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Residues of holomorphic sections and Lelong currents

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Abstract. Let Z be the zero set of a holomorphic section f of a Hermitian vector bundle. It is proved that the current of integration over the irreducible components of Z of top degree, counted with multiplicities, is a product of a residue factor R^f and a "Jacobian factor". There is also a relation to the Monge–Ampère expressions $(dd^c \log|f|)^k$, which we define for all positive powers k.

1. Introduction

Let $f=(f_1, ..., f_m)$ be a holomorphic mapping on a complex manifold X of dimension n and let $Z=\{z:f(z)=0\}$. If f is a complete intersection, i.e., $\operatorname{codim} Z=m$, and

$$R^f_{ch} = \left[\bar\partial\frac{1}{f_m}\wedge...\wedge\bar\partial\frac{1}{f_1}\right]$$

is the classical Coleff–Herrera current, then

$$R^f\wedge\frac{df_1\wedge...\wedge df_m}{(2\pi i)^m}=\sum_j \alpha_j[Z_j], \tag{1.1}$$

where Z_j are the irreducible components of Z and α_j are multiplicities related to the mapping f.

Let $F\to X$ be a holomorphic Hermitian vector bundle of rank m. Given $f\in \mathcal{O}(X,F)$, we defined in [1] the residue current R^f, which is a section of

$$\bigoplus_l \mathcal{D}'_{0,l}(X,\Lambda^l F^*)$$

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(considered as a subbundle of $\mathcal{D}'(X, \Lambda(T^*(X)\ominus F^*)))$ with support on Z. If $p=\operatorname{codim} Z$, then

$$R^f = R^f_p + \ldots + R^f_m, \tag{1.2}$$

where R^f_l is the component in $\mathcal{D}'_{0,l}(X, \Lambda^l F^*)$, see Section 2.

If f is a complete intersection, then locally

$$R^f = R^f_m = \left[\bar\partial \frac{1}{f_m} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}\right] \wedge e_1^* \wedge \ldots \wedge e_m^*, \tag{1.3}$$

if e_j is a local holomorphic frame for F, e_j^* is the dual frame, and $f = f_1 e_1 + \ldots + f_m e_m$. Notice that the duality of F and F^* induces a duality between the exterior algebra bundles $\Lambda^k F$ and $\Lambda^k F^*$. If D is any connection on F, then the factorization (1.1) can be written invariantly as

$$\frac{R^f \cdot (Df/2\pi i)^m}{m!} = \sum_j \alpha_j [Z_j]. \tag{1.4}$$

In fact, in a local holomorphic frame $Df = \sum_j df_j \wedge e_j + \mathcal{O}(f)$, the latter expression denoting smooth terms that contain some factor f_j; it is well known that $f_j R^f = 0$, and so (1.4) follows from (1.1) and (1.3). (In [6] a factorization like (1.4) is found when Z locally is a complete intersection but not necessarily the zero set of a holomorphic section.)

Our main result is the following more general statement.

Theorem 1.1. *Let f be a holomorphic section of the Hermitian vector bundle $F \to X$ and let $p=\operatorname{codim} Z$. If R^f is the residue current and D the Chern connection, then*

$$\frac{R^f_p \cdot (Df/2\pi i)^p}{p!} = \sum_j \alpha_j [Z^p_j], \tag{1.5}$$

where Z^p_j are the irreducible components of top dimension (codimension p) of Z and α_j are the multiplicities of Z^p_j.

Let a be a given point on the regular part of some Z^p_j. If $f_1, \ldots, f_p$ are the first p coefficients with respect to a generic holomorphic frame at a, then α_j is the multiplicity of the restriction of the mapping $(f_1, \ldots, f_p)$ to a generic complex p-plane through a (with respect to some local holomorphic coordinates), see, e.g., [5].

Corollary 1.2. *Under the hypothesis in the theorem it follows that $R^f_{p,p}$ is not identically zero.*

Given a local frame e_j and its dual frame e^*_j, then

$$R^f_p = \sum_{|I|=p}{}' (R^f_p)_I \wedge e^*_{I_1} \wedge \ldots \wedge e^*_{I_p}.$$

If F is a trivial bundle equipped with the trivial metric then

$$Df = D(f_1 e_1 + \ldots + f_m e_m) = df_1 \wedge e_1 + \ldots + df_m \wedge e_m,$$

and the theorem then means that

$$\sum_{|I|=p}{}' (R^f_p)_I \wedge \frac{df_{I_1} \wedge \ldots \wedge df_{I_p}}{(2\pi i)^p} = \sum_j \alpha_j [Z^p_j].$$

It follows from Remark 2 in Section 3 that one can replace D by any Chern connection associated with some Hermitian metric, but unlike the case with a complete intersection, the theorem is (probably) not true with any (holomorphic) connection. In the case $X = \mathbf{P}^n$ and f is a homogeneous polynomials, a formula related to (1.5) appeared in [4].

The proof of Theorem 1.1 relies on the possibility to resolve singularities by Hironaka's theorem and the technique with toric resolutions developed in [3] and [9]. The starting point in the proof of Theorem 1.1 is King's formula, [7], [5], which states that if f is as above and we have the trivial metric, then

$$(dd^c \log |f|)^p \mathbf{1}_Z = \sum_j \alpha_j [Z^p_j]. \tag{1.6}$$

Recently Méo, [8], proved (1.6) and some related formulas for an arbitrary Hermitian metric. As a by-product of the proof of our main theorem we obtain new proofs of these results. We also introduce a meaning to the Monge–Ampère expression $(dd^c \log |f|)^k$ for any positive power k, and discuss its connection to the residue current R^f as well as to some other related currents. In particular we have the factorization

$$(dd^c \log |f|)^k \mathbf{1}_Z = \frac{R^f_k \cdot (Df/2\pi i)^k}{k!}$$

for any k if the bundle F is trivial and equipped with the trivial metric.

