On the Valiron deficiencies of integral functions of infinite order

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1. Introduction

Let $f(z)$ be meromorphic in the plane. We define in the normal way the order ϱ and the characteristic $T(r, f)$ of $f(z)$ and also the quantities $m(r, a)$ and $N(r, a)$ for any a in the closed plane.[1])

The Valiron deficiency is defined to be

$$\Delta(a) = \overline{\lim_{r\to\infty}} \frac{m(r, a)}{T(r, f)} = 1 - \underline{\lim_{r\to\infty}} \frac{N(r, a)}{T(r, f)} .$$

We are concerned in this paper with the question of how large the set of a can be for which $\Delta(a, f) > 0$ [3, problem 1.2]. For functions of finite order this problem has recently been completely solved by Hyllengren. He proved [4, Theorem 1] the following

THEOREM A. *Let E be any plane point set. Then the following two conditions are equivalent*

a) *There exists a positive number k and an infinite sequence $a_1, a_2, \ldots$ of complex numbers, so that each $a \in E$ satisfies the inequality*

$$|a - a_n| < \exp\{-\exp(nk)\}$$

for infinitely many n.

b) *There exists a real number x, $0 < x < 1$ and a meromorphic function $f(z)$ of finite order in $|z| < \infty$, so that*

$$\Delta(a, f) > x$$

for every a in E.

In fact $f(z)$ can be chosen to be an integral function.

[1]) for the notation see e.g. [5, p. 158].

For functions of infinite order the situation is rather different. The strongest result in the positive direction is due to Ahlfors [1, see also 5, p. 264], who proved

THEOREM B. *Suppose that $f(z)$ is meromorphic in the plane. Then given $\varepsilon > 0$, we have for all a outside a set E of capacity zero*

$$m(r, a) < T(r, f)^{1/2+\varepsilon} \tag{1.1}$$

for all sufficiently large r. In particular $\Delta(a) = 0$ outside a set of capacity zero.

As Hyllengren points out, while all sets satisfying his condition a) have capacity zero, the converse is false, and in fact sets satisfying a) are metrically substantially smaller than general sets of capacity zero.

2.

In this paper we shall give examples to prove that for functions of infinite order Theorem B is more or less best possible, by proving

THEOREM 1. *Let E be an arbitrary F_σ set of capacity zero. Then there exists an integral function $f(z)$, such that $\Delta(a, f) = 1$ for $a \in E$.*

This result is an immediate consequence of the following more precise

THEOREM 2. *Let $\Phi_1(r)$ and $\Phi_2(r)$ be continuous increasing functions of r for $r > r_0$, which tend to $+\infty$ with r. Let E_m be an expanding sequence of compact sets of capacity zero, having the origin as an isolated point. Then there exists an integral function $f(z)$ with $f(0) = 0$, and a sequence $r_m \to \infty$ with m, such that for $m = 1, 2, \ldots,$ we have*

$$n(r_m, a, f) \leq \Phi_1(r_m), \quad a \in E_m , \tag{2.1}$$

$$N(r_m, a, f) \leq \Phi_1(r_m) \log r_m, \quad a \in E_m , \tag{2.2}$$

and

$$T(r_m, f) \geq \Phi_2(r_m) . \tag{2.3}$$

We note that $\Phi_1(r)$ can tend to infinity as slowly and $\Phi_2(r)$ as rapidly as we please. Taking for instance $\Phi_1(r) = \log r$, $\Phi_2(r) = r$, we see that all values a in $E = \cup E_m$, satisfy $\Delta(a, f) = 1$. It is also interesting to note that the lower growth of $N(r, a)$ for $a \in E$ may be as slow as we please, subject to being more rapid than $\log r$. Using the second fundamental theorem, we deduce from (2.2) that

$$T(r_m/2, f) < 4\Phi_1(r_m) \log (r_m), \quad m > m_0 ,$$

which contrasts with (2.3). We also deduce from (2.2) and (2.3) that the sequence r_m is exceptional for Nevanlinna's second fundamental theorem. For that theorem implies as $r \to \infty$ outside an exceptional set of finite measure [5, p. 241]

$$(q-2)T(r,f) \leq (1+o(1)) \sum_{\nu=1}^{q} N(r, a_\nu, f)$$

for any q distinct values a_ν. If this were true for r_m we should deduce $\Phi_2(r_m) < (q+o(1))(q-2)^{-1}\Phi_1(r_m)$ which is false in general. Thus the exceptional set in Nevanlinna's second fundamental theorem can really occur [see 3, problem 1.22].

