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Distance near the origin between elements of a strongly continuous semigroup

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Abstract. Set

$$\theta(s/t) := (s/t-1)(t/s)^{\frac{s/t}{s/t-1}} = (s-t)\frac{t^{t/(s-t)}}{s^{s/(s-t)}}$$

if $0<t<s$. The key result of the paper shows that if $(T(t))_{t>0}$ is a nontrivial strongly continuous quasinilpotent semigroup of bounded operators on a Banach space then there exists $\delta>0$ such that $\|T(t)-T(s)\|>\theta(s/t)$ for $0<t<s\leq\delta$. Also if $(T(t))_{t>0}$ is a strongly continuous semigroup of bounded operators on a Banach space, and if there exists $\eta>0$ and a continuous function $t\mapsto s(t)$ on $[0,\eta]$, satisfying $s(0)=0$, and such that $0<t<s(t)$ and $\|T(t)-T(s(t))\|<\theta(s/t)$ for $t\in(0,\eta]$, then the infinitesimal generator of the semigroup is bounded. Various examples show that these results are sharp.

1. Introduction

In the present paper we are interested in the behavior of $\|T(t)-T(s)\|$ near 0 when the infinitesimal generator A of a strongly continuous semigroup $(T(t))_{t>0}$ of bounded operators on a Banach space X is not bounded on its domain D_A. It is a standard fact that A is bounded on D_A if and only if there exists $P\in\mathcal{B}(X)$ such that $\lim_{t\to0^+}\|P-T(t)\|=0$. In this situation we have $P^2=P$, the projection P is the unit element of the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup and there exists $u\in\mathcal{A}_T$ such that $T(t)=Pe^{tu}$ for $t>0$, so that $D_A=PX$ and $A=u|_{PX}$, see Theorem 9.4.2 in [10].

In particular the semigroup can then be extended to the analytic group of operators $(T(z))_{z\in\mathbf{C}}:=(Pe^{zu})_{z\in\mathbf{C}}$, and we have, for $s,t>0$,

$$(1.1)\qquad \|T(t)-T(s)\| = |s-t|(\|u\|+M(s,t)|s-t|),$$

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where $\sup_{0<s,t\leq 1}|M(s,t)|<+\infty$.

Set

$$\theta(s/t):=(s/t-1)(t/s)^{\frac{s/t}{s/t-1}}=(s-t)\frac{t^{t/(s-t)}}{s^{s/(s-t)}}$$

if $0<t<s$. Notice that $\theta(s/t)$ is the maximum value of x^t-x^s on $[0,1]$ and that it occurs at $m(s,t)=(t/s)^{1/(s-t)}$, so that $\|T(t)-T(s)\|=\theta(s/t)$ for $0<t<s$ if $(T(t))_{t>0}$ is a semigroup of positive operators on the Hilbert space, continuous on $(0,+\infty)$ with respect to the norm of $\mathcal{B}(H)$, for which $\mathrm{Spec}(T(1))=[0,1]$. We will see that this quantity $\theta(s/t)$ plays a crucial role for the behavior of $\|T(t)-T(s)\|$ when the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t>0}$ is not bounded: In this situation, for each continuous function $t\mapsto s(t)$, defined on $[0,\delta]$ for some $\delta>0$ and satisfying $s(0)=0$ and $0<t<s(t)$ for $0<t<\delta$, there exists a sequence $(t_n)_{n\geq 1}$ of positive real numbers such that $\lim_{n\to+\infty}t_n=0$ and such that $\|T(t_n)-T(s(t_n))\|\geq \theta(s_n/t_n)$ for $n\geq 1$. This gives a sharp quantitative formulation of the intuitive fact that $T(t)$ cannot be uniformly too close to $T(s)$ for $s\neq t$, s and t small, when the generator of the semigroup is unbounded.

Let $\hat{\mathcal{A}}_T$ be the character space of the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup and set $\sigma_T=\{|\phi(T(1))|\}_{\phi\in\hat{\mathcal{A}}_T}\cup\{0\}$. In the case when $\hat{\mathcal{A}}_T=\emptyset$, the semigroup is quasinilpotent, i.e. $T(t)$ is quasinilpotent for all (or, equivalently, some) $t>0$. We can distinguish between four situations:

(1) 0 is an isolated point of σ_T, and the semigroup is not quasinilpotent;

(2) there exists $\delta>0$ such that $[0,\delta]\subset\sigma_T$;

(3) 0 is not an isolated point of σ_T, but there exists no $\delta>0$ such that $[0,\delta]\subset\sigma_T$;

(4) $\sigma_T=\{0\}$ (so that the semigroup is quasinilpotent).

It is easy to see that in situation (2) there is $\eta>0$ such that $\|T(t)-T(s)\|\geq \theta(s/t)$ for $0<t<s\leq\eta$.

Our key result (Theorem 2.5) shows that such an inequality still holds, in a slightly stronger form, in situation (4): we prove in Section 2 that if a nontrivial strongly continuous semigroup $(T(t))_{t>0}$ is quasinilpotent, then there exists $\eta>0$ such that $\|T(t)-T(s)\|>\theta(s/t)$ for $0<t<s\leq\eta$, and examples show that this inequality is sharp. In other terms, the fact that elements of a strongly continuous quasinilpotent semigroups have a small norm for large values of t implies that $T(t)$ cannot be too close to $T(s)$ for $s\neq t$ when s and t are sufficiently small.

Easy observations in Section 3 concerning situations (1) and (3) and elementary results from [6] allow then to deduce from Theorem 2.5 the following results:

(1) If the algebra $\mathcal{A}_T$ has no nonzero idempotent then there exists $\eta>0$ such that $\|T(t)-T(s)\|\geq\theta(s,t)$ for $0<t<s\leq\eta$.

(2) More precisely, if there exists two sequences $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$ of positive real numbers such that $0<t_n<s_n$, $\lim_{n\to+\infty}s_n=0$, and such that $\|T(t_n)-T(s_n)\|<$

$\theta(s_n/t_n)$, then there exists a nondecreasing sequence $(P_n)_{n\geq 1}$ of idempotents of $\mathcal{A}_T$ such that $\phi(P_n)=1$ when n is sufficiently large for every $\phi\in\hat{\mathcal{A}}_T$.

If, further, the semigroup is norm-continuous, i.e. continuous on $(0,+\infty)$ with respect to the norm of $\mathcal{B}(X)$, then the sequence $(P_n)_{n\geq 1}$ satisfies also the following two conditions:

(i) $\bigcup_{n\geq 1} P_n\mathcal{A}_T$ is dense in $\mathcal{A}_T$;

(ii) $\lim_{t\to 0^+}\|P_nT(t)-P_n\|=0$ for $n\geq 1$, so that the infinitesimal generator of the semigroup $(P_nT(t))_{t>0}$ is bounded for $n\geq 1$.

(3) If there exists $\delta>0$ and a continuous function $s\colon[0,\delta]\to[0,+\infty]$ satisfying $s(0)=0$, $0<t<s(t)$ and $\|T(t)-T(s(t))\|<\theta(s(t)/t)$ for $0<t\leq\delta$, then the infinitesimal generator of the semigroup is bounded, and so $\|T(t)-T(s)\|$ satisfies (1.1).

