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# Coefficient estimates for negative powers of the derivative of univalent functions

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**Abstract.** Let $f$ be a one-to-one analytic function in the unit disc with $f'(0)=1$. We prove sharp estimates for certain Taylor coefficients of the functions $(f')^p$, where $p<0$. The proof is similar to de Branges' proof of Bieberbach's conjecture, and thus relies on Löwner's equation. A special case leads to a generalization of the usual estimate for the Schwarzian derivative of $f$. We use this to improve known estimates for integral means of the functions $|f'|^p$ for integers $p\leq -2$.

## 1. Introduction and results

Let $f$ be a univalent (i.e., one-to-one analytic) function in the unit disc $|z|<1$ with $f(0)=0$ and $f'(0)=1$. Let $S$ denote the set of such $f$. We equip $S$ with the topology of locally uniform convergence; thus $S$ is compact [6, p. 9]. Let $p$ be a real number, and consider the coefficients of the function

$$(f'(z))^p=\sum_{n=0}^{\infty} c_{n,p}z^n, \quad \text{where } c_{0,p}=1. \tag{1.1}$$

The continuous functional $f\mapsto |c_{n,p}|$ assumes a maximum on $S$. The problem of determining (or estimating) this maximum is an interesting problem, which is related to estimates for the integral means of the function $|f'|^{2p}$. This follows from the identity

$$\int_0^{2\pi} |f'(re^{i\theta})|^{2p}\,d\theta=2\pi\sum_{n=0}^{\infty} |c_{n,p}|^2r^{2n}. \tag{1.2}$$

For instance, Brennan's conjecture [3]

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2}\,d\theta=O((1-r)^{-1-\varepsilon}) \quad \text{for all } \varepsilon>0$$

is equivalent to

$$\sum_{n=0}^{N} |c_{n,-1}|^2 = O(N^{1+\varepsilon}) \quad \text{for all } \varepsilon > 0. \tag{1.3}$$

Our main result concerns negative $p$.

**Theorem 1.** *Let $c_{n,p}$ be the $n$th coefficient of $(f')^p$, as in (1.1). Assume that $p<0$ and that $1 \le n \le -2p+1$. Then the maximum of $|c_{n,p}|$ over $f \in S$ is attained when $f$ is the Koebe function $f(z)=z/(1+z)^2$.*

*In other words, if $f \in S$, then*

$$|c_{n,p}| \le C_{n,p}, \tag{1.4}$$

*where $C_{n,p}$ is defined by*

$$\left(\frac{1-z}{(1+z)^3}\right)^p = \sum_{k=0}^{\infty} C_{k,p} z^k, \quad \text{where } C_{0,p} = 1. \tag{1.5}$$

*($C_{k,p}>0$ for all $k \ge 0$, if $p<-\frac{1}{8}$.)*

*Moreover, equality is attained in (1.4) if and only if*

$$f(z) = \frac{z}{(1+\lambda z)^2}, \quad \text{where } |\lambda| = 1.$$

Our proof of Theorem 1 is parallel to de Branges' proof of Milin's conjecture [5], [7]: We consider a Löwner family of single-slit mappings $f_t$. Löwner's equation leads to a linear system of differential equations for the coefficients of $(f_t')^p$. In this way the problem of maximizing $|c_{n,p}|$ becomes a problem in optimal control theory. This problem is solved by proving that a certain quadratic expression in the coefficients is an increasing function of $t$. The computations in this last step are more involved than in de Branges' proof. In de Branges' proof this was easy, since one could use known inequalities for Jacobi polynomials.

Unfortunately, it is not true for every $f \in S$ that $|c_{n,-1}| \le C_{n,-1}$ for all $n$. (This would imply Brennan's conjecture (1.3).) An example is given by the function

$$f(z) = \int_0^z \exp\left(-1.267 \sum_{k=0}^{\infty} \zeta^{40^k}\right) d\zeta,$$

which has $c_{n,p} \ne O(n^{0.064})$ if $p \le -1$, see [8, proof of Corollary 3]. On the other hand, Cauchy's formula for the coefficients of the derivative of (1.5) gives

$$C_{n,p} = O(n^{|p|-1}) \quad \text{if } p<0. \tag{1.6}$$

Thus, if $-1.064 \le p \le -1$, then there is a function $f \in S$ such that $|c_{n,p}| > C_{n,p}$ for infinitely many $n$. This also holds if $-1 < p < 0$, since then there is a function $f \in S$ with

$$\liminf_{r \to 1} \frac{\log \int_0^{2\pi} |f'(re^{i\theta})|^{2p} \, d\theta}{\log\left(\frac{1}{1-r}\right)} > \max\{2|p|-1, 0\}, \tag{1.7}$$

see [4, p. 34] and [13, Theorem 8.6]. (If we had $|c_{n,p}| \le |C_{n,p}|$ for all $n$, then (1.6) and (1.2) would give a contradiction to (1.7).)

The case $n = -2p+1$ of Theorem 1 turns out to be especially interesting. Consider the differential operator

$$S_n f = (f')^{(n-1)/2} D^n (f')^{-(n-1)/2}.$$

Theorem 1 states that

$$|S_n f(0)| \le n! \, C_{n,-(n-1)/2}. \tag{1.8}$$

The operator $S_2$ is just $-\frac{1}{2}$ times the Schwarzian derivative. Like the Schwarzian derivative, $S_n$ has an invariance property,

$$S_n(f \circ \tau) = ((S_n f) \circ \tau)(\tau')^n \quad \text{if } \tau \text{ is a Möbius transformation.} \tag{1.9}$$

See [9] for a general discussion of operators with this property. Using a disc automorphism $\tau$ we can thus "move" the estimate (1.8) to an arbitrary point of the unit disc.

