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Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems

Gerd Grubb

1. Introduction

One of the purposes of this paper is to prove asymptotic expansions of heat traces

$$\text{(1.1)} \qquad \begin{gathered} \operatorname{Tr}(\varphi e^{-t\Delta_i}) \sim \sum_{-n\le k<0} a_{i,k} t^{k/2} + \sum_{k=0}^{\infty} (a_{i,k}\log t + a'_{i,k}) t^{k/2} \quad \text{for } t\to 0, \\ \Delta_1 = D_B^* D_B, \quad \Delta_2 = D_B D_B^*, \end{gathered}$$

for general realizations D_B of first-order differential operators D (e.g. Dirac-type operators) on a manifold X with pseudodifferential boundary conditions: $B(u|_{X'})=0$ at the boundary $\partial X = X'$. In (1.1), φ denotes a compactly supported morphism. The coefficients without primes are locally determined, the primed coefficients global.

Such realizations were considered first by Atiyah, Patodi and Singer in [APS] who showed an interesting index formula in the so-called product case, when X is compact. We say that D is *of Dirac-type* when $D=\sigma(\partial_{x_n}+A_1)$ on a collar neighborhood of X', with a unitary morphism σ and a first-order differential operator A_1 such that $A_1 = A + x_n P_1 + P_0$ with A selfadjoint on X' and constant in x_n and the P_j of order j; the *product case* is where $P_1=P_0=0$. The operator B was in [APS] taken equal to the orthogonal projection $\Pi_{\geq}$ onto the eigenspace for A associated with eigenvalues ≥ 0.

For Dirac-type operators on compact manifolds, finite expansions (1.1) (up to $k=0$, with $\varphi=1$ and $a_{i,0}=0$) were shown in [G4], implying the index formula

$$\text{(1.2)} \qquad \operatorname{index} D_B = a'_{1,0} - a'_{2,0}, \quad \text{when } \varphi=1 \text{ and } X \text{ is compact.}$$

Full expansions were established in Grubb and Seeley [GS1], with precisions for the product case in [GS2]. Here $B=\Pi_{\geq}+B_0$ with special finite rank perturbations B_0.

Booss-Bavnbek and Wojciechowski studied, for the compact product case, the index of D_B in [BW2] and other works with $B=C^+ +S$, where C^+ is the Calderón projector for D (having the same principal part as $\Pi_{\geq}$) and S is a pseudodifferential operator (ψdo) of order -1. One of our motivations for the present work was to establish (1.1) for such problems too. A different type of boundary condition was introduced by Brüning and Lesch in [BL] (in a study of the gluing problem for the eta invariant), where they showed heat trace expansions in the product case but with B principally different from $\Pi_{\geq}$ (Example 4.2 below). For this type, we obtain (1.1) without the product assumption.

Actually, we find that there are many more boundary conditions, different from the above, for which (1.1) can be obtained. In fact, D need not even be of Dirac-type, but can be *any first-order elliptic differential operator*. The operator B need not be closely linked to the Calderón projector but can be *any ψdo that is well-posed for D* in the sense defined by Seeley in [S2, Chapter VI]. We obtain (1.1) and (1.2) in all these cases (including those previously known) for compact X, and generalize (1.1) to suitable noncompact situations.

The freedom to choose more general B seems to be useful e.g. for variational studies. It is also interesting to allow general D that are not tied, by the requirement of (principal) selfadjointness of the tangential part, to a specific choice of Hermitian structures.

In our method to establish (1.1), we imbed D_B and D_B^*, which are in themselves only injectively elliptic, into a truly elliptic system $\mathcal{D}_{\mathcal{B}}$, which we treat by use of the Calderón projector for $\mathcal{D}+\mu$ and by an elaboration of the calculus of weakly polyhomogeneous ψdos introduced in [GS1]. This treatment works also for general elliptic systems P of order $d\geq 1$ with appropriate pseudo-normal ψdo boundary conditions $S\varrho u=0$ ($\varrho u=\{(D_{x_n}^j u)|_{X'}\}_{0\leq j<d}$). We show a general result on resolvent and heat operator trace expansions for such realizations,

(1.3)
$$\operatorname{Tr}\varphi\partial_\lambda^m(P_S-\lambda)^{-1}\sim\sum_{-n\leq k<0}\tilde{c}_k(-\lambda)^{k/d-m-1}+\sum_{k=0}^{\infty}(\tilde{c}_k\log(-\lambda)+\tilde{c}_k')(-\lambda)^{k/d-m-1},$$
$$\operatorname{Tr}\varphi e^{-tP_S}\sim\sum_{-n\leq k<0}c_kt^{k/d}+\sum_{k=0}^{\infty}(c_k\log t+c_k')t^{k/d}\quad\text{for } t\to 0;$$

in the first formula, $\lambda\to\infty$ on a ray in $\mathbf{C}$, and the second formula follows, when $(P_S-\lambda)^{-1}$ exists and the expansion holds for $\lambda\to\infty$ in an obtuse keyhole region

$W=\{\lambda \mid |\lambda|\le r \text{ or } |\arg\lambda-\pi|\le\frac{1}{2}\pi+\varepsilon\}$, from the formula

$$\operatorname{Tr}\varphi e^{-tP_S}=\frac{\mathrm{i}}{2\pi}\int_{\partial W}(-t)^{-m}e^{-t\lambda}\operatorname{Tr}\varphi\partial_\lambda^m(P_S-\lambda)^{-1}\,d\lambda. \tag{1.4}$$

Such expansions were shown in cases where S is a differential operator by Seeley [S3] and Greiner [Gr]; then there are no logarithmic terms and all the coefficients are locally determined. The crucial step in the analysis is to find the symbol structure of the resolvent. We do this not only for compact manifolds but also in noncompact situations with global estimates; here we use the calculi established in [GK] (with Kokholm), [G5], [G3].

The plan of the paper is as follows: The hypotheses on general systems $\{P,S_\varrho\}$ are explained in Section 2. Well-posed first-order problems are introduced in Section 3, with examples in Section 4 and the imbedding into elliptic systems in Section 5. In Section 6 we show a technical result on spectral invariance of the weakly polyhomogeneous calculus from [GS1] (drawing on [G5]), and in Section 7 we establish the necessary results on Calderón projectors. In Section 8 we determine the structure of the resolvent, and in Section 9 we derive the trace estimates by use of [GS1].

2. The general set-up

On an n-dimensional C^∞ manifold X with boundary $\partial X=X'$ we consider an elliptic differential operator of order d, $P\colon C^\infty(X,E_1)\to C^\infty(X,E_2)$, between sections of Hermitian C^∞ vector bundles E_1 and E_2 of dimension N. The manifold X is provided with a smooth volume element $v(x)\,dx$ defining a Hilbert space structure on the sections.

In order to include noncompact manifolds such as $\mathbf{R}^n$, $\overline{\mathbf{R}}^n_+$ and exterior domains $\mathbf{R}^n\setminus Y$, $\overline{\mathbf{R}}^n_+\setminus Y$ (Y smooth compact), we take X to be *admissible* as defined in [GK], [G3]; this means that X is the union of a compact piece and finitely many conical pieces of the form $\{x=tx_0 \mid x_0\in M\subset S^{n-1},\ t>r\}$. The manifold X is covered by a finite system of local coordinate patches diffeomorphic to either bounded or conical open subsets of $\overline{\mathbf{R}}^n_+$. The use of such manifolds is worked out in detail in [GK], [G5], [G3], so we can be brief here. The crucial assumption is that the admissible coordinate changes $\varkappa$ are such that $|\varkappa(x)-\varkappa(y)|/|x-y|$ is bounded above and below by positive constants, and all derivatives of $\varkappa$ and $\varkappa^{-1}$ are bounded. Admissible vector bundles are likewise defined. The differential operators and ψdos considered in this context are defined by reference to the admissible local coordinate systems; their symbols are assumed to have global estimates in the space variable x, as in Hörmander [H2, Section 18.1]. The concepts are extended to pseudodifferential boundary operators in [GK], [G5], [G3]. An advantage is that the calculus has rather precise composition rules, where all remainders lie inside the calculus. For brevity, we shall call such operators admissible (in [G3] they are called uniformly estimated or globally estimated), and we always assume in the following when working with admissible manifolds that the operators are of this type.—A reader who is mainly interested in the case of compact manifolds can just disregard this generality.

The Sobolev space of order s of sections of E_i is denoted by $H^s(X,E_i)$ or just $H^s(E_i)$; it is defined by use of admissible local coordinates.

We denote $E_i|_{X'}$ by E_i'. We assume that a normal coordinate x_n has been chosen in a neighborhood U of the boundary X' such that the points are represented as $x=(x',x_n)$ there, with $x'\in X'$, $x_n\in[0,c(x')[$, the E_i are isomorphic to the pull-backs of the E_i' there, and there is a normal derivative ∂_{x_n}. The boundary X' is provided with the volume element $v(x',0)\,dx'$ induced by $v(x',x_n)\,dx'\,dx_n$ on U. For a compact manifold, we take U as a collar neighborhood $X_c=X'\times[0,c[$; more generally this is used for the compact part and extended conically in the conical parts (cf. [G3, Section A.5]).

Let $\varrho=\{\gamma_0,\dots,\gamma_{d-1}\}$ with $\gamma_j u=(-\mathrm{i}\partial_{x_n})^j u|_{x_n=0}$ (i denotes the imaginary unit $\sqrt{-1}$). For $s>d-\frac12$, ϱ maps $H^s(E_i)$ into $\mathcal{H}^s(E_i'^d)=\prod_{0\le j<d}H^{s-j-1/2}(E_i')$ ($E_i'^d=\bigoplus_{0\le j<d}E_i'$). The sections u of E_1 and w of E_2 in H^s ($s>d-\frac12$) satisfy Green's formula

$$(2.1)\qquad \begin{gathered}(Pu,w)_X-(u,P^*w)_X=(\mathcal{A}\varrho u,\varrho w)_{X'},\\ \mathcal{A}=(\mathcal{A}_{jk})_{j,k=0,\dots,d-1}\quad\text{with } \mathcal{A}_{jk} \text{ of order } d-1-j-k.\end{gathered}$$

Here the $\mathcal{A}_{jk}$ are differential operators; those with $k>d-1-j$ are 0 ($\mathcal{A}$ is upper skew-triangular) and those with $k=d-1-j$ are isomorphisms, so $\mathcal{A}$ has an inverse of a similar type, just lower skew-triangular.

When S is an operator on $\mathcal{H}^d(E_1'^d)$, the boundary condition

$$(2.2)\qquad S\varrho u=0$$

determines the realization P_S of P, defined as the operator acting like P and with domain

$$(2.3)\qquad D(P_S)=\{u\in H^d(X,E_1)\mid S\varrho u=0\}.$$

We shall study boundary conditions that are *pseudo-normal* in the following sense.

Assumption 2.1. (Pseudo-normality) *The operator S is a matrix of admissible classical ψdos S_{jk} going from E_1' to admissible bundles F_j over X' such that*

$$S = (S_{jk})_{j,k=0,\dots,d-1}, \quad \text{with } S_{jk} \text{ of order } j-k,\ S_{jk}=0 \text{ for } j<k, \\ S_{jj} \text{ surjective and uniformly surjectively elliptic.} \tag{2.4}$$

For convenience of notation, we here include bundles F_j of dimension 0. We set $F = \bigoplus_{0 \le j<d} F_j$. It will often be tacitly understood in the following that symbols and operators are taken admissible when the manifolds and bundles are so.

The new generality in comparison with the *normal* boundary conditions considered in [G3] (for compact manifolds, the information is found also in [G2], this will not be repeated), is that the S_{jj} are now allowed to be ψdos; this is needed in our application to first-order operators. The normal boundary conditions have just surjective *morphisms* as the S_{jj}, hence regularity $\nu>0$, whereas the present boundary conditions have regularity $\nu=0$, in the sense of the regularity concept from [G3]. (There is a discussion in [G3, Remark 1.5.8]. In other ways the conditions in the book are more general.)

Our basic hypothesis for the resolvent analysis is the following assumption.

Assumption 2.2. (Resolvent growth condition) *Let $E_1=E_2=E$. There is an open sector $\Gamma = \{\lambda \in \mathbf{C}\setminus\{0\} \mid \arg\lambda \in J\}$ (for an open interval $J \subset [0,2\pi]$) such that the following holds:*

(1) *The operator P is elliptic, and for the principal symbol p^0 of P, $p^0(x,\xi)-\lambda$ is invertible for all (x,ξ,λ) with $\lambda\in\Gamma\cup\{0\}$, $|\xi|^2+|\lambda|^{2/d}\ge 1$, the inverse being $O((|\xi|^d+|\lambda|)^{-1})$ on closed subsectors Γ', uniformly in x.*

(2) *The bundle F has dimension $\frac{1}{2}Nd$, the system $\{P, S\varrho\}$ is elliptic, and for any closed subsector Γ' there is an $r\ge 0$ such that the resolvent $R_\lambda=(P_S-\lambda)^{-1}$ exists as a bounded operator in L_2 and is $O(\lambda^{-1})$ for $\lambda\in\Gamma_r'$,*

$$\Gamma_r' = \{\lambda \in \Gamma' \mid |\lambda| \ge r\}. \tag{2.5}$$

The first property means uniform parameter-ellipticity of $P-\lambda$, as defined in [G3, Section 3.1].

The second property contains a global requirement of invertibility. If $S\varrho$ is *normal*, such invertibility for large λ is assured by a condition on principal symbols, namely uniform parameter-ellipticity of $\{P-\lambda, S\varrho\}$ as defined in [G3, Section 3.1]. This means that the associated model problem on $\mathbf{R}_+$ for each (x',ξ',λ) with $|\xi'|^2+|\lambda|^{2/d}=1$ is uniquely solvable with uniform bounds in x' for the solution operator, for λ in closed subsectors of Γ. Then the results of [G3, Section 3.3] imply invertibility with the $O(\lambda^{-1})$ estimate for large λ. When S is merely pseudo-normal, property (2) depends not just on principal symbols but on the full structure; it is verified e.g. if P_S is selfadjoint.

The resolvent R_λ will now be supplied with a Poisson operator K_λ to define an inverse of the full system $\{P-\lambda, S\varrho\}$. In the following lemma, $K_{\varrho,\lambda}$ denotes an auxiliary Poisson operator such that $\varrho K_{\varrho,\lambda}=I$, constructed e.g. as in [G3, Lemma 1.6.4] with $\langle\xi\rangle$ replaced by $\langle(\xi,|\lambda|^{1/d})\rangle$. (We use the notation $\langle x\rangle=(|x_1|^2+\ldots+|x_\nu|^2+1)^{1/2}$ for $x=(x_1,\ldots,x_\nu)$.) In its dependence on $\mu=|\lambda|^{1/d}$, $K_{\varrho,\lambda}$ is strongly polyhomogeneous on all rays, cf. Section 6, [GS1, Appendix]. If holomorphy in λ is desired, one can instead take the Poisson operator $K_{\varrho,\lambda}\colon\varphi\mapsto u$ solving the following Dirichlet problem, where Λ^{2d} is a positive differential operator with principal symbol $\langle\xi\rangle^{2d}$ and $|\arg\lambda-\omega|<\frac{1}{2}\pi$,

$$(\Lambda^{2d}+(e^{-i\omega}\lambda)^2)u=0 \text{ on } X, \quad \varrho u=\varphi \text{ on } X'.$$

Lemma 2.3. *Let Assumptions* 2.1 *and* 2.2 *hold. For the λ such that R_λ is defined, there exists a unique Poisson operator K_λ such that*

$$\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}^{-1} = \begin{pmatrix} R_\lambda & K_\lambda \end{pmatrix}. \tag{2.6}$$

In a neighborhood of each ray in Γ, K_λ equals

$$K_\lambda=[I-R_\lambda(P-\lambda)]K_{\varrho,\lambda}S'; \tag{2.7}$$

here $S'=(S'_{jk})_{j,k=0,\ldots,d-1}$ is a right inverse of S, constructed such that for all j, k, S'_{jk} is a classical ψdo of order $j-k$, $S'_{jk}=0$ for $j<k$, and S'_{jj} is injective and injectively elliptic; *and $K_{\varrho,\lambda}$ is an auxiliary right inverse of ϱ as described above.*

Proof. Let us first explain the construction of S'. We can write $S=S_{\text{diag}}+S_{\text{sub}}$, where $S_{\text{diag}}=(\delta_{jk}S_{jk})_{j,k=0,\ldots,d-1}$ and S_{sub} is subtriangular (has zero entries in and above the diagonal). Here S_{diag} is surjective and surjectively elliptic of order 0 from $E_1'^d$ to F, hence $S_{\text{diag}}S^*_{\text{diag}}$ is bijective and elliptic of order 0 in F and therefore has an (elliptic) inverse $[S_{\text{diag}}S^*_{\text{diag}}]^{-1}$. Then S_{diag} has the right inverse $S'_{\text{diag}}=S^*_{\text{diag}}[S_{\text{diag}}S^*_{\text{diag}}]^{-1}$; again a classical ψdo of order 0. Finally, since $SS'_{\text{diag}}=I+S_{\text{sub}}S'_{\text{diag}}$, where $S_{\text{sub}}S'_{\text{diag}}$ is subdiagonal and hence nilpotent, S has the right inverse

$$S'=S'_{\text{diag}}(I+S_{\text{sub}}S'_{\text{diag}})^{-1}=S'_{\text{diag}}\sum_{0\le l<d}(-S_{\text{sub}}S'_{\text{diag}})^l;$$

it is of the asserted form. (Admissibility follows from [G5, Theorem 1.12].)

