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Comparison theorems for the one-dimensional Schrödinger equation

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Dedicated to the memory of Matts Essén

Abstract. Using rearrangements of matrix-valued sequences, we prove that with certain boundary conditions the solution of the one-dimensional Schrödinger equation increases or decreases under monotone rearrangements of its potential.

1. Introduction

Let $L_+^\infty[0,1]$ be the set of nonnegative bounded measurable functions on the interval $[0,1]$. The decreasing rearrangement of a function $p \in L_+^\infty[0,1]$ is defined by

$$p^*(x) = \sup\{t : \mathcal{L}^1\{y : p(y) > t\} > x\},$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure. The increasing rearrangement of p is then defined by $p_*(x) = p^*(1-x)$. Here and in what follows the words "increasing" and "decreasing" are used in nonstrict sense; however, "positive" means "strictly positive".

Given $p \in L_+^\infty[0,1]$, consider the stationary one-dimensional Schrödinger equation

$$u''(x) - p(x)^2 u(x) = 0, \quad x \in [0,1]. \tag{1.1}$$

By a solution of (1.1) we mean a function with absolutely continuous first derivative that satisfies (1.1) for almost every $x \in [0,1]$. The question that is answered in the present paper is: under which boundary conditions the solution of (1.1) changes in a predictable manner when p is replaced by p^* or p_*? The first results of this kind were established by M. Essén [3], [4] in connection with estimates of harmonic measure and the growth of subharmonic functions. Later developments can be found in [5]–[8]. J. M. Luttinger [9] obtained related results in the form of integral, rather than pointwise, estimates for the solutions of (1.1).

We propose a unified approach to this problem, which is based on the analysis of permutations of state transition matrices (Theorems 2.1 and 3.1). The main tools used are polarization (see Section 2) and a certain partial order of 2×2 real matrices (Section 4). The following two theorems are easily derived from our results on state transition matrices.

Theorem 1.1. *Let $p\in L^\infty_+[0,1]$ and let u and v be the solutions of the initial value problems*

$$u''(x)-p(x)^2u(x)=0,\quad u(0)=1,\ u'(0)=\alpha;$$
$$v''(x)-p_*(x)^2v(x)=0,\quad v(0)=1,\ v'(0)=\alpha,$$

where $-1\le\alpha\le 1$. Then $u(1)\ge v(1)$. This result is no longer true if $|\alpha|>1$.

The case $\alpha=0$ of Theorem 1.1 was proved by M. Essén [3], see also [7].

Theorem 1.2. *Let $p\in L^\infty_+[0,1]$ and let u and v be the solutions of the boundary value problems*

$$u''(x)-p(x)^2u(x)=0,\quad u(0)=0,\ \alpha u(1)+u'(1)=1;$$
$$v''(x)-p^*(x)^2v(x)=0,\quad v(0)=0,\ \alpha v(1)+v'(1)=1,$$

where $-1\le\alpha\le 1$. Then $u(x)\le v(x)$ for all $x\in[0,1]$. This result is no longer true if $|\alpha|>1$.

Using Theorem 2.1, one can prove further inequalities concerning the values of solutions at the endpoints of $[0,1]$. However, the boundary conditions in Theorem 1.2 are (essentially) the only conditions of the form

$$\alpha_j u(j)+\beta_j u'(j)=\gamma_j,\quad j=0,1,$$

under which the global majorization $u(x)\le v(x)$, $x\in[0,1]$, holds. This fact also follows from Theorem 2.1; we do not present its proof here (although some indication is given in the proof of Theorem 1.2).

2. Preliminaries

Definition 2.1. For any subinterval $[a,b]\subset[0,1]$ the corresponding *state transition matrix* $T(a,b;p)$ is defined by the equation $(u(a),u'(a))T(a,b;p)=(u(b),u'(b))$, where u is any solution of (1.1).

Below we record two elementary properties of $T(a,b;p)$. Here and in what follows A_{ij} stands for the entry of the matrix A in the ith row and the jth column.

Lemma 2.1. *Let* $[a,b]\subset[0,1]$, $p\in L_+^\infty[0,1]$. *Then*

$$T(a,b;p)_{21}\le T(a,b;p)_{22}; \tag{2.1}$$

$$\det T(a,b;p)=1. \tag{2.2}$$

Proof. It suffices to prove (2.1) and (2.2) for continuous p. Let u be the solution of (1.1) with $u(a)=0$, $u'(a)=1$. Since u is convex, it follows that

$$u(b)=\int_a^b u'(x)\,dx\le\int_a^b u'(b)\,dx\le u'(b).$$

This implies (2.1) because $T(a,b;p)_{21}=u(b)$ and $T(a,b;p)_{22}=u'(b)$. In order to prove (2.2), let v be the solution of (1.1) with $v(a)=1$ and $v'(a)=0$. The Wronskian of u and v is constant, since $(u'v-uv')'=p^2uv-p^2uv=0$. Thus $\det T(a,b;p)=u'(b)v(b)-u(b)v'(b)=u'(a)v(a)-u(a)v'(a)=1$. □

Our main result for state transition matrices is the following.

Theorem 2.1. *Let* $\Phi\colon\mathbf{R}^{2\times2}\to\mathbf{R}$ *be a linear function.*

(a) *The inequality* $\Phi(T(0,1;p^*))\le\Phi(T(0,1;p))$ *holds for every* $p\in L_+^\infty[0,1]$ *if and only if* Φ *has the form* $\Phi(A)=\alpha A_{21}+\beta A_{22}$, *where* $|\alpha|\le\beta$.