**Theorem 2.** *For functions $f$ univalent in the unit disc we have the sharp estimate*

$$|S_n f(z)| = \left| (f'(z))^{(n-1)/2} \left(\frac{d}{dz}\right)^n (f'(z))^{-(n-1)/2} \right| \le K_n (1-|z|^2)^{-n},$$

*where $K_n = (n-1)(n+1)(n+3) \dots (3n-3)$ and $n$ is a positive integer.*

When $n=2$ this is the usual estimate for the Schwarzian derivative [6, p. 263]. Using this estimate for the Schwarzian derivative, Pommerenke [13, Theorem 8.5] proved that

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-1} \, d\theta = O((1-r)^{-0.601}) \quad \text{if } f \in S. \tag{1.10}$$

We use Pommerenke's argument and Theorem 2 to prove the following estimates for integral means.

**Theorem 3.** *Let $E_n$ be the positive root of*

$$E(E+1)(E+2)\dots(E+2n-1)=K_n^2,$$

*where $K_n$ is as in Theorem 2, and $n>1$ is an integer.*

*If $f$ is a univalent function in the unit disc, then*

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-n+1}\,d\theta = O((1-r)^{-E_n-\varepsilon}) \quad \textit{for all } \varepsilon>0.$$

*In particular,*

$$\int_0^{2\pi} |f'(re^{i\theta})|^{-2}\,d\theta = O((1-r)^{-1.547}),$$
$$\int_0^{2\pi} |f'(re^{i\theta})|^{-3}\,d\theta = O((1-r)^{-2.530}).$$

These estimates are small improvements of the known estimates

$$\int_0^{2\pi} |f'(re^{i\theta})|^{p}\,d\theta = O\left(\left(\frac{1}{1-r}\right)^{|p|-0.399}\right) \quad \text{for } p\le -1,$$

which follow from (1.10) and the elementary estimate $|f'(z)|>\frac{1}{8}|f'(0)|(1-|z|)$. Note that as $n\to+\infty$ we have $E_n=n-\frac{3}{2}+o(1)$. This should be compared with Carleson's and Makarov's result [4, Corollary 1]

$$\int_0^{2\pi} |f'(re^{i\theta})|^{p}\,d\theta = O\left(\left(\frac{1}{1-r}\right)^{|p|-1}\right) \quad \text{for large negative } p.$$

The exponent $|p|-1$ is best possible (take $f$ as a Koebe function).

In the next two sections we prove Theorem 1, except for the characterization of the extremal functions, which will be proved in Section 4. We prove Theorems 2 and 3 in Sections 5 and 6, respectively.

## 2. The proof of Theorem 1

Let $\Gamma\colon[0,+\infty)\to\mathbf{C}$ be a parametrization of a Jordan arc such that, for some $T>0$, the arc $\Gamma([T,+\infty))$ is an interval $[\Gamma(T),+\infty)$ on the positive real axis. Assume that $f_0\in S$ maps onto the complement of the arc $\Gamma([0,+\infty))$. The set of mappings of the type $f_0$ is dense in $S$ [6, p. 81]. Thus it is sufficient to prove Theorem 1 for $f=f_0$. For $t>0$, let $f_t$ be the Riemann mapping of the unit disc onto the complement of the arc $\Gamma([t,+\infty))$, normalized so that $f_t(0)=0$ and $f'_t(0)>0$. We can choose the parametrization $\Gamma$ so that $f'_t(0)=e^t$. Löwner's differential equation [1, Chapter 6] then relates the mappings $f_t$,

$$\dot{f}_t(z)=f'_t(z)z\frac{1+\omega(t)z}{1-\omega(t)z} \quad \text{for } t\geq 0, \tag{2.1}$$

where $\omega$ is a complex-valued continuous function with $|\omega(t)|=1$, and the dot denotes differentiation with respect to $t$.

Fix an integer $n$ and a negative number $p$ such that $1\leq n\leq -2p+1$. We are interested in the coefficients of the functions

$$g_t(z)=(e^{-t}f'_t(z))^p=\sum_{k=0}^{\infty}c_k(t)z^k, \quad \text{where } c_0(t)=1.$$

Specifically, our task is to prove the inequality

$$|c_n(0)|\leq C_n, \tag{2.2}$$

where $C_n=C_{n,p}$ is defined by (1.5). That $C_n>0$ will be proved after Lemma 4 in Section 3. To prove (2.2) we use two facts. First, the function $e^{-T}f_T\in S$ maps onto the complement of an interval $[x,+\infty)$ on the positive real axis. Thus $e^{-T}f_T(z)=z/(1+z)^2$, which implies

$$c_k(T)=C_k \quad \text{for } k\geq 0.$$

Second, differentiation of Löwner's equation (2.1) gives

$$\dot{g}_t(z)=g'_t(z)z\frac{1+\omega(t)z}{1-\omega(t)z}+pg_t(z)\left(\frac{2\omega(t)z}{1-\omega(t)z}+\frac{2\omega(t)z}{(1-\omega(t)z)^2}\right),$$

which leads to a linear system of differential equations for the coefficients of $g_t$,

$$\dot{c}_k(t)=kc_k(t)+\sum_{j=0}^{k-1}(2j+2p(k-j+1))\omega(t)^{k-j}c_j(t) \quad \text{for } k\geq 0. \tag{2.3}$$

We can write this in matrix form using the vectors

$$c(t)=(c_0(t),c_1(t),\dots,c_n(t))^T \quad \text{and} \quad C=(C_0,C_1,\dots,C_n)^T$$

and the lower triangular matrix $M(\omega)$ with elements

$$M(\omega)_{kj}=\begin{cases}(2j+2p(k-j+1))\omega^{k-j} & \text{for } 0\leq j<k\leq n;\\ k & \text{for } 0\leq j=k\leq n;\\ 0 & \text{for } 0\leq k<j\leq n.\end{cases}$$

We have thus reduced our task to the following control theory problem.

