

The measure μ on E is said to be non-degenerate if E does not contain a proper closed subspace of measure 1 or equivalently if $i(H_\mu)$ is a dense subspace of E.

Now suppose E is a quasi-complete locally convex space over the complex numbers and let E_R denote the underlying real locally convex space. By a *complex Gaussian measure* on E we mean a Radon Gaussian measure μ on E_R such that $\mu(e^{i\theta}B)=\mu(B)$ for every θ in $(0, 2\pi)$ and every Borel subset B of E.

One can show (5) that a Radon Gaussian measure μ on E_R is a complex Gaussian measure on E if and only if the (real) covariance operator of μ, S_μ, is a complex linear operator.

In the construction of certain complex Gaussian measures we are obliged to restrict ourselves to locally convex spaces with the following property:

$(*)$ every sequence is contained in the linear span of a compact set.

It is not difficult to show that a locally convex space E satisfies $(*)$ if and only if it satisfies

$(**)$ for each sequence $(x_n)_n$ in E there exists a sequence of non-zero scalars $(\lambda_n)_n$ such that $\lambda_n x_n \to 0$ as $n\to\infty$.

Every metrizable space satisfies $(*)$ and $\sum \mathbf{C}$ is an exemple of a space which does not satisfy $(**)$. It is also worth noting that every Gaussian measure on $\sum_n \mathbf{C}$ is degenerate.

§ 2.

For the remainder of this paper E will denote a locally convex space over the field of complex number.

A function $v=E\to[-\infty, +\infty]$ is said to be plurisubharmonic if it is upper semi-continuous and

$$v(a) \leqq 1/2\pi \int_0^{2\pi} v(a+be^{i\theta})\, d\theta$$

for all $a, b\in E$.

A subset A of E is a *complete polar set* if there exists a plurisubharmonic function v on E, $v\not\equiv -\infty$, such that $A=\{x\in E;\ v(x)=-\infty\}$.

A subset of a complete polar set is called a polar set. We refer to [8] for details of the theory of plurisubharmonic functions on locally convex spaces.

The following two results, included for the sake of completeness, can be found in [9].

Lemma 1. *If the plurisubharmonic function v is bounded on $B_p(x, r)=\{y\in E;\ p(x-y)<r\}$ where p is a continuous semi-norm on E and μ is a complex Gaussian measure on E then*

$$v(x) \leqq \frac{1}{\mu[B_p(0, r)]}\int_{p(y)\leqq r} v(x+y)\mu(dy)$$

for any x in E.

Proof. The integral

$$\int_E v(x+ye^{i\theta})\chi_{B_p(0,r)}(y)\,\mu(dy)$$

is well defined and does not depend on θ.
Hence

$$\begin{aligned}\int_{p(y)\leqq r} v(x+y)\,\mu(dy) &= 1/2\pi\int_0^{2\pi} d\theta\int_E v(x+ye^{i\theta})\chi_{B_p(0,r)}(y)\mu(dy)\\ &= \int_E\Big[1/2\pi\int_0^{2\pi} v(x+ye^{i\theta})\,d\theta\Big]\,\chi_{B_p(0,r)}(y)\mu(dy)\\ &\geqq \int_E v(x)\chi_{B_p(0,r)}(y)\mu(dy) = v(x)\mu[B_p(0,r)],\end{aligned}$$

Proposition 2. *Let A denote a complete polar set of E and let μ denote a complex Gaussian measure on E. Then, if $B=\{x\in\operatorname{supp}\mu$ s.t. $\mu[B_p(x,1)\cap A]>0$ for some semi-norm $p\}$, then*

$$B+H_\mu\subset A$$

and $\mu(A)>0$ implies $H_\mu\subset A$ and $\mu(A)=1$.

Proof. If $B=\emptyset$ the inclusion is obvious. If $B\neq\emptyset$, for any b in B there exists a continuous semi-norm p such that the function V defining the polar set is bounded from above in $B_p(b,2)$. Therefore, if h belong to $H_\mu\cap B_p(b,1)$, we have

$$\begin{aligned}\mu[B_p(0,1)]v(b+h) &\leqq \int_{B_p(0,1)} v(b+x+h)\mu(dx)\\ &= \int_{B_p(h,1)} v(b+x)\mu_h(dx) = -\infty\end{aligned}$$

since $b\in B_p(h,1)\subset B_p(b,2)$ and $\mu_h\sim\mu$. Hence $B+H_\mu\subset A$. It is not difficult to see that $\mu(B)=\mu(A)$ hence $\mu(B+H_\mu)=\mu(A)$. But $\mu(B+H_\mu)$ equals 0 or 1, and the proof is complete.