2. The residue current of a holomorphic section

Given the vector bundle $F \to X$ we consider the exterior algebra

$$\Lambda = \Lambda(T^*(X) \oplus F \oplus F^*).$$

Any section γ of $\Lambda T^*(X) \otimes (F \oplus F^*)$ induces a section $\tilde{\gamma}$ to Λ, just by identifying elements like $\xi \otimes \eta$ with $\xi \wedge \eta$ and extending bilinearly (one just has to keep track of the order). A connection D_F on F induces a natural connection D on $\Lambda(F \oplus F^*)$, and it induces a mapping on $\mathcal{E}(X, \Lambda(T^*(X) \oplus F \oplus F^*))$ which we also denote by D, via

$$D\tilde{\xi} = \widetilde{D\xi}.$$

It is an antiderivation, i.e., $D(\xi \wedge \eta) = D\xi \wedge \eta + (-1)^{\deg \xi} \xi \wedge D\eta$, where $\deg \xi$ refers to the total degree of ξ (with respect to both F, F^*, and $T^*(X)$). A form-valued endomorphism $a \in \mathcal{E}_k(X, \mathrm{End}(F))$ can be identified with

$$\tilde{a} = \sum_{jk} a_{jk} \wedge e_j \wedge e_k^*,$$

if e_j is a local frame, e_j^* is its dual frame, and $a = \sum_{jk} a_{jk} \otimes e_j \otimes e_k^*$ with respect to these frames. For instance, if I is the identity mapping on E, then $\tilde{I} = \sum_j e_j \wedge e_j^*$. If $D_{\mathrm{End}(F)}$ denotes the induced connection on the bundle $\mathrm{End}(F)$, then

$$(D_{\mathrm{End}\, F} a)^{\sim} = D\tilde{a}. \tag{2.1}$$

If $\Theta = D^2$ is the curvature tensor, then by Bianchi's identity, $D_{\mathrm{End}\, F}\Theta = 0$. Thus,

$$D\tilde{\Theta} = 0 \quad \text{and} \quad D\tilde{I} = 0. \tag{2.2}$$

Assume now that F is a Hermitian vector bundle and let D be the associated Chern connection. Given the holomorphic section f of F, we let $\delta_f : \mathcal{E}(X, \Lambda^{k+1} F^*) \to \mathcal{E}(X, \Lambda^k F^*)$ be interior multiplication (contraction) with f. It clearly extends to a mapping on $\mathcal{E}(X, \Lambda)$ and it anticommutes with $\bar{\partial}$. If we let $\nabla_f'' = \delta_f - \bar{\partial}$ we therefore have that $(\nabla_f'')^2 = 0$. Let s be the section of F^* that is dual to f with respect to the Hermitian metric, so that in particular $\delta_f s = |f|^2$. Then

$$\frac{s}{\nabla_f'' s} = \frac{s}{|f|^2} + \frac{s \wedge \bar{\partial} s}{|f|^4} + \dots + \frac{s \wedge (\bar{\partial} s)^{m-1}}{|f|^{2m}}$$

in $X \setminus Z$. Since $(\nabla_f'')^2 = 0$,

$$\nabla_f'' \frac{s}{\nabla_f'' s} = 1$$

in $X\setminus Z$. The form $|f|^{2\lambda}s/\nabla''_f s$ is well defined in X if $\operatorname{Re}\lambda$ is large, it has an analytic continuation as a current to $\operatorname{Re}\lambda>-\varepsilon$, see [1], and

$$U^f=|f|^{2\lambda}\wedge\frac{s}{\nabla''_f s}\bigg|_{\lambda=0}$$

is a current extension of $s/\nabla''_f s$ across Z. Moreover, $\nabla''_f U^f=1-R^f$, where

$$R^f=\bar\partial|f|^{2\lambda}\wedge\frac{s}{\nabla''_f s}\bigg|_{\lambda=0}$$

(therefore) is a current with support on Z, which we call the residue of f. It turns out that the components of degree less than $(0,p)$ vanishes, so (1.2) holds, see [1]. It is also proved there that $R^f=R^f_{m,m}$ is independent of the metric when f is a complete intersection. When the metric is trivial, $R^f_{m,m}$ is the so-called Bochner–Martinelli residue current, and it was first proved in [9] that it coincides with the Coleff–Herrera current, i.e., (1.3). A simplified proof appeared in [1].

Now let

$$\nabla_f=\delta_f-D.$$

Since D is the Chern connection,

$$Ds=\bar\partial s,$$

see, e.g., [1], and it is also pointed out there that one can replace $\bar\partial|f|^{2\lambda}$ by $d|f|^{2\lambda}$ in the definition of R^f; thus we have

$$R^f=d|f|^{2\lambda}\wedge\frac{s}{\nabla_f s}\bigg|_{\lambda=0}. \tag{2.3}$$

However, it is *not* true that $\nabla_f^2=0$; in fact, [1],

$$\nabla_f^2 s=\delta_s(Df-\widetilde\Theta), \tag{2.4}$$

where δ_s denotes contraction with s. Moreover,

$$\nabla_f(Df-\widetilde\Theta)=0,\quad \nabla_f\tilde I=-f,\quad\text{and}\quad \delta_s\tilde I=s. \tag{2.5}$$

3. Proof of the main theorem

We are primarily interested in the current $R^f_{p,p}\cdot(Df/2\pi i)_p$, but to begin with we have to consider a somewhat more general current. We let $\tilde{I}_m=\tilde{I}^m/m!$, and we use the same notation for other forms in the sequel. Any form α with values in Λ can be uniquely written as $\alpha=c\wedge\tilde{I}_m+\alpha'$, where α' does not have full degree in e_j and e_j^*. If we define

$$\int_e \alpha = c,$$

then this integral is of course linear and

$$d\int_e \alpha = \int_e D\alpha = -\int_e \nabla_f\alpha. \tag{3.1}$$

We can now define the current

$$M^f = \int_e R^f\wedge e^{(Df-\tilde{\Theta})/2\pi i+\tilde{I}}. \tag{3.2}$$

Since Θ has bidegree $(1,1)$, a simple consideration of degrees, using (1.2), reveals that

$$M^f = M^f_p+\ldots+M^f_m,$$

where M^f_k is the (k,k)-current

$$M^f_k = \int_e \sum_{l=p}^{k} R^f_{l,l}\wedge(Df/2\pi i)_l\wedge\Big(\frac{i}{2\pi}\tilde{\Theta}\Big)_{k-l}\wedge\tilde{I}_{m-k}.$$