The operator K_λ required in (2.6) is the solution operator for the problem

$$(P-\lambda)u=0 \text{ on } X, \quad S\varrho u=\varphi \text{ on } X'. \tag{2.8}$$

First note that since R_λ is injective, the problem has at most one solution u for any φ. Define K_λ by (2.7), then check that $u=K_\lambda\varphi$ solves (2.8) by observing

$$(P-\lambda)[I-R_\lambda(P-\lambda)]=0 \quad \text{since } (P-\lambda)R_\lambda=I,$$

and, using that $S\varrho R_\lambda=0$,

$$S\varrho K_\lambda=S\varrho K_{\varrho,\lambda}S'=I. \quad \square$$

For each fixed λ, the inverse $(\,R_\lambda \quad K_\lambda\,)$ belongs to the pseudodifferential boundary operator calculus ([B2], [G3]), but to start with, we in general only have a rough information on the behavior of R_λ with respect to λ that comes from its definition as a resolvent. Before showing this in an elementary lemma, let us recall the definition of parameter-dependent Sobolev spaces (used e.g. in [G3], [GS1]).

For $s\in\mathbf{R}$, the space $H^{s,\mu}(\mathbf{R}^n)$ is the Sobolev space provided with the norm

$$\|u\|_{H^{s,\mu}}=\|\langle(\xi,\mu)\rangle^s\hat{u}(\xi)\|_{L_2(\mathbf{R}^n)}. \tag{2.9}$$

The notion is extended to sections of a Hermitian bundle F over X by use of a finite family of admissible local coordinate systems (the space is then denoted $H^{s,\mu}(X,F)$ or $H^{s,\mu}(F)$). Note that $H^{0,\mu}(F)\simeq L_2(F)$, and that for $s\geq 0$, the norm is equivalent with $(\|u\|^2_{H^s}+\langle\mu\rangle^{2s}\|u\|^2_{L_2})^{1/2}$.

Lemma 2.4. *Let R_λ and K_λ be as in Lemma 2.3. For any $s\geq 0$, R_λ and K_λ define continuous mappings (where $\mathcal{H}^{s+d,\mu}(F)=\prod_{0\leq j<d}H^{s+d-j-1/2,\mu}(F_j)$, $\mu=|\lambda|^{1/d}$)*

$$R_\lambda\colon H^{s,\mu}(E)\longrightarrow H^{s+d,\mu}(E), \quad K_\lambda\colon \mathcal{H}^{s+d,\mu}(F)\longrightarrow H^{s+d,\mu}(E), \tag{2.10}$$

uniformly for λ in subsectors Γ'_r (as in Assumption 2.2).

Proof. From the elliptic regularity for the λ-independent system $\{P,S\varrho\}$ and from the resolvent growth condition follows that for $k\geq 1$, $v\in D(P_S)\cap H^{kd}(E_1)$,

$$\|v\|_{H^{kd}}\leq c_{1,k}(\|P_Sv\|_{H^{(k-1)d}}+\|v\|_{H^{(k-1)d}}), \quad |\lambda|\,\|R_\lambda f\|_{L_2}\leq c_2\|f\|_{L_2}, \tag{2.11}$$

uniformly for $\lambda\in\Gamma'_r$. We use this first with $v=R_\lambda f$ and $k=0$ to see that on the ray $\lambda=\mu^d e^{i\theta}$, $\mu\geq r^{1/d}$,

$$\begin{aligned}\|R_\lambda f\|_{H^{d,\mu}}&\leq c_3(\|R_\lambda f\|_{H^d}+\langle\lambda\rangle\|R_\lambda f\|_{L_2})\\ &\leq c_4(\|(P_S-\lambda)R_\lambda f\|_{L_2}+\langle\lambda\rangle\|R_\lambda f\|_{L_2}+\|R_\lambda f\|_{L_2})\leq c_5\|f\|_{L_2};\end{aligned} \tag{2.12}$$

in other words, R_λ is continuous from $L_2(E)$ to $H^{d,\mu}(E)$, uniformly for $\mu \ge r^{1/d}$.

Next, combining (2.11) with (2.12) we find for $k=1$,

$$\begin{aligned}\|R_\lambda f\|_{H^{2d,\mu}} &\le c_3'(\|R_\lambda f\|_{H^{2d}} + \langle\lambda\rangle^2 \|R_\lambda f\|_{L_2}) \\ &\le c_4'(\|(P_S-\lambda)R_\lambda f\|_{H^d} + |\lambda|\, \|R_\lambda f\|_{H^d} + \|R_\lambda f\|_{H^d} + \langle\lambda\rangle^2 \|R_\lambda f\|_{L_2}) \\ &\le c_5'(\|f\|_{H^d} + \langle\lambda\rangle \|f\|_{L_2}) \le c_6 \|f\|_{H^{d,\mu}}.\end{aligned}$$

This can be continued to give $H^{(k+1)d,\mu}$ estimates of $R_\lambda f$ in terms of $H^{kd,\mu}$ estimates of f for $k=2,3,\dots$, and we conclude that the first statement in (2.10) holds for $s=dk$, $k=0,1,2,\dots$. The remaining values of $s\ge 0$ are included by interpolation.

For the second statement we argue as follows. When C is a parameter-independent ψdo on X' of order $l\ge 0$, it is bounded from $H^{s,\mu}$ to $H^{s-l,\mu}$ for all $s\in\mathbf{R}$, uniformly in μ; cf. Section 2.5 in [G3] (using that C is of regularity $\nu=l\ge 0$). It follows that S' maps $\mathcal{H}^{s,\mu}(E'^d)=\prod_{0\le j<d} H^{s-j-1/2,\mu}(E')$ into $\mathcal{H}^{s,\mu}(F)$ with uniform bounds in μ for $s\in\mathbf{R}$. [G3] also shows that ϱ maps $H^{s,\mu}(E)$ into $\mathcal{H}^{s,\mu}(E'^d)$ for $s>d-\frac{1}{2}$ and that $K_{\varrho,\lambda}$ is continuous in the opposite direction, with uniform bounds in μ. Applying these facts to the factors in (2.7) and using what we just found for R_λ, we obtain the statement for K_λ in (2.10). □

Remark 2.5. There do exist boundary conditions other than those satisfying the assumption of pseudo-normality, for which the resolvent is $O(\lambda^{-1})$ on rays in $\mathbf{C}$. One example is the condition $\Lambda'^{-1} D_{x_1}\gamma_1 u + \Lambda'\gamma_0 u = 0$ for Δ on $\mathbf{R}^n_+$ studied in [G3, Example 1.7.17] (here $\Lambda'=(I-\Delta_{x'})^{1/2}$); the coefficient of γ_1 is not surjective.

For another type of example containing negative-order ψdos on X' and defining a realization P_S that is skew-selfadjoint and hence has many rays where the resolvent is $O(\lambda^{-1})$, see Remark 5.2 later. We expect that such cases may still be handled by variants of the present methods, but will give extra log terms at some of the negative powers of t in (1.3).

A third example is $D_B^* D_B$ considered below; here the surjectiveness is missing in the boundary condition $B\gamma_0 u=0$, $(I-B^*)\sigma^*\gamma_0(\partial_{x_n}+A_1)u=0$; but the questions for this operator are dealt with in a different way, as will be shown.

3. First order well-posed boundary problems

For first-order operators (and odd-order operators more generally) it may not be possible to fulfill Assumptions 2.1 and 2.2 that lead to good resolvents—already the condition in Assumption 2.2, that Nd be even, needs not hold. However, for compact manifolds it is known that there exist ψdo boundary conditions (not pseudo-normal)

$$B\gamma_0 u = 0, \tag{3.1}$$

such that the realization P_B is a Fredholm operator with a similar adjoint P_B^*. In this case there is an interest in studying the positive selfadjoint operator $P_B^* P_B$, which does have a resolvent. We now consider such problems in detail.

Let D be a first-order elliptic operator on X; $D\colon C^\infty(E_1) \to C^\infty(E_2)$, where E_1 and E_2 are N-dimensional Hermitian vector bundles over X. D can be represented on U as

$$D = \sigma\left(\frac{\partial}{\partial x_n} + A_1\right), \tag{3.2}$$

where σ is an isomorphism from $E_1|_U$ to $E_2|_U$ and A_1 is a first order differential operator that acts in the x' variable at $x_n=0$. The restriction $A_1|_{x_n=0}$ has the principal symbol $a_1^0(x', \xi')$. For these operators,

$$\mathcal{A} = -\sigma \text{ on } X' \quad \text{and} \quad \varrho = \gamma_0 \quad \text{in (2.1)}. \tag{3.3}$$

Definition 3.1. (1) We say that D is of *Dirac-type* when σ is a unitary morphism, and

$$A_1 = A + x_n P_1 + P_0 \quad \text{on } U, \tag{3.4}$$

where A is an elliptic first-order differential operator in $C^\infty(E_1')$ which is *selfadjoint* with respect to the Hermitian metric in E_1', and the P_j are differential operators of order $\leq j$.

(2) The *product case* is the case where D is of Dirac-type and, moreover, $v(x)\,dx = v(x', 0)\,dx'\,dx_n$ on U, σ is constant in x_n, and $P_1 = P_0 = 0$.

As explained in [G4, p. 2036], unitarity of σ in (3.2) can be obtained by a simple homotopy near X', whereas the assumption on A_1 in (1) is an essential restriction in comparison with arbitrary first-order elliptic systems; it means that the principal symbol $a_1^0(x', \xi')$ of A_1 at $x_n=0$ is Hermitian symmetric. The operators P_1 and P_0 can be taken arbitrary near X', but for larger x_n, P_1 is subject to the requirement that D be elliptic.

To begin with, let X be compact. When (1) holds, $a_1^0(x', \xi')$ equals the principal symbol $a^0(x', \xi')$ of A. Since A is selfadjoint and elliptic of order 1, it has a discrete spectrum consisting of eigenvalues of finite multiplicity going to $\pm\infty$. Along with A one considers the orthogonal projections $\Pi_{\geq}$, $\Pi_{>}$, $\Pi_{\leq}$, $\Pi_{<}$ and Π_λ onto the closed spaces $V_\geq$, $V_>$, $V_\leq$, $V_<$ and V_λ spanned by the eigenvectors belonging to eigenvalues of A that are ≥ 0, >0, ≤ 0, <0 resp. $=\lambda$. These operators are classical ψdos of order 0; Π_λ is of order $-\infty$.

Atiyah, Patodi and Singer considered in [APS] the product case. It is also studied e.g. in [GS2], [BW1], [BW2], [BL], whereas the case where only (1) holds is studied in [G4], [GS1] and other works. Cases where not even (1) holds, have to our knowledge not been studied for the purpose of heat trace expansions for boundary problems before.

We shall study boundary problems satisfying the condition of *well-posedness* introduced by Seeley in [S2]. This uses the Calderón projector C^+ associated with D (as defined in [S2]). The reader is kindly asked to consult Section 7 for notation and a general explanation of Calderón projectors. Since $d=1$, C^+ is a classical ψdo of order 0 in E_1' that projects $H^{s-1/2}(X', E_1')$ onto the space N_+^s of boundary values of null-solutions for all $s\in\mathbf{R}$;

$$(3.5)\qquad N_+^s=\gamma_0 Z_+^s\subset H^{s-1/2}(X',E_1'),\quad Z_+^s=\{z\in H^s(X,E_1)\mid Dz=0 \text{ on } X\};$$

$C^-=I-C^+$. The analogous construction for the model operator

$$d^0(x',0,\xi',D_{x_n})=\sigma(x')\left(\frac{d}{dx_n}+a_1^0(x',\xi')\right)$$

on $\mathbf{R}_+\subset\mathbf{R}$ (defined from the principal symbol at each boundary point) leads to the principal symbols $c^\pm(x',\xi')$ of $C^\pm$; they are the projections in $\mathbf{C}^N$ onto the spaces $N_\pm(x',\xi')$ of boundary values of the bounded solutions of $d^0(x',0,\xi',D_{x_n})z(x_n)=0$ on $\mathbf{R}_\pm$, resp. One finds e.g. by changing $a_1^0(x',\xi')$ to Jordan normal form that the spaces $N_\pm(x',\xi')$ are the generalized eigenspaces for $a_1^0(x',\xi')$ associated with the eigenvalues having real part $\gtrless 0$, resp. Moreover, one has the formulas

$$(3.6)\qquad c^\pm(x',\xi')=\frac{1}{2\pi}\int_{\mathcal{L}_\pm}(\mathrm{i}\tau I+a_1^0(x',\xi'))^{-1}\,d\tau,$$

where the integration is over curves $\mathcal{L}_\pm$ in $\mathbf{C}_\pm=\{\tau\in\mathbf{C}\mid\operatorname{Im}\tau\gtrless 0\}$ encircling the τ-roots of $\det(\mathrm{i}\tau I+a_1^0(x',\xi'))$ (the poles of $(d^0)^{-1}$) there, resp.

Remark 3.2. When D is of Dirac-type, so that $a_1^0(x',\xi')$ equals $a^0(x',\xi')$, $N_+(x',\xi')$ and $N_-(x',\xi')$ are orthogonal complements and are spanned by the eigenvectors belonging to the positive, resp. negative eigenvalues of $a^0(x',\xi')$. The projections $c^\pm(x',\xi')$ onto $N_\pm(X',\xi')$ along $N_\mp(x',\xi')$ are then orthogonal, and they are the principal symbols of $\Pi_\geq$ resp. $\Pi_<$. Thus

(3.7) $\qquad C^+-\Pi_\geq$ is a classical ψdo of order -1 when D is of Dirac-type.

Definition 3.3. (Well-posedness) Let X be compact and let D be an elliptic first-order differential operator from $C^\infty(E_1)$ to $C^\infty(E_2)$. A classical ψdo B in E_1' of order 0 is *well-posed* for D when:

(i) the mapping defined by B in $H^s(E_1')$ has closed range for each $s\in\mathbf{R}$;

(ii) for each (x',ξ') with $|\xi'|=1$, the principal symbol $b^0(x',\xi')$ maps $N_+(x',\xi')$ injectively onto the range of $b^0(x',\xi')$ in $\mathbf{C}^N$.

A generalization to admissible manifolds will be included at the end of Section 5.

In comparison with the general choices of $S\colon H^s(E_1')\to H^s(F)$ (for $d=1$) discussed in Section 7 from (7.7) on, $F=E_1'$ here, so $M=N$. Condition (ii) assures that the system $\{D, B\gamma_0\}$ is *injectively elliptic*; see the explanation around (7.15)–(7.16). But (ii) is stronger than injective ellipticity, since the range of $b^0(x',\xi')$ for general injectively elliptic problems can have a larger dimension than $b^0(x',\xi')N_+(x',\xi')$ has. (One can say that (ii) means injective ellipticity with smallest possible range dimension for b^0.)

Observe that when B satisfies Definition 3.3, $\{D, B\gamma_0\}$ cannot be surjectively elliptic if $n\geq 3$, since N is then even and strictly larger than $\dim N_+(x',\xi')=\frac{1}{2}N$. (If $n=2$, this lack of surjective ellipticity holds when $\dim N_+(x',\xi')<N$.) Therefore, the system $\{D, B\gamma_0\}$ is *not* elliptic in the standard terminology, and, e.g., its range does not have a smooth complement. The word "well-posed" does not conflict with this and was well chosen by Seeley. (Some authors use the dangerous notation "globally elliptic" for these boundary problems—sometimes even abbreviated to "elliptic".)

When Definition 3.3 holds, one can replace (3.1) by an equivalent condition

$$B_1\gamma_0 u=0, \tag{3.8}$$

where B_1 is a *projection* in the H^s-spaces, in addition to being well-posed for D; cf. [S2]. The range of B_1 in $H^s(E_1')$ is closed for each s, since it is the nullspace of the complementing projection $I-B_1$ which is likewise a ψdo of order 0. Thus it is no restriction to assume that B in (3.1) is a projection; we shall often do that.

Seeley shows in [S2] that for each boundary condition (3.1) with B well-posed for D, the realization D_B defined as in (2.3) (with domain $D(D_B)=\{u\in H^1(X,E_1)\mid B\gamma_0 u=0\}$) is a Fredholm operator from $D(D_B)$ to $L_2(E_2)$. Moreover, when B is a projection, the adjoint D_B^* (when D_B is considered as an unbounded operator from $L_2(E_1)$ to $L_2(E_2)$) is the realization of D^* with domain

$$D(D_B^*)=\{u\in H^1(X,E_2)\mid (I-B^*)\sigma^*\gamma_0 u=0\}=D((D^*)_{(I-B^*)\sigma^*}); \tag{3.9}$$

here $(I-B^*)\sigma^*$ is well-posed for D^*. The nullspaces $Z(D_B)$ and $Z(D_B^*)$ are finite dimensional spaces of C^∞ sections, defining $\operatorname{index} D_B=\dim Z(D_B)-\dim Z(D_B^*)$.

It is useful to know that when B has been replaced by a projection B_1, then furthermore, B_1 can be replaced by a projection B_2 that is *orthogonal* in $L_2(E_1')$. This may possibly be inferred from [S2] which leaves out details on the proof of the relevant Lemma VI.3, but it certainly follows by a formula from Birman and Solomyak [BS] recalled in [BW2].

Lemma 3.4. *When R is a projection in a Hilbert space H, then the operator $RR^*+(I-R^*)(I-R)$ is invertible and*

$$R_{\mathrm{ort}}=RR^*[RR^*+(I-R^*)(I-R)]^{-1} \tag{3.10}$$

is an orthogonal projection in H with $R(H)=R_{\mathrm{ort}}(H)$.

Here if $H=L_2(F)$, where F is an admissible vector bundle over a manifold X', and R is an admissible classical ψdo of order 0 *in F, then the same holds for R_{ort}, and the principal symbol is determined by a formula similar to* (3.10) *on the principal symbol level.*

Proof. The formulas are verified in detail in [BW2, Lemma 12.8]. For the last statement, the invertibility of $RR^*+(I-R^*)(I-R)$ implies, by the spectral invariance shown in [G5] (and in the proof of Theorem 6.5 below), that it is uniformly elliptic and its inverse is likewise admissible, classical and uniformly elliptic of order 0. Then, since the principal symbol of R is a projection, the formulas likewise hold on the principal symbol level. $\square$

Remark 3.5. Since the range of R in $H^s(F)$ equals the nullspace of $I-R$ there, it follows from the fact that $I-R$ and $I-R_{\mathrm{ort}}$ have the same nullspace in $L_2(F)$ that they also have the same nullspace in $H^s(F)$, $s\geq 0$. Hence

$$R(H^s(F))=R_{\mathrm{ort}}(H^s(F)) \tag{3.11}$$

for $s\geq 0$. This property extends to negative s by consideration of the adjoint R^*, which is likewise a projection and a classical ψdo of order 0, when one uses that the nullspace of $I-R$ in $H^{-s}(F)$ $(s\geq 0)$ is the annihilator of the range of $R'=I-R^*$ in $H^s(F)$.