The assumption that E_m is compact and has the origin as an isolated point is no real restriction. For if the E_n are arbitrary closed sets we may write E'_m for the part of $\bigcup_{\nu=1}^{m} E_\nu$ in $m^{-1} \leq |z| \leq m$, together with the origin. If the E_n all have capacity zero, so does E'_m, which is compact and

$$E = \bigcup_{m=1}^{\infty} E'_m = \bigcup_{m=1}^{\infty} E_m \cup \{0\}\,. \tag{2.4}$$

Thus any F_σ set E containing the origin can be written in the form (2.4). If we now choose for instance $\Phi_1(r) = r$, $\Phi_2(r) = r^2$ in Theorem 2, we deduce that for every a in E

$$\frac{N(r_m, a)}{T(r_m, f)} = \frac{O(r_m \log r_m)}{r_m^2} \to 0, \quad \text{as} \quad r_m \to \infty\,,$$

so that $\Delta(a,f) = 1$. Thus Theorem 1 follows immediately from Theorem 2.

Theorem 2 also shows that if E is any F_σ set of capacity zero and $\Phi_1(r)$, $\Phi_2(r)$ are the functions of Theorem 2, then there exists a sequence $r_m \to \infty$ and an integral function $f(z)$ such that (2.3) holds and for any $a \in E$ we have

$$N(r_m, a) \leq \Phi_1(r_m) \log r_m,\ m \geq m_0(a)\,. \tag{2.5}$$

In this form the result is best possible. In fact the set of a satisfying (2.5) for a given m_0 is an intersection of closed sets and so is closed. Thus the set E of all a satisfying (2.5) for a given sequence r_m is an F_σ set. It follows from a result of Nevanlinna [5, formula 18, p. 171] that any closed subset of E, and so E itself, must have capacity zero if (2.5) holds, as soon as

$$\Phi_2(r_m) - \Phi_1(r_m) \log r_m \to +\infty, \quad \text{as} \quad m \to \infty\,. \tag{2.6}$$

Thus it is remarkable that once the very weak condition (2.6) is satisfied, we do not restrict the set E further by decreasing $\Phi_1(r)$ or increasing $\Phi_2(r)$.

3. Some preliminary results

We complete the paper by proving Theorem 2. In order to do this we need to reproduce a situation in a finite disk for a sequence of values $r = r_m$ in the plane. We need two subsidiary results.

LEMMA 1. *Let E be a compact set of capacity zero, p a positive integer and x a positive number. Then there exists a_1, such that $x < a_1 < 10x$ and a function*

$$F(z) = a_1 z + a_{p+1} z^{p+1} + \dots, \tag{3.1}$$

regular in $|z| < 1$, univalent in $|z| < \sqrt{2} - 1$, having unbounded characteristic and assuming no value of E more than once in $|z| < 1$.

We assume initially that E does not meet the real axis, except perhaps at $w = 0$. Let E_0 be the set E with the points $0, \mp x, \infty$, added where x is a positive number. Let R be the infinite covering surface over the complement of E_0. We cut a copy of R from x to $+\infty$ along the real axis thus obtaining two surfaces R_1, R_2 and another copy of R from $-x$ to $-\infty$ obtaining two surfaces R_3 and R_4, all of which are simply connected. Let R_5 be the plane cut from x to ∞ and from $-x$ to $-\infty$ along the real axis and let R_0 be obtained by joining R_1, R_2 to R_5 on the segments (x, ∞), and R_3 and R_4 to R_5 along the segments $(-\infty, -x)$, so that R_0 is a Riemann surface containing none of the points $\mp x, \infty$ in any sheet and containing points over E exactly once, namely in the sheet R_5. Thus R_0 is simply connected, and since R_0 does not contain the points $\mp x, \infty$, R_0 is hyperbolic. Thus we may map $|z| < 1$ (1, 1) conformally onto R_0 by a function

$$F_0(z) = b_1 z + b_2 z^2 + \dots$$

where $b_1 > 0$.