For degree reasons no factors $\tilde{\Theta}$ occur in the term M^f_p and therefore

$$M^f_p = R^f_{p,p}\cdot(Df/2\pi i)_p. \tag{3.3}$$

Proposition 3.1. *The current M^f is closed and has order zero (i.e., measure coefficients).*

Proof. Let

$$M^f_\lambda = \int_e d|f|^{2\lambda}\wedge\frac{s}{\nabla s}\wedge e^{(Df-\tilde{\Theta})/2\pi i+\tilde{I}}.$$

Then each term is like

$$d|f|^{2\lambda}\wedge\frac{s\wedge(\bar{\partial}s)^{l-1}\wedge(Df)^l\wedge\text{smooth}}{|f|^{2l}}.$$

Locally we write $f=\sum_j f_je_j$ as before in a local holomorphic frame e_j, after an appropriate desingularization, using Hironaka's theorem and toric resolution, following [3] and [9], we may assume that one of the functions f_j divides all the other ones. Thus there is a holomorphic function f_0 such that $f=f_0f'$ where f' is a non-vanishing holomorphic section. Then $s=\bar{f}_0s'$ and thus $s\wedge(\bar\partial s)^{l-1}=\bar{f}_0^l s'\wedge(\bar\partial s')^{l-1}$. Moreover, $(Df)^l=f_0^{l-1}\alpha$, where α is smooth, and $|f|=|f_0|u$, where u is smooth and strictly positive, so we get

$$d|f_0u|^{2\lambda}\wedge\frac{\text{smooth}}{f_0}$$

which is locally integrable for $\operatorname{Re}\lambda>0$ and a current of order zero when $\lambda=0$.

For large $\operatorname{Re}\lambda$ we have, by (2.5), that

$$\begin{aligned}dM_\lambda^f &= \int_e d|f|^{2\lambda}\wedge\nabla_f\Big(\frac{s}{\nabla s}\Big)\wedge e^{(Df-\tilde\Theta)/2\pi i+\tilde I}\\ &\quad+\int_e d|f|^{2\lambda}\wedge\frac{s}{\nabla s}\wedge e^{(Df-\tilde\Theta)/2\pi i+\tilde I}\wedge f = I_1+I_2.\end{aligned}$$

The expression I_2 is a sum of terms like

$$d|f|^{2\lambda}\wedge\frac{s\wedge(\bar\partial s)^{l-1}\wedge f\wedge(Df)^{l-1}\wedge\text{smooth}}{|f|^{2l}};$$

after the desingularization, $f\wedge(Df)^{l-1}=f_0^lf'\wedge(Df')^{l-1}$, so the entire singularity in the denominator is cancelled and therefore I_2 vanishes when $\lambda=0$. Now,

$$\nabla_f\frac{s}{\nabla_f s}=1-\frac{s}{(\nabla_f s)^2}\nabla_f^2 s=1-\frac{s}{(\nabla_f s)^2}\delta_s(Df-\tilde\Theta)$$

by (2.4), so I_1 gives rise to two terms. The first one contains no singularities at all and it therefore vanishes when $\lambda=0$. The second term is

$$\begin{aligned}&\int_e d|f|^{2\lambda}\wedge\frac{s}{(\nabla_f s)^2}\delta_s(Df-\tilde\Theta)\wedge e^{(Df-\tilde\Theta)/2\pi i+\tilde I}\\ &\qquad=2\pi i\int_e d|f|^{2\lambda}\wedge\frac{s}{(\nabla_f s)^2}\delta_s e^{(Df-\tilde\Theta)/2\pi i}\wedge e^{\tilde I},\end{aligned}$$

where we have used (2.5) again. An integration by parts puts δ_s on the factor $e^{\tilde I}$, which yields a factor s, and thus the integral vanishes since $s\wedge s=0$. Thus $dM^f=dM_\lambda^f|_{\lambda=0}=0$ as claimed. $\square$

Remark 1. Let $D_{F/S}$ be the Chern connection on the vector bundle $F/S\to X\setminus Z$, where F/S is equipped with the induced metric, and let $c(D_{F/S})$ be the

Chern form. It turns out that its natural extension $C=c(D_{F/S})\mathbf{1}_{X\setminus Z}$ is locally integrable in X and that

$$C=\int_e e^{(i/2\pi)\tilde{\Theta}+\tilde{I}+Df/2\pi i}\wedge f\wedge U^f. \tag{3.4}$$

Moreover, if

$$A=-\int_e e^{(i/2\pi)\tilde{\Theta}+\tilde{I}+Df/2\pi i}\wedge U^f, \tag{3.5}$$

then $dA=c(D)-C-M^f$, i.e., $dA_k=c_k(D)-C_k-M_k^f$, where $c_k(D)$ is the kth Chern form of D. Since $c_m(D_{F/S})=0$ we have that $C_m=0$, so it follows in particular that M_m^f represents the top Chern class $c_m(F)$; this was proved already in [1]. Moreover, one can even find a current W such that $dd^cW=(i/\pi)\partial\bar\partial W=c(D)-C-M^f$. This is proved in [2]; the special case $k=p$ of this formula is also obtained in [8]. It is also proved in [2] that $M^f=\lim_{\lambda\to0^+}(i/2\pi)\lambda|f|^{2\lambda-4}\partial|f|^2\wedge\bar\partial|f|^2\wedge C$, and that M^f is a positive current if F^* is negative in Nakano's sense.

Let

$$\mathcal{A}_{k,\lambda}^f=\bar\partial|f|^{2\lambda}\wedge\frac{\partial|f|^2\wedge(\bar\partial\partial|f|^2)^{k-1}}{(2\pi i)^p|f|^{2k}}.$$

Proposition 3.2. *If the metric is trivial, then for any k, the form $\mathcal{A}_{k,\lambda}^f$ is locally integrable in X for each $\lambda>0$, and*

$$M_k^f=\frac{R_k^f\cdot(Df/2\pi i)^k}{k!}=\lim_{\lambda\to0^+}\mathcal{A}_{k,\lambda}^f.$$

Since $|f|^2$ is plurisubharmonic when the metric is trivial, it follows that M_k^f is a positive (k,k)-current.