Partial results in this direction were obtained recently by A. Mokhtari and the author in [7], where it is shown that any nontrivial semigroup $(a^t)_{t>0}$ in a Banach algebra satisfying $\limsup_{t\to 0^+}\|a^t-a^{(n+1)t}\|<n/(n+1)^{(1+1/n)}=\theta(n+1)$ for some integer $n\geq 1$ has a limit in norm at the origin (the case $n=2$ for norm-continuous semigroups bounded near the origin goes back to [13], and related results concerning distances between powers of bounded approximate identities can be found in [1]). These results are based on the fact that $\|u-u^{n+1}\|>n/(n+1)^{(1+1/n)}$ for any quasinilpotent element of a Banach algebra such that $\|u\|\geq 1/(n+1)^{1/n}$, proved in [7] for $n\geq 3$ (see [5] for the case $n=2$). The idea to prove this inequality is very simple: there exists a sequence $(a_p)_{p\geq 1}$ of nonnegative real numbers such that we have, for $|x|\leq 1/(n+1)^{1/n}$,

$$x=\sum_{p=1}^{+\infty} a_p(x-x^{n+1})^p.$$

Now if u is quasinilpotent, we have

$$u=\sum_{p=1}^{+\infty} a_p(u-u^{n+1})^p.$$

Since $h(x):=\sum_{n=1}^{+\infty} a_px^p$ is not a polynomial, we obtain, when $\|u-u^{n+1}\|\leq\theta(n+1)$,

$$\|u\|<h(\theta(n+1))=\frac{1}{(n+1)^{1/n}}$$

(similar ideas have been used independently by N. Kalton, S. Montgomery-Smith, K. Oleskiewicz and Y. Tomilov in [11] to obtain other types of inequalities). This implies immediately that if $(T(t))_{t>0}$ is a nontrivial quasinilpotent strongly continuous semigroup then there exists $\delta>0$ such that $\|T(t)-T((n+1)t)\|>\theta(n+1)$ for $0<t\leq\delta$.

The starting point of the proof of Theorem 2.5 is similar, but the result seems deeper. Let $\alpha>1$. There exists a sequence $(\mu_{\alpha,p})_{p\geq 1}$ of nonnegative real numbers such that we have, for $x\in[0,1/\alpha^{\alpha/(\alpha-1)}]$

$$x=(x-x^{\alpha})\sum_{p=0}^{+\infty}\mu_{\alpha,p}(x-x^{\alpha})^{p(\alpha-1)}.$$

Now let $(T(t))_{t>0}$ be a nontrivial quasinilpotent semigroup, and let $0<t<s$. It is easy to define $(T(t)-T(s))^r$ in a natural way for $r>0$, and if we set $\alpha=s/t$, we obtain the formula

$$T(t)=[T(t)-T(s)]\left(\sum_{p=0}^{+\infty}\mu_{\alpha,p}(T(t)-T(s))^{p(\alpha-1)}\right). \tag{1.2}$$

The difficulty when α is not an integer is that to control $\|T(t)-T(s)\|$ gives no control on $\|T(t)-T(s)\|^{\alpha-1}$, and no condition on $\|T(t)-T(s)\|$ will guarantee that $\|T(t)\|$, or $\|T(s)\|$, is small. To circumvent that fact we first use a classical method due to Feller [8] to reduce the problem to the case of quasinilpotent semigroups of contractions, and then use the fact that the semigroup acts smoothly on elements of the domain of its infinitesimal generator. Since

$$\frac{\theta(s/t)}{s-t}\geq\frac{1}{es},$$

this allows to compensate the loss of information which appears when we control $\|(T(t)-T(s))^{\lfloor p(\alpha-1)\rfloor}\|$ for $p\geq 1$ instead of $\|(T(t)-T(s))^{p(\alpha-1)}\|$. We can then construct nonzero vectors y in X for which there exists a constant $C(y)$ satisfying $\|T(s)y\|\leq C(y)s$ for all $s\in(0,1]$ such that there exists $t\in(0,s)$ for which $\|T(t)-T(s)\|\leq\theta(s/t)$, and Theorem 2.5 follows.

Various examples show that these results are optimal for the behavior near the origin of $\|T(t)-T(s)\|$ for norm-continuous semigroups of bounded operators on the Hilbert space. Other phenomena may occur on special Banach spaces. For example if X is a hereditarily indecomposable Banach space in the sense of Gowers and Maurey [9], Räbiger and Ricker showed in [15] that the infinitesimal generator of any C_0-group of bounded operators on X is bounded (this result extends to some classes of C_0-semigroups on X, see [16]).

When a semigroup $(T(t))_{t>0}$ is strongly continuous but not norm-continuous on $(0,+\infty)$, different phenomena appear: it follows from a simple argument of Blake [2] that if we set $\Delta(t)=\limsup_{h\to 0^+}\|T(t+h)-T(t)\|$, then $\Delta(s+t)\leq\Delta(s)\Delta(t)$ for $s,t>0$, and so $\liminf_{t\to 0^+}\Delta(t)\geq 1$ (in fact an elementary argument shows that

$\liminf_{t\to 0^+} \Delta(t) \geq \sqrt{3}$, see [6]), while $\theta(s/t)<1$ for $0<t<s$ and $\lim_{h\to 0^+} \theta((t+h)/t)=0$ for $t>0$. Notice that a nice characterization in terms of the resolvent of the infinitesimal generator of C_0-semigroups of bounded operators on the Hilbert space which are norm-continuous on $(0,+\infty)$ is given in [18], see also [3], and that a well-known result of Kato [12] shows that a strongly continuous semigroup $(T(t))_{>0}$ satisfying $\limsup_{t\to 0^+} \|I-T(t)\|<2$ admits a holomorphic extension to a sector $U_\theta=\{z\neq 0 \,|\, |\arg(z)|<\theta\}$ for some $\theta>0$.

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2. Quasinilpotent semigroups

We begin this section by using a well-known method due to Feller [8] which allows us to restrict attention to quasinilpotent C_0-semigroups of contractions (a semigroup $(T(t))_{t>0}$ will be said to be nontrivial if $T(t)\neq 0$ for some $t>0$).

Lemma 2.1. *Let $(T(t))_{t>0}$ be a nontrivial, quasinilpotent, strongly continuous semigroup of bounded operators on a Banach space X, let $t_0>0$ such that $T(t_0)\neq 0$, and let $\omega>0$. Then there exists a Banach space Y and a strongly continuous semigroup $(T_1(t))_{t>0}$ of bounded operators on Y satisfying the following conditions*:

(1) $T_1\left(\frac{1}{3}t_0\right)\neq 0$;
(2) $\|T_1(t)\|\leq e^{-\omega t}$ *for* $t>0$;
(3) $\lim_{t\to 0^+} \|T_1(t)y-y\|=0$ *for* $y\in Y$;
(4) $\|T_1(t)-T_1(s)\|\leq \|T(t)-T(s)\|$ *for* $s,t>0$.

Proof. Let $F:=\bigcap_{t>0} \ker T(t)$ and let further $X_0:=X/F$ be the quotient Banach space equipped with the quotient norm $\|x+F\|_0=\inf_{f\in F} \|x+f\|$. The formula $T_0(t)(x+F):=T(t)x+F$ defines a semigroup on X_0 such that $\|T_0(t)\|_0\leq \|T(t)\|$ and $\|T_0(t)-T_0(s)\|_0\leq \|T(t)-T(s)\|$ for $s,t>0$. The semigroup $(T_0(t))_{t>0}$ is strongly continuous, quasinilpotent, and satisfies $\bigcap_{t>0} \ker T_0(t)=\{0\}$.