The function $F_0(z)$ never assumes the values $\mp x, \infty$, and so is subordinate to the function $G(z)$ which maps $|z| < 1$ onto the infinite covering surface S over the plane with these 3 points removed and satisfying

$$G(0) = 0, \quad G'(0) > 0\,.$$

This latter function maps $z = 1, i, -1, -i$ onto $w = x, i\,\infty, -x, -i\infty$ so that the sheet R_5 corresponds to a »quadrilateral» Q in the unit disk bounded by 4 quarter circles joining these points $(1, i)$, $(i, -1)$, $(-1, -i)$ and $(-i, 1)$ and orthogonal to $|z| = 1$. Clearly Q contains the disk $|z| < \sqrt{2} - 1$, and since $F_0(z)$ is subordinate to $G(z)$, the disk $|z| < \sqrt{2} - 1$ corresponds to a subset of the sheet R_5 by $F_0(z)$, so that $F_0(z)$ is univalent in $|z| < \sqrt{2} - 1$.

It now follows from Koebe's theorem that $x > b_1(\sqrt{2} - 1)/4$. On the other hand the inverse function $z = \Phi(w)$ of $F_0(z)$ maps the disk $|w| < x$ into the disk $|z| < 1$, so that by Schwarz's Lemma $b_1^{-1} = \Phi'(0) < x^{-1}$. Thus we deduce that

$$x < b_1 < \frac{4}{\sqrt{2} - 1}\, x < 10x\,. \tag{3.2}$$

Thus $F_0(z)$ has the required development (3.1), when $p = 1$.

We next note that $F_0(z)$ has unbounded characteristic in $|z| < 1$. In fact $F_0(z)$ cannot have any radial limits other than points of E_0. It follows from a classical theorem of Frostman and Nevanlinna [5, p. 198] that if $F_0(z)$ had bounded characteristic then the total set of radial limits of $F_0(z)$ would have a positive capacity, giving a contradiction. Thus $F_0(z)$ has unbounded characteristic.

This proves Lemma 1 for the case $p = 1$. If $p > 1$, we proceed as follows. Let E_p be the set consisting of all complex numbers w^p, such that $w \in E$. We may say that E_p is the p-th power of E. Let $F_p(z)$ be defined as above with E_p instead of E, x^p instead of x, and set

$$F(z) = \{F_p(z^p)\}^{\frac{1}{p}} = b_1^{\frac{1}{p}}(z + (b_2 z^{p+1}/b_1 p) + \ldots) .$$

Since $F_p(z) \neq 0$ for $z \neq 0$, $F(z)$ is regular. Also if Δ_0 is the part of $|z| < 1$ which corresponds to the sheet R_5 by $F_p(z)$, then $F_p(z)$ is univalent in Δ_0. Thus if Δ_p is the p'th root of Δ_0, i.e. the set of all z, such that z^p lies in Δ_0, then $F(z)$ is univalent in Δ_p. In fact if z_1, z_2 lie in Δ_p and $F(z_1) = F(z_2)$, we deduce that z_1^p, z_2^p lie in Δ_0, $F_p(z_1^p) = F_p(z_2^p)$, so that $z_1^p = z_2^p$. Thus we have, for some integer k, $z_2 = z_1 \exp(2\pi i k/p)$. This implies $F(z_1) = F(z_2) \exp(2\pi i k/p)$, so that $z_2 = z_1$, and $F(z)$ is univalent in Δ_p, which includes the disk $|z| < (\sqrt{2} - 1)^{1/p}$, and so certainly the disk $|z| < \sqrt{2} - 1$. We also see that if $F(z) = w$ in E, then $F_p(z^p) = w^p$ in E_p, and this is possible only for z^p in Δ_0, i.e. z in Δ_p, where F is univalent. Thus $F_p(z)$ assumes no value of E more than once in $|z| < 1$. Finally we see that $x^p < b_1 < 10x^p$, so that $F(z)$ has the development (3.1).

The above argument assumes that E_p does not meet the real axis, except perhaps at the origin. However, since E_p has capacity and so linear measure zero, E_p will not meet every straight line through the origin, at points other than $w = 0$. If E_p does not meet $\arg z = \alpha, \alpha + \pi$, we apply the above argument with the set $E_p(\alpha)$ instead of E_p where $E_p(\alpha)$ is obtained by rotating E_p by an angle $-\alpha$ around the origin. We then consider $e^{i\alpha}F_p(ze^{-i\alpha})$ instead of $F(z)$. The argument showing that $F_p(z)$ has unbounded characteristic also shows that $F(z)$ has unbounded characteristic and Lemma 1 is proved.