**Control problem.** *Let $c(t)$ be the solution of the initial value problem*

$$\dot{c}(t)=M(\omega(t))c(t), \quad 0\le t\le T,\ c(T)=C, \tag{2.4}$$

*where $\omega(t)$ is a continuous function of modulus 1. Prove that $|c_n(0)|\le C_n$.*

Note that the choice $\omega(t)\equiv 1$ corresponds to $f_t(z)=e^t z/(1+z)^2$ and $c_n(t)=C_n$. Following de Branges, we solve the above problem by studying the quadratic expression

$$H(t)=\sum_{k=0}^{n} h_k(t)|c_k(t)|^2, \tag{2.5}$$

where the weights $h_k(t)$ are real-valued and depend on $t$, but not on the function $\omega$. We get

$$\begin{aligned}\dot{H}(t)&=\sum_{k=0}^{n} \dot{h}_k(t)|c_k(t)|^2+h_k(t)2\operatorname{Re}(\overline{c_k(t)}\dot{c}_k(t))\\ &=\sum_{k=0}^{n} \dot{h}_k(t)|\gamma_k(t)|^2+h_k(t)2\operatorname{Re}\left(\overline{\gamma_k(t)}\sum_{j=0}^{k} M(1)_{kj}\gamma_j(t)\right),\end{aligned} \tag{2.6}$$

where we have introduced $\gamma_j(t)=\omega(t)^{-j}c_j(t)$ and used equation (2.3). Using the diagonal matrix $D(t)$ with diagonal elements $h_0(t),\dots,h_n(t)$ and the vector $\gamma(t)=(\gamma_0(t),\dots,\gamma_n(t))^T$, we can write this as

$$\dot{H}(t)=\overline{\gamma(t)}^T(\dot{D}(t)+M(1)^T D(t)+D(t)M(1))\gamma(t).$$

We now try to determine the functions $h_k$ so that the following conditions are satisfied:

$$\dot{H}(t)\ge 0 \quad \text{for } t\ge 0 \text{ and for all choices of } \omega, \tag{2.7}$$

$$\dot{H}(t)=0 \quad \text{if } \omega=1, \tag{2.8}$$

$$h_k(0)=0 \quad \text{for } k<n \quad \text{and} \quad h_n(0)=1. \tag{2.9}$$

If these conditions are satisfied, our problem is solved, since we get $|c_n(0)|^2=H(0)\le H(T)=\sum_{k=0}^{n} h_k(T)C_k^2$, with equality if $\omega(t)\equiv 1$; hence $|c_n(0)|\le C_n$.

When establishing (2.7), we will forget that the $\gamma_k(t)$ are not arbitrary, and prove the stronger statement that the matrix

$$P(t)=\dot{D}(t)+M(1)^T D(t)+D(t)M(1) \quad \text{is positive semi-definite for } t\ge 0. \tag{2.10}$$

Let us first show that conditions (2.8)–(2.10) determine $h_k$ uniquely. In the case $\omega(t)\equiv 1$ we have $\gamma(t)=C$, so condition (2.8) gives $C^T P(t)C=0$. Condition (2.10) now yields $\sqrt{P(t)}\,C=0$, whence

$$P(t)C=\dot{D}(t)C+M(1)^T D(t)C+D(t)M(1)C=0.$$

However, $M(1)C=0$ (the case $\omega(t)\equiv 1$ of equation (2.4)). Thus we get

$$\dot{y}(t)=-M(1)^T y(t), \tag{2.11}$$

where $y(t)=D(t)C=(y_0(t),\dots,y_n(t))^T$. Condition (2.9) implies the initial condition

$$y(0)=(0,0,\dots,0,C_n)^T. \tag{2.12}$$

Thus

$$h_k(t)=y_k(t)/C_k, \quad k=0,1,\dots,n, \tag{2.13}$$

are uniquely determined.

Now, define $y(t)$ and $h_k(t)$ by equations (2.11)–(2.13). Then we have $P(t)C=0$, so condition (2.8) is satisfied, as well as (2.9). It remains to prove condition (2.10), that is, that the Hermitian form $\eta(t)=\overline{\gamma}^T P(t)\gamma$, also given by the right-hand side of (2.6), is positive semi-definite.