(b) *The inequality* $\Phi(T(0,1;p_*))\le\Phi(T(0,1;p))$ *holds for every* $p\in L_+^\infty[0,1]$ *if and only if* $\Phi(A)=\alpha A_{21}+\beta A_{11}$, *where* $|\alpha|\le\beta$.

We will first prove Theorem 2.1 for piecewise constant functions p. It is straightforward to verify that if $p(x)\equiv y>0$ on an interval $[a,b]$, then

$$T(a,b;p)=\begin{pmatrix}\cosh y(b-a) & y\sinh y(b-a)\\ y^{-1}\sinh y(b-a) & \cosh y(b-a)\end{pmatrix}. \tag{2.3}$$

Therefore, if a piecewise constant potential p takes a value $p_i>0$ on $[(i-1)/n,i/n)$, $i=1,\dots,n$, then

$$T(0,1;p)=\prod_{i=1}^{n}\begin{pmatrix}\cosh(p_i/n) & p_i\sinh(p_i/n)\\ p_i^{-1}\sinh(p_i/n) & \cosh(p_i/n)\end{pmatrix}.$$

Our goal is to describe how the matrix product is affected by rearrangements of the numbers $p_1,\dots,p_n$. In doing so, we may factor out the scalar term $\prod_{i=1}^n\cosh(p_i/n)$, since it does not depend on the order of $p_1,\dots,p_n$. Thus we define

$$M_n(y)=\begin{pmatrix}1 & y\,\mathrm{th}(y/n)\\ y^{-1}\,\mathrm{th}(y/n) & 1\end{pmatrix},\quad y>0,\ n=1,2,\dots, \tag{2.4}$$

where th is the hyperbolic tangent, more commonly denoted tanh. It is easy to see that such matrices do not commute; in fact,

$$M_n(y)M_n(z)-M_n(z)M_n(y)=\left(\frac{y}{z}-\frac{z}{y}\right)\operatorname{th}\frac{y}{n}\operatorname{th}\frac{z}{n}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.5}$$

If $\mathbf{p}=(p_1,\dots,p_n)$ is an n-tuple of positive numbers, we write $\mathbf{p}^*=(p_1^*,\dots,p_n^*)$ to denote the same numbers rearranged in decreasing order. It seems difficult to compare $\prod_{i=1}^n M_n(p_i)$ and $\prod_{i=1}^n M_n(p_i^*)$ directly, so we choose a step-by-step approach whose origins can be traced back to the papers [10] and [1]. One can transform the sequence $(p_1,\dots,p_n)$ into $(p_1^*,\dots,p_n^*)$ be successively applying an operation called *polarization*, which is described below. First we extend the finite sequence $\mathbf{p}$ by letting $p_i=\infty$ for $i<1$ and $p_i=-\infty$ for $i>n$. For an integer m between 2 and $2n$, the polarized sequence $\pi_m\mathbf{p}$ is defined by

$$(\pi_m\mathbf{p})_i=\begin{cases}\max\{p_i,p_{m-i}\}, & 1\le i\le \frac{1}{2}m;\\ \min\{p_i,p_{m-i}\}, & \frac{1}{2}m\le i\le n.\end{cases} \tag{2.6}$$

Polarization has the effect of moving larger elements toward the beginning of a sequence, so $\mathbf{p}$ can be transformed into $\mathbf{p}^*$ by a finite sequence of polarizations $\pi_2,\dots,\pi_{2n}$.

3. Rearrangement of state transition matrices

Theorem 3.1. *Let $\Phi:\mathbf{R}^{2\times2}\to\mathbf{R}$ be a linear function. The following are equivalent*:

(i) *For any $p_1,\dots,p_n>0$, $\Phi\bigl(\prod_{i=1}^n M_n(p_i)\bigr)$ decreases when $p_1,\dots,p_n$ are rearranged in the decreasing order*;

(ii) *For any $p_1,\dots,p_n>0$, $\Phi\bigl(\prod_{i=1}^n M_n(p_i)\bigr)$ decreases when $(p_1,\dots,p_n)$ is polarized*;

(iii) *There exist real numbers α and β such that $|\alpha|\le\beta$ and $\Phi(A)=\alpha A_{21}+\beta A_{22}$ for any $A\in\mathbf{R}^{2\times2}$.*

Note that implication (ii) $\Rightarrow$ (i) holds because the decreasing rearrangement can be obtained by a sequence of polarizations.

Proof of (i) $\Rightarrow$ (iii). Suppose that Φ is such that (i) holds. We will identify Φ in three steps which involve little besides straightforward computations.

Step 1. *$\Phi(A)$ does not depend on A_{12}.* Suppose that A_{12} appears in Φ with a positive coefficient. Let $a>b>c>0$. Using (2.5), it is easy to compute

$$\begin{aligned} A &= M_3(a)M_3(b)M_3(c)-M_3(b)M_3(a)M_3(c) \\ &= \left(\frac{a}{b}-\frac{b}{a}\right)\operatorname{th}\frac{a}{3}\operatorname{th}\frac{b}{3}\begin{pmatrix} 1 & c\operatorname{th}\dfrac{c}{3} \\ -\dfrac{1}{c}\operatorname{th}\dfrac{c}{3} & -1 \end{pmatrix}. \end{aligned} \tag{3.1}$$

According to (i), $\Phi(A)\le 0$. Now let $a=c+2\sqrt{c}$, $b=c+\sqrt{c}$, and $c\to\infty$. Since $A_{12}\to\infty$ and $A_{ij}\to 0$ for $(i,j)\ne(1,2)$, it follows that $\Phi(A)\to\infty$, a contradiction.