Set $X_1:=\bigcup_{t>0} T_0(t)X_0$. For $x\in X_1$, we have $\lim_{t\to 0^+} T_0(t)x=x$. Since the semigroup $(T_0(t))_{t>0}$ is quasinilpotent, we have $\lim_{t\to +\infty} \|T_0(t)\|^{1/t}=0$, and we can define a norm on X_1 by using the formula

$$\|x\|_1 := \sup_{h>0} e^{\omega h} \|T_0(h)x\|_0.$$

We have $\|T_0(t)x\|_1 \le e^{-\omega t}\|x\|_1$ and

$$\|T_0(t)x - T_0(s)x\|_1 \le \|T_0(t) - T_0(s)\|_0 \|x\|_1 \le \|T(t) - T(s)\|\,\|x\|_1 \quad \text{for } x \in X_1.$$

Define $T_0(0)$ to be the identity map on X_1. For $s,t \ge 0$, $x \in X_1$, we have

$$\|T_0(t)x - T_0(s)x\|_1 \le \max\Big\{ e^{\omega} \sup_{0<h<1} \|T_0(t+h)x - T_0(s+h)x\|_0, \sup_{h \ge 1} e^{\omega h} \|T_0(h)\|_0 \|T_0(t)x - T_0(s)x\|_0 \Big\}.$$

A uniform continuity argument shows then that the map $t \mapsto T_0(t)(x)$ is a continuous map from $[0, +\infty)$ into $(X_1, \|\cdot\|_1)$ for $x \in X_1$.

Let $x \in X$ be such that $T(\frac{1}{3}t_0)T(\frac{2}{3}t_0)x = T(t_0)x \ne 0$. Set $y := T(\frac{1}{3}t_0)x + F \in X_1$. Then $T_0(\frac{1}{3}t_0)y = T(\frac{2}{3}t_0)x + F$. Since $T(\frac{2}{3}t_0)x \notin F$, we have $T_0(\frac{1}{3}t_0)y \ne 0$.

Now denote by Y the completion of $(X_1, \|\cdot\|_1)$, and for $t>0$ denote by $T_1(t)$ the continuous extension of $T_0(t)$ to Y. The semigroup $(T_1(t))_{t>0}$ clearly satisfies the conditions of the lemma. □

Let $\alpha > 1$, and consider the function $x \mapsto x - x^{\alpha}$, which is strictly increasing on the interval $[0, 1/\alpha^{1/(\alpha-1)}]$. The inverse function $h: [0, (\alpha-1)/\alpha^{\alpha/(\alpha-1)}] \to [0, 1/\alpha^{1/(\alpha-1)}]$ satisfies the identity

$$h(x) - h(x)^{\alpha} = x.$$

Now since $h'(0) = 1$, we can set $h(x) = xe^{v(x)}$. We obtain $e^{v(x)} - x^{\alpha-1}e^{\alpha v(x)} = 1$. If we set $g(x) = v(x^{1/(\alpha-1)})$, then g is defined on $[0, (\alpha-1)^{\alpha-1}/\alpha^{\alpha}]$ and satisfies

$$e^{g(x)} - xe^{\alpha g(x)} = 1.$$

The following elementary lemma shows in particular that g admits an analytic extension to the open unit disc $D(0, (\alpha-1)^{\alpha-1}/\alpha^{\alpha})$.

Lemma 2.2. *Let $\alpha > 1$, and set $r_\alpha = \log(\alpha/(\alpha-1))$ and $R_\alpha = (\alpha-1)^{\alpha-1}/\alpha^{\alpha}$. There exists an analytic function $g_\alpha: D(0, R_\alpha) \to D(0, r_\alpha)$ such that $g_\alpha(0) = 0$ and $e^{g_\alpha(z)} - ze^{\alpha g_\alpha(z)} = 1$ for $|z| < R_\alpha$. Moreover $g_\alpha^{(k)}(0) > 0$ for every $k \ge 1$ and*

$$\sum_{k=1}^{+\infty} \frac{g_\alpha^{(k)}(0)}{k!} R_\alpha^k = r_\alpha.$$

Proof. Consider the entire function $F: z \mapsto e^{-(\alpha-1)z} - e^{-\alpha z}$. Since $F(0)=0$ and $F'(0)=1$, there exist open neighborhoods V and W of 0 such that F maps V onto W conformally. Denote by $G:=F^{-1}: W \to V$ the inverse map, so that $G(0)=0$, $G'(0)=1$, and for $|z| \le R$, set

$$g_\alpha(z) = \sum_{n=1}^{+\infty} \frac{G^{(n)}(0)}{n!} z^n,$$

where R denotes the radius of convergence of the series above.

By analytic continuation, we have $e^{-(\alpha-1)g_\alpha(z)} - e^{-\alpha g_\alpha(z)} = z$ for $|z| < R$.

The fact that $R = R_\alpha$ can be deduced from Lemma 3.1 of [7], but we will here give a shorter and more natural argument. Denote by f the restriction of F to $(-\infty, r_\alpha]$. A computation shows that $f(r_\alpha) = R_\alpha$, $f'(r_\alpha) = 0$ and that $f'(x) > 0$ for $x < r_\alpha$. So f is a bijection from $(-\infty, r_\alpha]$ onto $(-\infty, R_\alpha]$. Denote by $g: (-\infty, R_\alpha] \to (-\infty, r_\alpha]$ the inverse map. We have $g(x) = F^{-1}(x)$ when $|x|$ is sufficiently small, and so $g^{(n)}(0) = G^{(n)}(0)$ for $n \ge 1$.

For $x < R_\alpha$, we obtain

$$e^{g(x)} = 1 + x e^{\alpha g(x)} \quad \text{and} \quad g'(x) = \alpha x g'(x) e^{(\alpha-1)g(x)} + e^{(\alpha-1)g(x)}.$$

Now for $n \ge 2$, an immediate induction shows that we have

$$\begin{aligned} g^{(n)}(x) &= \alpha x g^{(n)}(x) e^{(\alpha-1)g(x)} \\ &\quad + e^{(\alpha-1)g(x)} \big(P_n(g'(x), \dots, g^{(n-1)}(x)) + x Q_n(g'(x), \dots, g^{(n-1)}(x))\big), \end{aligned}$$

where $P_n(X_1, \dots, X_{n-1})$ and $Q_n(X_1, \dots, X_{n-1})$ are polynomials in $n-1$ variables with nonnegative coefficients.

For $0 \le x < R_\alpha$, we have $\alpha x e^{(\alpha-1)g(x)} < \alpha R_\alpha e^{(\alpha-1)r_\alpha} = 1$. We thus see by induction that $g^{(n)}(x) > 0$ for $n \ge 1$ and $x \in [0, R_\alpha)$. Using the Taylor–Lagrange formula, we obtain, for $p \ge 1$,

$$\sum_{n=1}^{p} \frac{g^{(n)}(0)}{n!} R_\alpha^n = \lim_{x \to R_\alpha -} \sum_{n=1}^{p} \frac{g^{(n)}(0)}{n!} x^n \le \lim_{x \to R_\alpha -} g(x) = g(R_\alpha) = r_\alpha.$$

We thus see that $R \ge R_\alpha$, and the series $\sum_{n=1}^{+\infty} g^{(n)}(0) z^n / n!$ is absolutely convergent for $|z| = R_\alpha$. Since $\lim_{x \to R_\alpha^-} g'(x) = +\infty$, we have $R = R_\alpha$. If we extend g_α by continuity to the closed disc $\overline{D}(0, R_\alpha)$ we see that $g(x) = g_\alpha(x)$ for $x \in [-R_\alpha, R_\alpha]$, and so $g_\alpha(R_\alpha) = r_\alpha$. □

We see that $x = (x - x^\alpha) e^{g_\alpha((x-x^\alpha)^{\alpha-1})}$ for $x \in [0, 1/\alpha^{1/(\alpha-1)}]$, and we wish to obtain an analogous formula allowing to compute $T(t)$ as a function of $T(t) - T(\alpha t)$ when $(T(t))_{t>0}$ is a quasinilpotent strongly continuous semigroup. We will need the following observation.