We can deduce

LEMMA 2. *Suppose that* $a_1, \ldots, a_p$ *are preassigned complex numbers, not all zero, and let* $M = \sum_{\nu=1}^{p} |a_\nu|$. *Let* E *be the set of Lemma* 1. *Then there exists* $F_p(z)$ *regular in* $|z| < 1$, *assuming no value of* E *more than* $2p$ *times there, having unbounded characteristic in* $|z| < 1$ *and a power series development*

$$F_p(z) = a_1 z + a_2 z^2 + \ldots . a_p z^p + O(z^{p+1}) \tag{3.3}$$

near $z = 0$. *Further*

$$|F_p(z)| < 10M, \quad \text{for} \quad |z| \leq (\sqrt{2} - 1)/2 . \tag{3.4}$$

Suppose that $\mu > M$ and write

$$\omega(z) = \frac{\sum_{\nu=1}^{p} a_\nu z^\nu + \mu z^{2p}}{\mu + \sum_{\nu=1}^{p} \bar{a}_\nu z^{2p-\nu}} .$$

Then $|\omega(z)| = 1$ for $|z| = 1$, and $\omega(z)$ has precisely $2p$ zeros and no poles in $|z| < 1$ by Rouché's Theorem. Let $F(z)$ be the function whose existence is asserted in Lemma 1, with $a_1 = \mu$, where $M < \mu < 10M$ and set

$$F_p(z) = F\{\omega(z)\} .$$

We proceed to show that $F_p(z)$ has the required properties. It is evident that

$$\omega(z) = \mu^{-1} \sum_{\nu=1}^{p} a_\nu z^\nu + O(z^{p+1}) ,$$

and so

$$F_p(z) = \mu\omega(z) + O(z^{p+1})$$

has the required power series development at the origin. Next it is evident from the same argument concerning radial limits that $F_p(z)$ has unbounded characteristic in $|z| < 1$. Also the equation $\omega(z) = \zeta$ has precisely $2p$ roots in $|z| < 1$ for any $|\zeta| < 1$, and so, since $F(z) = w$ has at most one root z for w in E, it follows that $F_p(z) = w$ has at most $2p$ roots for w in E.

Finally, since $F(z)$ is univalent in $|z| < r_0 = \sqrt{2} - 1$ it follows from a classical inequality for univalent function [2, p. 4] that $|F(z)| < \mu r_0^2 |z| (r_0 - |z|)^{-2}$, $|z| < r_0$. Also by Schwarz's lemma $|\omega(z)| \leq |z|$, for $|z| < 1$, and so we deduce that

$$|F_p(z)| \leq \mu r_0^2 |\omega(z)| (r_0 - |\omega(z)|)^{-2} < 10 M r_0^2 |z| (r_0 - |z|)^{-2} < 10M, \quad \text{if} \quad |z| \leq r_0/2 ,$$

This completes the proof of Lemma 2.

4. Proof of Theorem 2

We shall proceed to construct the function of Theorem 2

$$f(z) = \sum_{n=1}^{\infty} b_n z^n , \tag{4.1}$$

by successively constructing its coefficients b_n. We set $b_1 = 1$, and assume that p_k is a strictly increasing sequence of positive integers such that $p_1 = 1$. We assume that b_n is known for $n \leq p_k$ and proceed to construct b_n for $p_k \leq n \leq p_{k+1}$.

To do this, we shall inductively define a sequence ϱ_k of positive numbers, increasing rapidly to infinity and such that $\varrho_0 = 1$. Suppose that ϱ_k has been chosen and let E_k be the set of Theorem 2. Let $F_k(z)$ be the function defined as in Lemma 2 with $p = p_k$, $E = E_k$ and

$$a_n = b_n \varrho_k^n . \tag{4.2}$$

Then if a_n are the coefficients of $F_k(z)$ for all n, we define b_n by (4.2) for $p_k < n \leq p_{k+1}$. It remains to show that the sequences ϱ_k, p_k can be chosen inductively so that $f(z)$ given by (4.1) satisfies all the conditions of Theorem 2.