Proof. Since $\Theta=0$ for a trivial metric, $M_k^f=R_k^f\cdot(Df/2\pi i)^k/k!$. From (the proof of) Proposition 3.1 it follows that

$$M_{k,\lambda}^f=\bar\partial|f|^{2\lambda}\wedge\int_e\frac{s\wedge(\bar\partial s)^{k-1}}{|f|^{2k}}\wedge(Df/2\pi i)_k\wedge\tilde{I}_{m-k} \tag{3.6}$$

is locally integrable for each $\lambda>0$ and by definition $M_k^f=M_{k,\lambda}^f|_{\lambda=0}$. Thus actually $M_k^f=\lim_{\lambda\to0}M_{k,\lambda}^f$. If $f=\sum_j f_je_j$ in a trivial holomorphic frame for F, then

$$s=\sum_j\bar f_je_j^*,\quad \bar\partial s=\sum_j d\bar f_j\wedge e_j^*,\quad Df=\sum_j df_j\wedge e_j, \tag{3.7}$$

and

$$\partial|f|^2=\sum_j \bar{f}_j df_j, \quad \bar{\partial}\partial|f|^2=\sum_j d\bar{f}_j\wedge df_j. \tag{3.8}$$

Moreover, a simple combinatorical argument yields that

$$\begin{aligned}\int_e \sum \bar{f}_j e_j^*\wedge\Big(\sum_j d\bar{f}_j\wedge e_j^*\Big)^{k-1}\wedge\Big(\sum_j df_j\wedge e_j\Big)_k\wedge\tilde{I}_{m-k}\\ =\sum \bar{f}_j df_j\wedge\Big(\sum_j d\bar{f}_j\wedge df_j\Big)^{k-1}.\end{aligned} \tag{3.9}$$

Combining (3.9) and (3.7) we get that

$$M_{k,\lambda}^f=\bar{\partial}|f|^{2\lambda}\wedge\frac{\sum_j \bar{f}_j df_j\wedge(\sum_j d\bar{f}_j\wedge df_j)^{k-1}}{(2\pi i)^k|f|^{2k}}. \tag{3.10}$$

In view of (3.8) we therefore have that $M_{k,\lambda}^f=\mathcal{A}_{k,\lambda}^f$, and hence the proposition follows. □

It is easy to see now that M_p^f is positive even for a general metric and we give a direct argument here, although it is also a consequence of the main theorem.

Proposition 3.3. *The current M_p^f is positive ($p=\operatorname{codim} Z$ as usual) for any Hermitian metric.*

Proof. With the formula (3.6) for $M_{p,\lambda}^f$ it follows that $M_p^f=\lim_{\lambda\to 0^+} M_{p,\lambda}^f$, cf., (3.3). In a neighborhood of a fixed point 0 we can choose a local holomorphic frame e_j such that the metric $h_{j\bar{k}}(z)$ is $\delta_{jk}+\mathcal{O}(|z|^2)$. Then $s=\sum_j \bar{f}_j e_j^*+\mathcal{O}(|z|^2)$. Moreover, $D_F=d+h^{-1}\partial h=d+\mathcal{O}(|z|)$, and hence $D_F f=\sum_j df_j\wedge e_j+\mathcal{O}(|z|)$. Thus (3.10) holds at $z=0$ (for $k=p$) as in the previous case, and therefore $M_{p,\lambda}^f$ is positive there. Since the point is arbitrary, the form is positive, and letting $\lambda\to 0$ we conclude that M_p^f is a positive current. □

As was mentioned in the introduction, our proof of Theorem 1.1 will rely on King's formula which we now recall. Let $d^c=(i/2\pi)(\bar{\partial}-\partial)$. If we have the trivial metric, so that $\log|f|$ is a plurisubharmonic function, then it is well known that $\log|f|(dd^c\log|f|)^{k-1}\mathbf{1}_{X\setminus Z}$ is locally integrable for all $k<p$, and that

$$dd^c(\log|f|(dd^c\log|f|)^{k-1}\mathbf{1}_{X\setminus Z})=(dd^c\log|f|)^k\mathbf{1}_{X\setminus Z} \tag{3.11}$$

for $k<p$. Moreover, for $k=p$ we have King's formula, [7] and [5],

$$dd^c(\log|f|(dd^c\log|f|)^{p-1}\mathbf{1}_{X\setminus Z})=(dd^c\log|f|)^p\mathbf{1}_{X\setminus Z}+\sum_j \alpha_j[Z_j], \tag{3.12}$$

where Z_j^p are the irreducible components of Z of codimension p, and α_j are the multiplicity numbers described after Theorem 1.1 above.

Lemma 3.4. *For the trivial metric we have that*

$$dd^c(\log|f|(dd^c\log|f|)^{p-1})\mathbf{1}_Z = \lim_{\lambda\to 0^+} \mathcal{A}^f_{p,\lambda}.$$

Proof. Since $\log|f|(dd^c\log|f|\mathbf{1}_{X\setminus Z})^{p-1}\mathbf{1}_{X\setminus Z}$ is locally integrable in X, and $\lambda\mapsto(|f|^{2\lambda}-1)$ is increasing for $\lambda>0$ we have by dominated convergence that

$$\int\log|f|(dd^c\log|f|)^{p-1}\wedge dd^c\phi = \lim_{\lambda\to 0^+}\int\frac{1}{2\lambda}(|f|^{2\lambda}-1)(dd^c\log|f|)^{p-1}\wedge dd^c\phi.$$

The current $(dd^c\log|f|)^{p-1}\mathbf{1}_{X\setminus Z}$ is closed in the current sense according to (3.11), and an integration by parts therefore gives

(3.13)
$$\lim_{\lambda\to 0^+}\int\bar\partial|f|^{2\lambda}\wedge\frac{\partial|f|^2}{2\pi i|f|^2}\wedge\left(\bar\partial\frac{\partial|f|^2}{2\pi i|f|^2}\right)^{p-1}\wedge\phi+\lim_{\lambda\to 0^+}\int|f|^{2\lambda}\left(\bar\partial\frac{\partial|f|^2}{2\pi i|f|^2}\right)^{p}\wedge\phi,$$

which proves the lemma since the second term in (3.13) is precisely

$$\int(dd^c\log|f|)^p\mathbf{1}_{X\setminus Z}\wedge\phi$$

(the finiteness of the limit is ensured by King's formula). $\square$

It is now a simple matter to obtain our main result.