Lemma 2.3. *Let $\mathcal{A}$ be a commutative unital Banach algebra, and let $\alpha>1$. Then the map $u\mapsto e^{-(\alpha-1)u}-e^{-\alpha u}$ is one-to-one on* $\mathrm{Rad}(\mathcal{A})$.

Proof. Set again $F(z)=e^{-(\alpha-1)z}-e^{-\alpha z}$. The function G defined by the formulae $G(z_1,z_2):=(F(z_1)-F(z_2))/(z_1-z_2)$ if $z_1\neq z_2$ and $G(z_1,z_1):=F'(z_1)$ is analytic on $\mathbf{C}^2$ and satisfies $F(z_1)-F(z_2)=(z_1-z_2)G(z_1,z_2)$ for $(z_1,z_2)\in\mathbf{C}^2$. Now let $u,v\in \mathrm{Rad}(\mathcal{A})$. We have $F(u)-F(v)=(u-v)G(u,v)$. Since $\chi(G(u,v))=G(\chi(u),\chi(v))=F'(0)\neq 0$ for every $\chi\in\hat{\mathcal{A}}$, $G(u,v)$ is invertible in $\mathcal{A}$, and the result follows. □

Proposition 2.4. *Let $(T(t))_{t>0}$ be a strongly continuous quasinilpotent semigroup of bounded operators on a Banach space X. The formula*

$$(T(t)-T(s))^{\tau}:=T(\tau t)\left(I+\sum_{k=1}^{+\infty}(-1)^k\frac{\tau\ldots(\tau-k+1)}{k!}T(k(s-t))\right)$$

defines a strongly continuous quasinilpotent semigroup

$$((T(t)-T(s))^{\tau})_{\tau>0}$$

of bounded operators on X, and we have, for $0<t<s$,

$$T(t)=(T(t)-T(s))\exp g_{s/t}((T(t)-T(s))^{s/t-1}),$$

where $g_{s/t}$ is the function given by Lemma 2.2.

Proof. We have $\log(1-z):=-\sum_{k=1}^{+\infty}z^k/k$ for $|z|<1$. We get

$$1+\sum_{k=1}^{+\infty}(-1)^k\frac{\tau\ldots(\tau-k+1)}{k!}z^k=e^{\tau\log(1-z)}.$$

Hence if $S\in\mathcal{B}(X)$, and if $\varrho(S)<1$, the formula

$$(I-S)^{\tau}=\left(I+\sum_{k=1}^{+\infty}(-1)^k\frac{\tau\ldots(\tau-k+1)}{k!}S^k\right)\quad(\tau\in\mathbf{C})$$

defines an analytic group of operators on X, and $((T(t)-T(s))^{\tau})_{\tau>0}$ is a strongly continuous quasinilpotent semigroup of bounded operators on X.

Set $\alpha=s/t$, $U=T(t)-T(s)$, $V=\sum_{k=1}^{+\infty}T(k(s-t))/k$, $R:=(T(t)-T(s))^{s/t-1}$ and $W:=g_{s/t}(R)$. It follows from Lemma 2.2 that $e^W-Re^{\alpha W}=I$. Moreover $UR=U^{s/t}=T(s)e^{-sV/t}$, and $U=U^1=T(t)e^{-V}$. We obtain $U=Ue^V-URe^{sV/t}$, and so $U(e^V-Re^{sV/t}-I)=0$. Notice that V and W are quasinilpotent.

Let $\mathcal{A}_T$ be the closed unital subalgebra of $\mathcal{B}(X)$ generated by $(T(t))_{t\geq 0}$ and set $\mathcal{I}:=\{S\in\mathcal{A}_T \mid US=0\}$, so that $\mathcal{I}$ is a closed ideal of $\mathcal{A}_T$. Denote by $\pi: \mathcal{A}_T\to \mathcal{A}_T/\mathcal{I}$ the canonical map and set $r:=\pi(R)$, $v:=\pi(V)$, $w:=\pi(W)$ and $1:=\pi(I)$. Then $1=e^v-re^{\alpha v}=e^w-re^{\alpha w}$, and $r=e^{-(\alpha-1)v}-e^{-\alpha v}=e^{-(\alpha-1)w}-e^{-\alpha w}$. Since v and w are quasinilpotent elements of the commutative unital Banach algebra $\mathcal{A}_T/I$, Lemma 2.3 shows that $v=w$, i.e. $U(V-W)=0$. So $U(e^V-e^W)=0$, and $T(t)=Ue^W$. □

We are now ready to prove the key result of the paper.

Theorem 2.5. *Let $(T(t))_{t>0}$ be a nontrivial strongly continuous semigroup of bounded operators on a Banach space X. If $(T(t))_{t>0}$ is quasinilpotent, then there exists $\delta>0$ such that $\|T(t)-T(s)\|>\theta(s/t)$ for $0<t<s<\delta$.*

Proof. Apply Lemma 2.1 with $\omega=3/t_0$ and set $S(t):=T_1(t/\omega)$ for $t>0$. Then $(S(t))_{t>0}$ is a strongly continuous quasinilpotent semigroup on Y which satisfies the following properties:

(1) $S(1)\neq 0$;
(2) $\|S(t)\|\leq e^{-t}$ for $t>0$;
(3) $\lim_{t\to 0^+}\|S(t)y-y\|$ for $y\in Y$;
(4) $\|S(\omega t)-S(\omega s)\|\leq\|T(t)-T(s)\|$ for $t,s>0$.

Let $D(A)$ be the domain of the infinitesimal generator A of the semigroup $(S(t))_{t>0}$. Then $D(A^2):=\{y\in D(A)\mid Ay\in D(A)\}$ is a dense subspace of Y (see for example Proposition 1.8 in [4]). If $y\in D(A)$, we have $(S(t)-S(s))y=\int_s^t S(u)Ay\,du$, and so $\|(S(t)-S(s))y\|\leq\|Ay\|(s-t)$ for $s>t>0$. Since $A(S(t)-S(s))=(S(t)-S(s))A$, we see that $\|(S(t)-S(s))^2y\|\leq\|A^2y\|(s-t)^2$ for $y\in D(A^2)$ and $s>t>0$.

Fix $0<t<s<1$, and set $U=S(t)-S(s)$, $\alpha=s/t$ and $\gamma=\alpha-1$. It follows from Proposition 2.4 that $S(t)=U\exp g_\alpha(U^\gamma)$. Set $h=e^{g_\alpha}$. It follows from Lemma 2.2 that $h(z)=\sum_{k=0}^{+\infty}a_kz^k$ for $|z|\leq R_\alpha$, with $a_k>0$ for $k\geq 0$, and that $\sum_{k=0}^{+\infty}a_kR_\alpha^k=e^{r_\alpha}$. We obtain

$$S(t)=U\sum_{k=0}^{+\infty}a_k(U^\gamma)^k=\sum_{k=0}^{+\infty}a_kU^{k\gamma+1}=\sum_{k=0}^{+\infty}a_kU^{\lfloor k\gamma\rfloor+2-\lambda_k},$$

where $\lfloor k\gamma\rfloor\in\mathbf{Z}^+$ satisfies $\lfloor k\gamma\rfloor\leq k\gamma<\lfloor k\gamma\rfloor+1$. Hence $\lambda_k:=\lfloor k\gamma\rfloor-k\gamma+1\in]0,1]$.