To exclude the possibility that A_{12} has a negative coefficient, consider

$$\begin{aligned} A &= M_3(a)M_3(b)M_3(c)-M_3(a)M_3(c)M_3(b) \\ &= \left(\frac{b}{c}-\frac{c}{b}\right)\operatorname{th}\frac{b}{3}\operatorname{th}\frac{c}{3}\begin{pmatrix} 1 & -a\operatorname{th}\dfrac{a}{3} \\ \dfrac{1}{a}\operatorname{th}\dfrac{a}{3} & -1 \end{pmatrix} \end{aligned} \tag{3.2}$$

with a, b, and c as above. As $c\to\infty$, we have $A_{12}\to-\infty$ and $A_{ij}\to 0$ for $(i,j)\ne(1,2)$. Thus A_{12} cannot appear in Φ with a negative coefficient either.

Step 2. *$\Phi(A)$ does not depend on A_{11}.* Suppose that A_{11} appears in Φ with a positive coefficient. Consider

$$\begin{aligned} A &= M_4(a)M_4(b)M_4(c)M_4(d)-M_4(a)M_4(b)M_4(d)M_4(c) \\ &= \left(\frac{c}{d}-\frac{d}{c}\right)\operatorname{th}\frac{c}{4}\operatorname{th}\frac{d}{4}\begin{pmatrix} 1+\dfrac{a}{b}\operatorname{th}\dfrac{a}{4}\operatorname{th}\dfrac{b}{4} & -a\operatorname{th}\dfrac{a}{4}-b\operatorname{th}\dfrac{b}{4} \\ \dfrac{1}{a}\operatorname{th}\dfrac{a}{4}+\dfrac{1}{b}\operatorname{th}\dfrac{b}{4} & -1-\dfrac{b}{a}\operatorname{th}\dfrac{a}{4}\operatorname{th}\dfrac{b}{4} \end{pmatrix}. \end{aligned} \tag{3.3}$$

Let $a=d^3$, $b=d+2$, and $c=d+1$. For d sufficiently large we have $a>b>c>d$, hence $\Phi(A)\le 0$. On the other hand, as $d\to\infty$, we have $A_{11}\to\infty$ and $A_{21}, A_{22}\to 0$. Since Φ does not depend on A_{12}, it follows that $\Phi(A)\to\infty$, a contradiction. Next, consider

$$\begin{aligned} A &= M_4(a)M_4(b)M_4(c)M_4(d)-M_4(a)M_4(c)M_4(b)M_4(d) \\ &= \left(\frac{b}{c}-\frac{c}{b}\right)\operatorname{th}\frac{b}{4}\operatorname{th}\frac{c}{4}\begin{pmatrix} 1-\dfrac{a}{d}\operatorname{th}\dfrac{a}{4}\operatorname{th}\dfrac{d}{4} & d\operatorname{th}\dfrac{d}{4}-a\operatorname{th}\dfrac{a}{4} \\ \dfrac{1}{a}\operatorname{th}\dfrac{a}{4}-\dfrac{1}{d}\operatorname{th}\dfrac{d}{4} & 1-\dfrac{d}{a}\operatorname{th}\dfrac{a}{4}\operatorname{th}\dfrac{d}{4} \end{pmatrix} \end{aligned} \tag{3.4}$$

with the same values of a, b, c and d. As $d\to\infty$, we have $A_{11}\to-\infty$ and $A_{21}, A_{22}\to 0$. It follows that A_{11} cannot enter Φ with a negative coefficient.

Step 3. So far we know that $\Phi(A)=\alpha A_{21}+\beta A_{22}$ for some $\alpha,\beta\in\mathbf{R}$. It remains to prove that $|\alpha|\le\beta$. First note that for any positive integer k and any $y>0$ we have

$$\begin{aligned} M_n(y)^k &= C_1\begin{pmatrix}\cosh(y/n) & y\sinh(y/n)\\ y^{-1}\sinh(y/n) & \cosh(y/n)\end{pmatrix}^k \\ &= C_1\begin{pmatrix}\cosh(yk/n) & y\sinh(yk/n)\\ y^{-1}\sinh(yk/n) & \cosh(yk/n)\end{pmatrix} \\ &= C_2\begin{pmatrix}1 & y\operatorname{th}(yk/n)\\ y^{-1}\operatorname{th}(yk/n) & 1\end{pmatrix}, \end{aligned} \tag{3.5}$$

where $C_1=\cosh(y/n)^{-k}$ and $C_2=C_1\cosh(yk/n)$. Now for $a>b>c>0$ we have

$$M_n(a)M_n(b)M_n(c)^{n-2}-M_n(b)M_n(a)M_n(c)^{n-2} = C\begin{pmatrix}1 & c\operatorname{th}\dfrac{c(n-2)}{n}\\ -\dfrac{1}{c}\operatorname{th}\dfrac{c(n-2)}{n} & -1\end{pmatrix}$$