The lemma and remark imply that when R is a classical ψdo in E_1' which acts as a projection in $H^s(E_1')$, then R_{ort} defined by (3.10) is a projection which is orthogonal in $L_2(E_1')$ and has the same range as R in $H^s(E_1')$ for all s. When we apply this construction to $R=I-B_1$, (3.8) can be replaced by the condition $B_2\gamma_0u=0$ with the orthogonal projection $B_2=I-R_{\mathrm{ort}}$. It is not hard to check that B_2 again satisfies Definition 3.3.

Only the *orthogonal* projection defining a boundary condition is uniquely determined from it; without the orthogonality there can be many choices of projection that give the same condition.

4. Examples of well-posed problems

We here give examples with increasing generality, still taking X compact.

Clearly, the choice $B=C^+$ is well-posed, and so is $B=\Pi_{\geq}$ when D is of Dirac-type, in view of Remark 3.2. The first situation that was considered for index questions, in [APS], was the choice $B=\Pi_{\geq}$ in the product case. This choice is convenient because it permits construction of the heat operators (in a good approximation) by easy functional calculus for the selfadjoint operator A.

Grubb and Seeley consider in [GS2] the product case with $B-\Pi_{\geq}$ ranging in the nullspace of A, and in [GS1] Dirac-type operators with $B-\Pi_{\geq}$ ranging in the eigenspace for eigenvalues of A of modulus $\leq a$ (some $a>0$), showing full heat trace expansions.

Booss-Bavnbek and Wojciechowski [BW2] consider, for the product case, index questions for the full set of projections B of the form

$$B=C^+ + S, \quad S \text{ of order } -1; \tag{4.1}$$

likewise well-posed. This includes the preceding cases, and moreover allows infinite rank perturbations of $\Pi_{\geq}$.

Before leaving the case (4.1) we observe that (3.7) can be sharpened in the product case; this is of interest for the trace estimates (cf. Corollary 9.5 below).

Proposition 4.1. *In the product case, when X is compact,*

$$C^+ - \Pi_{\geq} \textit{ is a } \psi do \textit{ of order } -\infty. \tag{4.2}$$

Proof. We shall compare D, extended as $\sigma(\partial_{x_n}+A)$ on $X'\times\,]-c,0]$, with the operator σD^0, where

$$D^0=\partial_{x_n}+A', \quad A'=A+\Pi_0,$$

on $X^0=X'\times\mathbf{R}_+$ and $\widetilde{X}^0=X'\times\mathbf{R}$ provided with the volume element $v(x',0)\,dx'\,dx_n$. The operator D^0 acts in E_1^0 and in $\widetilde{E}_1^0$, the pull-backs of E_1' to X^0 and $\widetilde{X}^0$; in Green's formula (cf. (2.1) and (3.3)), $\mathcal{A}=-I$. The operator D^0 has an inverse Q^0 on $\widetilde{X}^0$, easily described by its action on functions of x_n taking values in the eigenspaces V_λ' of A' (here $V_0'=\{0\}$, $V_1'=V_1\oplus V_0$, $V_\lambda'=V_\lambda$ for $\lambda\neq 0,1$). When $f(x_n)$ has values in V_λ', Q^0 acts on f as the ψdo in x_n with symbol $(i\xi_n+\lambda)^{-1}$; more generally when f has an expansion $f(x)=\sum_{\lambda\in\operatorname{spec}A'} g_\lambda(x_n)u_\lambda(x')$ in terms of eigenfunctions $u_\lambda\in V_\lambda'$, then $Q^0 f=\sum_\lambda \mathcal{F}^{-1}_{\xi_n\to x_n}[(i\xi_n+\lambda)^{-1}\hat{g}_\lambda(\xi_n)]u_\lambda(x')$. For D^0, the Calderón projector is constructed exactly as in the differential operator case; it equals $\gamma_0^+ r^+ Q^0\widetilde{\gamma}_0^*$ as in (7.5). It acts on a $u_\lambda\in V_\lambda'$ like the Calderón projector for $\partial_{x_n}+\lambda$, so

$$\gamma_0^+ r^+ Q^0 \widetilde{\gamma}_0^* u_\lambda = \begin{cases} u_\lambda & \text{if } \lambda\geq 0, \\ 0 & \text{if } \lambda<0. \end{cases}$$

(One may also consult (3.6).) This implies that $\gamma_0^+ r^+ Q^0 \widetilde{\gamma}_0^* = \Pi_\geq$.

On $\widetilde{X}_c = X' \times]-c, c[$, σD^0 and D differ only by the term $\sigma \Pi_0$. Let Q be a parametrix of D on $\widetilde{X} = X \cup \widetilde{X}_c$; then $C^+ = \gamma_0^+ r^+ Q \widetilde{\gamma}_0^* \sigma + \mathcal{T}_3$, where $\mathcal{T}_3$ is of order $-\infty$, cf. (7.6). Let χ and $\chi_1 \in C_0^\infty(]-c, c[)$, equal to 1 on a neighborhood of 0 and satisfying $\chi\chi_1 = \chi$, then

$$
(4.3) \qquad C^+ - \Pi_\geq = \gamma_0^+ r^+ Q \widetilde{\gamma}_0^* \sigma + \mathcal{T}_3 - \gamma_0^+ r^+ Q^0 \widetilde{\gamma}_0^* = \gamma_0^+ r^+ \chi (Q - (\sigma D^0)^{-1}) \chi \widetilde{\gamma}_0^* \sigma + \mathcal{T}_3.
$$

If $\Pi_0 = 0$ (i.e., $\dim \ker A = 0$), $\chi(Q - (\sigma D^0)^{-1})\chi$ is a ψdo on $\widetilde{X}_c$ with symbol 0, hence of order $-\infty$, so $C^+ - \Pi_\geq$ is a ψdo on X' of order $-\infty$ by the boundary operator calculus; this ends the proof. If $\Pi_0 \neq 0$, we need a further effort since Π_0 on $\widetilde{X}_c$ is not a ψdo.

In view of (7.1), we have on $\widetilde{X}_c$,

$$
(4.4) \qquad \begin{aligned}
\chi(Q - (\sigma D^0)^{-1})\chi &= \chi Q \chi_1 \sigma D^0 Q^0 \sigma^{-1} \chi_1 \chi - \chi\chi_1 (QD - \mathcal{T}_2) \chi_1 Q^0 \sigma^{-1} \chi \\
&= \chi Q [\chi_1 \sigma D^0 - D \chi_1] Q^0 \sigma^{-1} \chi + \chi \mathcal{T}_2 \chi_1 Q^0 \sigma^{-1} \chi \\
&= \chi Q [\chi_1 \sigma \Pi_0 - (\partial_{x_n} \chi_1) \sigma] Q^0 \sigma^{-1} \chi + \chi \mathcal{T}_2 \chi_1 Q^0 \sigma^{-1} \chi.
\end{aligned}
$$

Define the anisotropic spaces $H^{(s,t)}(X' \times \mathbf{R})$ and $H^{(s,t)}(X' \times]-c, c[)$, via local coordinates and a partition of unity on X', from the spaces $H^{(s,t)}(\mathbf{R}^{n-1} \times \mathbf{R})$ with norm $\|\langle\xi\rangle^s \langle\xi'\rangle^t \hat{u}(\xi)\|$. The operators have the continuity properties:

$$
\begin{aligned}
\chi Q \chi_1 &: H^{(s,t)}(E_2|_{\widetilde{X}_c}) \longrightarrow H^{(s+1,t)}(E_1|_{\widetilde{X}_c}), & Q^0 &: H^{(s,t)}(\widetilde{E}_1^0) \longrightarrow H^{(s+1,t)}(\widetilde{E}_1^0), \\
\chi \mathcal{T}_2 \chi_1 &: H^{(s,t)}(E_2|_{\widetilde{X}_c}) \longrightarrow H^{(s_1,t_1)}(E_1|_{\widetilde{X}_c}), & \Pi_0 &: H^{(s,t)}(\widetilde{E}_1^0) \longrightarrow H^{(s,t_1)}(\widetilde{E}_1^0), \\
\gamma_0^+ &: H^{(1,t)}(X_c) \longrightarrow H^{1/2+t}(X'), & \widetilde{\gamma}_0^* &: H^{-1/2+t}(X') \longrightarrow H^{(-1,t)}(\widetilde{X}_c)
\end{aligned}
$$

for all $s, s_1, t, t_1 \in \mathbf{R}$. Such properties are easy to show and are e.g. dealt with in [G2], [G3, Section 2.5] (used with fixed μ). Then the operator in (4.4) is continuous from $H^{(-1,t)}(E_1|_{\widetilde{X}_c})$ to $H^{(1,t_1)}(E_1|_{\widetilde{X}_c})$ for all $t, t_1 \in \mathbf{R}$, and when we compose it to the left with $\gamma_0^+ r^+$ and to the right with $\widetilde{\gamma}_0^*$, we get an operator that is continuous from $H^t(E_1')$ to $H^{t_1}(E_1')$ for all $t, t_1 \in \mathbf{R}$. Then this is a ψdo of order $-\infty$ on X'. Thus finally, $C^+ - \Pi_\geq$ in (4.3) is a ψdo of order $-\infty$ on X'. $\square$

Defining C_{ort}^+ by formula (3.10), we find as a corollary that $C_{\mathrm{ort}}^+ - \Pi_\geq$ is likewise a ψdo of order $-\infty$. For selfadjoint Dirac operators on spin manifolds, this was shown in the case $\dim \ker A = 0$ by Scott in [Sc, Proposition 2.2] by a rather different argument.

Example 4.2. A well-posed B need not be of the type (4.1). One example was introduced by Brüning and Lesch [BL], in the product case and under the additional hypotheses that D is formally selfadjoint and $\sigma A=-A\sigma$, $\sigma^2=-I$, $\tau A=-A\tau$, $\tau^2=I$, $\tau\sigma=-\sigma\tau$, where τ is an auxiliary morphism or ψdo of order 0. The prototype is, for $\cos\theta\neq 0$,

$$B_\theta=\cos^2\theta\,\Pi_>+\sin^2\theta\,\Pi_<-\cos\theta\sin\theta\,\tau(\Pi_>+\Pi_<)+B', \tag{4.5}$$

with a suitable projection B' in V_0. Here B_θ is principally different from $\Pi_\geq$ when $\cos^2\theta\neq 1$. The operator D_{B_θ} is selfadjoint.

For the analysis it is useful to observe that the hypotheses imply a *spectral symmetry* of A; in fact τ (as well as σ) defines isometries of the eigenspaces V_j^+ for positive eigenvalues λ_j^+ (ordered increasingly) onto the eigenspaces V_j^- for negative eigenvalues $\lambda_j^-=-\lambda_j^+$ and vice versa (in particular, $\eta(A,s)=\operatorname{Tr}(A|A|^{-s-1})\equiv 0$). Then the nullspace of B_θ in $V_0^\perp$ is a "shifted version" of $V_<$,

$$\overline{\operatorname{span}}\{e_{j,k}^-+\tan\theta e_{j,k}^+ \mid j>0,\ k=1,\dots,\nu_j\}; \tag{4.6}$$

here the $e_{j,k}^-$, $1\leq k\leq\nu_j$, are an orthonormal basis of V_j^-, and $e_{j,k}^+=\tau e_{j,k}^-$.

For $B=B_\theta$, [BL] shows a precise version of (1.1), related to that of [GS2] (see also Grubb [G6, Remark 7.14]). The present study allows generalizations to the non-product case and perturbations of order -1. The same holds for the more abstractly formulated well-posed conditions in [BL].

Example 4.3. Without assuming spectral symmetry, we can give general examples of well-posed B for Dirac-type operators by taking

$$B=\Pi_\geq+\Pi_\geq S\Pi_<, \tag{4.7}$$

where S is a classical ψdo of order 0 in E_1'. The operator B is a projection, since $\Pi_<\Pi_\geq=0$; so (i) in Definition 3.3 is satisfied. For the principal symbols, the injectiveness (7.16) is obvious for $b^0(x',\xi')=c^+(x',\xi')+c^+(x',\xi')s^0(x',\xi')c^-(x',\xi')$. Moreover,

$$b^0(x',\xi')N_+(x',\xi')\subset b^0(x',\xi')\mathbf{C}^N\subset N_+(x',\xi'),$$

so since the former has the same dimension as $N_+(x',\xi')$, there must be equality. Then also (ii) of Definition 3.3 is satisfied.

To compare this with earlier cases, we replace B by the orthogonal projection $B_1=I-(I-B)_{\mathrm{ort}}$ defining the same boundary condition. Write S and B in blocks according to the decomposition $L_2(E_1')=V_\geq\oplus V_<$; $S=\begin{pmatrix} S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}$, $B=\begin{pmatrix} I & S_{12}\\ 0 & 0\end{pmatrix}$. Then with $R=I-B$, we find from (3.10) that

$$R_{\mathrm{ort}}=\begin{pmatrix} S_{12}S_{12}^*(I+S_{12}S_{12}^*)^{-1} & -S_{12}(I+S_{12}^*S_{12})^{-1}\\ -S_{12}^*(I+S_{12}S_{12}^*)^{-1} & (I+S_{12}^*S_{12})^{-1}\end{pmatrix}. \tag{4.8}$$

Here $B_1=I-R_{\mathrm{ort}}$ is principally different from $\Pi_{\geq}=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ as soon as S_{12} has non-vanishing principal symbol, which is the generic case (when $0<\dim N_+(x',\xi')<N$, in particular when $n\geq 3$). One can also allow lower order perturbations.

Let us remark that if there is a spectral symmetry, $A\tau=-\tau A$ for some zero-order ψdo τ with $\tau^2=I$, then the choice $B=\Pi_{\geq}+\beta\tau\Pi_{<}$ for some $\beta\in\mathbf{R}$, is of the above type with $S=\beta\tau$, since $\tau\Pi_{<}=\tau\Pi_{<}\Pi_{<}=\Pi_{>}\tau\Pi_{<}$. The condition defined by this B is similar to that defined by (4.5); in fact the nullspace of B in $V_0^{\perp}$ equals (4.6) with $\tan\theta=-\beta$.

Still other examples can be found by replacing $(\Pi_{\geq},\Pi_{<})$ in (4.7) by (C^+,C^-) or by $(C^+_{\mathrm{ort}},C^{+\perp}_{\mathrm{ort}})$ or $(C^{-\perp}_{\mathrm{ort}},C^-_{\mathrm{ort}})$ (with $C^{\pm\perp}_{\mathrm{ort}}=I-C^{\pm}_{\mathrm{ort}}$); these choices have a meaning for an arbitrary D. Since the $c^{\pm}$ are orthogonal projections when P is of Dirac-type,

$$C^+-C^+_{\mathrm{ort}} \text{ and } C^+-C^{-\perp}_{\mathrm{ort}} \text{ are of order } -1 \text{ when } P \text{ is of Dirac-type,} \tag{4.9}$$

so the resulting problems are just perturbations of order -1 of the previous types. However, c^+ is not orthogonal in general (examples with non-symmetric a_1^0 are easy to give).

Example 4.4. Denote the principal symbols and range spaces of C^+_{ort} and $C^{+\perp}_{\mathrm{ort}}$ by $c^+_{\mathrm{ort}}(x',\xi')$, $c^{+\perp}_{\mathrm{ort}}(x',\xi')=I-c^+_{\mathrm{ort}}(x',\xi')$, $N_+(x',\xi')$, $N_+^{\perp}(x',\xi')=\mathbf{C}^N\ominus N_+(x',\xi')$. As noted above, the following operators are well-posed for D,

$$B=C^+_{\mathrm{ort}}+C^+_{\mathrm{ort}}SC^{+\perp}_{\mathrm{ort}}. \tag{4.10}$$

(We can add S_1 of order -1, as long as B remains a projection.) This is, in a *microlocal* sense, the most general possible example. When B defines the condition $B\gamma_0u=0$, so does CB for any invertible classical elliptic ψdo C of order 0; in this sense, B and CB can be regarded as equivalent. Now if B satisfies Definition 3.3, we can *for* (x',ξ') *in a neighborhood of each* (x_0',ξ_0') $(|\xi'|=1)$ find a smooth family of bijective matrices $c(x',\xi')$ such that $c(x',\xi')b^0(x',\xi')$ is of the form $c^+_{\mathrm{ort}}(x',\xi')+c^+_{\mathrm{ort}}(x',\xi')s(x',\xi')c^{+\perp}_{\mathrm{ort}}(x',\xi')$, as follows. Note that $\mathbf{C}^N$ has the two decompositions (depending smoothly on (x',ξ'))

$$\mathbf{C}^N=N_+(x',\xi')\dot{+}N_+^{\perp}(x',\xi')=R(b^0(x',\xi'))\dot{+}Z(b^0(x',\xi')); \tag{4.11}$$

the latter denotes the range and nullspace of b^0 (we now omit the indication (x',ξ')). Here b^0 defines a homeomorphism c_1 of N_+ onto $R(b^0)$. Let $c_2=c_1^{-1}$ and let c_3 be a homeomorphism of $Z(b^0)$ onto $N_+^{\perp}$ (it can be chosen to depend smoothly on (x',ξ') in a neighborhood of (x_0',ξ_0')); then $c_4=c_2b^0+c_3(I-b^0)$ is a bijection in $\mathbf{C}^N$. Now its inverse $c_5=c_4^{-1}$ does the job, it is a bijection in $\mathbf{C}^N$ that maps $R(b^0)$ to N_+ as an inverse of b^0 from N_+ to $R(b^0)$. So $c_5 b^0$ ranges in N_+ and is the identity on N_+, and hence

$$c_5 b^0 = c^+_{\mathrm{ort}} c_5 b^0 (c^+_{\mathrm{ort}} + c^{+\perp}_{\mathrm{ort}}) = c^+_{\mathrm{ort}} + c^+_{\mathrm{ort}} c_5 b^0 c^{+\perp}_{\mathrm{ort}}; \tag{4.12}$$

it is of the desired form and is equivalent with b^0.—Similar considerations hold with $(C^+_{\mathrm{ort}}, C^{+\perp}_{\mathrm{ort}})$ and $(c^+_{\mathrm{ort}}, c^{+\perp}_{\mathrm{ort}})$ replaced by (C^+, C^-) and (c^+, c^-).