From now on we consider $\gamma_k$ as free variables (independent of $t$ and $\omega$). The quantity $\eta(t)$ takes a simple form expressed in the new variables

$$\alpha_k=\sum_{j=0}^{k}(k-j+1)\gamma_j, \quad k=0,1,\dots,n.$$

We can write $\gamma_k=\alpha_k-2\alpha_{k-1}+\alpha_{k-2}$ and

$$\sum_{j=0}^{k} M(1)_{kj}\gamma_j=k\alpha_k+(4p-2)\alpha_{k-1}+(-2p-k)\alpha_{k-2},$$

where $\alpha_{-1}=\alpha_{-2}=0$. Substituting this into (2.6) and collecting terms one gets

$$\eta(t)=\sum_{k=0}^{n} u_k(t)|\alpha_k|^2+v_k(t)2\operatorname{Re}(\alpha_k\overline{\alpha_{k-1}})+w_k(t)2\operatorname{Re}(\alpha_k\overline{\alpha_{k-2}}), \tag{2.14}$$

where

$$u_k(t)=\dot h_k(t)+4\dot h_{k+1}(t)+\dot h_{k+2}(t)+2kh_k(t) +(-16p+8)h_{k+1}(t)+(-4p-4-2k)h_{k+2}(t), \tag{2.15}$$

$$v_k(t)=-2\dot h_k(t)-2\dot h_{k+1}(t)+(4p-2-2k)h_k(t)+(8p+2k)h_{k+1}(t), \tag{2.16}$$

$$w_k(t)=\dot h_k(t)-2ph_k(t), \tag{2.17}$$

and $h_{n+1}(t)=h_{n+2}(t)=0$. We want to complete the squares and write

$$\eta(t)=\sum_{k=0}^{n} p_k(t)|\alpha_k+q_k(t)\alpha_{k-1}+r_k(t)\alpha_{k-2}|^2. \tag{2.18}$$

First, we prove that this is possible if $t$ is large.

**Lemma 1.** *If $n<-2p+1$ then*

$$h_k(t)=\sum_{j=k}^{n} h_{kj}e^{-jt}>0 \quad \textit{for } t>0,\ k=0,1,\dots,n, \tag{2.19}$$

*where $h_{kj}$ are constants and $h_{kk}>0$.*

*Proof.* By (2.13) and (2.11) we have

$$\dot h_k(t)+kh_k(t)=C_k^{-1}\sum_{j=k+1}^{n}(-2k-2p(j-k+1))C_jh_j(t), \tag{2.20}$$

where the coefficient of $h_j(t)$ is positive thanks to the assumption $n<-2p+1$. A descending induction on $k$, using also the initial condition (2.9), now proves (2.19). Moreover, $h_{kk}=h_k(0)+\int_0^\infty e^{kt}(\dot h_k(t)+kh_k(t))\,dt>0$. $\square$

By continuity, we can henceforth assume that $n<-2p+1$. Lemma 1 implies that $u_k(t), v_k(t)$ and $w_k(t)$ are polynomials in $e^{-t}$,

$$u_k(t)=kh_{kk}e^{-kt}+\dots\,,$$
$$v_k(t)=(4p-2)h_{kk}e^{-kt}+\dots\,,$$
$$w_k(t)=(-2p-k)h_{kk}e^{-kt}+\dots\,,$$

where the omitted terms are of smaller order as $t\to+\infty$. This shows that for large $t$ we can successively complete squares in (2.14), beginning with terms containing $\alpha_n$. In each step of this process the coefficient of terms of type $|\alpha_k|^2$ will have leading term $kh_{kk}e^{-kt}$. Thus we get (2.18) with $p_k(t)>0$ for $k=1,2,\dots,n$, for large $t$. Since $P(t)$ is a singular matrix, $\eta(t)$ is also singular, so we must have $p_0(t)=0$ for large $t$.

The functions $p_k(t)$, $q_k(t)$, $r_k(t)$, now defined for large $t$, are rational functions of $e^{-t}$. We now want to prove that these functions have no poles for $0<e^{-t}\le 1$, and that $p_k(t)>0$ for $t\ge 0$, $k=1,2,\dots,n$. Thanks to the special structure of $\eta(t)$, we can derive a one-step recursion formula for $p_k(t)$. Identifying coefficients in (2.14) and (2.18) we get, for large $t$,

$$(2.21)\qquad u_k(t)=p_k(t)+p_{k+1}(t)q_{k+1}(t)^2+p_{k+2}(t)r_{k+2}(t)^2,\quad 0\le k\le n,$$

$$(2.22)\qquad v_k(t)=p_k(t)q_k(t)+p_{k+1}(t)q_{k+1}(t)r_{k+1}(t),\qquad 1\le k\le n,$$

$$(2.23)\qquad w_k(t)=p_k(t)r_k(t),\qquad 2\le k\le n,$$

where $p_{n+1}=p_{n+2}=q_{n+1}=r_{n+1}=r_{n+2}=0$.

In the case $\gamma=C$ we have $\eta(t)=C^TP(t)C=0$, so (2.18) together with $p_k(t)>0$ for $1\le k\le n$ and $p_0(t)=0$ gives, for large $t$,

$$(2.24)\qquad A_k+q_k(t)A_{k-1}+r_k(t)A_{k-2}=0,\quad 1\le k\le n,$$

where $A_k=\sum_{j=0}^{k}(k-j+1)C_j$. Solve (2.24) for $q_k(t)$, plug this into (2.22) and use (2.23). This gives the recursion formula

$$(2.25)\qquad p_k(t)=-\frac{A_{k-2}}{A_k}w_k(t)-\frac{A_{k-1}A_{k+1}}{A_k^2}w_{k+1}(t)-\frac{A_{k-1}}{A_k}v_k(t)-\frac{A_{k-1}^2}{A_k^2}\frac{w_{k+1}(t)^2}{p_{k+1}(t)}$$

for $1\le k\le n-1$, but still only for large $t$. In the next section we use this formula to prove the following lemma.