Proof of Theorem 1.1. Let Z^p be the union of all irreducible components of Z of codimension p. Then $Z\setminus Z^p$ is a union of regular submanifolds of codimensions more than p. Since M^f_p is a closed (p,p)-current of order zero it must vanish there, and thus M^f_p has support on Z^p. Therefore, see, e.g., [5],

$$M^f_p=\sum_j\alpha'_j[Z^p_j] \tag{3.14}$$

for some nonnegative numbers α'_j. It is easy to see that these numbers α_j are independent of the metric on F. In fact, the definition of M^f_p in a neighborhood of a given point only depends on the metric in that neighborhood. In view of (3.14), M^f_p will not be affected if we change the metric locally on some given irreducible component Z^p_j. Since we can choose a metric which is equal to any two prescribed metrics close to two given distinct points on Z_j it follows that actually (3.14) is independent of the metric. For a direct argument, see Remark 2 below. However, when we have the trivial metric, Proposition 3.2, Lemma 3.4 and King's formula together show that α'_j actually are equal to the multiplicities α_j. Thus the theorem is proved. □

Remark 2. Here we provide a direct argument for that $M_p^f = R_p^f \cdot (Df/2\pi i)_p$ is independent of the metric. Let $\widehat{R}^f$ be the residue current with respect to another metric. It is enough to show that

$$A = \int_e (R_p^f - \widehat{R}_p^f) \wedge (Df)_p \wedge \tilde{I}_{m-p} = 0.$$

Let $u = s/\nabla_f s$ (here $\nabla_f = \delta_f - \bar\partial$) and let $\hat u$ be the corresponding form with respect to the other metric. Then

$$R^f - \widehat{R}^f = \nabla_f (\bar\partial |f|^{2\lambda} \wedge u \wedge \hat u)|_{\lambda=0}$$

(here $|f|$ can be the norm with respect to any metric), and therefore

$$R_p^f - \widehat{R}_p^f = \delta_f (\bar\partial |f|^{2\lambda} \wedge u \wedge \hat u)_{p+1,p}|_{\lambda=0}$$

(lower indices denote degrees in e_j^* and $d\bar z_j$, respectively). This is because terms of lower degree than p in $d\bar z_j$ of the current $\bar\partial |f|^{2\lambda} \wedge u \wedge \hat u|_{\lambda=0}$ must vanish, see [1] Proposition 2.2. Therefore,

$$\begin{aligned} A &= \int_e \delta_f (\bar\partial |f|^{2\lambda} \wedge u \wedge \hat u)_{p+1,p} \wedge (Df)_p \wedge \tilde{I}_{m-p} \\ &= \int_e (\bar\partial |f|^{2\lambda} \wedge u \wedge \hat u)_{p+1,p} \wedge (Df)_p \wedge f \wedge \tilde{I}_{m-p-1}, \end{aligned}$$

where the equality follows from an integration by parts. After desingularization, $(\bar\partial |f|^{2\lambda} \wedge u \wedge \hat u)_{p+1,p}$ is a smooth form times $\bar\partial |f|^{2\lambda}/f_0^{p+1}$, but on the other hand $(Df)_p \wedge f$ is like f_0^{p+1} so the singularity is cancelled out, and hence the expression vanishes when $\lambda=0$.

For any metric and any k, $R^f \cdot (Df/2\pi i)^k/k!$ is a positive (k,k)-current (it is proved as Proposition 3.3) and M_k^f is a closed (k,k)-current, but in general they do not coincide.

It was recently proved by Méo, [8], that ($p = \operatorname{codim} Z$ as usual)

$$\sum_j \alpha_j [Z_j] = dd^c (\log |f| (dd^c \log |f| \mathbf{1}_{X \setminus Z})^{p-1}) \mathbf{1}_Z = \lim_{\lambda \to 0^+} \mathcal{A}_{p,\lambda}^f \tag{3.15}$$

for an arbitrary metric. In view of Theorem 1.1, they are also all equal to M_p^f. In the general case $\log |f|$ is no longer plurisubharmonic so one cannot rely on the usual theory for the Monge–Ampère operator acting. In fact, it is not clear a priori that any of the last two currents in (3.15) is well-defined, let alone positive. The equalities (3.15) are consequences of more general results in the next section.

4. The Monge–Ampère operator and residue currents

A crucial point in the proof of the main theorem was the relation between the currents $(dd^c \log|f|)^p$ and $R^f_p \cdot (Df/2\pi i)_p$. In this section we discuss their relation for general k. To begin with we introduce a meaning to the Monge–Ampère expression $(dd^c \log|f|)^k$ for an arbitrary positive power k.

Proposition 4.1. *Let f be a holomorphic section of a Hermitian vector bundle $F \to X$ and let $Z = \{z : f(z) = 0\}$. Then the form*

$$\log|f|(dd^c \log|f|)^{k-1}\mathbf{1}_{X\setminus Z}$$

is locally integrable in X for any k, $(dd^c \log|f|)^{k-1}\mathbf{1}_{X\setminus Z}$ is closed, and

$$dd^c(\log|f|(dd^c \log|f|\mathbf{1}_{X\setminus Z})^{k-1})$$

is a current of order zero. Moreover,

$$\mathcal{A}^f_{k,\lambda} = \bar\partial|f|^{2\lambda} \wedge \frac{\partial|f|^2 \wedge (\bar\partial\partial|f|^2)^{k-1}}{(2\pi i)^k |f|^{2k}}$$

is locally integrable in X for each $\lambda > 0$, and

$$dd^c(\log|f|(dd^c \log|f|)^{k-1}\mathbf{1}_{X\setminus Z})\mathbf{1}_Z = \lim_{\lambda\to 0^+} \mathcal{A}^f_{k,\lambda}. \tag{4.1}$$

Proof. Since the statement is local in X we may assume that $f = \sum_j f_j e_j$ in some local holomorphic frame in U. By a desingularization we may assume that $f = f_0 f'$, where f_0 is a holomorphic function and $f' \neq 0$. Then outside the zero set,

$$dd^c \log|f| = dd^c \log|f'|$$

since $dd^c \log|f_0| = 0$ there. Therefore, outside the singularity we have that

$$\log|f|(dd^c \log|f|)^{k-1} = (\log|f_0| + \log|f'|)(dd^c \log|f'|)^{k-1}.$$

The right-hand side is integrable since $\log|f_0|$ is integrable and $\log|f'|$ is smooth. Since the desingularization is a biholomorphism outside a set of measure zero it follows that the original form is locally integrable as well.