5. Imbedding of well-posed problems into elliptic systems

We shall now show how the resolvents of the operators

$$(\Delta_1 - \lambda)^{-1},\ (\Delta_2 - \lambda)^{-1}, \quad \text{where } \Delta_1 = D_B^* D_B,\ \Delta_2 = D_B D_B^*, \tag{5.1}$$

can be treated within the framework of Section 2. In fact, there is a nice trick of replacing the study of the injectively elliptic first-order system $\{D, B\gamma_0\}$ by a truly elliptic first-order system $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ satisfying the resolvent growth condition, in such a way that the second-order resolvents (5.1) are found from the resolvent construction for $\mathcal{D}_{\mathcal{B}}$.

Let B be a well-posed projection for D and let us denote

$$\mathcal{D} = \begin{pmatrix} 0 & -D^* \\ D & 0 \end{pmatrix}, \quad \mathcal{D}_{\mathcal{B}} = \begin{pmatrix} 0 & -D_B^* \\ D_B & 0 \end{pmatrix}. \tag{5.2}$$

The operator $\mathcal{D}$ in (5.2) is formally skew-selfadjoint on X. The operator $\mathcal{D}_{\mathcal{B}}$ is skew-selfadjoint as an unbounded operator in $L_2(E)$, $E = E_1 \oplus E_2$. It then has a resolvent $\mathcal{R}_\mu = (\mathcal{D}_{\mathcal{B}} + \mu)^{-1}$ for $\mu \in \mathbf{C} \setminus \mathrm{i}\mathbf{R}$. A calculation shows that

$$\begin{gathered} \mathcal{R}_\mu = (\mathcal{D}_{\mathcal{B}} + \mu)^{-1} = \begin{pmatrix} \mu R_{1,\mu} & D_B^* R_{2,\mu} \\ -D_B R_{1,\mu} & \mu R_{2,\mu} \end{pmatrix}, \\ \text{where} \quad R_{1,\mu} = (\Delta_1 + \mu^2)^{-1}, \quad R_{2,\mu} = (\Delta_2 + \mu^2)^{-1}; \end{gathered} \tag{5.3}$$

this shows how the resolvents (5.1) can be recovered from $\mathcal{R}_\mu$. Also $D_B R_{1,\mu}$ and $D_B^* R_{2,\mu}$ are determined. When $\mu \in \Gamma_0 = \{ z \in \mathbf{C} \mid |\arg z| < \frac{1}{2}\pi \}$, then $\lambda = -\mu^2$ runs through $\mathbf{C} \setminus \overline{\mathbf{R}}_+$, so it suffices for (5.1) to let $\mu \in \Gamma_0$.

Now $\mathcal{D}_{\mathcal{B}}$ is the realization of $\mathcal{D}$ in $L_2(E)$ of the boundary condition

$$\mathcal{B}\gamma_0 u = 0, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{5.4}$$

where $\mathcal{B}$ is the row matrix (cf. (3.9))

$$\mathcal{B} = (\,B \quad (I-B^*)\sigma^*\,)\,, \tag{5.5}$$

going from $L_2(E_1')\times L_2(E_2')$ to $L_2(E_1')$. Since the ranges of B and $I-B^*$ are orthogonal complements in $L_2(E_1')$, $\mathcal{B}$ is *surjective*; note that the dimension N of E_1' is half of the dimension $2N$ of $E'=E_1'\oplus E_2'$. Moreover, $\mathcal{B}$ has as a right inverse the ψdo $\mathcal{C}$ of order 0,

$$\mathcal{C} = \begin{pmatrix} B^* \\ (\sigma^*)^{-1}(I-B) \end{pmatrix} [BB^*+(I-B^*)(I-B)]^{-1} \tag{5.6}$$

(cf. Lemma 3.4); in particular, $\mathcal{B}$ is surjectively elliptic. Now $\{\mathcal{D}+1, \mathcal{B}\gamma_0\}$ has the inverse $(\,\mathcal{R}_1 \quad \mathcal{K}_1\,)$ with $\mathcal{K}_1=[I-\mathcal{R}_1(\mathcal{D}+1)]K_{\gamma_0,1}\mathcal{C}$ as in (2.7). Since the inverse is continuous from $L_2(E)\times H^{1/2}(E_1')$ to $H^1(E)$, $\{\mathcal{D}+1, \mathcal{B}\gamma_0\}$ and hence also $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ is elliptic. Thus all the conditions in Assumptions 2.1 and 2.2 are satisfied by $\{\mathcal{D}, \mathcal{B}\varrho\}$, with N replaced by $2N$, $d=1$, $\varrho=\gamma_0$, $F=F_0=E_1'$!

Then the consequences we draw later for the general systems in Section 2 apply in particular to $\mathcal{D}_{\mathcal{B}}$.

Example 5.1. By Theorem 7.5 below, the Calderón projector for D^* equals $C'^+=(\sigma^*)^{-1}(I-C^{+*})\sigma^*$, when D has an invertible extension. (More generally, this holds modulo smoothing operators.) Then in view of (3.9), the adjoint of D_{C^+} is the realization of D^* determined by the boundary condition $C'^+\gamma_0 u=0$. Here $\mathcal{B}$ is the surjective operator $\mathcal{B}=(\,C^+ \quad (I-C^{+*})\sigma^*\,)=(\,C^+ \quad \sigma^* C'^+\,)$. (We observe moreover that if $\sigma^*=\sigma^{-1}$, one finds by (3.10) that $C'^+_{\mathrm{ort}}=\sigma(I-C^+_{\mathrm{ort}})\sigma^*$, generalizing [BW1, Corollary 3.3].)

Remark 5.2. The trick of considering the "doubled-up" system (5.2) will be restricted to first-order operators in this paper. Well-posed boundary conditions can also be defined for higher order systems, cf. [S2]. But here when one takes the example of $B=C^+$, one gets an operator on the boundary with entries of negative order that are generally nontrivial, and these exist also in the doubled-up version and violate the requirement concerning order ≥ 0 in Assumption 2.1. Manipulations with order-reducing operators do not seem to help; they cannot at the same time remove a singularity in ξ' and be strongly polyhomogeneous in (ξ',μ). (See also Remark 2.5 and the calculations after (8.2).)

The analysis of (5.4)–(5.6) moreover tells us how to include admissible manifolds in the study of first-order systems. Here we need a uniformity in x' in the well-posedness condition. We restrict the attention to projections B.

Definition 5.3 (Uniform well-posedness). Let D be an admissible, uniformly elliptic first-order differential operator from E_1 to E_2 (admissible vector bundles over an admissible manifold X). Let B be an admissible classical ψdo of order 0 in E_1' with $B^2=B$. We say that B is *uniformly well-posed* for D, when B satisfies Definition 3.3(ii) and in addition, $\mathcal{B}$ defined by (5.5) is uniformly surjectively elliptic and $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ (cf. (5.2)) is uniformly elliptic.

When Definition 5.3 is satisfied, the realization $\mathcal{D}_{\mathcal{B}}$ is seen by Green's formula to be skew-symmetric. It is skew-selfadjoint since $(\mathcal{D}_{\mathcal{B}})^*$ acts like $\mathcal{D}^*$ and $u \in D((\mathcal{D}_{\mathcal{B}})^*)$ implies $u \in L_2(E)$ with $\mathcal{D}^* u \in L_2(E)$ and $\mathcal{B}\gamma_0 u=0$ as an element of $H^{-1/2}(E_1')$, hence by use of a parametrix of $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ it is seen that $u \in H^1(E)$ and thus $u \in D(\mathcal{D}_{\mathcal{B}})$.

It follows that Assumptions 2.1 and 2.2 are satisfied, with $\Gamma=\Gamma_0$; so (5.3) exists and gives the resolvents of the Δ_i as in the compact case.

Examples are constructed as in Section 4, most easily when D has an invertible extension to a boundaryless manifold so that Theorem 7.1 defines an exact projection C^+; then $B=C^+ + C^+ S C^-$ and $B=C^+_{\mathrm{ort}} + C^+_{\mathrm{ort}} S C^{+\perp}_{\mathrm{ort}}$ are examples. (Otherwise there is a question of modifying B to be a projection.)

6. Spectral invariance of weakly polyhomogeneous ψdos

For use in the fine analysis of the resolvents, we now recall the definition of weakly polyhomogeneous ψdo classes from Grubb and Seeley [GS1], presently allowing non-compact admissible manifolds and globally estimated operators as in [G5], [G3].

The symbol space $S^m(\mathbf{R}^\nu \times \mathbf{R}^n)$ consists of the functions $p(x,\xi) \in C^\infty(\mathbf{R}^\nu \times \mathbf{R}^n)$ such that

$$\partial_x^\beta \partial_\xi^\alpha p = O(\langle \xi \rangle^{m-|\alpha|}) \quad \text{for all } \alpha \in \mathbf{N}^n,\ \beta \in \mathbf{N}^\nu; \tag{6.1}$$

$\mathbf{N}=\{\text{integers} \geq 0\}$. The basic rules of calculus for this space are well known from Hörmander [H2, Section 18.1]. (When we are only interested in symbols with estimates valid over compact subsets of $\mathbf{R}^n$, we can use the results of the global calculus by introducing suitable cut-off functions.) A symbol $p \in S^m(\mathbf{R}^\nu \times \mathbf{R}^n)$ is called classical (or classical polyhomogeneous) of degree m if it has an expansion $p \sim \sum_{j \in \mathbf{N}} p_j$, where the p_j are homogeneous in ξ of degree $m-j$ for $|\xi| \geq 1$, and $p - \sum_{j<J} p_j \in S^{m-J}(\mathbf{R}^\nu \times \mathbf{R}^n)$ for $J \in \mathbf{N}$.

Grubb and Seeley [GS1] introduced a class of symbols p depending on a parameter μ varying in a sector $\Gamma \subset \mathbf{C} \setminus \{0\}$, in a special way. Here it is the behavior for $|\mu| \to \infty$ that is important; it is often described in terms of the behavior of $p(x,\xi,1/z)$ for $z \to 0$, $1/z=\mu \in \Gamma$. For brevity of notation, we write $\partial_z^j p(x,\xi,1/z)$ (or just $\partial_z^j p$) for the jth z-derivative of the composite function $z \mapsto p(x,\xi,1/z)$.

Definition 6.1. Let n and ν be positive integers, and let m and $d\in\mathbf{R}$. Let Γ be a sector in $\mathbf{C}\setminus\{0\}$. The space $S^{m,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)$ consists of the functions $p(x,\xi,\mu)\in C^\infty(\mathbf{R}^\nu\times\mathbf{R}^n\times\Gamma)$ that are holomorphic in $\mu\in\overset{\circ}{\Gamma}$ for $|(\xi,\mu)|\geq\varepsilon$ (some $\varepsilon>0$) and satisfy, for all $j\in\mathbf{N}$,

$$(6.2)\quad \partial_z^j p\Big(\cdot,\cdot,\frac{1}{z}\Big) \text{ is in } S^{m+j}(\mathbf{R}^\nu\times\mathbf{R}^n) \text{ for } \frac{1}{z}\in\Gamma,$$

$$\text{with estimates valid uniformly for } |z|\leq 1,\ \frac{1}{z}\in\text{closed subsectors of }\Gamma.$$

Moreover, we set $S^{m,d}=\mu^d S^{m,0}$ (so $p\in S^{m,d}$ means that $z^d p\in S^{m,0}$).

Sometimes the symbols are only defined for $|\mu|\geq$ a constant depending on the subsector of Γ; this requires obvious modifications. We can inject $S^m(\mathbf{R}^\nu\times\mathbf{R}^n)\subset S^{m,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\mathbf{C})$. Asymptotic expansions and polyhomogeneous subclasses are introduced as follows.

Definition 6.2. (1) Let $p\in S^{m-d,d}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)$ and let p_j be a sequence of symbols in $S^{m-j-d,d}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)$ such that $p-\sum_{j<J}p_j\in S^{m-J-d,d}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)$ for any $J\in\mathbf{N}$; then we say that $p\sim\sum_{j\in\mathbf{N}}p_j$ in $S^{m-d,d}$.

(2) If, moreover, the p_j are weakly homogeneous of degree $m-j$, i.e.,

$$(6.3)\qquad p_j(x,t\xi,t\mu)=t^{m-j}p_j(x,\xi,\mu)\quad\text{for } |\xi|\geq 1,\ t\geq 1,\ (\xi,\mu)\in\mathbf{R}^n\times\Gamma,$$

we say that p is *weakly polyhomogeneous.*

(3) If, furthermore, the p_j are strongly homogeneous of degree $m-j$, i.e.,

$$(6.4)\quad p_j(x,t\xi,t\mu)=t^{m-j}p_j(x,\xi,\mu)\quad\text{for } |\xi|^2+|\mu|^2\geq 1,\ t\geq 1,\ (\xi,\mu)\in\mathbf{R}^n\times(\Gamma\times\{0\}),$$

and $\partial_x^\beta\partial_\xi^\alpha\partial_\mu^k(p-\sum_{j<J}p_j)=O(\langle(\xi,\mu)\rangle^{m-J-|\alpha|-k})$ for all indices α,β,J, then we say that p is *strongly polyhomogeneous.*

(For simplicity, we leave out the possibility of noninteger steps between the degrees of the p_j, included in [GS1].) It is shown in [GS1] that the conditions in (3) imply those in (1) and (2). Thus the strongly polyhomogeneous symbol can be thought of as the case where μ enters as an extra cotangent variable, on a par with the others, in a classical symbol. For example, for $m\in\mathbf{Z}$,

$$(6.5)\qquad (|\xi|^2+|\mu|^2+1)^{m/2}\in\begin{cases} S^{m,0}+S^{0,m} & \text{for } m\geq 0,\\ S^{m,0}\cap S^{0,m} & \text{for } m\leq 0,\end{cases}$$

is strongly polyhomogeneous, whereas the function $(|\mu|^2+(\xi_1^4+\xi_2^4)/(\xi_1^2+\xi_2^2+1))^{-1}$ is weakly polyhomogeneous and belongs to $S^{-2,0}\cap S^{0,-2}$. (Cf. [GS1, Lemma 1.13 and Theorem 1.17].)

We shall use a special name (as in [G6]) for symbols of the latter kind.

Definition 6.3. Let r be an integer ≥ 0. A symbol $s(x,\xi,\mu)$ (and the operator it defines) is called *special parameter-dependent of order* $-r$, when

$$\begin{aligned} &s(x,\xi,\mu)\in S^{-r,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\cap S^{0,-r}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\\ \text{with}\quad &\partial_\mu^m s(x,\xi,\mu)\in S^{-r-m,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\cap S^{0,-r-m}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\end{aligned}\tag{6.6}$$

for any m, all $\partial_\mu^m s(x,\xi,\mu)$ being weakly polyhomogeneous.

By [GS1, Theorem 1.16], a strongly polyhomogeneous symbol of order $-r$ has this property.

The rules of calculus for the symbol spaces and the associated operators are described in detail in [GS1]. Let us here just recall a few elements. A symbol $p(x,\xi,\mu)$ with x and $\xi\in\mathbf{R}^n$ defines a family of ψdos on $\mathbf{R}^n$ depending on $\mu\in\Gamma$,

$$P_\mu f(x)=\mathrm{OP}(p)f(x)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}p(x,\xi,\mu)\hat{f}(\xi)\,d\xi.\tag{6.7}$$

There holds the composition rule

$$\begin{aligned} &P_\mu\in\mathrm{OP}(S^{m,d}),\ P'_\mu\in\mathrm{OP}(S^{m',d'})\quad\Longrightarrow\quad P_\mu P'_\mu\in\mathrm{OP}(S^{m+m',d+d'}),\\ &\text{with symbol } (p\circ p')(x,\xi,\mu)\sim\sum_{\alpha\in\mathbf{N}^n}\frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi,\mu)(-i\partial_x)^\alpha p'(x,\xi,\mu).\end{aligned}\tag{6.8}$$

Theorem 1.23 in [GS1], formulated there for symbols with local estimates in x, extends without difficulty to symbols with global estimates in x, and to one-sided ellipticity.

Theorem 6.4. (1) *Let* $p(x,\xi,\mu)\in S^{0,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\otimes\mathcal{L}(\mathbf{C}^N,\mathbf{C}^N)$ *be such that* $p=p_0+p_{-1}$ *with* $p_{-1}\in S^{-1,0}$ *and with* $p_0^{-1}\in C^\infty$ *bounded uniformly in* $(x,\xi,\mu)\in\mathbf{R}^n\times\mathbf{R}^n\times\Gamma'_1$, *for any closed subsector* Γ' *of* Γ *and* $\Gamma'_1=\{\mu\in\Gamma'\mid|\mu|\geq1\}$. *Then there exists a parametrix symbol* $q(x,\xi,\mu)\in S^{0,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)$ *such that* $p\circ q\sim I$ *in* $S^{0,0}$; *here*

$$q\sim q_0\circ\sum_{k\in\mathbf{N}}r^{\circ k},\quad \textit{where } q_0=p_0{}^{-1},\ r=I-p\circ q_0,\ r^{\circ k}=r\circ r\circ\ldots\circ r\ (k\textit{ factors}).\tag{6.9}$$

(2) *Let* $p(x,\xi,\mu)\in S^{0,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\otimes\mathcal{L}(\mathbf{C}^N,\mathbf{C}^M)$ *be such that* $p=p_0+p_{-1}$ *with* $p_{-1}\in S^{-1,0}$ *and with* p_0 *having a right inverse* $q_0\in C^\infty$ *that is bounded uniformly in* $(x,\xi,\mu)\in\mathbf{R}^n\times\mathbf{R}^n\times\Gamma'_1$, *for any closed truncated subsector* Γ'_1 *of* Γ. *Then there exists a right parametrix symbol* $q(x,\xi,\mu)\in S^{0,0}(\mathbf{R}^\nu\times\mathbf{R}^n,\Gamma)\otimes\mathcal{L}(\mathbf{C}^M,\mathbf{C}^N)$ *such that* $p\circ q\sim I$ *in* $S^{0,0}$; *here*

$$q\sim p^*\circ(p\circ p^*)^{\circ-1},\tag{6.10}$$

where $(p\circ p^*)^{\circ -1}$ *is a parametrix symbol for* $p\circ p^*$ *according to* (1).