**Lemma 2.** *The meromorphic functions $p_k(t)$ are regular for $t\ge 0$, and for $t\ge 0$ we have*

$$(2.26)\qquad p_n(t)=nh_n(t)>0,\quad p_0(t)=0,$$

$$(2.27)\qquad p_k(t)>\frac{A_{k-2}}{A_k}w_k(t)+\frac{A_{k-1}}{A_k}2(k+1)h_k(t)\ge 0\quad \textit{for } k=1,2,\dots,n-1.$$

By (2.23) and (2.24) the functions $r_k$ and $q_k$ are also regular for $t\ge 0$. Hence (2.18) holds for $t\ge 0$, and thus $\eta(t)\ge 0$.

*Remark.* If $n>-2p+1$ and $p<-\frac{1}{8}$ the proof breaks down: We get $u_{n-1}(t)<0$ for large $t$; hence $P(t)$ is not positive semi-definite.

*Remark.* Casting de Branges' proof of Milin's conjecture into our notation we note the following differences. Corresponding to (2.14) one gets the simpler expression

$$\tilde{\eta}(t)=\sum_{k=0}^{n}\tilde{u}_k(t)|\tilde{\alpha}_k|^2+\tilde{v}_k(t)2\operatorname{Re}(\tilde{\alpha}_k\overline{\tilde{\alpha}_{k-1}}).$$

This stems from the fact that one studies the functions $\tilde{g}_t(z)=\log(e^{-t}f_t(z)/z)$, which contain no derivative of $f_t$. The simpler structure of $\tilde{\eta}(t)$ implies that the functions $\tilde{p}_k(t)$ corresponding to our functions $p_k(t)$ become polynomials in $e^{-t}$.

## 3. The proof of Lemma 2

For convenience, introduce the parameter $q=-2p+1>n$. We need some lemmas concerning the constants $C_k$, $A_k$ and $B_k=A_k-A_{k-1}=C_0+\ldots+C_k$.

**Lemma 3.** *The following recursion formulas hold for all integers $k$:*

$$kC_k=(2q-2)C_{k-1}-(q-k+1)C_{k-2}, \tag{3.1}$$

$$kB_k=(2q-1)B_{k-1}-(q-k)B_{k-2}, \tag{3.2}$$

$$kA_k=2qA_{k-1}-(q-k-1)A_{k-2}. \tag{3.3}$$

*Here, $C_k=B_k=A_k=0$ for $k<0$.*

*Proof.* These formulas follow from applying $(1-z^2)d/dz$ on the generating functions

$$(1-z)^p(1+z)^{-3p}=\sum_{k=0}^{\infty}C_kz^k,$$

$$(1-z)^{p-1}(1+z)^{-3p}=\sum_{k=0}^{\infty}B_kz^k,$$

$$(1-z)^{p-2}(1+z)^{-3p}=\sum_{k=0}^{\infty}A_kz^k. \quad \square$$

**Lemma 4.** *If $q\geq 3$, then $kC_k>qC_{k-1}>0$ for $1\leq k\leq q+1$.*

*Proof.* The case $k=1$ is clear. Inductively, assume that $kC_k>qC_{k-1}>0$, where $1\leq k\leq q$. Together with equation (3.1) this implies

$$(k+1)C_{k+1}\geq(2q-2)C_k-(q-k)kC_k/q>qC_k. \quad \square$$

This shows that $C_k>0$ if $4\leq k\leq q+1$. Moreover, for $q>\frac{5}{4}$, $C_0=1$, $C_1=2q-2$, $C_2=(q-1)\left(2q-\frac{5}{2}\right)$ and $C_3=\frac{1}{3}(q-1)(4q^2-11q+9)$ are all positive. Equation (3.1) now implies that $C_k>0$ for all $k\geq 0$, if $q>\frac{5}{4}$.

**Lemma 5.** *If $q \geq 3$, then $C_{k-1}/C_k < C_k/C_{k+1}$ for $0 \leq k \leq q$.*

*Proof.* The case $k=0$ is trivial. Inductively, assume that $C_k C_{k-2} < C_{k-1}^2$, where $1 \leq k \leq q$. Equation (3.1) implies

$$(k+1)C_{k+1}C_{k-1} - kC_k^2 = (q+1-k)C_k C_{k-2} - (q-k)C_{k-1}^2.$$

But Lemma 4 implies $C_k > C_{k-2}$, so we get

$$(k+1)(C_{k+1}C_{k-1} - C_k^2) < (q-k)(C_k C_{k-2} - C_{k-1}^2) \leq 0. \quad \square$$

**Lemma 6.** *If $q \geq 3$, then $A_{k-1}/A_k < B_{k-1}/B_k$ for $1 \leq k \leq q$.*

*Proof.* By Lemma 5 we have

$$\frac{B_{j-1}}{B_j} = \frac{C_0 + \ldots + C_{j-1}}{C_0 + \ldots + C_j} < \frac{C_0 + \ldots + C_j}{C_0 + \ldots + C_{j+1}} = \frac{B_j}{B_{j+1}} \quad \text{for } 1 \leq j \leq q.$$

Thus

$$\frac{A_{k-1}}{A_k} = \frac{B_0 + \ldots + B_{k-1}}{B_0 + \ldots + B_k} < \frac{B_{k-1}}{B_k} \quad \text{for } 1 \leq k \leq q. \quad \square$$

**Lemma 7.** *If $q \geq 3$ and $1 \leq k < q$, then*

$$\frac{C_k B_{k+1}}{C_{k+1} B_k} < 1 + \frac{1}{q-k}. \tag{3.4}$$

*Proof.* The case $k=1$ is easily checked. Inductively, assume that

$$\frac{C_{k-1} B_k}{C_k B_{k-1}} < 1 + \frac{1}{q+1-k},$$

where $2 \leq k < q$. Using $C_k = B_k - B_{k-1}$ and the recursion formula (3.2) we can write this as

$$-k(q-k+1)B_k^2 + (2q(q-k) + 2q - 1)B_k B_{k-1} - (q-k+2)(q-k)B_{k-1}^2 > 0. \tag{3.5}$$