Set $S:=S(1)$, and let $\lambda\in]0,1]$ and $y\in D(A^2)$. A simple computation gives

$$U^{2-\lambda}Sy=S(1-\lambda t)(I-S(s-t))^{-\lambda}U^2y.$$

We know that $\|U^2y\| \le \|A^2y\|(s-t)^2$, and we have

$$\begin{aligned}\|(I-S(s-t))^{-\lambda}\| &= \left\| I+\sum_{k=1}^{+\infty}(-1)^k \frac{(-\lambda)\dots(-\lambda-k+1)}{k!}S(k(s-t))\right\| \\ &= \left\| I+\sum_{k=1}^{+\infty}\frac{\lambda\dots(\lambda+k-1)}{k!}S(k(s-t))\right\| \\ &\le 1+\sum_{k=1}^{+\infty}\frac{\lambda\dots(\lambda+k-1)}{k!}e^{-k(s-t)} \\ &= (1-e^{-(s-t)})^{-\lambda} \\ &\le e^{\lambda(s-t)}(s-t)^{-\lambda}.\end{aligned}$$

We obtain

$$\|U^{2-\lambda}Sy\| \le \|A^2y\|e^{\lambda t-1}e^{\lambda(s-t)}(s-t)^{2-\lambda} \le \|A^2y\|(s-t)^{2-\lambda}$$

for $0<t<s<1$, $\lambda\in]0,1]$ and $y\in D(A^2)$.

If $\|U\|\le\theta(\alpha)$, we have

$$\begin{aligned}\|S(t)Sy\| &\le \|A^2y\|\sum_{k=0}^{+\infty}a_k\theta(\alpha)^{\lfloor k\gamma\rfloor}(s-t)^{2-\lambda_k} \\ &= \|A^2y\|(s-t)\sum_{k=0}^{+\infty}a_k\theta(\alpha)^{k\gamma}\left(\frac{s-t}{\theta(\alpha)}\right)^{1-\lambda_k}.\end{aligned}$$

We have $(s-t)/\theta(s/t)\le e$. Also $\theta(\alpha)^\gamma = R_\alpha$, and so $\sum_{k=0}^{+\infty}a_k\theta(\alpha)^{k\gamma}=e^{r_\alpha}=s/(s-t)$.

We obtain

$$\|S(t)Sy\|\le e\|A^2y\|s$$

if $y\in D(A^2)$, $s\in(0,1)$, and if $t\in(0,s)$ satisfies $\|S(t)-S(s)\|\le\theta(s/t)$.

Since $S\ne0$ and since $D(A^2)$ is dense in Y, there exists $y\in D(A^2)$ such that $Sy\ne0$. Since the semigroup $(S(t))_{t>0}$ is strongly continuous, there exists $\eta>0$ such that $\|S(t)Sy\|\ge\frac{1}{2}\|Sy\|$ for $t\in]0,\eta[$. Set $\delta:=\min\{\eta,1,\|Sy\|/2e(\|A^2y\|+1)\}>0$. For $0<t<s<\delta$, we have $\|S(t)Sy\|>e\|A^2y\|s$, and so $\|S(t)-S(s)\|>\theta(s/t)$. Hence $\|T(t)-T(s)\|\ge\|S(3t/t_0)-S(3s/t_0)\|=\theta(s/t)$ if $0<t<s<\frac{1}{3}t_0\delta$. □

The following example shows that the lower estimate for $\|T(t)-T(s)\|$ near the origin for quasinilpotent strongly continuous semigroups given by Theorem 2.5 is sharp.

Theorem 2.6. *Let $\varepsilon\colon(0,1)\to(0,+\infty)$ be a nondecreasing function. Then there exists a nontrivial quasinilpotent, norm-continuous semigroup $(T_\varepsilon(t))_{t>0}$ of bounded operators on the separable Hilbert space which satisfies, for $0<t<s\le 1$,*

$$\|T_\varepsilon(t)-T_\varepsilon(s)\|\le\theta(s/t)+(s-t)\varepsilon(s).$$

Proof. Let $\mathcal{A}(\mathbf{D})$ be the usual disc algebra, i.e. the algebra of functions analytic on the open unit disc $\mathbf{D}$ which admit a continuous extension to the closed unit disc $\overline{\mathbf{D}}$, equipped with the norm $\|\phi\|=\max_{z\in\overline{\mathbf{D}}}|\phi(z)|=\sup_{z\in\mathbf{D}}|\phi(z)|$. The Banach algebra $\mathcal{A}(\mathbf{D})$ is a closed subalgebra of the Banach algebra $H^\infty(\mathbf{D})$ of bounded holomorphic functions on $\mathbf{D}$.

If h is a nonnegative function on $(0,1)$, we will use the notation

$$\Omega_h:=\{z=x+iy\in\mathbf{C}\mid 0<x<1 \text{ and } 0<y<h(x)\}.$$

We will first associate a nontrivial norm-continuous semigroup in $\mathcal{A}(\mathbf{D})$ to each continuous function $f\colon[0,1]\to[0,+\infty)$ such that $f(0)=f(1)=0$ and $f(x)>0$ for $x\in(0,1)$. Set $g(x)=x+if(x)$ for $x\in(0,1)$, and $g(x)=2-x$ for $x\in(1,2)$. We obtain a Jordan curve, and there exists a conformal mapping G from the open unit disc $\mathbf{D}$ onto the interior Ω_f of this Jordan curve. By Caratheodory's theorem, G extends to a homeomorphism from $\overline{\mathbf{D}}$ onto $\overline{\Omega}_f$, which maps the unit circle onto $\partial\Omega_f=g([0,2])$. Since $0\in\overline{\Omega}_f$, $|G^{-1}(0)|=1$. Set $F(z)=G(G^{-1}(0)z)$ for $z\in\overline{\mathbf{D}}$, so that $F(1)=0$. Then F is also a homeomorphism from $\overline{\mathbf{D}}$ onto $\overline{\Omega}_f$, and the restriction of F to $\mathbf{D}$ is a conformal mapping from $\mathbf{D}$ onto Ω_f. Using the principal determination of the logarithm we now define $F^t(z)$ for $z\in\overline{D}$, $t>0$, by the formula

$$F^t(z)=\begin{cases} e^{t\log F(z)}, & z\in\overline{D}\setminus\{1\},\\ 0, & z=1.\end{cases}$$

It follows from the definition of $F^t(z)$ that $F^{s+t}(z)=F^s(z)F^t(z)$ for $s>0$, $t>0$ and $|z|\le 1$. The function F^t is clearly continuous on $\overline{\mathbf{D}}\setminus\{1\}$ and analytic on $\mathbf{D}$. Since $|F^t(z)|=|F(z)|^t$ for $t>0$ and $|z|\le 1$, and since $F(1)=0$, we see that F^t is also continuous at 1, and $F^t\in\mathcal{M}:=\{H\in\mathcal{A}(\mathbf{D})\,|\,H(1)=0\}$ for $t>0$.