In particular, $(dd^c \log|f|)^{k-1}\mathbf{1}_{X\setminus Z}$ is locally integrable, and in the desingularization it is just $(dd^c \log|f'|)^{k-1}$ outside the singularity, and therefore it is closed. It follows that $(dd^c \log|f|)^{k-1}\mathbf{1}_{X\setminus Z}$ is closed in X.

We have that $dd^c(\log|f|(dd^c\log|f|)^{k-1}\mathbf{1}_{X\setminus Z})$ is equal to

(4.2)
$$dd^c[(\log|f_0|+\log|f'|)(dd^c\log|f'|)^{k-1}]=[f_0=0]\wedge(dd^c\log|f'|)^{k-1}+(dd^c\log|f'|)^k,$$

where $[f_0=0]=dd^c\log|f_0|$ is the current of integration over the zero set of f counted with multiplicities. Notice that

$$\bar\partial|f|^{2\lambda}\wedge\frac{\partial|f|^2\wedge(\bar\partial\partial|f|^2)^{k-1}}{(2\pi i)^k|f|^{2k}}=\bar\partial|f|^{2\lambda}\wedge\frac{\partial|f|^2}{2\pi i|f|^2}\wedge\left(\bar\partial\frac{\partial|f|^2}{2\pi i|f|^2}\right)^{k-1},$$

which in the desingularization becomes

$$\bar\partial(|f_0|^{2\lambda}|f'|^{2\lambda})\wedge\left(\frac{\partial|f_0|^2}{2\pi i|f_0|^2}+\frac{\partial|f'|^2}{2\pi i|f'|^2}\right)\wedge\left(\bar\partial\frac{\partial|f'|^2}{2\pi i|f'|^2}\right)^{k-1}. \tag{4.3}$$

It is locally integrable since

$$\frac{\bar\partial|f_0|^2\wedge\partial|f_0|^2}{|f_0|^{4-2\lambda}}=\frac{d\bar f_0\wedge df_0}{|f_0|^{2-2\lambda}} \tag{4.4}$$

is locally integrable for $\lambda>0$. Moreover, it is well known that (4.4) tends to $[f_0=0]$ when $\lambda\to0^+$, and hence (4.1) follows from (4.2) and (4.3). $\square$

If we think of $\log|f|$ as being equal to zero on Z, then the proposition says that the usual iterative definition

$$(dd^c\log|f|)^k=dd^c(\log|f|(dd^c\log|f|)^{k-1})$$

can be extended to all k, and that

$$(dd^c\log|f|)^k=(dd^c\log|f|)^k\mathbf{1}_{X\setminus Z}+\lim_{\lambda\to0^+}\mathcal{A}_{k,\lambda}.$$

When we have the trivial metric, so that $\log|f|$ is plurisubharmonic, it follows that $(dd^c\log|f|)^k$ is a positive (k,k)-current.

Remark 3. There are other ways to express the residue current $(dd^c\log|f|)^k\mathbf{1}_Z$. With essentially the same proof it follows that

$$\mathcal{B}^f_{k,\lambda}=\frac{\lambda}{k}|f|^{2\lambda}\wedge\frac{(\bar\partial\partial|f|^2)^k}{(2\pi i)^k|f|^{2k}}$$

is locally integrable for each $\lambda>0$ and that $(dd^c\log|f|)^k\mathbf{1}_Z=\lim_{\lambda\to 0^+}\mathcal{B}^f_{k,\lambda}$. One can also deduce the equality

$$\lim_{\lambda\to 0^+}\mathcal{A}_{k,\lambda}=\lim_{\lambda\to 0^+}\mathcal{B}_{k,\lambda} \tag{4.5}$$

directly, in the following elementary way. For large λ we have

$$\begin{aligned}\mathcal{A}^f_{k,\lambda}&=\frac{\lambda}{\lambda-k}\frac{\bar\partial|f|^{2\lambda-2k}\wedge\partial|f|^2\wedge(\bar\partial\partial|f|^2)^{k-1}}{(2\pi i)^k}\\&=\frac{1}{(2\pi i)^k}\frac{\lambda}{k-\lambda}(\bar\partial(|f|^{2\lambda-2k}\partial|f|^2\wedge(\bar\partial\partial|f|^2)^{k-1})-|f|^{2\lambda-2k}(\bar\partial\partial|f|^2)^k).\end{aligned}$$

The second term within the brackets gives rise to the limit $\lim_{\lambda\to 0^+}\mathcal{B}_{k,\lambda}$ when $\lambda\to 0^+$. The first term is $\bar\partial$ of $\mathcal{O}(\lambda)|f|^{2\lambda-2k}\partial|f|^2\wedge(\bar\partial\partial|f|^2)^{k-1}$ and it follows easily by a desingularization that this form tends to zero. Thus (4.5) follows.

From Propositions 4.1 and 3.2 we get the following result.

Corollary 4.2. *If the metric is trivial, then $(dd^c\log|f|)^k$ is a positive (k,k)-current for any k, and*

$$(dd^c\log|f|)^k\mathbf{1}_Z=\frac{R^f\cdot(Df/2\pi i)^k}{k!}.$$

It is now easy to obtain (3.15).