Theorem 3. *Let E denote a quasi-complete locally convex space satisfying $*$ (or $**$) and let $(A_n)_n$ denote a sequence of complete polar subsets of E. There exists a complex Gaussian measure μ on E such that $\mu(\cup A_n)=0$.*

Proof. For each n let $x_n\in E\setminus A_n$. Let K denote an absolutely convex compact subset of E whose span contains the sequence (x_n). The space E_K is a Banach space and it is possible ([5]) to construct a complex Gaussian measure μ on E_K such that $(x_n)\subset H_\mu$. If $\tilde\mu$ is the complex Gaussian measure on E defined by $\tilde\mu(B)=\mu(B\cap E_K)$ for any Borel subset B of E then Proposition 2 implies that $\tilde\mu(A_n)=0$ for each n and hence $\mu(\cup A_n)=0$.

As a corollary we obtain the following result of ([3]).

Corollary 4. *In a quasi complete locally convex space satisfying $(*)$ the countable union of polar sets has empty interior.*

Theorem 4. *Suppose E satisfies $(*)$ and is quasi-complete. Then there exists a centered non-degenerate complex Gaussian measure μ on E such that $\mu(A)=1$ if and only if A contains a dense separable subspace of E.*

The assumption "subspace" cannot by replaced by "countable subset whose span is dense in E" as the following example shows: Let (e_n) be an orthonormal basis in a separable Hilbert space E, then the function

$$v\colon\ z \to \sum_{n=1}^{\infty} \frac{1}{n^2} \log \frac{\|z-e_n\|}{n}$$

is plurisubharmonic in E. The complete polar set $A=\{v=-\infty\}$ contains the basis (e_n) but no infinite dimensional subspace so $\mu(A)=0$ for any non degenerate centered Gaussian measure.

Proof of Theorem 4. $\Rightarrow$) follows from Proposition 2 and the separability of H_μ since $\operatorname{supp}\mu=\overline{H_\mu}$.
$\Leftarrow$) Let $(e_n)\subset A$ be a linearly independant sequence in A whose span is dense in E. By assumption, (e_n) belongs to E_K for an appropriate absolutely convex compact subset K of E. Without loss of generality we may assume that

$$\sum_0^\infty \|e_n\|_K < +\infty,$$

so the linear map

$$f\colon l^1 \to E_K, \quad x \to \sum_0^\infty x_n e_n$$

is continuous. Let

$$g = i \circ f \quad \text{where } i \text{ is the injection} \quad E_K \hookrightarrow E.$$

Since A is a G_δ-set that contains the span of (e_n) we can find a sequence (δ_n), $\delta_n>0$, such that

$$g^{-1}(A) \supset L$$

where $L=\{x\in l^\infty,\ |x_n|\leqq\delta_n,\ n\in\mathbf{N}\}$. We now choose a complex Gaussian Radon measure ν on $F=(l^\infty, \sigma(l^\infty, l^1))$ such that $\nu(L)>0$ and $c_{00}\subset H_\nu$ where c_{00} the sequences having a finite number of non zero elements. Since F is σ-compact we deduce that $\mu=g(\nu)$ is a complex Gaussian Radon measure on E. The measure μ is non degenerate since (e_n) belongs to H_μ. Moreover

$$\mu(A) = \nu\big(g^{-1}(A)\big) \geqq \nu(L) > 0$$

so Proposition 2 tells us that $\mu(A)=1$.

A more elementary but longer proof was given in a conference at Lelong-s seminar (february 77).

This completes our discussion of problem (a). We now look at polar sets which are Gaussian null sets.

A Borel subset B of a locally convex space E is said to be a *Gaussian null set* if there exists a complex Gaussian measure μ on E such that $\mu(A-x)=0$ for every x in E.

Theorem 4 easily implies that a non-dense complete polar set is a Gaussian null set with respect to any non degenerate Gaussian measures.

If A is a Borel subset of E and μ is a complex Gaussian measure on E we let $A_\mu=\{x\in E;\ \mu(A-x)=1\}$. Note that $A_\mu+H_\mu=A_\mu$ and that A_μ is a G_δ-set.

Lemma 5. *If μ is a complex Gaussian measure on E and if A is a complete polar subset of E then:*

a) $A_\mu\subset A$

b) *there exists a complex Gaussian measure ν such that* $\nu(A_\mu)=0$.

Proof. (a) follow from proposition 1. (b) Let $x\in E\setminus A$ and let ν denote a complex Gaussian measure on E such that $x\in H_\nu$. Then $\nu(A_\mu)\leqq\nu(A)=0$ by proposition 2.

We remark that (b) of the preceeding lemma can be modified to show that for every complete polar set there exists a complex Gaussian measure ν such that $\nu(A_\nu)=0$.