with $C>0$. The assumption (i) implies

$$\frac{\alpha}{c}\operatorname{th}\frac{c(n-2)}{n}+\beta>0.$$

Letting $c\to0$ and $n\to\infty$, we obtain $\alpha+\beta\ge0$. Next, the identity

$$M_n(a)^{n-2}M_n(b)M_n(c)-M_n(a)^{n-2}M_n(c)M_n(b) = C\begin{pmatrix}1 & -a\operatorname{th}\dfrac{a(n-2)}{n}\\ \dfrac{1}{a}\operatorname{th}\dfrac{a(n-2)}{n} & -1\end{pmatrix}$$

and assumption (i) imply

$$\frac{\alpha}{a}\operatorname{th}\frac{a(n-2)}{n}-\beta<0.$$

As $a\to0$ and $n\to\infty$, we obtain $\alpha-\beta\le0$. In conclusion, $|\alpha|\le\beta$. $\square$

In order to complete the proof of Theorem 3.1 we must establish the implication (iii) $\Rightarrow$ (ii). This is the main part of the proof, and it relies on the properties of a particular partial order of 2×2 matrices, which are derived in the next section.

4. Monotonicity of state transition matrices

Definition 4.1. Define a partial order on $\mathbf{R}^{2\times 2}$ as follows

$$A \le B \iff A_{12} \le B_{12},\ A_{21} \le B_{21},\ A_{22} \le B_{22} \text{ and } \operatorname{tr} A \le \operatorname{tr} B.$$

It is easy to see that this order is not stable under multiplication. However, it turns out to be well suited to state transition matrices for the Schrödinger equation. The following theorem asserts that a state transition matrix $T(a,b;p)$ decreases (in the sense of the above partial order) when the potential p is polarized with respect to the midpoint of $[a,b]$. For the remainder of this section, we fix a positive integer n and write

$$M(y) = M_n(y) = \begin{pmatrix} 1 & y\,\mathrm{th}(y/n) \\ y^{-1}\,\mathrm{th}(y/n) & 1 \end{pmatrix}, \quad y > 0.$$

Theorem 4.1. *Given positive numbers $p_0, \dots, p_s$, define*

$$\tilde{p}_r = \begin{cases} \max\{p_r, p_{s-r}\}, & r \le \frac{1}{2}s; \\ \min\{p_r, p_{s-r}\}, & r > \frac{1}{2}s. \end{cases}$$

Then

$$\prod_{r=0}^{s} M(\tilde{p}_r) \le \prod_{r=0}^{s} M(p_r). \tag{4.1}$$

The proof of Theorem 4.1 is based on four lemmas that follow. The first of them describes what happens when state transition matrices are multiplied in the reverse order. To simplify formulas, we use the notation $q_r = \mathrm{th}(p_r/n)$, $r = 0, \dots, s$.

Lemma 4.1. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *denote the product* $\prod_{r=0}^{s} M(p_r)$. *Then*

$$\prod_{r=0}^{s} M(p_{s-r}) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Proof. Let $D = ad - bc = \prod_{r=0}^{s} \det M(p_r)$. Then

$$\begin{aligned}
\left(\prod_{r=0}^{s} M(p_{s-r})\right)^{-1} &= \prod_{r=0}^{s} M(p_r)^{-1} = \frac{1}{D} \prod_{r=0}^{s} \begin{pmatrix} 1 & -p_r q_r \\ -p_r^{-1} q_r & 1 \end{pmatrix} \\
&= \frac{1}{D} \prod_{r=0}^{s} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(p_r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\
&= \frac{1}{D} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\prod_{r=0}^{s} M(p_r) \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{1}{D} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}^{-1}. \quad \square
\end{aligned}$$

In the course of proving Theorem 4.1 it will be necessary to compare the upper-left and lower-right entries of a state transition matrix. The following lemma makes this possible.

Lemma 4.2. *Suppose that $p_0 \le p_s$. Let a, b, c and d be nonnegative numbers such that $a \le d$. Then*

$$\left(M(p_0)\begin{pmatrix} a & b \\ c & d \end{pmatrix} M(p_s)\right)_{11} \le \left(M(p_0)\begin{pmatrix} a & b \\ c & d \end{pmatrix} M(p_s)\right)_{22}. \tag{4.2}$$

Proof. Routine calculation yields

$$\begin{aligned} &M(p_0)\begin{pmatrix} a & b \\ c & d \end{pmatrix} M(p_s) \\ &= \begin{pmatrix} a+cp_0q_0+p_s^{-1}q_s(b+dp_0q_0) & p_sq_s(a+cp_0q_0)+b+dp_0q_0 \\ c+ap_0^{-1}q_0+p_s^{-1}q_s(d+bp_0^{-1}q_0) & p_sq_s(c+ap_0^{-1}q_0)+d+bp_0^{-1}q_0 \end{pmatrix}. \end{aligned} \tag{4.3}$$

It is easy to check that the function $x \mapsto x^{-1}\operatorname{th}(x/n)$ is decreasing for $x>0$. Since $p_0 \le p_s$, it follows that $p_0^{-1}q_0 \ge p_s^{-1}q_s$. Hence

$$bp_s^{-1}q_s \le bp_0^{-1}q_0, \quad cp_0q_0 \le cp_sq_s \quad \text{and} \quad a(1-p_0^{-1}p_sq_0q_s) \le d(1-p_0p_s^{-1}q_0q_s).$$

Together with (4.3) these inequalities imply (4.2). $\square$

What follows is a crucial observation: the partial order introduced above is stable under two-sided multiplication by state transition matrices, provided that the matrix on the left corresponds to a higher level of potential.