(3) *When the assumptions in* (2) *hold with "right" replaced by "left," there exists a left parametrix symbol* $q\sim(p^*\circ p)^{\circ -1}\circ p^*\in S^{0,0}$ *such that* $q\circ p\sim I$ *in* $S^{0,0}$.

In (1)–(3), *if* p *is weakly resp. strongly polyhomogeneous, so is* q.

Proof. For (1), the proof of [GS1, Theorem 1.23] extends readily; it is in fact simplified because the compositions can be carried out directly, without cut-off functions, in the global calculus. Cases (2) and (3) follow from (1), when we note that $p^*\circ p$ in case (2), resp. $p\circ p^*$ in case (3), satisfies the hypotheses of (1). The last statement is seen from the formulas. $\square$

We shall not introduce a general ellipticity definition but just say that the operators with symbol satisfying the hypotheses of Theorem 6.4(1), (2) resp. (3) are *uniformly parameter-elliptic, uniformly surjectively parameter-elliptic*, resp. *uniformly injectively parameter-elliptic, in the sense of Theorem* 6.4.

For our application to the resolvent analysis we need to show *spectral invariance* of our calculus (briefly expressed this means that when a ψdo has an inverse in some operator sense, then the inverse belongs to the calculus, and both operators are elliptic). We even need a one-sided version. In the earlier work [G5], results were shown both for parameter-independent ψdos and for parameter-dependent ψdos of a slightly different type than here. The following proof uses the parameter-*independent* results.

Theorem 6.5. *Let* E_1 *and* E_2 *be admissible vector bundles of dimensions* N *and* M *over an admissible boundaryless manifold* $\widetilde{X}$, *and let* P_μ *(depending on* μ *in a sector* Γ *of* $\mathbf{C}$*) be a weakly polyhomogeneous* ψ*do with symbol in* $S^{0,0}$ *in admissible coordinate systems.*

(1) *Assume that* $N=M$ *and for some* $l\in\mathbf{Z}$, $P_\mu\colon H^{l,\mu}(E_1)\to H^{l,\mu}(E_2)$ *(which is bounded uniformly for* μ *in closed truncated subsectors* Γ'_r*) has an inverse* P_μ^{-1} *that is likewise* $H^{l,\mu}$*-bounded uniformly for* μ *in subsectors* Γ'_r. *Then* P_μ^{-1} *is a weakly polyhomogeneous* ψ*do with symbol in* $S^{0,0}$. *Moreover,* P_μ *and* P_μ^{-1} *are uniformly parameter-elliptic in the sense of Theorem* 6.4. *If* P_μ *is strongly polyhomogeneous, so is* P_μ^{-1}. *If* P_μ *is special parameter-dependent of order* 0 *(cf. Definition* 6.3*), so is* P_μ^{-1}.

(2) *Assume that for some* $l\in\mathbf{Z}$, $P_\mu\colon H^{l,\mu}(E_1)\to H^{l,\mu}(E_2)$ *has a right inverse* R_μ *that is likewise bounded uniformly for* μ *in truncated closed subsectors* Γ'_r. *Then* P_μ *has a right inverse* R'_μ *that is a weakly polyhomogeneous* ψ*do with symbol in* $S^{0,0}$. *If* P_μ *is strongly polyhomogeneous, so is* R'_μ. *If* P_μ *is special parameter-dependent of order* 0, *so is* R'_μ.

(3) *A similar statement holds with "right" replaced by "left."*

Proof. (1) Consider a Γ'_r. First let $l=0$, so that $H^{l,\mu}$ is simply L_2. Consider a fixed μ. Here we can draw on [G5, Theorem 1.14], which shows that P_μ^{-1} is a classical elliptic ψdo with globally estimated symbol. The details in [G5] are given for a Green operator; for a ψdo on $\widetilde{X}$, the proof is a simpler variant: Using that

$$c\|u\|^2_{L_2(E_1)} \leq \|P_\mu u\|^2_{L_2(E_2)} \leq C\|u\|^2_{L_2(E_1)}, \tag{6.11}$$

with $0<c\leq C$, one can define $B_\mu = I - C^{-1}P_\mu^* P_\mu \geq 0$ with norm $\|B_\mu\| \leq (C-c)/C = \delta < 1$. Its principal symbol $b^0(x,\xi,\mu)$ then has $|b^0(x,\xi,\mu)| \leq \delta$. (In fact, when $\chi(x) \in C_0^\infty$, the essential spectrum of $\chi B_\mu \chi$ equals the union over x and $|\xi| \geq 1$ of the spectra of $\chi(x)^2 b^0(x,\xi,\mu)$.) Now $I-B_\mu$ is elliptic and has the inverse $\sum_{k\in\mathbf{N}} B_\mu^k$ (converging in norm); it belongs to the globally estimated calculus by [G5, Theorem 1.12] (using also the localization worked out in Theorem 1.7 there). Finally,

$$P_\mu^{-1} = (I-B_\mu)^{-1} C^{-1} P_\mu^* \tag{6.12}$$

belongs to the calculus by the composition rules, the principal symbol $(p^0)^{-1}$ satisfying $|p^0(x,\xi,\mu)^{-1}| \leq c^{-1}$.

This shows that P_μ^{-1} is in the calculus with symbols in $S^0(\mathbf{R}^n \times \mathbf{R}^n) \otimes \mathcal{L}(\mathbf{C}^N)$ in admissible coordinates, for each $\mu \in \Gamma'_r$. We now study the μ-dependence. Here we use that the constants c and C can be taken independent of $\mu \in \Gamma'_r$ and the S^0-estimates for P_μ hold uniformly in μ. Then *the whole analysis of the inverse works uniformly in* $\mu \in \Gamma'_r$, so we can conclude that the S^0-estimates for P_μ^{-1} are likewise uniform in $\mu \in \Gamma'_r$. Thus the requirement for $j=0$ in (6.2) is satisfied. For derivatives ∂_z^j we use successively the formulas

$$\partial_z^j P_\mu^{-1} = -P_\mu^{-1} \sum_{l<j} \binom{j}{l} \partial_z^{j-l} P_\mu \partial_z^l P_\mu^{-1}, \quad j>0, \tag{6.13}$$

that follow from $\partial_z^j(P_\mu P_\mu^{-1}) = 0$ by the Leibniz formula; they lead to the conclusion that $\partial_z^j P_\mu^{-1}$ has symbol in S^j uniformly in $\mu \in \Gamma'_r$, and thus finally P_μ^{-1} has symbol in $S^{0,0}$. Inspection of the construction shows that strong polyhomogeneity of P_μ carries over to P_μ^{-1}. The preservation of special parameter-dependence follows by a version of (6.13) with ∂_z replaced by ∂_μ.

If $l \neq 0$, we reduce to the preceding case as follows. For any admissible vector bundle F over $\widetilde{X}$ there exists a family of isomorphisms $\Lambda_{F,\mu}^m$ from $H^{s,\mu}(F)$ to $H^{s-m,\mu}(F)$ ($m \in \mathbf{Z}$) with principal symbol essentially $\langle(\xi,\mu)\rangle^m I$ ($\Lambda_{F,\mu}^0 = I$, $\Lambda_{F,\mu}^{-m} = (\Lambda_{F,\mu}^m)^{-1}$), such that the operator norm of $\Lambda_{F,\mu}^m$ for any s is uniformly bounded in μ, for $\arg\mu$ in an interval $]\theta_1, \theta_2[$. (These order-reducing operators are a standard tool in [G2], [G3], [G5]; to get holomorphicness in μ for $|\arg\mu - \omega| < \delta$, say, one can for $m>0$ take an operator as in [G3, Corollary 3.2.12] with $\langle(\xi,\mu)\rangle$ replaced by $(|\xi|^{2m}+(e^{-i\omega}\mu)^{2m}+1)^{1/2}$ that is well-defined when $\delta\leq\pi/2m$; for $-m$ one takes the inverse.) Now we replace P_μ and P_μ^{-1} on suitable subsectors by

$$P_{1,\mu}=\Lambda^l_{E_2,\mu}P_\mu\Lambda^{-l}_{E_1,\mu},\quad P^{-1}_{1,\mu}=\Lambda^l_{E_1,\mu}P^{-1}_\mu\Lambda^{-l}_{E_2,\mu}. \tag{6.14}$$

Here $P_{1,\mu}$ and $P^{-1}_{1,\mu}$ are uniformly bounded with respect to L_2 norms. Assume e.g. that $l>0$. In view of (6.5) and (6.8), $P_\mu\Lambda^{-l}_{E_1,\mu}$ has symbol in $S^{-l,0}\cap S^{0,-l}$; subsequently $P_{1,\mu}=\Lambda^l_{E_2,\mu}P_\mu\Lambda^{-l}_{E_1,\mu}$ has symbol in

$$(S^{l,0}+S^{0,l})\circ(S^{-l,0}\cap S^{0,-l})\subset(S^{0,0}\cap S^{l,-l})+(S^{-l,l}\cap S^{0,0})\subset S^{0,0}. \tag{6.15}$$

It is seen in a similar way that the mth μ-derivative of $P_{1,\mu}$ has symbol in $S^{-m,0}\cap S^{0,-m}$. This $P_{1,\mu}$ satisfies the hypotheses with $l=0$, so the already proved part of the theorem shows that $P^{-1}_{1,\mu}$ is as asserted. We get back to P^{-1}_μ by considerations as in (6.15). This completes the proof of (1).

(2). One can reduce to the case $l=0$ in the same way as in the preceding proof. The identity $P_\mu R_\mu=I$ implies $R^*_\mu P^*_\mu=I$. Since R_μ is uniformly L_2-bounded for $\mu\in\Gamma'_r$, its adjoint R^*_μ has norm $\leq C_1$ for some fixed $C_1>0$. Insertion of $u=P^*_\mu v$ for an arbitrary $v\in L_2(E_2)$ gives

$$\|v\|^2_{L_2(E_2)}=\|R^*_\mu P^*_\mu v\|^2_{L_2(E_2)}\leq C_1^2\|P^*_\mu v\|^2_{L_2(E_1)}=C_1^2(P_\mu P^*_\mu v,v)_{L_2(E_2)}.$$

This shows that the selfadjoint operator $P_\mu P^*_\mu$ in $L_2(E_2)$ has lower bound $\geq C_1^{-2}$, so it has an inverse $(P_\mu P^*_\mu)^{-1}$ with L_2-operator norm $\leq C_1^{-2}$ for $\mu\in\Gamma'_r$. Now (1) applies to $P_\mu P^*_\mu$, since it has symbol in $S^{0,0}$ by the composition rules (cf. (6.8)). Then $(P_\mu P^*_\mu)^{-1}$ is a weakly polyhomogeneous ψdo with symbol in $S^{0,0}$, and since $P_\mu P^*_\mu(P_\mu P^*_\mu)^{-1}=I$, $R'_\mu=P^*_\mu(P_\mu P^*_\mu)^{-1}$ is a right inverse of P_μ; it is likewise a ψdo with symbol in $S^{0,0}$. Also strong polyhomogeneity and special parameter-dependence is preserved. This shows (2).

Finally, (3) follows by obvious modifications of the proof of (2). $\square$

Note that (2) does not say anything about the structure of R_μ itself. However, we shall use it in Section 8 in a situation where we can also infer that the given right inverse is a weakly polyhomogeneous ψdo.

7. Calderón projectors and their construction for resolvents

We recall, and extend to admissible manifolds, the definition and application of the Calderón projector C^+ for an elliptic differential operator $P\colon C^\infty(X,E_1)\to$

$C^\infty(X, E_2)$ of order d, as introduced by Calderón [C], Seeley [S1], [S2], see also Hörmander [H1], Boutet de Monvel [B1], Grubb [G1]. It is used in the discussion of well-posed boundary conditions for first-order operators in Sections 3–5, and a parameter-dependent version enters as a tool in the resolvent analysis in Section 8.

The manifold X is taken to be compact or, more generally, admissible as defined in [GK], [G3], see the introduction to Section 2; P is assumed to be admissible and uniformly elliptic. We can assume that X is smoothly imbedded in an n-dimensional admissible boundaryless manifold $\widetilde{X}$ such that X' is an $(n-1)$-dimensional hypersurface in $\widetilde{X}$ and E_1 and E_2 are restrictions to X of N-dimensional bundles $\widetilde{E}_1$ and $\widetilde{E}_2$ over $\widetilde{X}$; one such choice is to double up the neighborhood U (cf. Section 2) along X', augmenting X by the reflected piece U_-. In $U\cup U_-$ we write $x=(x', x_n)$, where $|x_n|<c(x')$, $c(x')\geq c>0$. In the compact case one can add another piece to $X\cup U_-$ to get a compact $\widetilde{X}$.

If P extends to a uniformly elliptic operator (also denoted P) from $C^\infty(\widetilde{E}_1)$ to $C^\infty(\widetilde{E}_2)$, we let Q denote an admissible parametrix of P on $\widetilde{X}$; then

$$PQ=I+\mathcal{T}_1,\quad QP=I+\mathcal{T}_2\quad \text{on } \widetilde{X}, \tag{7.1}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are admissible ψdos on $\widetilde{X}$ of order $-\infty$. The use of Calderón projectors is simplest if $\widetilde{X}$ and the extension P can be chosen so that P is *invertible* on $\widetilde{X}$; then Q stands for the inverse (necessarily admissible by the spectral invariance proved in [G5]), and $\mathcal{T}_1$ and $\mathcal{T}_2$ are zero.

Define $X^\circ=X_+$, $\widetilde{X}\setminus X=X_-$, $\widetilde{E}_i|_{X_\pm}=E_{i,\pm}$. The mapping $\varrho=\{\gamma_0, \dots, \gamma_{d-1}\}$ can be regarded as a mapping either from functions on $\overline{X}_+$, or from functions on $\overline{X}_-$, or from functions on $\widetilde{X}$, to functions on X'; to distinguish between the three versions, we denote them ϱ^+, ϱ^- resp. $\tilde{\varrho}$ (so $\varrho=\varrho^+$). When $F=F_0\oplus\dots\oplus F_{d-1}$ are vector bundles over X' we define

$$\mathcal{H}^s(F)=\prod_{0\leq j<d} H^{s-j-1/2}(F_j),\quad \widetilde{\mathcal{H}}^s(F)=\prod_{0\leq j<d} H^{s+j+1/2}(F_j)=(\mathcal{H}^{-s}(F))'. \tag{7.2}$$

Writing $\bigoplus_{0\leq j<d} E_i'=E_i'^d$, we have that $\varrho^\pm$ and $\tilde{\varrho}$ map the respective H^s spaces into $\mathcal{H}^s(E_i'^d)$ for $s>d-\frac{1}{2}$. The mapping $\tilde{\varrho}\colon H^s(\widetilde{E}_i)\to\mathcal{H}^s(E_i'^d)$ has the adjoint mapping $\tilde{\varrho}^*\colon \widetilde{\mathcal{H}}^{-s}(E_i'^d)\to H^{-s}(\widetilde{E}_i)$ for $s>d-\frac{1}{2}$; it ranges in distributions supported in X'. We use the notation $A_\pm$ for the truncation of a ψdo A on $\widetilde{X}$ to $X_\pm$,

$$A_\pm=r^\pm A e^\pm,\quad \text{when } A \text{ is a } \psi\text{do on } \widetilde{X}; \tag{7.3}$$

here $r^\pm$ means restriction to $X_\pm$ and $e^\pm$ means extension by zero on $X_\mp$.

Define the spaces

$$
\begin{aligned}
&Z^s_\pm = \{z \in H^s(X_\pm, E_{1,\pm}) \mid Pz = 0 \text{ on } X_\pm\},\\
(7.4)\qquad &N^s_\pm = \varrho^\pm Z^s_\pm \subset \mathcal{H}^s(E_1'^d),\\
&Z_0 = \{z \in C^\infty(\widetilde{X}, \widetilde{E}_1) \cap H^d(\widetilde{X}, \widetilde{E}_1) \mid Pz = 0,\ \operatorname{supp} z \subset X\};
\end{aligned}
$$

here Z_0 is identified with a subspace of the Z^s_+ and has finite dimension when X is compact. Although the trace operators $\varrho^\pm$ are defined on $H^s(E_{1,\pm})$ for $s > d - \frac{1}{2}$ only, the definition of the spaces $N^s_\pm$ of Cauchy data for null solutions can be extended to all $s \in \mathbf{R}$, by results in Lions and Magenes [LM] or by the arguments in [S1], [S2].