We start by showing that if ϱ_k is chosen sufficiently large, when ϱ_{k-1}, p_k have been chosen, we shall have

$$b_n < (2\varrho_{k-1})^{-n}, \quad p_k < n \leq p_{k+1} . \tag{4.3}$$

In fact it follows from Lemma 2, (3.4) and Cauchy's inequality that

$$|a_n| < 10(\{\sqrt{2} - 1\}/2)^n M, \quad p_k < n \leq p_{k+1} , \tag{4.4}$$

where

$$M = \sum_{\nu=1}^{p_k} |b_\nu| \varrho_k^\nu < \varrho_k^{p_k} \sum_{\nu=1}^{p_k} |b_\nu| .$$

Writing $A_0 = (\sqrt{2} - 1)/2$, $B_k = \sum_{\nu=1}^{p_k} |b_\nu|$, we deduce from (4.2), (4.4) that for $p_k < n \leq p_{k+1}$ we have

$$|b_n| \leq 10 \varrho_k^{p_k - n} A_0^{-n} B_k .$$

Thus (4.3) holds if $\varrho_k^{n-p_k} > 10(2\varrho_{k-1}/A_0)^n B_k$, i.e. $\varrho_k > (10B_k)^{\frac{1}{n-p_k}} (2\varrho_{k-1}/A_0)^{\frac{n}{n-p_k}}$, and this condition is certainly satisfied for all $n > p_k$, if

$$\varrho_k > 10B_k(2\varrho_{k-1}/A_0)^{p_k+1} . \tag{4.5}$$

Here we use the fact that $B_k \geq |b_1| = 1$. We assume that, if p_k and ϱ_{k-1} are known, ϱ_k is chosen to satisfy (4.5) so that (4.3) holds. Since $\varrho_k \to \infty$ with k, we deduce at once that $f(z)$ given by (4.1) is an integral function.

We note that (4.3) implies in particular that $|b_n| \leq 1$ for all n. Thus for $|z| = \varrho \leq 1/2$, we have

$$|f(z)| \leq \sum_{n=1}^{\infty} \varrho^n \leq \frac{\varrho}{1-\varrho} \leq 2\varrho . \tag{4.6}$$

Let E'_k consist of all points of E_k other than the origin, so that by hypothesis E'_k has a positive distance from the origin. We choose δ_k to be positive decreasing, less than half this distance and less than 1/2. Then it follows from (4.6) that $f(z)$ assumes no value of E'_k in $|z| < \delta_k$. Also for $|z| = \varrho$, where $0 < \varrho < \delta_k$, we have

$$|f(z)| \geq \varrho - \sum_{2}^{\infty} \varrho^\nu = \varrho - \frac{\varrho^2}{1-\varrho} = \frac{\varrho - 2\varrho^2}{1-\varrho} > 0 .$$

Thus $f(z) \neq 0$, for $0 < |z| < \delta_k$, and so in this annulus $f(z)$ assumes no value of E_k. Also $f(z)$ has a simple zero at the origin.

The function $F_k(z/\varrho_k)$ by our construction approximates very closely to $f(z)$ and the coefficients of both functions are the same, namely a_n, for $n \leq p_{k+1}$. Also for $n > p_{k+1}$ we have in view of (4.3)

$$|b_n| < (2\varrho_k)^{-n} .$$

This will enable us to show that $f(z)$ and $F_k(z/\varrho_k)$ behave similarly for $|z| = r_k < \varrho_k$, provided that p_{k+1} is large enough. However before constructing r_k and p_{k+1} we need some further conditions on ϱ_k, which like (4.5) will be satisfied if ϱ_k is sufficiently large. Accordingly we choose ϱ_k so large that in addition to (4.5) we have

$$2p_k < \Phi_1(\tfrac{1}{2}\varrho_k) , \tag{4.7}$$

and

$$2p_k \log\left(\frac{\varrho_k}{\delta_k}\right) < \Phi_1(\tfrac{1}{2}\varrho_k) \log(\tfrac{1}{2}\varrho_k) . \tag{4.8}$$