For $\eta>0$ set $V_\eta=\{z\in\overline{\mathbf{D}}\,\big|\,|z-1|<\eta\}$. Fix $t>0$ and $\varepsilon>0$. There exists $\eta>0$ such that $|F^s(z)|<\frac{1}{2}\varepsilon$ for $z\in V_\eta$, $s\in\left(\frac{1}{2}t,\frac{3}{2}t\right)$. Since $\min_{z\in\overline{\mathbf{D}}\setminus V_\eta}|F(z)|>0$, the set $\{\log F(z)\,|\,z\in\overline{\mathbf{D}}\setminus V_\eta\}$ is compact, and $\lim_{s\to t}\sup_{z\in\overline{\mathbf{D}}\setminus V_\eta}|F^s(z)-F^t(z)|=0$. These two observations show that the map $t\mapsto F^t$ is continuous on $(0,+\infty)$ with respect to the norm of $\mathcal{A}(\mathbf{D})$.

Consider the singular inner function $\Psi(z)=e^{(z+1)/(z-1)}$, and denote by P the orthogonal projection of $H^2(\mathbf{D})$ onto $H^2(\mathbf{D})\ominus\Psi H^2(\mathbf{D})$.

Now set

$$T(t)f = PF^t f, \quad f \in H^2(\mathbf{D}) \ominus \Psi H^2(\mathbf{D}).$$

A standard verification shows that we have $\|T(t)\| = \|\pi(F^t)\|$, where $\pi: \mathcal{M} \to \mathcal{M}/\Psi\mathcal{M}$ is the canonical surjection. So $(T(t))_{t>0}$ is norm-continuous on $(0, +\infty)$. Since $\bigcap_{n=1}^{\infty} \psi^n \mathcal{M} = \{0\}$, and since the quotient algebra $\mathcal{M}/\Psi\mathcal{M}$ is radical, $(T(t))_{t>0}$ is a nontrivial quasinilpotent semigroup. So in order to prove the proposition it suffices to construct a function f such that $|z^t - z^s| \le \theta(s/t) + (s-t)\varepsilon(s)$ for $z \in \bar{\Omega}_f$ and $0 < t < s \le 1$.

To perform some elementary computations we will use polar coordinates. Set $\Delta := \{z \in \mathbf{C} \,|\, |z| \le 1 \text{ and } 0 \le \arg z \le \frac{1}{20}\pi\}$, and let $z = x + iy = re^{i\alpha} \in \Delta$, with $0 \le \alpha \le \frac{1}{20}\pi$. We have, for $0 < t < s \le 1$,

$$\begin{aligned} |z^t - z^s|^2 &= r^{2t}|1 - z^{s-t}|^2 \\ &= r^{2t}\big((1 - r^{s-t})^2 + 4r^{s-t}\sin^2\big(\tfrac{1}{2}(s-t)\alpha\big)\big) \le (r^t - r^s)^2 + r^{s+t}(s-t)^2\alpha^2. \end{aligned}$$

We obtain

$$|z^t - z^s| \le r^t - r^s + (s-t)\alpha \le (s-t)\big[\log(1/r) + \tfrac{1}{20}\pi\big].$$

Since $\theta(s/t) \ge (s-t)/es \ge (s-t)/e$, and as $\frac{1}{20}\pi < \frac{1}{6} < 1/2e$, we have $|z^t - z^s| \le \theta(s/t)$ if $r \ge e^{-1/6s}$. In particular, $|z^t - z^s| \le \theta(s/t)$ if $x \ge e^{-1/6s}$, which is satisfied for $s \in (0, 1]$ if $x \ge e^{-1/6}$.

Also, since $r^t - r^s \le \theta(s/t)$, and as $\alpha \le \tan\alpha$, we obtain

$$|z^t - z^s| \le \theta\left(\frac{s}{t}\right) + (s-t)\varepsilon(s)$$

if $\tan\alpha \le \varepsilon(s)$. But $\varepsilon(s) \ge \varepsilon(-1/6\log x)$ if $0 < x \le e^{-1/6s}$. So if $z \in \Delta$, and if $\tan\alpha \le \varepsilon(-1/6\log x)$, we have $|z^t - z^s| \le \theta(s/t) + (s-t)\varepsilon(s)$. Finally if we set $\varepsilon_1(0) = 0$,

$$\begin{aligned} \varepsilon_1(x) &= x\min\left\{\varepsilon\left(\frac{-1}{6\log x}\right), \tan\frac{\pi}{20}\right\} \quad \text{for } x \in (0, e^{-1/6}), \\ \varepsilon_1(x) &= x\min\left\{\sqrt{1-x^2}, \tan\left(\frac{\pi}{20}\right)\right\} \quad \text{for } x \in [e^{-1/6}, 1], \end{aligned}$$

we see that $|z^t - z^s| \le \theta(s/t) + (s-t)\varepsilon(s)$ for $z \in \bar{\Omega}_{\varepsilon_1}$, $0 < t < s \le 1$. Since ε_1 is nondecreasing on $[0, e^{-1/6})$, decreasing on $[e^{-1/6}, 1]$ and strictly positive on $(0, 1)$ it is then easy to construct a continuous function f on $[0, 1]$ such that $f(0) = f(1) = 0$ which satisfies $0 < f(x) < \varepsilon_1(x)$ for $x \in (0, 1)$. Then $|z^t - z^s| \le \theta(s/t) + (s-t)\varepsilon(s)$ for $z \in \bar{\Omega}_f$, $0 < t < s < 1$, which concludes the proof of the theorem. □

Corollary 2.7. *Let* $(s_n)_{n\geq 1}$ *and* $(t_n)_{n\geq 1}$ *be two sequences of real numbers such that* $0<t_n<s_n\leq 1$ *for* $n\geq 1$, *and such that* $\lim_{n\to+\infty} s_n=0$. *Then for every sequence* $u=(u_n)_{n\geq 1}$ *of positive real numbers there exists a nontrivial norm-continuous quasinilpotent semigroup* $(T_u(t))_{t>0}$ *of bounded operators on the separable Hilbert space such that* $\|T_u(t_n)-T_u(s_n)\|<\theta(s_n/t_n)+u_n$ *for* $n\geq 1$.

Proof. We can construct a strictly increasing continuous piecewise affine function $\varepsilon\colon[0,1]\to[0,1]$ such that $\varepsilon(0)=0$ and such that $\varepsilon(s_n)<u_n/(s_n-t_n)$ for $n\geq 1$, and the semigroup $(T_\varepsilon(t))_{t>0}$ provided by Theorem 2.6 satisfies the required condition. □

3. The general case

We now wish to discuss the behavior of strongly continuous semigroups near the origin in a general situation. For $0<t<s$, set

$$m(s,t)=\left(\frac{t}{s}\right)^{1/(s-t)}$$

We have $m(s,t)^t-m(s,t)^s=\theta(s/t)$, and $x^t-x^s<\theta(s/t)$ for $x\in[0,1]$, $x\neq m(s,t)$. Also

$$e^{-1/t}\leq m(s,t)\leq e^{-1/s}\quad(0<t<s). \tag{3.1}$$

If $(T(t))_{t>0}$ is a strongly continuous semigroup of bounded operators on a Banach space X we will denote by $\mathcal{A}_T$ the closed subalgebra of $\mathcal{B}(X)$ generated by the semigroup. Set $\sigma_T=\{0\}\cup\{|\phi(T(1))|\,|\,\phi\in\hat{\mathcal{A}}_T\}$, and denote by $\varrho(u)$ the spectral radius of u for $u\in\mathcal{A}_T$. A standard well-known application of the Banach–Steinhaus theorem shows that the function $t\mapsto\|T(t)\|$ is bounded on $[\alpha,\beta]$ for $0<\alpha<\beta<+\infty$, and it follows from Theorem 4.7.12 in [10] that there exists for each $\phi\in\hat{\mathcal{A}}_T$ a real number $a(\phi)$ such that $|\phi(T(t))|=e^{ta(\phi)}$ for $t>0$. In particular $|\phi(T(t))|=|\phi(T(1))|^t$ for $\phi\in\mathcal{A}_T$ and $t>0$. We obtain the following easy result.