Proposition 4.3. *For any metric, if $k=p=\operatorname{codim} Z$, then*

$$(dd^c\log|f|)^p\mathbf{1}_Z=\sum_j\alpha_j[Z_j]=\frac{R^f\cdot(Df/2\pi i)^p}{p!}.$$

Proof. The second equality is precisely Theorem 1.1 so it remains to prove the first one. However, from Proposition 4.1 we know that $(dd^c\log|f|)^p\mathbf{1}_Z$ is a closed (p,p)-current of order zero with support on Z, and by the corollary the equalities hold when the metric is trivial. As in the proof of Theorem 1.1 we can vary the metric locally and conclude that the equalities hold everywhere. □

The current $(dd^c\log|f|)^k$ is robust and it can also be defined as a limit of smooth forms in the following way.

Proposition 4.4. *If f is as in Proposition* 4.1, *then*

$$(dd^c \log|f|)^k = \lim_{\varepsilon\to 0^+} (dd^c \log(|f|^2+\varepsilon)^{1/2})^k.$$

Proof. By a desingularization as before we may assume that $f=f_0 f'$, where f_0 is holomorphic, even a monomial, and f' is nonvanishing. In view of (4.2) we are to prove that then

$$\lim_{\varepsilon\to 0^+}\left(\frac{1}{2\pi i}\bar\partial\partial\log(|f|^2+\varepsilon)\right)^k = [f_0=0]\wedge(dd^c\log|f'|)^{k-1}+(dd^c\log|f'|)^k. \tag{4.6}$$

To simplify notation we let $h=f_0$ and $\alpha=|f'|^2$. We will only use that α is a strictly positive smooth function. A straightforward computation gives that

$$\begin{aligned}\bar\partial\partial\log(|h|^2\alpha+\varepsilon) &= \bar\partial\frac{|h|^2\partial\alpha}{|h|^2\alpha+\varepsilon}+\bar\partial\frac{\alpha\partial|h|^2}{|h|^2\alpha+\varepsilon}\\ &=\varepsilon\frac{\bar\partial|h|^2\wedge\partial\alpha}{(|h|^2\alpha+\varepsilon)^2}+\varepsilon\frac{\bar\partial\alpha\wedge\partial|h|^2}{(|h|^2\alpha+\varepsilon)^2}+\frac{|h|^2\bar\partial\partial\alpha}{|h|^2\alpha+\varepsilon}-\frac{|h|^4\bar\partial\alpha\wedge\partial\alpha}{(|h|^2\alpha+\varepsilon)^2}\\ &\quad+\varepsilon\frac{\alpha d\bar h\wedge dh}{(|h|^2\alpha+\varepsilon)^2}.\end{aligned} \tag{4.7}$$

Notice that the last two terms on the second line are bounded. Some further calculations give

$$\begin{aligned}(\bar\partial\partial\log(|h|^2\alpha+\varepsilon))^k &= k\varepsilon\frac{\alpha^k|h|^{2k-2}d\bar h\wedge dh}{(|h|^2\alpha+\varepsilon)^{k+1}}\wedge\left(\bar\partial\frac{\partial\alpha}{\alpha}\right)^{k-1}+\frac{|h|^{2k}}{(|h|^2\alpha+\varepsilon)^k}(\bar\partial\partial\alpha)^k\\ &\quad-k\frac{|h|^{2k+2}}{(|h|^2\alpha+\varepsilon)^{k+1}}\bar\partial\alpha\wedge\partial\alpha\wedge(\bar\partial\partial\alpha)^{k-1}\\ &\quad+\varepsilon\frac{\bar\partial|h|^2\wedge\partial\alpha}{(|h|^2\alpha+\varepsilon)^2}\wedge\mathcal{O}(1)+\varepsilon\frac{\partial|h|^2\wedge\bar\partial\alpha}{(|h|^2\alpha+\varepsilon)^2}\wedge\mathcal{O}(1).\end{aligned} \tag{4.8}$$

The last two terms vanish when ε tends to 0, and the terms on the middle line tend to $(\bar\partial(\partial\alpha/\alpha))^k=(2\pi i)^k(dd^c\log|f'|)^k$ by dominated convergence. Moreover, if, say, $h(z)=z_1^{m_1}\ldots z_l^{m_l}$, then terms occurring from $d\bar h\wedge dh$ with "mixed variables" will vanish when $\varepsilon\to 0$. In view of the simple Lemma 4.5 below, the expression on the first line tends to

$$(m_1[z_1=0]+\ldots+m_l[z_l=0])\wedge\left(\bar\partial\frac{\partial\alpha}{\alpha}\right)^{k-1},$$

and since $\alpha=\log|f'|^2$, the equality (4.6) follows. This concludes the proof. □

Lemma 4.5. *If α is a strictly positive smooth function in* $\mathbf{C}$ *and* $h(z)=z^m$, *then*

$$\frac{\varepsilon k}{2\pi i}\frac{\alpha^k|h|^{2k-2}d\bar{h}\wedge dh}{(|h|^2\alpha+\varepsilon)^{k+1}}\to m[0].$$

Proof. The form on the left-hand side is positive and tends to zero outside the origin. Therefore it is enough to see that the total mass tends to 1. By the m-to-one change of coordinates $z\mapsto w=h(z)$ in $\mathbf{C}$ we have that

$$\frac{k}{2\pi i}\int_z\frac{\alpha^k|h|^{2k-2}d\bar{h}\wedge dh}{(|h|^2\alpha+\varepsilon)^{k+1}}=m\frac{k}{2\pi i}\int_w\frac{\alpha^k|w|^{2k-2}d\bar{w}\wedge dw}{(|w|^2\alpha+\varepsilon)^{k+1}}.$$

The nonholomorphic change of variables $w=\sqrt{\alpha}\,\zeta$ now gives

$$m\frac{k}{2\pi i}\int_\zeta\frac{|\zeta|^{2k-2}d\bar{\zeta}\wedge d\zeta}{(|\zeta|^2+\varepsilon)^{k+1}}+\mathcal{O}(\varepsilon)=1+\mathcal{O}(\varepsilon),$$

and thus the lemma follows. □

5. Further remarks and examples

We begin with a simple example.