Lemma 4.3. *Suppose that $p_0 \ge p_s$. For $i=1,2$ let a_i, b_i, c_i and d_i be nonnegative numbers such that*

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \le \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then

$$M(p_0)\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} M(p_s) \le M(p_0)\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} M(p_s).$$

Proof. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

We need to prove that

$$M(p_0)\begin{pmatrix} a & b \\ c & d \end{pmatrix} M(p_s) \le \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In view of (4.3) this relation is equivalent to the following four inequalities:

$$\begin{aligned} p_s q_s(a+cp_0q_0)+b+dp_0q_0 &\le 0;\\ c+ap_0^{-1}q_0+p_s^{-1}q_s(d+bp_0^{-1}q_0) &\le 0;\\ p_sq_s(c+ap_0^{-1}q_0)+d+bp_0^{-1}q_0 &\le 0;\\ a+cp_0q_0+p_s^{-1}q_s(b+dp_0q_0)+p_sq_s(c+ap_0^{-1}q_0)+d+bp_0^{-1}q_0 &\le 0. \end{aligned}$$

Since $b,c,d\le 0$ and $a+d\le 0$, the above inequalities follow immediately. □

The final lemma contains a special case $s=2$ of Theorem 4.1.

Lemma 4.4. *Suppose that $p_0\le p_s$. Let a, b, c and d be nonnegative numbers such that $a\ge d$. Then*

$$M(p_s)\begin{pmatrix} a & b\\ c & d\end{pmatrix}M(p_0)\le M(p_0)\begin{pmatrix} a & b\\ c & d\end{pmatrix}M(p_s). \tag{4.4}$$

Proof. In view of (4.3) the relation (4.4) is equivalent to the following four inequalities:

$$\begin{aligned} &p_0q_0(a+cp_sq_s)+b+dp_sq_s\le p_sq_s(a+cp_0q_0)+b+dp_0q_0;\\ &c+ap_s^{-1}q_s+p_0^{-1}q_0(d+bp_s^{-1}q_s)\le c+ap_0^{-1}q_0+p_s^{-1}q_s(d+bp_0^{-1}q_0);\\ &p_0q_0(c+ap_s^{-1}q_s)+d+bp_s^{-1}q_s\le p_sq_s(c+ap_0^{-1}q_0)+d+bp_0^{-1}q_0;\\ &a+cp_sq_s+p_0^{-1}q_0(b+dp_sq_s)+p_0q_0(c+ap_s^{-1}q_s)+d+bp_s^{-1}q_s\\ &\qquad\le a+cp_0q_0+p_s^{-1}q_s(b+dp_0q_0)+p_sq_s(c+ap_0^{-1}q_0)+d+bp_0^{-1}q_0. \end{aligned}$$

One can easily verify the above inequalities by comparing the coefficients of a, b, c and d on both sides and using the inequality $a\ge d$ if necessary. □

Proof of Theorem 4.1. If $\tilde{p}_r=p_r$ for every r, there is nothing to prove. Otherwise there exists t such that $f_t<f_{s-t}$ and $f_r\ge f_{s-r}$ whenever $r<t$. Obviously $0\le t\le\left[\frac{1}{2}(s-1)\right]$. We use backward induction on t.

Base of induction. Suppose $t=\left[\frac{1}{2}(s-1)\right]$. Applying Lemma 4.4 with

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{cases} M(f_{s/2}), & s \text{ is even};\\ \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, & s \text{ is odd};\end{cases}$$

we obtain

$$\prod_{r=t}^{s-t}M(\tilde{p}_r)\le\prod_{r=t}^{s-t}M(p_r).$$

Applying Lemma 4.3 t times, we obtain (4.1).

Step of induction. Let $A=\prod_{r=t+1}^{s-t-1} M(p_r)$. Consider two possibilities.

Case 1. $A_{11}\geq A_{22}$. Let $g_r=p_{s-r}$ for $r\in\{t,s-t\}$, and $g_r=p_r$ otherwise. By Lemma 4.4

$$\prod_{r=t}^{s-t} M(g_r)\leq\prod_{r=t}^{s-t} M(p_r).$$

Lemma 4.3, applied t times, yields

$$\prod_{r=0}^{s} M(g_r)\leq\prod_{r=0}^{s} M(p_r).$$

By the induction hypothesis

$$\prod_{r=0}^{s} M(\tilde{p}_r)\leq\prod_{r=0}^{s} M(g_r),$$

and we are done.