Proposition 3.1. *Let* $(T(t))_{t>0}$ *be a strongly continuous semigroup of bounded operators on a Banach space* X, *let* $\eta>0$ *and let* $s\colon[0,\eta]\to[0,+\infty]$ *be a continuous function such that* $s(0)=0$ *and such that* $0<t<s(t)$ *for* $0<t\leq\eta$.

(i) *Assume that* $[0,\delta]\subset\sigma_T$ *for some* $\delta>0$. *Then* $\varrho(T(t)-T(s))\geq\theta(s/t)$ *for* $0<t<s\leq-1/\log\delta$.

(ii) *Assume that* 0 *is not an isolated point of* σ_T. *Then there exists a sequence* $(t_n)_{n\geq 1}$ *of elements of* $(0,\eta]$ *such that* $\lim_{n\to+\infty}t_n=0$, *and such that*

$$\varrho(T(t_n)-T(s(t_n)))\geq\theta\left(\frac{s(t_n)}{t_n}\right)\quad \textit{for } n\geq 1.$$

Proof. (i) It follows from (3.1) that $m(s,t)\in(0,\delta]$ for $0<t<s\leq -1/\log\delta$. Let $\phi\in\hat{\mathcal{A}}_T$ satisfy $|\phi(T(1))|=m(s,t)$. We have $|\phi(T(s))|=m(t,s)^s$, $|\phi(T(t))|=m(t,s)^t$, and so

$$\begin{aligned}\varrho(T(t)-T(s)) &\geq |\phi(T(t))-\phi(T(s))| \geq |\phi(T(t))|-|\phi(T(s))| \\ &= m(t,s)^t - m(t,s)^s = \theta(s/t).\end{aligned}$$

(ii) Let ϕ_n be a sequence of characters of $\mathcal{A}_T$ such that $\lim_{n\to+\infty}|\phi_n(T(1))|=0$. We can assume that $|\phi_n(T(1))|\leq m(s(\eta),\eta)$ for $n\geq 1$. Hence there exists a sequence $(t_n)_{n\geq 1}$ of elements of $(0,\eta]$ such that $m(s(t_n),t_n)=|\phi_n(T(1))|$ for $n\geq 1$. It follows from inequality (3.1) that $\lim_{n\to+\infty}t_n=0$, and we see as above that $\varrho(T(t_n)-T(s(t_n)))\geq\theta(s(t_n)/t_n)$ for $n\geq 1$. □

A sequence $(P_n)_{n\geq 1}$ of nonzero idempotents of a commutative Banach algebra $\mathcal{A}$ will be said to be *exhaustive* if $P_{n+1}P_n=P_{n+1}$ for $n\geq 1$ and if for every $\phi\in\hat{\mathcal{A}}$ there exists $n(\phi)\geq 1$ such that $\phi(P_n)=1$ for $n\geq n(\phi)$. The following corollary shows in particular that if the closed algebra $\mathcal{A}_T$ generated by a nontrivial strongly continuous semigroup $(T(t))_{t>0}$ has no nonzero idempotent, then there exists $\delta>0$ such that $\|T(t)-T(s)\|\geq\theta(s/t)$ for $0<t<s\leq\delta$.

Corollary 3.2. *Let $(T(t))_{t>0}$ be a nontrivial strongly continuous semigroup of bounded operators on a Banach space X. If there exists two sequences $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$, with $0<t_n<s_n$ for $n\geq 1$, such that $\lim_{n\to+\infty}s_n=0$, and such that*

$$\|T(t_n)-T(s_n)\|<\theta\left(\frac{s_n}{t_n}\right) \quad \textit{for } n\geq 1,$$

then the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup $(T(t))_{t>0}$ is not radical, and $\mathcal{A}_T$ possesses an exhaustive sequence $(P_n)_{n\geq 1}$ of nonzero idempotents.

Proof. It follows from Theorem 2.5 that the semigroup $(T(t))_{t>0}$ is not quasinilpotent, and it follows from Proposition 3.1(i) that there exists a decreasing sequence $(\delta_n)_{n\geq 1}$ of elements of $(0,\varrho(T(1))]$ such that $\lim_{n\to+\infty}\delta_n=0$ and such that $\lambda\neq\delta_n$ for $\lambda\in\operatorname{Spec}(T(1))$, $n\geq 1$. Set $U_n:=\{\phi\in\hat{\mathcal{A}}_T \,|\, |\phi(T(1))|\geq\delta_n\}=\{\phi\in\hat{\mathcal{A}}_T \,|\, |\phi(T(1))|>\delta_n\}$. Then U_n is a nonempty compact subset of $\mathcal{A}_T$ for $n\geq 1$, and it follows from Theorems 3.6.3 and 3.6.6 of [17] that there exists an idempotent P_n of $\mathcal{A}_T$ such that $\phi(P_n)=1$ for $\phi\in U_n$ and $\phi(P_n)=0$ for $\phi\in\hat{\mathcal{A}}_T\backslash U_n$ (it is also possible to define P_n directly by the formula

$$P_n=\frac{1}{2i\pi}\int_{C(0,1+\varrho(T(1)))}(T(1)-zI)^{-1}\,dz-\frac{1}{2i\pi}\int_{C(0,\delta_n)}(T(1)-zI)^{-1}\,dz,$$

where we denote by $C(0,r)$ the circle of radius r centered at the origin, oriented counterclockwise, for $r>0$). An immediate verification shows then that $(P_n)_{n\geq 1}$ is an exhaustive sequence of idempotents of $\mathcal{A}_T$. $\square$

We get a more precise result for norm-continuous semigroups, which shows in particular that the infinitesimal generator of the semigroup $(P_nT(t))_{t>0}$ is then bounded for $n\geq 1$.

Corollary 3.3. *Let $T(t)_{t>0}$ be a nontrivial norm-continuous semigroup of bounded operators on a Banach space X. If there exists two sequences $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$, with $0<t_n<s_n$ for $n\geq 1$, such that $\lim_{n\to+\infty} s_n=0$, and such that $\|T(t_n)-T(s_n)\|<\theta(s_n/t_n)$ for $n\geq 1$, then the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup $T(t)_{t>0}$ possesses an exhaustive sequence $(P_n)_{n\geq 1}$ of nonzero idempotents satisfying the following conditions:*

(i) $\bigcup_{n=1}^{\infty} P_n\mathcal{A}_T$ is dense in $\mathcal{A}_T$;

(ii) $\lim_{t\to 0^+} \|P_nT(t)-P_n\|=0$ for every $n\geq 1$, so that the infinitesimal generator of the semigroup $(P_nT(t))_{t>0}$ is bounded for $n\geq 1$.