We now suppose that ϱ_k satisfies (4.5), (4.7) and (4.8), and proceed to define r_k. It follows from Lemma 2 that $F_k(z/\varrho_k)$ has unbounded characteristic in $|z| < \varrho_k$. Thus we may choose r_k, such that

$$\tfrac{1}{2}\varrho_k < r_k < \varrho_k , \tag{4.9}$$

and in addition

$$T\{r_k, F_k(z/\varrho_k)\} > \Phi_2(\varrho_k) + 1 . \tag{4.10}$$

Next we note that sets of capacity zero have linear measure zero, and hence so do their inverse images by regular functions, since the inverse function is locally conformal except at isolated points. In particular the inverse image of E_k by $F_k(z/\varrho_k)$ meets $|z| = r$ only for a set of r of linear measure zero. Thus, by increasing r_k if necessary, we suppose in addition to (4.9) and (4.10) that $F_k(z/\varrho_k)$ does not meet E_k, for $|z| = r_k$. Since E_k is compact, this implies the existence of a quantity ε_k, such that $0 < \varepsilon_k < 1$ and

$$|F_k(z/\varrho_k) - a| > \varepsilon_k, \quad \text{for } a \in E_k \text{ and } |z| = r_k . \tag{4.11}$$

Having chosen r_k to satisfy (4.9) to (4.11), we proceed to show that if p_{k+1}, which has so far been left undetermined, is chosen suitably, then (2.1), (2.2) and (2.3) will be satisfied.

We write $F_k(z/\varrho_k) = \sum_{n=1}^{\infty} B_n z^n$, and note that the series is absolutely convergent for $|z| = r_k$. Thus we may choose p_{k+1} so large that $\sum_{n=p_{k+1}}^{\infty} |B^n||z|^n < \frac{1}{2}\varepsilon_k$, $|z| = r_k$, where ε_k is the quantity in (4.11). Next it follows from (4.3) and the fact that $r_k < \varrho_k$, that for $|z| = r_k$

$$\sum_{n=p_{k+1}+1}^{\infty} |b_n||z|^n \leq \sum_{n=p_{k+1}+1}^{\infty} 2^{-n} = 2^{-p_{k+1}},$$

and this is less than $\frac{1}{2}\varepsilon_k$ if p_{k+1} is large enough, which we assume. Thus we may choose p_{k+1} so large that (regardless of any later choices of ϱ_ν, r_ν, and p_ν for $\nu \geq k+1$) we have

$$|f(z) - F_k(z/\varrho_k)| = |\sum_{p_{k+1}+1}^{\infty} (b_n - B_n)z^n| < \varepsilon_k, \quad |z| = r_k. \tag{4.12}$$

It follows from this and (4.11) that for a in E_k the equations $f(z) = a$ and $F_k(z/\varrho_k) = a$ have equally many roots in $|z| < r_k$, i.e. at most $2p_k$, in view of Lemma 2. Now (2.1) follows at once from (4.7), (4.9) and the fact that $\Phi_1(r)$ increases.

Next if $n(t, a)$ denotes the number of zeros of $f(z) - a$ in $0 < |z| \leq t$, it follows from the definition of δ_k, that for $a \in E_k$

$$n(t, a) = 0, \quad t < \delta_k,$$

while from what we have just shown

$$n(t, a) \leq 2p_k, \quad \delta_k \leq t < r_k.$$

Thus if $a \neq 0$

$$N(r_k, a) = \int_0^{r_k} \frac{n(t, a)dt}{t} \leq 2p_k \log (r_k/\delta_k).$$

If $a = 0$, we recall that, since $f(z)$ has a simple zero at the origin, and no zeros in $0 < |z| < \delta_k$

$$N(r_k, 0) = \int_{\delta_k}^{r_k} n(t, 0) \frac{dt}{t} + \log \delta_k \leq 2p_k \log (r_k/\delta_k).$$

Thus for a in E_k we have in all cases

$$N(r_k, a) \leq 2p_k \log (r_k/\delta_k) < \Phi_1(\tfrac{1}{2}\varrho_k) \log (\tfrac{1}{2}\varrho_k) < \Phi_1(r_k) \log r_k,$$

in view of (4.8), (4.9) and the fact that $\Phi_1(t)$ increases with t. This proves (2.2).

Finally we have by (4.10), and the well-known inequality for the characteristic of the sum of two functions [5, p. 162],

$$\Phi_2(r_k) + 1 \leq \Phi_2(\varrho_k) + 1 < T\{r_k, F_k(z/\varrho_k)\} \leq T\{r_k, F_k(z/\varrho_k) - f(z)\} + T\{r_k, f(z)\} + 1$$
$$= T\{r_k, f(z)\} + 1,$$

in view of (4.12) and the fact that $\varepsilon_k < 1$. This proves (2.3) and completes the proof of Theorem 2.

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