Case 2. $A_{11}<A_{22}$. By Lemma 4.2

$$(M(p_t)AM(p_{s-t}))_{11}\leq(M(p_t)AM(p_{s-t}))_{22}. \tag{4.5}$$

Let $g_r=p_{s-r}$ for $t\leq r\leq s-t$, and $g_r=p_r$ otherwise. In view of (4.5) and Lemma 4.1

$$\prod_{r=t}^{s-t} M(g_r)\leq\prod_{r=t}^{s-t} M(p_r).$$

Lemma 4.3, applied s times, yields

$$\prod_{r=0}^{s} M(g_r)\leq\prod_{r=0}^{s} M(p_r).$$

By the induction hypothesis

$$\prod_{r=0}^{s} M(\tilde{p}_r)\leq\prod_{r=0}^{s} M(g_r),$$

and we are done. $\square$

5. Completion of the proof of Theorem 3.1

Let us recall the setup. Given a sequence of positive numbers $\mathbf{p}=(p_1,\dots,p_n)$, let $\mathcal{M}(\mathbf{p})=\prod_{i=1}^{n} M_n(p_i)$, where the matrices $M_n(p_i)$ are defined by (2.4). Our goal is to prove that for any $|\alpha|\leq 1$ and $m=2,\dots,2n$,

$$\alpha\mathcal{M}(\pi_m\mathbf{p})_{21}+\mathcal{M}(\pi_m\mathbf{p})_{22}\leq\alpha\mathcal{M}(\mathbf{p})_{21}+\mathcal{M}(\mathbf{p})_{22}. \tag{5.1}$$

Here $\pi_m\mathbf{p}$ is the polarized sequence defined by (2.6).

The reason we are interested in $\mathcal{M}(\mathbf{p})$ is that

$$\left(\prod_{i=1}^{n}\cosh(p_i/n)\right)\mathcal{M}(\mathbf{p})=T(0,1;p),$$

where $T(0,1;p)$ is the state transition matrix for equation (1.1) with piecewise constant potential $p=\sum_{i=1}^{n} p_i\chi_{[(i-1)/n,i/n)}$. Since the scalar multiple on the left-hand side is invariant under rearrangements of $\mathbf{p}$, inequality (5.1) is equivalent to

$$\alpha T(0,1;\pi_m p)_{21}+T(0,1;\pi_m p)_{22}\leq\alpha T(0,1;p)_{21}+T(0,1;p)_{22},$$

where $\pi_m p=\sum_{i=1}^{n}(\pi_m\mathbf{p})_i\chi_{[(i-1)/n,i/n)}$ is the piecewise constant function corresponding to the polarized sequence $\pi_m p$. We will first prove (5.1) for $0\leq\alpha\leq 1$, and then use the connection with state transition matrices to extend this result to all $|\alpha|\leq 1$.

Proof of (iii) $\Rightarrow$ (ii). *Step* 1. *The inequality* (5.1) *holds when* $0\leq\alpha\leq 1$.

Fix an integer m between 2 and $2n$. From the definition of $\pi_m\mathbf{p}$ we see that in general not all of the elements of $\mathbf{p}$ are rearranged under polarization. When $m\leq n$, π_m affects only the elements with indices $1,\dots,m-1$; otherwise it affects only the elements with indices $m-n,\dots,n$. We treat these two cases separately.

First, suppose that $m\leq n$. Theorem 4.1 implies

$$\left(\prod_{i=1}^{m-1} M_n((\pi_m\mathbf{p})_i)\right)_{2,l}\leq\left(\prod_{i=1}^{m-1} M_n(p_i)\right)_{2,l},\qquad l=1,2.$$

Therefore, for $j=1,2$ we have

$$\begin{aligned}\mathcal{M}(\pi_m\mathbf{p})_{2,j}&=\sum_{l=1}^{2}\left(\prod_{i=1}^{m-1} M_n((\pi_m\mathbf{p})_i)\right)_{2,l}\left(\prod_{i=m}^{n} M_n(p_i)\right)_{l,j}\\&\leq\sum_{l=1}^{2}\left(\prod_{i=1}^{m-1} M_n(p_i)\right)_{2,l}\left(\prod_{i=m}^{n} M_n(p_i)\right)_{l,j}=\mathcal{M}(\mathbf{p})_{2,j}.\end{aligned}$$

We see that in this case (5.1) holds for any $\alpha \geq 0$.

Now suppose that $m>n$. Introduce the notation

$$A=\prod_{i=1}^{m-n-1} M_n(p_i), \quad B=\prod_{i=m-n}^{n} M_n(p_i) \quad \text{and} \quad \widetilde{B}=\prod_{i=m-n}^{n} M_n((\pi_m \mathbf{p})_i).$$

Obviously, $\mathcal{M}(\mathbf{p})=AB$ and $\mathcal{M}(\pi_m\mathbf{p})=A\widetilde{B}$. By Theorem 4.1 we have $\widetilde{B}\leq B$; furthermore, Lemma 2.1 implies $A_{21}\leq A_{22}$. Combining these inequalities and using the assumption $0\leq\alpha\leq 1$, we obtain

$$\begin{aligned}\alpha(A\widetilde{B})_{21}+(A\widetilde{B})_{22} &= \alpha A_{21}\widetilde{B}_{11}+\alpha A_{22}\widetilde{B}_{21}+A_{21}\widetilde{B}_{12}+A_{22}\widetilde{B}_{22}\\ &= \alpha A_{21}\operatorname{tr}\widetilde{B}+(A_{22}-\alpha A_{21})\widetilde{B}_{22}+A_{22}\widetilde{B}_{21}+A_{21}\widetilde{B}_{12}\\ &\leq \alpha A_{21}\operatorname{tr}B+(A_{22}-\alpha A_{21})B_{22}+A_{22}B_{21}+A_{21}B_{12}\\ &= \alpha A_{21}B_{11}+\alpha A_{22}B_{21}+A_{21}B_{12}+A_{22}B_{22}=\alpha(AB)_{21}+(AB)_{22},\end{aligned}$$

as required.