We want to prove the inequality (3.4), which in a similar way can be written

$$\begin{aligned}(2q-1-(k+1)(q-k+1))B_k^2 + ((2q-2)(q-k)+1)B_k B_{k-1} \\ -(q-k-1)(q-k)B_{k-1}^2 > 0.\end{aligned} \tag{3.6}$$

It suffices to prove that the difference between the left-hand sides of (3.6) and (3.5),

$$(q+k-2)B_k^2+(-4q+2k+2)B_kB_{k-1}+3(q-k)B_{k-1}^2 \\ =(B_k-B_{k-1})((q+k-2)B_k-3(q-k)B_{k-1})$$

is positive. Lemmas 5 and 4 imply

$$\frac{B_k}{B_{k-1}}=\frac{C_0+\ldots+C_k}{C_0+\ldots+C_{k-1}}>\frac{C_k}{C_{k-1}}>\frac{q}{k}.$$

Thus we need only prove that $(q+k-2)q/k-3(q-k)\geq 0$, which is a simple verification. □

We also need the following facts. By (2.20), (2.19) and $h_n(t)=e^{-nt}$ we have $\dot{h}_k(t)+kh_k(t)>0$ if $t\geq 0$ and $0\leq k\leq n-1$, and so equation (2.17) and Lemma 1 yields

$$w_k(t)>0 \quad \text{for } t\geq 0, \text{if } 0\leq k\leq n-1. \tag{3.7}$$

Together with Lemma 1, this proves the second inequality of (2.27). By (2.16) and (2.17) we have

$$v_k(t)=-2w_k(t)-2w_{k+1}(t)-2(k+1)h_k(t)-2(q-k-1)h_{k+1}(t),$$

which substituted into (2.25) gives the recursion formula

$$\begin{aligned} p_k(t)=\frac{A_{k-1}}{A_k}\Bigg[&\left(2-\frac{A_{k-2}}{A_{k-1}}\right)w_k(t)+\left(2-\frac{A_{k+1}}{A_k}\right)w_{k+1}(t)+2(k+1)h_k(t) \\ &+2(q-k-1)h_{k+1}(t)-\frac{A_{k-1}w_{k+1}(t)^2}{A_kp_{k+1}(t)}\Bigg], \quad 1\leq k\leq n-1. \end{aligned} \tag{3.8}$$

Now (2.26) follows from (2.21), (2.15) and (2.20),

$$p_n(t)=u_n(t)=\dot{h}_n(t)+2nh_n(t)=nh_n(t)=ne^{-nt}>0 \quad \text{for } t\geq 0. \tag{3.9}$$

We prove inequality (2.27) by descending induction over $k$.

**Induction base.** *For $t\geq 0$ and $2\leq n<q$,*

$$p_{n-1}(t)>\frac{A_{n-3}}{A_{n-1}}w_{n-1}(t)+\frac{A_{n-2}}{A_{n-1}}2nh_{n-1}(t).$$

*Proof.* By (3.8) and (3.9), this is equivalent to

$$2\left(1-\frac{A_{n-3}}{A_{n-2}}\right)w_{n-1}(t)+\left(2-\frac{A_n}{A_{n-1}}\right)w_n(t)+2(q-n)h_n(t)-\frac{A_{n-2}}{A_{n-1}}\frac{w_n(t)^2}{nh_n(t)}>0. \tag{3.10}$$

By (2.20) and (2.17) we have

$$w_{n-1}(t)=(q-n)(2C_nC_{n-1}^{-1}h_n(t)+h_{n-1}(t)) \quad \text{and} \quad w_n(t)=(q-n-1)h_n(t),$$

which substituted into (3.10) gives

$$2\left(1-\frac{A_{n-3}}{A_{n-2}}\right)(q-n)h_{n-1}(t)+\left[2\left(1-\frac{A_{n-3}}{A_{n-2}}\right)(q-n)\frac{2C_n}{C_{n-1}}+\left(2-\frac{A_n}{A_{n-1}}\right)(q-n-1)\right.$$
$$\left.+2(q-n)-\frac{A_{n-2}}{A_{n-1}}\frac{(q-n-1)^2}{n}\right]h_n(t)>0.$$

Since $A_{n-3}<A_{n-2}$ and $h_{n-1}(t)\geq 0$, we only have to prove that the coefficient of $h_n(t)$ is positive. Using the recursion formula (3.3) with $k=n$, we can write this coefficient as

$$4(q-n)\left[\left(1-\frac{A_{n-3}}{A_{n-2}}\right)\frac{C_n}{C_{n-1}}+1-\frac{q-1}{2n}\right]. \tag{3.11}$$

It follows from Lemma 4 and $C_2/C_1=q-\frac{5}{4}$ that

$$\frac{C_n}{C_{n-1}}>\frac{q-1}{2n}.$$

Lemma 5 implies that

$$\frac{A_{n-3}}{A_{n-2}}=\frac{C_{n-3}+2C_{n-4}+\ldots+(n-2)C_0}{C_{n-2}+2C_{n-3}+\ldots+(n-2)C_1+(n-1)C_0}<\frac{C_{n-1}}{C_n}.$$