Theorem 7.1. *Consider admissible manifolds, bundles and operators, and assume that P has the inverse Q on $\widetilde{X}$. Then the spaces $N^s_\pm$ are complementing subspaces of $\mathcal{H}^s(E_1'^d)$; $\mathcal{H}^s(E_1'^d) = N^s_+ \dot{+} N^s_-$. When we define (cf. (2.1))*

$$
K^\pm = \mp r^\pm Q \tilde{\varrho}^* \mathcal{A}, \quad C^\pm = \varrho^\pm K^\pm = \mp \varrho^\pm r^\pm Q \tilde{\varrho}^* \mathcal{A}, \tag{7.5}
$$

the Poisson operators $K^\pm : \mathcal{H}^s(E_1'^d) \to H^s(E_{1,\pm})$ have range equal to $Z^s_\pm$ and provide left inverses of $\varrho^\pm$ on $Z^s_\pm$, resp.; and the ψdos $C^\pm$ (the Calderón projectors for P) are the projections of $\mathcal{H}^s(E_1'^d)$ onto $N^s_\pm$ along $N^s_\mp$, resp. In particular,

$$
C^+ + C^- = I, \quad (C^+)^2 = C^+, \quad (C^-)^2 = C^-, \quad C^+C^- = 0.
$$

Proof. The proof is a generalization of the deduction in [S1], [S2] for the invertible case with $\widetilde{X}$ compact. In fact, the proof given in [G3, Example 1.3.5] carries over *verbatim* to the present admissible manifolds, when the operators are admissible and one allows the range bundle for P to be different from the initial bundle E. To save space, we refrain from repeating the details here. $\square$

When P merely satisfies (7.1), one can still define operators $K^\pm$ by formulas as in (7.5) supplied with smoothing terms, setting

$$
C^+ = \varrho^+ K^+ = -\varrho^+ r^+ Q \tilde{\varrho}^* \mathcal{A} + \mathcal{T}_3 \tag{7.6}
$$

and $C^- = I - C^+$ (with a ψdo $\mathcal{T}_3$ of order $-\infty$); then they have the listed mapping properties only modulo smoothing operators. Such a construction is worked out in [G1] for general multi-order operators P (on compact manifolds), with applications. For compact manifolds, Seeley gives in [S2] an *optimal* construction, where K^+ maps $\mathcal{H}^s$ injectively onto a subspace of Z^s_+ with complement Z_0, and where $C^+ = \varrho^+ K^+$ *is a projection of $\mathcal{H}^s$ onto N^s_+*; we use this in Sections 3 and 4.

The book of Booss-Bavnbek and Wojciechowski [BW2] goes through the proof of Theorem 7.1 for first-order operators as in Definition 3.1(2).

The Calderón projectors are used to treat boundary value problems for P,

$$Pu = f \text{ on } X, \quad S\varrho u = \varphi \text{ on } X', \tag{7.7}$$

where S is a system of ψdos S_{jk} of order $j-k$ ($j,k=0,\dots,d-1$) going from E_1' to bundles F_j of dimension ≥ 0 over X'; $M=\sum_{0\leq j<d} \dim F_j$. In the following we consider $\{P, S\varrho\}$ as a mapping from $H^s(E_1)$ to $H^{s-d}(E_2)\times\mathcal{H}^s(F)$ for some $s>d-\frac{1}{2}$, and discuss right/left inverses that are continuous in the opposite direction; here S is considered as a mapping from $\mathcal{H}^s(E_1'^d)$ to $\mathcal{H}^s(F)$ and the $C^\pm$ act in $\mathcal{H}^s(E_1'^d)$.

Theorem 7.2. *Hypotheses and definitions as in Theorem* 7.1.

(1) *If* SC^+ *has a right inverse* S_1, *then* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ *has the right inverse*

$$\begin{pmatrix} R_S & K_S \end{pmatrix} = \begin{pmatrix} Q_+ - K^+S_1S\varrho Q_+ & K^+S_1 \end{pmatrix}. \tag{7.8}$$

Conversely, if $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ *has a right inverse* $\begin{pmatrix} R_S & K_S \end{pmatrix}$, *then* SC^+ *has the right inverse*

$$S_1 = \varrho K_S. \tag{7.9}$$

(2) *If* $\begin{pmatrix} S \\ C^- \end{pmatrix}$ *has a left inverse* $\begin{pmatrix} S_1 & S_2 \end{pmatrix}$, *then* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ *has the left inverse* (7.8).

Conversely, if $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ *has a left inverse* $\begin{pmatrix} R_S & K_S \end{pmatrix}$, *then* $\begin{pmatrix} S \\ C^- \end{pmatrix}$ *has the left inverse*

$$\begin{pmatrix} S_1 & S_2 \end{pmatrix} = \begin{pmatrix} \varrho K_S & I - \varrho K_S S \end{pmatrix}. \tag{7.10}$$

Proof. We first observe some auxiliary formulas,

$$PQ_+ = I, \quad Q_+P = I - K^+\varrho, \quad K^+C^- = 0. \tag{7.11}$$

The first formula holds since $PQ=I$ on $\widetilde{X}$ and P is local. Next, we note that Green's formula (2.1) can be written in distributional form

$$e^+r^+P\tilde{u} = Pe^+r^+\tilde{u} + \tilde{\varrho}^*(\mathcal{A}\varrho u) \quad \text{for } \tilde{u}\in H^{s+d}(\widetilde{E}_1),\ u = r^+\tilde{u},\ s > -\tfrac{1}{2}. \tag{7.12}$$

The second formula follows from this by composition with r^+Q, using (7.5) and $QP=I$; it holds on $H^{s+d}(E_1)$, $s>-\frac{1}{2}$. Now the third formula follows from a calculation using also that $\varrho K^+=C^+$, $PK^+=0$,

$$K^+C^- = K^+ - K^+C^+ = K^+ - K^+\varrho K^+ = K^+ - (I - Q_+P)K^+ = 0.$$

For the first statement, let S_1 be a right inverse of SC^+. Then, by the above rules,

$$\begin{aligned} P(Q_+ - K^+ S_1 S\varrho Q_+) &= I, \\ S\varrho(Q_+ - K^+ S_1 S\varrho Q_+) &= S\varrho Q_+ - SC^+ S_1 S\varrho Q_+ = 0, \\ PK^+ S_1 &= 0, \\ S\varrho K^+ S_1 &= SC^+ S_1 = I. \end{aligned}$$

Conversely, when $(R_S \quad K_S)$ is a right inverse of $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$, then $PK_S = 0$, $S\varrho K_S = I$, so K_S maps into Z^s_+, whereby $C^- \varrho K_S = 0$ and consequently $SC^+ \varrho K_S = S\varrho K_S - SC^- \varrho K_S = I$. Thus ϱK_S is a right inverse of SC^+. This proves (1).

For (2), we check the composition of (7.8) to the left with $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ as follows, using (7.11) and the fact that $C^- C^+ = 0$,

$$\begin{aligned} (Q_+ - K^+ S_1 S\varrho Q_+ \quad K^+ S_1) \begin{pmatrix} P \\ S\varrho \end{pmatrix} &= (I - K^+ S_1 S\varrho) Q_+ P + K^+ S_1 S\varrho \\ &= (I - K^+ S_1 S\varrho)(I - K^+ \varrho) + K^+ S_1 S\varrho \\ &= I - K^+ (I - S_1 SC^+) \varrho \\ &= I - K^+ (I - (I - S_2 C^-) C^+) \varrho \\ &= I - K^+ C^- \varrho = I. \end{aligned} \tag{7.13}$$

Conversely, define $(S_1 \quad S_2)$ by (7.10) and check its left composition with $\begin{pmatrix} S \\ C^- \end{pmatrix}$,

$$(\varrho K_S \quad I - \varrho K_S S) \begin{pmatrix} S \\ C^- \end{pmatrix} = \varrho K_S S + C^- - \varrho K_S S C^- = \varrho K_S S C^+ + I - C^+. \tag{7.14}$$

When $w = K^+ C^+ \varphi$ for some $\varphi \in C^\infty(E_1'^d)$, then $Pw = 0$, $\varrho w = C^+ C^+ \varphi = C^+ \varphi$ and $S\varrho w = SC^+ \varphi$, so since $(R_S \quad K_S)$ is a left inverse of $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$,

$$w = K_S S \varrho w = K_S S C^+ \varphi.$$

It follows that $\varrho K_S S C^+ \varphi = \varrho w = C^+ \varphi$ for $\varphi \in C^\infty(E_1'^d)$. Then the expression in (7.14) equals I. This ends the proof of (2). □

The statements have generalizations where the word "inverse" is replaced by "parametrix", also when Q is merely a parametrix of P (here one can keep track of the smoothing terms as in [G1]). Moreover, the statements hold on the principal symbol level, i.e., for the model operator $\{p^0(x',0,\xi',D_{x_n}), s^0(x',\xi')\varrho\}$ defined on $\mathbf{R}_+\subset\mathbf{R}$ from the principal symbols at a boundary point; its Calderón projectors $c^\pm(x',\xi')$ are the principal symbols of $C^\pm$. It is standard terminology to call the systems with surjectiveness, resp. injectiveness, of the model operator (for all x', all $|\xi'|=1$) *surjectively elliptic*, resp. *injectively elliptic.* It follows that

$$\begin{aligned}
&\begin{pmatrix} P \\ S\varrho \end{pmatrix} \text{ is injectively elliptic} &&\iff \begin{pmatrix} S \\ C^- \end{pmatrix} \text{ is injectively elliptic,}\\
&\begin{pmatrix} P \\ S\varrho \end{pmatrix} \text{ is surjectively elliptic} &&\iff SC^+ \text{ is surjectively elliptic.}
\end{aligned} \tag{7.15}$$

The range spaces $N_\pm(x',\xi')$ for $c^\pm(x',\xi')$ in $\mathbf{C}^{Nd}$ have dimensions $m_\pm(x',\xi')$ (with sum Nd). By (7.15), the injectively resp. surjectively elliptic problems can also be characterized by *injectiveness resp. surjectiveness of* $s^0(x',\xi')$ *from* $N_+(x',\xi')$ *to* $\mathbf{C}^{Nd}$ *for all* x', $|\xi'|=1$. In particular, this requires $M\geq m_+(x',\xi')$ resp. $M\leq m_+(x',\xi')$. Thus for two-sided elliptic problems, M must equal $m_+(x',\xi')$ (which must then be constant in (x',ξ')). It is well known that when $n\geq 3$, $m_+(x',\xi')=m_-(x',\xi')=\frac{1}{2}Nd$ (the properly elliptic case).

Note that injective ellipticity holds if and only if

$$v\in\mathbf{C}^{Nd},\ s^0(x',\xi')v=0,\ c^-(x',\xi')v=0 \implies v=0; \tag{7.16}$$

i.e., the nullspaces of s^0 and c^- are *linearly independent.*

Example 7.3. The systems $\begin{pmatrix} P \\ \varrho \end{pmatrix}$ and $\begin{pmatrix} P \\ C^+\varrho \end{pmatrix}$ are injectively elliptic; they both have the left inverse $\begin{pmatrix} Q_+ & K^+ \end{pmatrix}$ (parametrix when Q is merely a parametrix of P). In fact, by (7.11),

$$Q_+P+K^+\varrho=I,\quad Q_+P+K^+C^+\varrho=I.$$

This left inverse is also found from (7.8), when we use that $\begin{pmatrix} I \\ C^- \end{pmatrix}$ and $\begin{pmatrix} C^+ \\ C^- \end{pmatrix}$ both have the left inverse $\begin{pmatrix} C^+ & C^- \end{pmatrix}$. The case $S=C^+$ is studied in Section 4 when $d=1$.

The formulas (7.11) also show that Q_+ is a right inverse of P *without boundary condition*; i.e., in the case $F=0$. This is also confirmed by the formulas in the theorem.

Although an elliptic operator P cannot always be extended to a boundaryless manifold $\widetilde{X}\supset X$ such that the extension is invertible, we do have such a fact for the $P-\lambda$ satisfying Assumption 2.2(1); this is essential for the resolvent analysis in Section 8.

Theorem 7.4. *Let P be such that Assumption 2.2(1) is satisfied. Let $\widetilde{X}$ be an admissible boundaryless n-dimensional manifold in which X is smoothly imbedded, the bundle E being extended to an admissible bundle $\widetilde{E}$ there; take $\widetilde{X}$ compact when X is compact.*

Each ray $re^{i\theta_0}$ in Γ has a neighborhood $\Gamma'=\{\lambda=re^{i\theta}\,|\,|\theta-\theta_0|\le\varepsilon,\ r>0\}$ in Γ so that for $\lambda\in\Gamma'$, there is an extension $\widetilde{P}_\lambda$ of $P-\lambda$ to $\widetilde{E}$ (acting like $P-\lambda$ on X), which is a uniformly parameter-elliptic strongly polyhomogeneous ψdo of degree d with respect to $\mu\in\widetilde{\Gamma}'=(-\Gamma')^{1/d}$ and has a parametrix $\widetilde{Q}_\lambda$ for $\lambda\in\Gamma'$ which is an inverse for $|\lambda|\ge r'$ (some $r'\ge 0$). Then when we define

$$K_\lambda^{\pm}=\mp r^{\pm}\widetilde{Q}_\lambda\tilde{\varrho}^*\mathcal{A},\quad C_\lambda^{\pm}=\varrho^{\pm}K_\lambda^{\pm}, \tag{7.17}$$

the assertions in Theorem 7.1 hold with $Z^s_{\lambda,+}=\{z\in H^s(X,E)\,|\,(P-\lambda)z=0$ on $X\}$, $Z^s_{\lambda,-}=\{z\in H^s(X_-,\widetilde{E}|_{X_-})\,|\,\widetilde{P}_\lambda z=0$ on $X_-\}$, $N^s_{\lambda,\pm}=\varrho^{\pm}Z^s_{\lambda,\pm}$.

Here $C_\lambda^{\pm}$ is a matrix of ψdos $(C^{\pm}_{\lambda,jk})_{j,k=0,\dots,d-1}$ with $C^{\pm}_{\lambda,jk}$ strongly polyhomogeneous of order $j-k$ with respect to $\mu\in\widetilde{\Gamma}'$, and $K_\lambda^{\pm}$ is a row of Poisson operators $(K^{\pm}_{\lambda,j})_{j=0,\dots,d-1}$ with $K^{\pm}_{\lambda,j}$ strongly polyhomogeneous of order $-j$ (all belonging to the global calculus).

Proof. We here use ideas from [S2], in particular from the appendix there. Define $\Gamma_{(\alpha)}=\{re^{i\theta}\,|\,r>0,\ |\theta|\le\alpha\}$. Consider a ray $re^{i\theta_0}$ in Γ; multiplying $P-\lambda$ by a complex constant we can obtain that $\theta_0=\pi$ and that $\Gamma_{(\delta)}\subset-\Gamma$ for some $\delta>0$. Then for $\varepsilon\le\frac{1}{2}\delta$,

$$\begin{aligned}-\lambda\in\Gamma_{(\varepsilon)},\ -\tau\in\Gamma_{(\varepsilon)}\quad&\Longrightarrow\quad |\xi|^{2d}+\lambda^2\in\Gamma_{(2\varepsilon)}\text{ and }-\lambda-\tau(|\xi|^{2d}+\lambda^2)^{1/2}\in\Gamma_{(2\varepsilon)}\\&\Longrightarrow\quad p(x,\xi)-\lambda-\tau(|\xi|^{2d}+\lambda^2)^{1/2}\text{ is invertible.}\end{aligned}$$

We can then, for $\lambda\in\Gamma'=-\Gamma_{(\varepsilon)}$ and $|\xi|^{2d}+|\lambda|^2\ge 1$, define a homotopy of $p^0-\lambda I$ to the symbol $\mathfrak{p}(\xi,\lambda)=(|\xi|^{2d}+\lambda^2)^{1/2}I$. Let $\mathcal{C}$ be a curve in $(-\Gamma_{(\varepsilon)}\cup\{|\tau|\le 1\})\setminus\overline{\mathbf{R}}_-$ encircling the eigenvalues of $\mathfrak{p}(\xi,\lambda)^{-1}(p^0(x,\xi)-\lambda^d I)$, and set

$$\tilde{p}^0(x,\xi,\lambda,\theta)=\mathfrak{p}(\xi,\lambda)\frac{i}{2\pi}\int_{\mathcal{C}}\lambda^\theta[\mathfrak{p}(\xi,\lambda)^{-1}(p^0(x,\xi)-\lambda I)-\tau I]^{-1}\,d\tau, \tag{7.18}$$

(note that λ^θ is well-defined on $\mathcal{C}$). Here $\tilde{p}^0(x,\xi,\lambda,\theta)$ equals $\mathfrak{p}(\xi,\lambda)I$ for $\theta=0$ and equals $p^0(x,\xi)-\lambda I$ for $\theta=1$, and it is homogeneous of degree d in $(\xi,|\lambda|^{1/d})$, holomorphic in λ, C^∞, and invertible for all $\theta\in[0,1]$, all $|\xi|^{2d}+|\lambda|^2\ge 1$ with $\lambda\in-\Gamma_{(\varepsilon)}$.

We can assume that $\widetilde{X}$ contains the neighborhood $U\cup U_-$ of X' (see the beginning of this section), where we can identify $\widetilde{E}$ with the pull-back of E'. In view of the uniform parameter-ellipticity, there is a neighborhood V of X with

$X\cup(X'\times[-c,0])\subset\overline{V}\subset X\cup U_-$ so that P extends to V as an admissible differential operator satisfying Assumption 2.2(1). Moreover, we can deform the symbol $p^0(x,\xi)-\lambda$ smoothly through uniformly parameter-elliptic ψdo symbols homogeneous in $(\xi,|\lambda|^{1/d})$ to $\mathfrak{p}(\xi,\lambda)I$ by use of (7.18) when x_n goes from $-\frac{1}{3}c$ to $-\frac{2}{3}c$, and then extend it as $\mathfrak{p}(\xi,\lambda)I$ on the rest of $\widetilde{X}$. This gives a principal symbol $p_1^0(x,\xi,\lambda)$ defined on all of $\widetilde{X}$, defining a uniformly parameter-elliptic ψdo $\widetilde{P}_{1,\lambda}$ of order d; it is strongly polyhomogeneous for $\mu\in\widetilde{\Gamma}'$. Now take

$$\widetilde{P}_\lambda=\varphi(P-\lambda I)\varphi+\psi\widetilde{P}_{1,\lambda}\psi, \tag{7.19}$$

where φ and ψ are admissible (bounded with bounded derivatives) C^∞ functions on $\widetilde{X}$ with $\varphi^2+\psi^2=1$, such that φ is 1 on $X\cup\big(X'\times\big[-\frac{1}{9}c,0\big]\big)$ and ψ is 1 on the complement of $X\cup\big(X'\times\big[-\frac{2}{9},0\big]\big)$. This $\widetilde{P}_\lambda$ is a uniformly parameter-elliptic and strongly polyhomogeneous ψdo of order d that acts like $P-\lambda$ on distributions supported in a neighborhood of X. The operator $\widetilde{P}_{\lambda,+}$ has the same Green's formula as P, (2.1).