Proof. Denote by $\mathcal{I}$ the closure of the ideal $\bigcup_{n=1}^{\infty} P_n\mathcal{A}_T$ in $\mathcal{A}_T$, and let $\pi\colon \mathcal{A}_T\to \mathcal{A}_T/\mathcal{I}$ be the canonical map. We can consider the semigroup $(\pi(T(t)))_{t>0}$ as a strongly continuous semigroup acting on $\mathcal{A}_T/\mathcal{I}$. It follows from Theorem 2.5 that $\pi(T(t))=0$ for $t>0$, and so $\mathcal{I}=\mathcal{A}_T$.

Now fix $n\geq 1$. Then, the notation being the same as in the proof of Corollary 3.2, P_n is the unit element of the Banach algebra $\mathcal{A}_n:=P_n\mathcal{A}_T$, and $\psi(P_nT(1))\geq \delta_n$ for $\psi\in\hat{\mathcal{A}}_n$, since the map $\phi\mapsto\phi|_{\mathcal{A}_n}$ is a surjection from U_n onto the character space of $\mathcal{A}_n$. Hence $P_nT(1)$ has an inverse S in $\mathcal{A}_n$, and

$$\limsup_{h\to 0^+}\|P_n-P_nT(h)\|\leq \|S\|\limsup_{h\to 0^+}\|T(1+h)-T(1)\|=0. \quad \square$$

Notice that we may have $\{\phi\in\hat{\mathcal{A}}_T \,|\, \phi(P_n)=0\}\neq\emptyset$ for $n\geq 1$ in Corollary 3.3(i) even if there exists $\delta>0$ such that $\|T(t)-T(s)\|\leq\theta(s/t)$ for $0<t<s\leq\delta$. A simple example of this situation is given by the semigroup $(T_\alpha(t))_{t>0}$ of positive operators on l^2 defined by the formula

$$T_\alpha(t)(u)=(e^{-\alpha_n t}u_n)_{n\geq 1},\quad u=(u_n)_{n\geq 1}\in l^2,\ t>0, \tag{3.2}$$

where $\alpha=(\alpha_n)_{n\geq 1}$ is a sequence of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$.

A. Mokthari and the author showed in [7] that if a semigroup in a Banach algebra $\mathcal{A}$ satisfies

$$\limsup_{t\to 0^+}\|T(t)-T((n+1)t)\|<\theta(n+1)$$

for some $n \geq 1$, then there exist an idempotent $P \in \mathcal{A}$ and $u \in \mathcal{A}$ such that $T(t) = Pe^{tu}$ for $t > 0$. The positivity of the Taylor coefficients at the origin of the function g_α introduced in Lemma 3.2, which was not noticed in [7], allows one to replace the condition $\limsup_{t \to 0^+} \|T(t) - T((n+1)t)\| < \theta(n+1)$ by the weaker condition $\|T(t) - T((n+1)t)\| < \theta(n+1)$ for t sufficiently small (we leave the details to the reader). In order to obtain a general result in this direction for strongly continuous semigroups, we will need the following variant of a classical result of Pazy [14], which shows that the generator of a strongly continuous group $(T(t))_{t \in \mathbf{R}}$ of operators satisfying $\limsup_{t \to 0^+} \|I - T(t)\| < 2$ is bounded (a detailed proof of the version given below can be found in [6]).

Proposition 3.4. *Let $(T(t))_{t>0}$ be a strongly continuous semigroup of bounded operators on a Banach space X, and assume that the closed subalgebra $\mathcal{A}_T$ of $\mathcal{B}(X)$ generated by the semigroup possesses a unit element P. Then either*

$$\lim_{t \to 0^+} \|P - T(t)\| = 0,$$

so that the infinitesimal generator of the semigroup $(T(t))_{t>0}$ is bounded, or

$$\lim_{t \to 0^+} \limsup_{h \to 0^+} \varrho(T(t+h) - T(t)) = 2.$$

Theorem 3.5. *Let $(T(t))_{t>0}$ be a nontrivial strongly continuous semigroup of bounded operators on a Banach space X. If there exist $\delta > 0$ and a continuous function $s\colon [0, \delta] \to (0, +\infty)$ such that $s(0) = 0$, and such that $0 < t < s(t)$ and*

$$\|T(t) - T(s(t))\| < \theta\left(\frac{s(t)}{t}\right) \quad \textit{for } 0 < t \leq \delta,$$

then the infinitesimal generator of the semigroup $T(t)_{t>0}$ is bounded, so that (1.1) *holds.*

Proof. An elementary computation shows that $\theta(s/t) < (s-t)/s < 1$ for $0 < t < s$. It follows from Theorem 2.5 that the semigroup is not quasinilpotent, and it follows from Proposition 3.1 that there exists $\delta > 0$ such that $[0, \delta) \cap \sigma_T = \{0\}$. Hence $\hat{\mathcal{A}}_T$ is compact and we see as in the proof of Corollary 3.2 that there exists an idempotent P of $\mathcal{A}_T$ such that $\phi(P) = 1$ for every $\phi \in \hat{\mathcal{A}}_T$. The semigroup $(S(t))_{t>0} := (PT(t))_{t>0}$ is strongly continuous and P is the unit element of the closed subalgebra $\mathcal{A}_S$ generated by this semigroup. Set $S(0) = P$, and denote by $S(-t)$ the inverse of $S(t)$ in $\mathcal{A}_S$ for $t > 0$. If $\limsup_{t \to 0^+} \|P - S(t)\| > 0$, it would follow from Proposition 3.4 that $\limsup_{t \to 0^+} \varrho(P - S(t)) = 2$. Now let $(r_n)_{n \geq 1}$ be a decreasing sequence of positive real numbers such that $\lim_{n\to+\infty} r_n=0$ and such that $\lim_{n\to+\infty} \varrho(P-PT(r_n))=2$. Since the map $t\mapsto s(t)$ is continuous on $[0,\delta]$, there would exist $n_0\geq 1$ and a sequence $(t_n)_{n\geq n_0}$ of elements of $[0,\delta)$ such that $\lim_{n\to+\infty} t_n=0$ and such that $s(t_n)-t_n=r_n$ for $n\geq n_0$. We would have

$$\begin{aligned}\lim_{n\to+\infty} \varrho(T(s(t_n))-T(t_n)) &\geq \lim_{n\to+\infty} \varrho(PT(t_n+r_n)-PT(t_n)) \\ &\geq \Big(\lim_{n\to+\infty} \varrho(S(-t_n))\Big)^{-1} \lim_{n\to+\infty} \varrho(PT(r_n)-P)=2.\end{aligned}$$

But this is impossible since

$$\varrho(T(s(t_n))-T(s_n)) \leq \|T(s(t_n))-T(t_n)\| < \theta\bigg(\frac{s(t_n)}{t_n}\bigg) < 1$$

for $n\geq 1$. Hence $\lim_{t\to 0^+} \|P-PT(t)\|=0$.

The subspace $Y:=PX$ is closed. If $R\in\mathcal{B}(X)$ satisfies $RP=PR$, set $\pi(R)y=Rx+Y$ for $y=x+Y\in X/Y$, so that $\pi(R)\in\mathcal{B}(X/Y)$ and $\|\pi(R)\|\leq\|R\|$. Then

$$\pi(T(t))=\pi(T(t)-PT(t))$$

is quasinilpotent, and it follows from Theorem 2.5 that $\pi(T(t))=0$ for $t>0$. Hence $T(t)(X)\subset Y$, so that $T(t)x=PT(t)x$ for $x\in X$, and

$$\lim_{t\to 0^+} \|P-T(t)\| = \lim_{t\to 0^+} \|P-PT(t)\| = 0. \quad \square$$

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