Thus (3.11) is positive. $\square$

**Induction step.** Assume that $1\leq k\leq n-2$, $t\geq 0$ and

$$p_{k+1}(t)>\frac{A_{k-1}}{A_{k+1}}w_{k+1}(t)+\frac{A_k}{A_{k+1}}2(k+2)h_{k+1}(t).$$

Hence, by (3.7) and Lemma 1,

$$p_{k+1}(t)>\frac{A_{k-1}}{A_{k+1}}w_{k+1}(t)>0. \tag{3.12}$$

We want to prove that $p_k(t)>(A_{k-2}/A_k)w_k(t)+(A_{k-1}/A_k)2(k+1)h_k(t)$. By the recursion formula (3.8) and (3.12) it is enough to prove

$$2\left(1-\frac{A_{k-2}}{A_{k-1}}\right)w_k(t)+2\left(1-\frac{A_{k+1}}{A_k}\right)w_{k+1}(t)+2(q-k-1)h_{k+1}(t)>0. \tag{3.13}$$

Using the functions $s_k(t)=\sum_{j=k}^{n}(j-k+1)y_j(t)$ we can write the differential equation (2.11) as

$$\dot{y}_k(t)=-ks_k(t)+2(q-1)s_{k+1}(t)+(-q+1+k)s_{k+2}(t). \tag{3.14}$$

Remember that $h_k(t)=y_k(t)/C_k\geq 0$. Substituting this, $y_k(t)=s_k(t)-2s_{k+1}(t)+s_{k+2}(t)$ and (3.14) into (2.17) we get

$$w_k(t)=C_k^{-1}((q-1-k)s_k(t)+ks_{k+2}(t)).$$

Putting this into (3.13) and using $h_{k+1}(t)\geq 0$, we see that it suffices to prove

$$(q-1-k)s_k(t)+ks_{k+2}(t)-\mu_k(q-2-k)s_{k+1}(t)-\mu_k(k+1)s_{k+3}(t)>0, \tag{3.15}$$

where

$$\mu_k=\frac{\dfrac{C_k}{C_{k+1}}\left(\dfrac{A_{k+1}}{A_k}-1\right)}{1-\dfrac{A_{k-2}}{A_{k-1}}}=\frac{C_kB_{k+1}A_{k-1}}{C_{k+1}B_{k-1}A_k}>0.$$

Note that $q>n\geq 3$. From Lemma 6 and Lemma 7 it follows that

$$\mu_k=\frac{C_kB_{k+1}}{C_{k+1}B_k}\frac{B_kA_{k-1}}{B_{k-1}A_k}<1+\frac{1}{q-k}.$$

Hence

$$\frac{s_k(t)}{s_{k+1}(t)}=\frac{y_k(t)+2y_{k+1}(t)+...+(n-k+1)y_n(t)}{y_{k+1}(t)+...+(n-k)y_n(t)}\geq\frac{n-k+1}{n-k}>\mu_k,$$

and similarly $s_{k+2}(t)>\mu_k s_{k+3}(t)$. Thus

$$(q-1-k)(s_k(t)-\mu_k s_{k+1}(t))+k(s_{k+2}(t)-\mu_k s_{k+3}(t))>0,$$

which implies (3.15), since $s_{k+1}(t)>s_{k+3}(t)$.

## 4. The extremal functions

Clearly, the Koebe functions $f(z)=z/(1+\lambda z)^2$, where $|\lambda|=1$, give equality in Theorem 1. Assume that $f(z)=z+a_2z^2+a_3z^3+...\in S$ is not a Koebe function. As in [7] we use an approximation argument to prove that this implies strict inequality in (1.4). The function $f$ is the limit of a sequence of functions $f_0^m$ of the type $f_0$ considered in Section 2. We denote the corresponding quantities with the same symbol as in Section 2, but with a superscript $m$. Assume first that $1 \le n < q = -2p+1$. Since $c_1^m(0) = 2pa_2^m$, Bieberbach's theorem $|a_2| < 2$ [12, Theorem 1.5] implies that

$$|c_1^m(0)| < 4|p| - \delta = C_1 - \delta \quad \text{for } m > m_0,$$

where $\delta > 0$. The differential equation (2.3) implies

$$|\dot{c}_1^m(t)| = |c_1^m(t) + 4p\omega(t)| \le C_1 + 4|p|.$$

Thus

$$|c_1^m(t)| < C_1 - \tilde{\delta} \quad \text{for } m > m_0,\ 0 < t < t_0,$$

where $\tilde{\delta} > 0$ and $t_0 > 0$ are independent of $m$. Now equation (2.18) gives, for $m > m_0$ and $0 < t < t_0$,

$$\dot{H}^m(t) \ge p_1(t)|\gamma_1^m(t) + 2\gamma_0^m(t) + q_1(t)\gamma_0^m(t)|^2 = p_1(t)|\omega(t)^{-1}c_1^m(t) - C_1|^2 > p_1(t)\tilde{\delta}^2.$$

By Lemma 2 we have $p_1(t) \ge h_1(t)/q$, so we get

$$|c_n^m(0)|^2 = H^m(0) \le \sum_{k=0}^{n} h_k(T)C_k^2 - \int_0^{t_0} \frac{\tilde{\delta}^2}{q} h_1(t)\,dt = C_n^2 - \int_0^{t_0} \frac{\tilde{\delta}^2}{q} h_1(t)\,dt.$$

In the limit $m \to \infty$ this shows that $|c_{n,p}| < C_n$. By the proof of Lemma 1, this conclusion also holds in the limit case $q = n$, except when $n = 2$, in which case $h_1(t) \equiv 0$. In this exceptional case we have $c_{2,-1/2} = \frac{3}{2}(a_2^2 - a_3)$. Thus it follows from the elementary estimate $|a_2^2 - a_3| < 1$ [12, Theorem 1.5] that we have strict inequality in (1.4). This concludes the proof of Theorem 1.