The operator $\widetilde{P}_\lambda$ has a parametrix $\widetilde{Q}'_\lambda$ for $\lambda\in-\Gamma_{(\varepsilon)}$, uniformly parameter-elliptic and strongly polyhomogeneous of order $-d$, by the usual formulas. Since $\widetilde{P}_\lambda\widetilde{Q}'_\lambda=I+\mathcal{S}_\lambda$, where $\mathcal{S}_\lambda$ is strongly polyhomogeneous of order -1, hence has an L_2 operator norm going to 0 for $|\lambda|\to\infty$ in $-\Gamma_{(\varepsilon)}$, $I+\mathcal{S}_\lambda$ can be inverted within the calculus (by a Neumann series) for sufficiently large λ; here $\widetilde{Q}'_\lambda$ can be modified to the true inverse $\widetilde{Q}_\lambda=\widetilde{Q}'_\lambda(I+\mathcal{S}_\lambda)^{-1}$. This is strongly polyhomogeneous with global spatial estimates, by Theorem 6.5.

We now simply define $K^\pm_\lambda$ and $C^\pm_\lambda$ by (7.17); then the verification that they have the mentioned mapping properties goes exactly as in Theorem 7.1. The resulting operators are strongly polyhomogeneous by [GS1, Lemma A.1, Theorem 1.16] and have global spatial estimates since $\widetilde{Q}_\lambda$ and $\mathcal{A}$ do so. □

For use later in Corollary 8.3 let us also note that $\varrho\widetilde{Q}_{\lambda,+}$ (as a function of $\mu=(-\lambda)^{1/d}\in\widetilde{\Gamma}'$) is a strongly polyhomogeneous trace operator of class 0, cf. [GS1, Lemma A.1(ii)].

Let us finally observe the following result on adjoints.

Theorem 7.5. *Under the assumptions of Theorem* 7.1, *denote by C'^+ the Calderón projector for P^* (defined according to Theorem* 7.1 *with Q replaced by Q^*). Then*

$$C'^+=(\mathcal{A}^*)^{-1}(I-C^{+*})\mathcal{A}^*. \tag{7.20}$$

Proof. The operator P^* has a Green's formula similar to (2.1) with $\mathcal{A}$ replaced by $-\mathcal{A}^*$, so the Calderón projector C'^+ and associated Poisson operator K'^+ for

P^* are $K'^+ = r^+ Q^* \tilde{\varrho}^* \mathcal{A}^*$, $C'^+ = \varrho r^+ Q^* \tilde{\varrho}^* \mathcal{A}^*$. Let K_ϱ be a Poisson operator lifting sections $\varphi \in \mathcal{H}^d(E_1'^d)$ to sections $u = K_\varrho \varphi \in H^d(E_1)$ such that $\varrho u = \varphi$, cf. e.g. [G3, Lemma 1.6.4] or the text before Lemma 2.3 above. Then (7.11) gives by application of ϱ,

$$K^+ \varrho u = u - Q_+ Pu, \quad C^+ \varphi = \varrho K^+ \varrho u = \varrho u - \varrho Q_+ Pu = \varphi - \varrho Q_+ P K_\varrho \varphi. \tag{7.21}$$

For the term $\varrho Q_+ Pu$ we note that when $\psi \in \widetilde{\mathcal{H}}^0(E_1'^d)$,

$$\begin{aligned}(\varrho Q_+ Pu, \psi)_{X'} &= (\tilde{\varrho} Q e^+ Pu, \psi)_{X'} = (e^+ Pu, Q^* \tilde{\varrho}^* \psi)_{\widetilde{X}} \\ &= (Pu, r^+ Q^* \tilde{\varrho}^* \psi)_X = (Pu, K'^+ (\mathcal{A}^*)^{-1} \psi)_X \\ &= (Pu, K'^+ (\mathcal{A}^*)^{-1} \psi)_X - (u, P^* K'^+ (\mathcal{A}^*)^{-1} \psi)_X \\ &= (\varphi, \mathcal{A}^* C'^+ (\mathcal{A}^*)^{-1} \psi)_{X'}.\end{aligned}$$

It is used here that $Qe^+ Pu \in H^d(\widetilde{E}_1)$ so that $\tilde{\varrho}$ and ϱr^+ give the same result, and that $P^* K'^+ = 0$. Taking this together with (7.21), we find

$$(C^+ \varphi, \psi)_{X'} = (\varphi, \psi) - (\varphi, \mathcal{A}^* C'^+ (\mathcal{A}^*)^{-1} \psi)_{X'} \quad \text{for all } \varphi,\ \psi;$$

this implies (7.20). $\square$

For systems without the invertibility assumption there are similar formulas with smoothing terms. For first order systems, the orthogonalized Calderón projector for P^* was investigated earlier by Booss and Wojciechowski in [BW1] (see also Example 5.1 above), playing an essential role in their analysis of the index.

8. Analysis of the resolvent

Consider P_S as defined in Section 2; in particular it can be equal to $\mathcal{D}_\mathcal{B}$ as introduced in Section 5. We shall find a constructive expression of its resolvent in a form that allows showing asymptotic expansions of traces.

The strategy in [GS1] for characterizing the resolvent $(\Delta_1 + \mu^2)^{-1}$ associated with a Dirac-type problem with a boundary condition $(\Pi_\geq + B_0)\gamma_0 u = 0$ was essentially to express the general resolvent as a suitable perturbation of the product case resolvent, by a term that is of lower order at the boundary. When P is not of Dirac-type, we do not have a simpler reference problem (like the product case) to depart from, so a new strategy is needed. Here we establish the analysis directly by use of a Calderón projector for $P - \lambda$.

Consider a system $\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}$ satisfying Assumptions 2.1 and 2.2. By Lemma 2.3, it is surjective from $H^d(E)$ to $L_2(E) \times \mathcal{H}^d(F)$ for each large $\lambda \in \Gamma$. For suitably small subsectors Γ' of Γ (covering Γ) we can define the Calderón projector C_λ^+ by Theorem 7.4.

Lemma 8.1. *Let $\lambda \in \Gamma'_r$ (with Γ' as in Theorem 7.4 and r so large that $\widetilde{Q}_\lambda = \widetilde{P}_\lambda^{-1}$ and Assumption 2.2 is satisfied). Then SC_λ^+ has the following right inverse, where K_λ is defined by Lemma 2.3,*

$$S'_\lambda = \varrho K_\lambda; \tag{8.1}$$

it is a ψdo mapping $\mathcal{H}^{s,\mu}(F)$ into $\mathcal{H}^{s,\mu}(E'^d)$ with norm uniformly bounded in $\mu = |\lambda|^{1/d}$ for any $s \geq d$.

Proof. By the converse part of Theorem 7.2(1), (8.1) is a right inverse of SC_λ^+. The mapping property follows from the second statement in (2.10) by composition with ϱ. □

We would like to use Theorem 6.5 to show that S'_λ is weakly polyhomogeneous in terms of $\mu = (-\lambda)^{1/d}$. One difficulty in this is that S'_λ is just a right inverse of SC_λ^+, not a two-sided inverse (and such right inverses are not uniquely determined). Another difficulty is that S and C_λ^+ are multi-order systems.

To eliminate the effects of the multi-order, we conjugate the operators with $\Theta_{F,\lambda} = (\delta_{jk} \Lambda_{F_j,\mu}^{d-j-1})_{j,k=0,\dots,d-1}$ and $\Theta_{E'^d,\lambda} = (\delta_{jk} \Lambda_{E',\mu}^{d-j-1})_{j,k=0,\dots,d-1}$ (in each subsector Γ'_r); the entries are defined as in the proof of Theorem 6.5. The following operators are of order 0,

$$\widetilde{S}_\lambda = \Theta_{F,\lambda} S \Theta_{E'^d,\lambda}^{-1}, \quad \widetilde{C}_\lambda^+ = \Theta_{E'^d,\lambda} C_\lambda^+ \Theta_{E'^d,\lambda}^{-1}. \tag{8.2}$$

Since C_λ^+ and the Θ_λ are strongly polyhomogeneous, so is $\widetilde{C}_\lambda^+$. Then by the remark after Definition 6.3, $\widetilde{C}_\lambda^+$ is special parameter-dependent of order 0. For $\widetilde{S}_\lambda$ it follows from the lower triangular form of S that $\widetilde{S}_\lambda$ is again lower triangular. The entries in and below the diagonal are of the form $\Lambda_{F_j,\mu}^{d-1-j} S_{jk} \Lambda_{E',\mu}^{k+1-d}$ with $j \geq k$ and thus, since $S_{jk} \in S^{j-k} \subset S^{j-k,0}$, they are seen to have symbols in $S^{0,0}$ with μ-derivatives of order m in $S^{-m,0} \cap S^{0,-m}$ for any m, by calculations as around (6.15). (For $k < j < d-1$ one needs the observation that $S^{j-k,k-j} \cap S^{j+1-d,d-1-j} \subset S^{0,0}$ by interpolation since $j-k>0$, $j+1-d<0$.) Thus $\widetilde{S}_\lambda$ is special parameter-dependent of order 0. We also define

$$\widetilde{S}'_\lambda = \Theta_{E'^d,\lambda} S'_\lambda \Theta_{F,\lambda}^{-1}. \tag{8.3}$$

Theorem 8.2. *Let P and S satisfy Assumptions 2.1 and 2.2. For λ in truncated subsectors Γ'_r of Γ (as in Lemma 8.1), the operator SC_λ^+ has a right inverse $S''_\lambda = \Theta_{E'^d,\lambda}^{-1} \widetilde{S}''_\lambda \Theta_{F,\lambda}$ where $\widetilde{S}''_\lambda$ is special parameter-dependent of order 0 (in terms of $\mu = (-\lambda)^{1/d}$).*

The right inverse S'_λ defined in Lemma 8.1 *equals $C^+_\lambda S''_\lambda$, and $\widetilde{S}'_\lambda$ defined by* (8.3) *is special parameter-dependent of order* 0.

Proof. The operator $\widetilde{S}_\lambda \widetilde{C}^+_\lambda$ is continuous from $H^{t,\mu}(E'^d)$ to $H^{t,\mu}(F)$ for any t. It has the right inverse $\widetilde{S}'_\lambda$, which is continuous from $H^{t,\mu}(F)$ to $H^{t,\mu}(E'^d)$, uniformly in μ, for $t \geq \frac{1}{2}$, in view of (8.1), (2.10) and the mapping properties of the $\Lambda^l_{F_j,\mu}$. In particular, the continuity holds with $t=1$. We can then apply Theorem 6.5(2) with $l=1$, which shows the existence of a right inverse $\widetilde{S}''_\lambda$ that is special parameter-dependent of order 0.

The right inverse we have constructed in this way need not be the same as $\widetilde{S}'_\lambda$ defined after Lemma 8.1 in (8.3). However, since $\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}$ is *bijective*, we infer from the converse parts of (1) and (2) in Theorem 7.2 that $\begin{pmatrix} S \\ C^-_\lambda \end{pmatrix}$ is injective and SC^+_λ is surjective, hence S defines a *bijection* of $N^s_{\lambda,+}$ onto $\mathcal{H}^s(F)$, and so does SC^+_λ. Then SC^+_λ has only one right inverse ranging in $N^s_{\lambda,+}$. Now S'_λ in (8.1) does map into $N^s_{\lambda,+}$ since $(P-\lambda)K_\lambda=0$, so it is *the* right inverse of SC^+_λ ranging in $N^s_{\lambda,+}$. When S'''_λ is an arbitrary right inverse, then $I=SC^+_\lambda S'''_\lambda = SC^+_\lambda C^+_\lambda S'''_\lambda$, so $C^+_\lambda S'''_\lambda$ is a right inverse ranging in $N_{\lambda,+}$; hence it must equal S'_λ. In particular, for the right inverse S''_λ found above,

$$S'_\lambda = C^+_\lambda S''_\lambda. \tag{8.4}$$

It then follows from the rules of calculus that also $\widetilde{S}'_\lambda = \Theta_{E'^d,\lambda} S'_\lambda \Theta^{-1}_{F,\lambda} = \widetilde{C}^+_\lambda \widetilde{S}''_\lambda$ is a special parameter-dependent ψdo of order 0. □

Since $\widetilde{Q}_\lambda$ is the inverse of $\widetilde{P}_\lambda$, we can now apply the direct part of Theorem 7.2(1) to describe the inverse of $\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}$. This gives an immediate corollary.

Corollary 8.3. *For λ in truncated subsectors Γ'_r of Γ (as in Lemma* 8.1*), the resolvent $R_\lambda = (P_S - \lambda)^{-1}$ and the Poisson solution operator K_λ in* (2.6) *satisfy*

$$R_\lambda = \widetilde{Q}_{\lambda,+} - G_\lambda \text{ with } G_\lambda = K^+_\lambda S'_\lambda S \varrho \widetilde{Q}_{\lambda,+}, \quad K_\lambda = K^+_\lambda S'_\lambda, \tag{8.5}$$

where S'_λ is as in Theorem 8.2.

In terms of $\mu = (-\lambda)^{1/d}$, K^+_λ resp. $\varrho \widetilde{Q}_{\lambda,+}$ are a strongly polyhomogeneous Poisson resp. trace operator, and $\Theta_{E'^d,\lambda} S'_\lambda \Theta^{-1}_{F,\lambda}$ and $\Theta_{E'^d,\lambda} S'_\lambda S \Theta^{-1}_{F,\lambda}$ are special parameter-dependent ψdos of order 0. *In particular, we can write*

(8.6)
$$G_\lambda = \mathcal{K}_\lambda \mathcal{S}_\lambda \mathcal{T}_\lambda \quad \text{with } \mathcal{K}_\lambda = K^+_\lambda \Theta^{-1}_{E'^d,\lambda},\ \mathcal{S}_\lambda = \Theta_{E'^d,\lambda} S'_\lambda S \Theta^{-1}_{E'^d,\lambda},\ \mathcal{T}_\lambda = \Theta_{E'^d,\lambda} \varrho \widetilde{Q}_{\lambda,+},$$

where $\mathcal{K}_\lambda$ is a strongly polyhomogeneous Poisson operator of order $1-d$, $\mathcal{S}_\lambda$ is a special parameter-dependent ψdo on X' of order 0, *and $\mathcal{T}_\lambda$ is a strongly polyhomogeneous trace operator of order -1 and class* 0.

Here S'_λ and $S'_\lambda S$ are covered by the analysis in Theorem 8.2, whereas K^+_λ and $\varrho\widetilde{Q}_{\lambda,+}$ were described in Theorem 7.4ff.

9. Trace formulas

We can finally obtain trace formulas, by the methods of [GS1].

Theorem 9.1. *Let P_S be the realization* (2.3) *defined from a differential operator P of order d in a bundle E over a manifold X together with a boundary condition* (2.2) (*all admissible*), *such that Assumptions* 2.1 *and* 2.2 *are satisfied. When $(m+1)d>n=\dim X$, the resolvent $R_\lambda=(P_S-\lambda)^{-1}$ satisfies for any compactly supported morphism φ in E,*

$$\begin{aligned}\operatorname{Tr}(\varphi\partial_\lambda^m(P_S-\lambda)^{-1})\sim a_0(-\lambda)^{n/d-m-1}+\sum_{j=1}^{\infty}(a_j+b_j)(-\lambda)^{(n-j)/d-m-1}\\ +\sum_{k=0}^{\infty}(c_k\log(-\lambda)+c'_k)(-\lambda)^{-k/d-m-1}\end{aligned}\tag{9.1}$$

for $\lambda\to\infty$ in closed subsectors of Γ. The coefficients a_j, b_j and c_k are integrals, $\int_{X_1}a_j(x)\,dx$, $\int_{X'_1}b_j(x')\,dx'$ and $\int_{X'_1}c_k(x')\,dx'$, of densities a_j locally determined by the symbols of P, resp. b_j and c_k locally determined by the symbols of P and S at X'; here X_1 is a smooth compact neighborhood of $\operatorname{supp}\varphi$ in X such that $X'_1=X_1\cap X'$ is a neighborhood of $\operatorname{supp}\varphi\cap X'$ in X'. The c'_k are in general globally determined.

Proof. The operator $\varphi\partial_\lambda^m R_\lambda$ is trace class, since it maps the space $L_2(E)$ into $H^{(m+1)d}(E|_{X_1})$ and the injection $H^{(m+1)d}(E|_{X_1})\hookrightarrow L_2(E|_{X_1})$ is trace class. The kernel is continuous and the trace is the integral of the fiber trace of the kernel on the diagonal, so one only has to integrate over X_1. Consider a truncated subsector Γ'_r as in Lemma 8.1. From Corollary 8.3 follows that

$$\begin{aligned}\partial_\lambda^m R_\lambda&=\partial_\lambda^m(P_S-\lambda)^{-1}=m!(P_S-\lambda)^{-m-1}=m!(\widetilde{Q}_{\lambda,+}-G_\lambda)^{m+1}\\ &=m!(\widetilde{Q}_{\lambda,+})^{m+1}+\sum_{k=1}^{m+1}\operatorname{pol}_k(\widetilde{Q}_{\lambda,+},G_\lambda)\\ &=m!(\widetilde{Q}_\lambda^{m+1})_+ +\widetilde{G}_\lambda+\sum_{k=1}^{m+1}\operatorname{pol}_k(\widetilde{Q}_{\lambda,+},G_\lambda),\end{aligned}\tag{9.2}$$

where the expressions pol_k are "polynomials" in the two (non-commuting) terms in R_λ, in the sense that they are linear combinations of compositions with $m-k$

factors $\widetilde{Q}_{\lambda,+}$ and k factors G_λ. The term $\widetilde{G}_\lambda$ is the singular Green operator (cf. e.g. [G3, (1.2.35)])

$$\widetilde{G}_\lambda = m!((\widetilde{Q}_{\lambda,+})^{m+1} - (\widetilde{Q}_\lambda^{m+1})_+). \tag{9.3}$$

In the dependence on $\mu = (-\lambda)^{1/d}$, we have in view of the rules of calculus of [GS1], [G3] that $\widetilde{Q}_\lambda^{m+1}$ is a strongly polyhomogeneous ψdo of order $-(m+1)d$ on $\widetilde{X}$, $\widetilde{G}_\lambda$ is a strongly polyhomogeneous singular Green operator of order $-(m+1)d$ on X, and the sum over k is a sum of compositions containing strongly polyhomogeneous operators (of all types) together with the special parameter-dependent ψdo $\mathcal{S}_\lambda$, cf. (8.6).