Example 1. Suppose that $p=1$, that we have the trivial metric and that (locally somewhere) there is a function f_0 such that $f=f_0f'$ with $f'\neq 0$. Thus the ideal is generated by f_0. Now, $s=\sum \bar{f}_j e_j^*=\bar{f}_0 s'$ and

$$R_l^f=\bar{\partial}|f|^{2\lambda}\wedge\frac{s'\wedge(\bar{\partial}s')^{l-1}}{f_0^l u^l}\bigg|_{\lambda=0}=\bar{\partial}\left[\frac{1}{f_0^l}\right]\wedge\frac{s'\wedge(\bar{\partial}s')^{l-1}}{u^l},$$

where $u=|f'|^2$. Since

$$[f_0=0]=\frac{\bar{\partial}\partial\log|f_0|^2}{2\pi i}=\bar{\partial}\left[\frac{1}{f_0^l}\right]\wedge f_0^{l-1}df_0/2\pi i,$$

we have

$$M_l^f=\int_e R_l^f\wedge(df/2\pi i)_l\wedge\tilde{I}_{m-l}=[f_0=0]\wedge\int_e\frac{s'\wedge(\bar{\partial}s')^{l-1}\wedge f'\wedge(df')_{l-1}\wedge\tilde{I}_{m-l}}{u^{2l}}.$$

If $l=1$ we get the current $[f_0=0]$ as expected. For higher l we find that M_l^f is equal to $[f_0=0]\wedge\alpha^l$ where α^l is smooth. In view of (4.2) we have that

$$\alpha^l=(dd^c\log|f'|)^{l-1}.$$

In general, M_l^f is nonvanishing on $Z=Z^1$ even for $l>1$. If we take, for instance, $f=(z,zw)$, then $f_0=z$ and $|f'|^2=1+|w|^2$. Thus

$$M_2^f=[z=0]\wedge\frac{d\bar{w}\wedge dw}{2\pi i(1+|w|^2)^2}$$

and this current is nonvanishing on $Z=\{(z,w):z=0\}$.

The example shows that M_k^f is not necessarily vanishing on Z^p for $k>p$. When the metric is nontrivial it is not even positive in general. However, in $X\setminus Z^p$ (recall that Z^p is the union of the irreducible components of Z of codimension p) we have that M_{p+1}^f is closed and positive by the same arguments as before. Therefore we can apply Theorem 1.1 in $X\setminus Z^p$ and conclude that $\widehat{M}_{p+1}^f=M_{p+1}^f\mathbf{1}_{X\setminus Z^p}$ is equal to $\sum_j\alpha_j^{p+1}[Z_j^{p+1}]$ in $X\setminus Z^p$. In general, if we let

$$\widehat{M}_k^f=M_k^f\mathbf{1}_{X\setminus(Z^p\cup Z^{p+1}\cup...\cup Z^{k-1})},$$

then $\widehat{M}_k^f$ is a positive closed (k,k)-current on X and more precisely

$$\widehat{M}_k^f=\sum_j\alpha_j^k[Z_j^k],$$

where Z_j^k are the irreducible components of Z of codimension precisely k. From our previous results it follows that

$$(dd^c\log|f|)^k\mathbf{1}_{Z^k}=\sum_j\alpha_j^k[Z_j^k].$$

Thus we have a full description of current M_k^f on $X\setminus(Z^p\cup Z^{p+1}\cup...\cup Z^{k-1})$.

Let I^f be the ideal generated by f and let $I^f=J_1\cap...\cap J_N$ be a minimal decomposition of I^f in primary ideals J_l. Then the prime ideals $\sqrt{J_k}$ and the corresponding irreducible varieties Y_j (i.e., their zero loci) are unique (except for the order). A primary ideal whose zero locus is a proper subvariety of some irreducible component Z_j^k is said to be embedded.

Example 2. Again we take the trivial metric and let $f=(z_1^2,z_1z_2)=z_1(z_1,z_2)$. Then the ideal $\langle z_1^2,z_2\rangle$ is the intersection of the primary ideals $\langle z_1\rangle$ and $\langle z_1^2,z_2\rangle$. Let us determine M_1^f and M_2^f by direct computation.

Let U be a neighborhood of the origin in $\mathbf{C}^2$ and let $\widetilde{U}$ be the blow up of U at the origin and let $\Pi:\widetilde{U}\to U$ be the natural map. The manifold $\widetilde{U}$ is covered by the coordinate systems τ_1, τ_2 and σ_1, σ_2, where $z_1=\tau_1\tau_2$, $z_2=\tau_2$ and $z_1=\sigma_1$, $z_2=\sigma_1\sigma_2$. Notice that in the τ-coordinates,

$$\log|\Pi^* f|^2 = \log|\tau_1\tau_2^2| + \log(1+|\tau_1|^2),$$

so that

$$\frac{\bar\partial\partial \log|\Pi^* f|^2}{2\pi i} = [\tau_1=0] + 2[\tau_2=0] + \frac{\bar\partial\partial \log(1+|\tau_1|^2)}{2\pi i}.$$

Thus

$$\begin{aligned}\int_U M_1^f \wedge \phi(z) &= \int_{\tilde U} \frac{\bar\partial\partial \log|f|^2}{2\pi i} \mathbf{1}_Z \wedge \Pi^*\phi \\ &= \int_{\tilde U} ([\tau_1=0]+2[\tau_2=0]) \wedge \Pi^*\phi = \int_{\tau_2} \phi(0,\tau_2),\end{aligned}$$

since the pullback of $\Pi^*\phi$ to $\{\tau_2=0\}$ vanishes. Thus $M_1^f=[z_1=0]$, which is in accordance with Theorem 1.1.

To compute M_2^f we choose a test function ϕ. In view of (4.2) we have

$$\begin{aligned}\int_U M_2^f \phi &= \int_{\tilde U} ([\tau_1=0]+2[\tau_2=0]) \wedge \frac{\bar\partial\partial \log(1+|\tau_1|^2)}{2\pi i} \phi(\tau_1\tau_2, \tau_2) \\ &= 2\phi(0) \int_{\tau_1} \frac{\bar\partial\partial \log(1+|\tau_1|^2)}{2\pi i} = 2\phi(0).\end{aligned}$$

and thus $M_2^f=2[0]$.

We do not know if it is true for any embedded prime ideal with zero locus Y^j and codimension k that $M_k^f=\alpha[Y^j]$ locally.

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