Step 2. *The inequality* (5.1) *holds when* $-1\leq\alpha<0$. For now we assume the inequality

$$\frac{\mathcal{M}(\pi_m\mathbf{p})_{21}}{\mathcal{M}(\pi_m\mathbf{p})_{22}} \geq \frac{\mathcal{M}(\mathbf{p})_{21}}{\mathcal{M}(\mathbf{p})_{22}}. \tag{5.2}$$

Since (5.1) is already proved for $\alpha=0$, we have $\mathcal{M}(\pi_m\mathbf{p})_{22}\leq\mathcal{M}(\mathbf{p})_{22}$. This inequality together with (5.2) yield

$$\begin{aligned}\alpha\mathcal{M}(\pi_m\mathbf{p})_{21}+\mathcal{M}(\pi_m\mathbf{p})_{22} &= \left(\alpha\frac{\mathcal{M}(\pi_m\mathbf{p})_{21}}{\mathcal{M}(\pi_m\mathbf{p})_{22}}+1\right)\mathcal{M}(\pi_m\mathbf{p})_{22}\\ &\leq \left(\alpha\frac{\mathcal{M}(\mathbf{p})_{21}}{\mathcal{M}(\mathbf{p})_{22}}+1\right)\mathcal{M}(\mathbf{p})_{22}=\alpha\mathcal{M}(\mathbf{p})_{21}+\mathcal{M}(\mathbf{p})_{22}\end{aligned}$$

for $-1\leq\alpha\leq 0$. (Note that the expression in parentheses is nonnegative by virtue of Lemma 2.1.) This completes the proof of Theorem 3.1, modulo inequality (5.2) which will be established in the next section (Remark 6.1). □

Proof of Theorem 2.1. Since the piecewise constant functions are dense in L_+^∞ with L^1 norm, Theorem 3.1 immediately implies part (a) of Theorem 2.1. The part (b) follows by applying (a) to the function $\tilde{p}(x)=p(1-x)$. Indeed, if we denote $T(0,1;p)=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, then $T(0,1;\tilde{p})=T(0,1;p)^{-1}=\left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)$, where the last equality follows from (2.2). □

Remark 5.1. One can define the polarization of functions in the same way as it was done above for sequences (see, e.g., [2]). By virtue of Theorem 3.1, the statements of Theorems 1.1, 1.2 and 2.1 can be expanded to include analogous results for polarized potentials.

6. Proofs of Theorems 1.1 and 1.2

We start with a maximum principle that will be applied to the solutions of the Schrödinger equation with polarized potential. Note that neither p_* nor p^* satisfy the conditions imposed on $\tilde{p}$ in Lemma 6.1, but a polarization of p does.

Lemma 6.1. *Suppose that $p, \tilde{p} \in L_+^\infty[0,1]$ are such that $p \leq \tilde{p}$ on the interval $[0,\gamma]$ and $p \geq \tilde{p}$ on the interval $[\gamma, 1]$, for some $\gamma \in [0,1]$. Suppose further that the following inequalities hold.*

$$T(0,1;\tilde{p})_{21} + T(0,1;\tilde{p})_{22} \leq T(0,1;p)_{21} + T(0,1;p)_{22}; \tag{6.1}$$

$$T(0,1;\tilde{p})_{22} \leq T(0,1;p)_{22}. \tag{6.2}$$

Let u be the solution of (1.1) with $u(0)=0$ and $u'(0)=1$. Let v be the solution of $v'' - \tilde{p}^2 v = 0$ with the same initial data. Then for any $\alpha \in [-1,1]$ one has

$$\frac{u(x)}{\alpha u(1) + u'(1)} \leq \frac{v(x)}{\alpha v(1) + v'(1)}, \quad x \in [0,1]. \tag{6.3}$$

Proof. First we consider the case $\alpha = 1$. By an approximation argument we may assume that p and $\tilde{p}$ are continuous, so that $u, v \in C^2[0,1]$. On the interval $[0,\gamma]$ we have $(u'v - uv')' = (p^2 - \tilde{p}^2)uv \leq 0$, which together with $(u'v - uv')(0) = 0$ imply $u'v - uv' \leq 0$ on $[0,\gamma]$. As a consequence,

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \leq 0 \quad \text{on } [0,\gamma],$$

hence $u(x)/v(x) \leq \lim_{x\to 0} u(x)/v(x) = 1$ whenever $0 \leq x \leq \gamma$. By assumption (6.1) $u(1) + u'(1) \geq v(1) + v'(1)$, so we conclude that inequality (6.3) holds on the interval $[0,\gamma]$ with $\alpha = 1$.