Consider the trace

$$\operatorname{Tr}_X \varphi\partial_\lambda^m R_\lambda = \operatorname{Tr}_X \varphi m!(\widetilde{Q}_\lambda^{m+1})_+ + \operatorname{Tr}_X \varphi\Big(\widetilde{G}_\lambda + \sum_{k=1}^{m+1} \operatorname{pol}_k(\widetilde{Q}_{\lambda,+}, G_\lambda)\Big).$$

By the construction of $\widetilde{P}_\lambda$ in Theorem 7.4, the restriction $(\widetilde{Q}_\lambda^{m+1})_+$ of $\widetilde{Q}_\lambda^{m+1}$ is the restriction of a strongly polyhomogeneous parametrix of $(P-\lambda)^{m+1}$ defined on a neighborhood of X, so $\operatorname{Tr}_X \varphi m!(\widetilde{Q}_\lambda^{m+1})_+$ contributes a well-known expansion $\sum_{j=0}^\infty a_j(-\lambda)^{(n-j)/d-m-1}$.

The singular Green operator $\varphi\widetilde{G}_\lambda$ is of order $-(m+1)d$ and strongly polyhomogeneous, hence of regularity $+\infty$ in the sense of [G3], so it contributes an expansion $\sum_{j=1}^\infty b_{0,j}(-\lambda)^{(n-j)/d-m-1}$, by the proof of [G3, Theorem 3.3.10ff.], also recalled in [G4, Appendix].

In view of (8.6), the terms in the polynomials pol_k contain $\mathcal{S}_\lambda$ as one or several factors. Here we use the invariance of the trace under cyclic permutation of the operators, to reduce to the study of an operator on X'. Since $\widetilde{Q}_{\lambda,+}$ composes with strongly polyhomogeneous Poisson and trace operators to give Poisson resp. trace operators that are again strongly polyhomogeneous, each term in pol_k has the structure

$$\mathcal{G}_\lambda = \varphi\mathcal{K}_{1,\lambda}\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda}, \tag{9.4}$$

with $\mathcal{G}_\lambda$ of total order $-(m+1)d$ and the $\mathcal{K}_{j,\lambda}$ and $\mathcal{T}_{j,\lambda}$ strongly polyhomogeneous Poisson and trace operators of order ≤ 0 and class 0. Let ψ denote a morphism over X' that is the identity over a neighborhood of $\operatorname{supp}\varphi\cap X'$ and is supported in X_1'; then $\varphi\mathcal{K}_{1,\lambda}(I-\psi)$ is strongly polyhomogeneous of order $-\infty$, so its norm in Sobolev spaces is $O(\langle\lambda\rangle^{-M})$, any M, and $\operatorname{Tr}_X \varphi\mathcal{K}_{1,\lambda}(I-\psi)\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda}$ is $O(\langle\lambda\rangle^{-M})$, any M. For the remaining part,

$$\begin{aligned} &\operatorname{Tr}_X \varphi\mathcal{K}_{1,\lambda}\psi\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda} = \operatorname{Tr}_{X'} \mathcal{S}_\lambda' \\ &\text{with}\quad \mathcal{S}_\lambda' = \psi\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda}\varphi\mathcal{K}_{1,\lambda}; \end{aligned} \tag{9.5}$$

here the factors $\mathcal{T}_{j,\lambda}\mathcal{K}_{j+1,\lambda}$ and $\mathcal{T}_{J,\lambda}\varphi\mathcal{K}_{1,\lambda}$ are strongly polyhomogeneous ψdos on X' of orders ≤ 0. It follows that the ψdo $\mathcal{S}'_\lambda$ is a special parameter-dependent ψdo of order $-(m+1)d$. We can now apply [GS1, Theorem 2.1] to this by integration over X'_1, using a reduction to local trivializations and a partition of unity. Since X' has dimension $n-1$ and the symbol has degrees $-(m+1)d-k$, $k\ge 0$, and μ-exponent $-(m+1)d$, we get an expansion in a series of locally determined terms $\tilde b_k(-\lambda)^{(n-k)/d-m-1}$, $k\ge 1$, together with a series of terms $(\tilde c_k \log(-\lambda)+\tilde c'_k)(-\lambda)^{k/d-m-1}$, $k\ge 0$, with $\tilde c_k$ locally determined.

Collecting all the contributions, we find (9.1). $\square$

We have as an immediate consequence.

Corollary 9.2. *When J in Assumption 2.2 contains $\left[\frac12\pi, \frac32\pi\right]$ in the interior, and R_λ exists on W (cf. Section 1), then the heat operator e^{-tP_S} has an expansion for $t\to 0$, when φ has compact support,*

$$\operatorname{Tr}(\varphi e^{-tP_S}) \sim \bar a_0 t^{-n/d} + \sum_{j\ge 1} (\bar a_j + \bar b_j) t^{(j-n)/d} + \sum_{k\ge 0} (\bar c_k \log t + \bar c'_k) t^{k/d}; \tag{9.6}$$

here the coefficients are proportional to those in (9.1) by universal factors.

Proof. The expansion (9.6) is shown by inserting in (1.4) sums of terms from (9.1) down to a certain order plus a remainder $O(\langle\lambda\rangle^{-N})$, and letting $N\to\infty$. Here one uses simple calculations such as

$$\begin{aligned}\int_{\partial W} e^{-t\lambda}(-\lambda)^s \log(-\lambda)\, d\lambda &= -\frac{d}{ds}\int_{\partial W} e^{-t\lambda}(-\lambda)^s\, d\lambda \\ &= -\frac{d}{ds} t^{-s-1}\int_{t\partial W} e^{-\varrho}(-\varrho)^s\, d\varrho \\ &= \text{const.}\, t^{-s-1}\log t. \quad \square\end{aligned} \tag{9.7}$$

Theorem 9.1 holds in particular for $(\mathcal{D}_{\mathcal{B}}+\mu)^{-1}$, giving expansions of the form

$$\operatorname{Tr}(\varphi\partial_\mu^m(\mathcal{D}_{\mathcal{B}}+\mu)^{-1}) \sim \sum_{j=0}^{n-1} c_{j-n}\mu^{n-j-m-1} + \sum_{k\ge 0}(c_k\log\mu + c'_k)\mu^{-k-m-1} \tag{9.8}$$

for $\mu\to\infty$ in closed subsectors of Γ_0. We apply this to (5.1) by use of (5.3) as in [GS1, Section 3.4], taking $\varphi=(\varphi_{kl})_{k,l=1,2}$ with just one block different from zero in order to get the traces of the individual blocks in (5.3), and setting $\lambda=-\mu^2$. This gives trace expansions of the mth derivatives of $\varphi(\Delta_i-\lambda)^{-1}$ $(i=1,2)$, $\psi D_B(\Delta_1-\lambda)^{-1}$ and $\psi D_B^*(\Delta_2-\lambda)^{-1}$, with consequences for heat trace expansions.

Theorem 9.3. *Let D_B be the realization of a first-order uniformly elliptic differential operator D from E_1 to E_2 with a uniformly well-posed boundary condition $B\gamma_0 u=0$ (manifolds, bundles and operators being admissible). Then when φ and ψ are compactly supported morphisms (in E_i resp. from E_j to E_i, $i,j=1,2$), there are resolvent trace expansions in closed truncated subsectors of $\mathbf{C}\setminus\overline{\mathbf{R}}_+$, for $m\geq n$,*

$$
\begin{aligned}
\operatorname{Tr}(\varphi\partial_\lambda^m(\Delta_i-\lambda)^{-1}) &\sim \sum_{j=0}^{n-1} \tilde a_{i,j-n}(-\lambda)^{(n-j)/2-m-1} \\
&\quad + \sum_{k\geq 0}(\tilde a_{i,k}\log(-\lambda)+\tilde a'_{i,k})(-\lambda)^{-k/2-m-1}, \\
\operatorname{Tr}(\psi D_B\partial_\lambda^m(\Delta_1-\lambda)^{-1}) &\sim \sum_{j=1}^{n-1} \tilde b_{1,j-n}(-\lambda)^{(n-j+1)/2-m-1} \\
&\quad + \sum_{k\geq 0}(\tilde b_{1,k}\log(-\lambda)+\tilde b'_{1,k})(-\lambda)^{(-k+1)/2-m-1},
\end{aligned}
\tag{9.9}
$$

with a similar formula for $\operatorname{Tr}(\psi D_B^*\partial_\lambda^m(\Delta_2-\lambda)^{-1})$ with coefficients $\tilde b_{2,k}$ and $\tilde b'_{2,k}$. If $\mathcal{D}_{\mathcal{B}}$ is bijective (so $\Delta_i>0$), or X is compact, there are heat trace expansions for $t\to 0+$,

$$
\begin{aligned}
\operatorname{Tr}(\varphi e^{-t\Delta_i}) &\sim \sum_{j=0}^{n-1} a_{i,j-n}t^{(j-n)/2}+\sum_{k\geq 0}(a_{i,k}\log t+a'_{i,k})t^{k/2}, \quad i=1,2; \\
\operatorname{Tr}(\psi D_B e^{-t\Delta_1}) &\sim \sum_{j=1}^{n-1} b_{1,j-n}t^{(j-n-1)/2}+\sum_{k\geq 0}(b_{1,k}\log t+b'_{1,k})t^{(k-1)/2},
\end{aligned}
\tag{9.10}
$$

with a similar formula for $\operatorname{Tr}(\psi D_B^* e^{-t\Delta_2})$ with coefficients $b_{2,k}$ and $b'_{2,k}$. The coefficients in (9.10) are proportional to those in (9.9) by universal factors. The unprimed coefficients are locally determined; the primed coefficients depend on the operators in a global way.

The terms $\tilde b_{i,-n}(-\lambda)^{1/2-m-1}$ and $b_{i,-n}t^{(n+1)/2}$ have been left out, since their coefficients are formed by integration in ξ of functions that are odd in ξ, which gives zero. When the $\Delta_i>0$, (1.4) is used to get (9.10). When X is compact, the resolvent has a pole at 0 when $\ker\mathcal{D}_{\mathcal{B}}\neq 0$, and we use [GS2, Corollary 2.10, Theorem 5.3] as in [GS1]. Then one also gets zeta expansions, with the same $a_{i,k}$, $a'_{i,k}$, $b_{i,k}$ and $b'_{i,k}$, (9.11)

$$\Gamma(s)\operatorname{Tr}(\varphi\Delta_i^{-s})\sim\sum_{j=0}^{n-1}\frac{a_{i,j-n}}{s+\frac12(j-n)}-\frac{\operatorname{Tr}\varphi\Pi_0(D_B)}{s}+\sum_{k\ge0}\Big(\frac{-a_{i,k}}{(s+\frac12k)^2}+\frac{a'_{i,k}}{s+\frac12k}\Big),$$
$$\Gamma(s)\operatorname{Tr}(\psi D_B\Delta_1^{-s})\sim\sum_{j=1}^{n-1}\frac{b_{1,j-n}}{s+\frac12(j-n-1)}+\sum_{k\ge0}\Big(\frac{-b_{1,k}}{(s+\frac12(k-1))^2}+\frac{b'_{1,k}}{s+\frac12(k-1)}\Big),$$

with a similar formula for $\operatorname{Tr}(\psi D_B^*\Delta_2^{-s})$ with coefficients $b_{2,k}$ and $b'_{2,k}$. (The left-hand side is meromorphic on $\mathbf{C}$ and the right-hand side gives the full pole structure.)

The results apply of course to all the cases presented in the examples in Section 4.

For comparison with earlier results it is of interest to see how the expansions vary under perturbations of B. Let us consider two choices B_1 and B_2 of B, setting $B'=B_2-B_1$. Let $\mathcal{B}_i=(\,B_i \quad (I-B_i^*)\sigma^*\,)$, for $i=1,2$; $\mathcal{B}'=\mathcal{B}_2-\mathcal{B}_1$. Let $(\,\mathcal{R}_{i,\mu} \quad \mathcal{K}_{i,\mu}\,)$ be the inverse of $\begin{pmatrix}\mathcal{D}+\mu\\ \mathcal{B}_i\gamma_0\end{pmatrix}$ for $\mu\in\mathbf{C}\setminus\mathrm{i}\,\mathbf{R}$. Then

$$\begin{aligned}(\,\mathcal{R}_{2,\mu}\quad\mathcal{K}_{2,\mu}\,)&=(\,\mathcal{R}_{1,\mu}\quad\mathcal{K}_{1,\mu}\,)\begin{pmatrix}\mathcal{D}+\mu\\ \mathcal{B}_1\gamma_0\end{pmatrix}(\,\mathcal{R}_{2,\mu}\quad\mathcal{K}_{2,\mu}\,)\\&=(\,\mathcal{R}_{1,\mu}\quad\mathcal{K}_{1,\mu}\,)\begin{pmatrix}I&0\\-\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu}&I-\mathcal{B}'\gamma_0\mathcal{K}_{2,\mu}\end{pmatrix},\end{aligned}$$

which implies that

$$\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu}=-\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu},\quad \mathcal{K}_{2,\mu}-\mathcal{K}_{1,\mu}=-\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{K}_{2,\mu}. \tag{9.12}$$

Theorem 9.4. *In the above notation, when B' is a ψdo of order -1,*

$$\begin{aligned}&\operatorname{Tr}\varphi\partial_\mu^m(\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu})\sim\sum_{j=2}^{n-1}c_{j-n}\mu^{n-m-1-j}+\sum_{k\ge0}(c_k\log\mu+c'_k)\mu^{-m-1-k},\\&\operatorname{Tr}\varphi\partial_\mu^m(\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu})\sim\sum_{k\ge0}c'_k\mu^{-m-1-k},\quad \textit{if } B' \textit{ is moreover of order } -\infty.\end{aligned}\tag{9.13}$$

Proof. We find by circular perturbation (as in Theorem 9.1) of the expression in (9.12),

$$\begin{aligned}\operatorname{Tr}_X\varphi\partial_\mu^m(\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu})&=\operatorname{Tr}_X\sum_{k\le m}\binom{m}{k}\varphi\partial_\mu^k\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\partial_\mu^{m-k}\mathcal{R}_{2,\mu}\\&=\operatorname{Tr}_{X'}\partial_\mu^m(\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu}\varphi\mathcal{K}_{1,\mu})=\operatorname{Tr}_{X'}S'_\mu,\\&\text{where } S'_\mu=\partial_\mu^m(\mathcal{B}'\gamma_0(\widetilde{Q}_{\mu,+}-K_\mu^+S'_{2,\mu}\mathcal{B}_2\gamma_0\widetilde{Q}_{\mu,+})\varphi K_\mu^+S'_{1,\mu});\end{aligned}$$

the $S'_{i,\mu}$ denote the right inverses of $\mathcal{B}_i C^+_\mu$ constructed for the respective problems in Lemma 8.1 and Theorem 8.2. It is found from the composition rules that S'_μ has symbol in $S^{-2-m,0} \cap S^{-1,-1-m}$ (in $S^{-\infty,-1-m}$ if B' is of order $-\infty$). Then [GS1, Theorem 2.1] implies (9.13), when $m \geq n-2$ (resp. for any m). □

In the case with X compact and a product structure near X', the Calderón projector differs from $\Pi_\geq$ by an operator of order $-\infty$ by Proposition 4.1, so for $B=C^+$, the expansions (9.9)–(9.11) only differ in the primed coefficients from the expansions known for $B=\Pi_\geq$, by (9.13). Here it was shown in [GS2] that all the logarithmic terms vanish when $n=\dim X$ is odd; when n is even, the logarithmic terms with k even >0 vanish, and the logarithm at the power zero vanishes if in addition $\varphi=I$ (exact formulas were also given). So we get the following corollary.

Corollary 9.5. *Consider the product case with X compact, $B=C^+$. Then the expansions* (9.9)–(9.11) *differ from those known for $B=\Pi_\geq$ only in the primed coefficients. In particular, when n is odd, all the logarithmic terms vanish, when n is even, the logarithmic terms with k even >0 vanish in* (9.9)–(9.10); *also the $\tilde{a}_{i,0}$ and $a_{i,0}$ vanish if $\varphi=I$. The same holds for smooth perturbations of $\Pi_\geq$ or C^+.*

Note that it is the global coefficients that may be changed when we replace $\Pi_\geq$ by C^+ in the product case, whereas the locally determined coefficients are unchanged. Their values are in principle determined from the precise formulas in [GS2].

Remark 9.6. Our results show that the boundary conditions considered in [BL] give heat operators with trace expansions (9.10) also when the structure is not of product type near X'; this is a new result. Comparison with perturbations as in Theorem 9.4ff.

Let us finally observe the resulting index formula.

Corollary 9.7. *Let X be compact and let B be well-posed for D. Let $\varphi=1$ in* (9.10). *Then the index of D_B equals*

$$\operatorname{index} D_B = a'_{1,0} - a'_{2,0}. \tag{9.14}$$

Furthermore, all the other coefficients coincide for $i=1$ and 2, $a_{1,k}=a_{2,k}$ for all $k \geq -n$ and $a'_{1,k}=a'_{2,k}$ for all $k>0$.

Proof. This follows from the well-known fact (cf. e.g. [G3, Section 4.3]) that $\operatorname{index} D_B = \operatorname{Tr} e^{-t\Delta_1} - \operatorname{Tr} e^{-t\Delta_2}$ for $t>0$. Since this expression is constant in t, the variable terms must vanish. (One can make a successive elimination of the terms $(a_{1,-n}-a_{2,-n})t^{-n/2}$, $(a_{1,1-n}-a_{2,1-n})t^{-(n-1)/2}$, etc., by order of magnitude.) □

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Gerd Grubb
Mathematics Department
Copenhagen University
Universitetsparken 5
DK-2100 Copenhagen
Denmark