We now give examples of dense polar sets that are Gaussian null sets.

Proposition 6. *If A is a complete polar subset of E and $E\neq A-A=\{(x-y\,|\,x\in A, y\in A\}$, then A is a Gaussian null set.*

Proof. Let $x_0\in E\setminus(A-A)$ and let μ denote a complex Gaussian measure such that x_0 lies in the reproducing kernel of μ.

Now suppose $x_1\in A_\mu$. Then $x_1+x_0=x_2\in A_\mu+H_\mu=A_\mu$ and $x_0=x_2-x_1\in A_\mu-A_\mu\subset A-A$. This is impossible and hence $A_\mu=\emptyset$ i.e. A is a Gaussian null set.

Proposition 7. *A complete circled polar subset A of a locally convex space E is a Gaussian null set. (A is said to circled if $e^{i\theta}A\subset A$ for all $\theta\in[0,2\pi]$).*

Proof. Since $A\neq E$ we can choose a point $x_0\in E\setminus A$ and a complex Gaussian measure μ such that $x_0\in H_\mu$.

If $x\in A_\mu$ and $|\lambda|=1$ then

$$\mu(\lambda(A-x)) = \mu(A-x) = \mu(A-\lambda x) = 1$$

and hence $\lambda x\in A_\mu$.

Since $x_0 \notin A$ we have $v(x_0) > -\infty$ where v is a plurisubharmonic function on E which defines A. Let $y \in A_\mu$. Then

$$v(x_0) \leqq \frac{1}{2\pi} \int_0^{2\pi} v(x_0 + e^{i\theta} y)\, d\theta$$

and hence $x_0 + e^{i\theta} y \notin A$ for almost all θ.

This contradicts the fact that

$$x_0 + e^{i\theta} y \in H_\mu + A_\mu = A_\mu \quad \text{for all } \theta.$$

Hence $A_\mu = \emptyset$ and A is a Gaussian null set.

Corollary 8. *If F is a polar subspace of a locally convex space then F is a Gaussian null set.*

Proof. Since F is polar there exists a plurisubharmonic function v on E such that $F \subset \{x \in E;\ v(x) = -\infty\}$. The function v_c defined by $v\ (x) = \sup_{|\lambda| \leqq 1} v(\lambda x)$ is easily seen to be plurisubharmonic on E and $F \subset \{x \in E;\ v_c(x) = -\infty\} = A$.

Since $e^{i\theta} A \subset A$ for all $\theta \in [0, 2\pi]$ an application of Proposition 7 completes the proof.

We now give a few applications of the preceeding results.

Theorem 9. *If U is an open subset of $\mathbf{C}^n$ and η is the subspace of $H(U)$ consisting of functions which can be analytically continued outside U then either $\eta = H(U)$ or η is a Gaussian null set.*

Proof. This result follows from the fact that every complex Gaussian measure on $H(U)$ is supported by a Banach subspace of $H(U)$ and from the fact that $\eta \cap E$ is a polar subspace of every Banach subspace of $H(U)$ ([7]) by applying Corollary 8.

Now, let $\hat{E}$ denote the completion of the locally convex space E. The largest subspace of $\hat{E}$ to which every holomorphic function on E can be continued as a holomorphic function is called the holomorphic completion of E and is denoted by E_θ. The space E_θ always exists ([8]) and we say that E is holomorphically complete if $E = E_\theta$.

Theorem 10. *Let E denote a Fréchet space and let $(a_i)_{i \in I}$ be an algebraic basis for E. Let $(b_i)_{i \in I}$ denote a set of linear forms on E such that $b_i(a_j) = \delta_{ij}$ for all $i, j \in I$. Then at most a finite number of the hyperplanes $H_i = b_i^{-1}(0)$, $i \in I$, are holomorphically complete.*

Proof. Suppose there exists an infinite set of holomorphically complete hyperplanes $(H_{i_n})_n$, H_{i_n} is a polar subset of E for each n ([18]). Let $E_n = \bigcap_{m > n} H_{i_m}$. Each E_n is polar in E and hence $\bigcup_n E_n \neq E$ by Corollary 4. Since each $x \in E$ lies in all except a finite number of H_i it follows that $E = \bigcup_n \bigcap_{m > n} H_{i_m} = \bigcap_n E_n$. This is impossible and completes the proof.

We remark that one can also use a result from (2) instead of Corollary 4.

Corollary 11. *If E is a Frechet space, then there exists a dense hyperplane H in E such that every holomorphic function on H can be extended to a holomorphic function on E.*

Proof. If H is a dense hyperplane then we have either $H=H_\theta$ or $E=H_\theta$. An application of Theorem 10 now completes the proof.

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