Suppose that the function $w(x) = u(x)/(u(1)+u'(1)) - v(x)/(v(1)+v'(1))$ attains a positive maximum at a point $x_0 \in [\gamma, 1]$. Since $w(1) + w'(1) = 0$, it follows that $x_0 < 1$. But then the inequality

$$w''(x_0) = \frac{p^2(x_0)u(x_0)}{u(1)+u'(1)} - \frac{\tilde{p}^2(x_0)v(x_0)}{v(1)+v'(1)} > 0$$

contradicts the maximality of x_0. Hence $w(x) \leq 0$ for $x \in [0,1]$, which means that (6.3) holds with $\alpha = 1$. In particular, we have

$$\frac{u(1)}{u(1)+u'(1)} \leq \frac{v(1)}{v(1)+v'(1)},$$

which implies

$$\frac{u(1)}{u'(1)} \le \frac{v(1)}{v'(1)}. \tag{6.4}$$

To extend the result to arbitrary $\alpha \in [-1,1]$, note that the function $\varphi(t) = (1+t)/(1+\alpha t)$ is increasing on $[0,1]$. Using (6.4), we deduce

$$\frac{u(x)}{\alpha u(1)+u'(1)} = \frac{u(x)\varphi(u(1)/u'(1))}{u(1)+u'(1)} \le \frac{v(x)\varphi(v(1)/v'(1))}{v(1)+v'(1)} = \frac{v(x)}{\alpha v(1)+v'(1)}$$

for any x in the interval $[0,1]$. $\square$

Remark 6.1. If $\mathbf{p}$ and $\pi_m \mathbf{p}$ are as in Theorem 3.1, then the functions

$$p = \sum_{i=1}^{n} p_i^2 \chi_{[(i-1)/n,i/n)} \quad \text{and} \quad \tilde{p} = \sum_{i=1}^{n} (\pi_m \mathbf{p})_i^2 \chi_{[(i-1)/n,i/n)}$$

satisfy the conditions of Lemma 6.1. Inequality (6.4) then reads as

$$\frac{T(0,1;p)_{21}}{T(0,1;p)_{22}} \le \frac{T(0,1;\tilde{p})_{21}}{T(0,1;\tilde{p})_{22}}, \tag{6.5}$$

which proves (5.2). Notice that there is no circular reasoning here, as we have used only a part of Theorem 3.1 that does not rely on (5.2).

Remark 6.2. Even though in Lemma 6.1 we cannot set $\tilde{p}$ to be p^*, the conclusion of Lemma 6.1 is still true in this case. To see this, take a piecewise constant function p and transform it into p^* by a sequence of polarizations, applying Lemma 6.1 at each step. This gives inequalities (6.3) and (6.5) with $\tilde{p}=p^*$. By approximation, (6.3) and (6.5) are true whenever $p \in L_+^\infty[0,1]$ and $\tilde{p}=p^*$.

Remark 6.3. It is evident that the functions $U(x)=u(x)/(\alpha u(1)+u'(1))$ and $V(x)=v(x)/(\alpha v(1)+v'(1))$ that appear in (6.3) satisfy the boundary conditions $U(0)=0=V(0)$ and $\alpha U(1)+U'(1)=1=\alpha V(1)+V'(1)$. Since Lemma 6.1 holds with $\tilde{p}=p^*$, inequality (6.3) contains a part of Theorem 1.2.

Proof of Theorem 1.1. Let u be the solution of (1.1) with initial values $u(0)=\beta$ and $u'(0)=\alpha$, where $|\alpha|+|\beta|>0$. Changing the sign of u if necessary, we may assume that $\beta \ge 0$. Since $(u(1),u'(1))=(u(0),u'(0))T(0,1;p)$, it follows that

$$u(1) = \beta T(0,1;p)_{11} + \alpha T(0,1;p)_{21}.$$

According to Theorem 2.1, $u(1)$ can change in either direction when p is replaced by p^*. If p is replaced by p_*, then $u(1)$ decreases provided $|\alpha|\leq\beta$. □

In the following proof we consider more general boundary conditions than those in the statement of Theorem 1.2. It turns out that the choice of conditions in Theorem 1.2 is the only possible one, up to normalization.

Proof of Theorem 1.2. Let $T=T(0,1;p)$, where $p\in L^\infty_+[0,1]$. Let u be a solution of (1.1) such that

$$\alpha_0u(0)+\beta_0u'(0)=0 \quad \text{and} \quad \alpha_1u(1)+\beta_1u'(1)=1. \tag{6.6}$$

Define v in the same way with p replaced by p^*. Without loss of generality we may assume that $\beta_1\geq 0$. Equations (6.6) together with $(u(0),u'(0))T=(u(1),u'(1))$ form a linear system, which we solve to find

$$u(0)=\beta_0(\alpha_1\beta_0T_{11}+\beta_0\beta_1T_{12}-\alpha_0\alpha_1T_{21}-\alpha_0\beta_1T_{22})^{-1}.$$

By Theorem 2.1 both of the inequalities $u(0)<v(0)$ and $u(0)>v(0)$ can occur unless $\beta_0=0$. Let $\beta_0=0$, then $u(0)=0=v(0)$ and $u'(0)=(\alpha_1T_{21}+\beta_1T_{22})^{-1}$. Since $u(0)=v(0)$, we must have $u'(0)\leq v'(0)$ in order for the inequality $u(x)\leq v(x)$ to hold for all $x\in[0,1]$. The above formula for $u'(0)$ together with Theorem 2.1 imply that $u'(0)\leq v'(0)$ holds when $|\alpha_1|\leq\beta_1$, but may fail when $|\alpha_1|>\beta_1$. Without loss of generality $\beta_1=1$ and $|\alpha_1|\leq 1$. Now it follows from Remark 6.3 that $u(x)\leq v(x)$ for all $x\in[0,1]$. □

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