*Remark.* Just like our estimate of $c_{2,-1/2}$, the estimate of $c_{3,-1}$ is not new. It was proved by Ozawa [11] in the form $|a_4 - 3a_2a_3 + 2a_2^3| \le 2$ using Schiffer's variational method.

## 5. Generalized Schwarzian derivatives

Define the differential operator $S_n$ for conformal maps $f$ by

$$S_n f = (f')^{(n-1)/2} D^n (f')^{-(n-1)/2},$$

where we use the same branch of $\sqrt{f'}$ at both occurrences. To prove the invariance property (1.9) we use the following lemma [2], [10].

**Lemma 8.** *Let $\tau$ be a Möbius transformation. If two analytic functions are related by $\tilde{g}=(g\circ\tau)(\tau')^{-(n-1)/2}$, then their nth derivatives have the relation $\tilde{g}^{(n)}=(g^{(n)}\circ\tau)(\tau')^{(n+1)/2}$.*

*Proof.* Since $\tau$ is a composition of transformations of the types $z\mapsto z+a$, $z\mapsto bz$ and $z\mapsto 1/z$, it suffices to prove the lemma when $\tau$ is one of these. The first two cases are rather trivial. In the third case $\tau(z)=1/z$, we can by continuity and linearity assume that $g(z)=z^k$, and then an easy calculation proves the lemma. □

Now let $g=(f')^{-(n-1)/2}$ and $\tilde{g}=((f\circ\tau)')^{-(n-1)/2}$. Since $\tilde{g}=(g\circ\tau)(\tau')^{-(n-1)/2}$, the lemma shows that

$$S_n(f\circ\tau)=\frac{\tilde{g}^{(n)}}{\tilde{g}}=\frac{g^{(n)}\circ\tau}{g\circ\tau}(\tau')^n=((S_nf)\circ\tau)(\tau')^n, \tag{5.1}$$

if $\tau$ is a Möbius transformation.

We now prove Theorem 2. Theorem 1 gives

$$|S_nf(0)|\le n!\,C_{n,-(n-1)/2} \tag{5.2}$$

if $f\in S$. Since $S_n$ is homogeneous this holds also if $f$ is just univalent in the unit disc. By (1.5), $C_{n,-(n-1)/2}$ is the $n$th coefficient of

$$(1-z)^{-(n-1)/2}(1+z)^{3(n-1)/2}=\sum_{k=0}^{\infty}\binom{-(n-1)/2}{k}(-z)^k\sum_{j=0}^{\infty}\binom{3(n-1)/2}{j}z^j,$$

so that

$$C_{n,-(n-1)/2}=\sum_{k=0}^{n}\binom{-(n-1)/2}{k}\binom{3(n-1)/2}{n-k}(-1)^k=\frac{K_n}{n!},$$

where $K_n=(n-1)(n+1)\dots(3n-3)$. Now let $\tau(z)=(z+\zeta)/(1+\bar{\zeta}z)$, where $|\zeta|<1$, and let $f$ be a univalent function in the unit disc. Then $f\circ\tau$ is also univalent in the unit disc, so (5.2) gives $|S_n(f\circ\tau)(0)|\le K_n$. Together with the invariance property (5.1), this shows that

$$|S_nf(\zeta)|=|S_nf(\tau(0))|=\left|\frac{S_n(f\circ\tau)(0)}{(\tau'(0))^n}\right|\le\frac{K_n}{(1-|\zeta|^2)^n},$$

since $\tau'(0)=1-|\zeta|^2$.

## 6. Estimates for integral means

We now use Theorem 2, following Pommerenke [13, p. 181] to prove Theorem 3.

**Lemma 9.** *If $g$ is analytic in the unit disc and $m(r)=\int_0^{2\pi}|g(re^{i\theta})|^2\,d\theta$, then*

$$m^{(2n)}(r)\le 4^n\int_0^{2\pi}|g^{(n)}(re^{i\theta})|^2\,d\theta.$$

*Proof.* Writing $g(z)=\sum_{k=0}^{\infty}b_kz^k$, we get $m(r)=2\pi\sum_{k=0}^{\infty}|b_k|^2r^{2k}$. The lemma is evident from a comparison of coefficients in

$$m^{(2n)}(r)=2\pi\sum_{k=n}^{\infty}|b_k|^2 2k(2k-1)\dots(2k-2n+1)r^{2k-2n}$$

and

$$\int_0^{2\pi}|g^{(n)}(re^{i\theta})|^2\,d\theta=2\pi\sum_{k=n}^{\infty}|b_kk(k-1)\dots(k-n+1)|^2r^{2k-2n}.\quad\square$$

Let $n>1$ be an integer and let $f$ be a univalent function in the unit disc. Using Lemma 9 with $g=(f')^{-(n-1)/2}$ we get

$$m^{(2n)}(r)\le 4^n\int_0^{2\pi}|g(re^{i\theta})S_nf(re^{i\theta})|^2\,d\theta.$$

Theorem 2 now gives the differential inequality

$$m^{(2n)}(r)\le 4^n\left(\frac{K_n}{(1-r^2)^n}\right)^2\int_0^{2\pi}|g(re^{i\theta})|^2\,d\theta\le\left(\frac{2}{1+r_0}\right)^{2n}\frac{K_n^2}{(1-r)^{2n}}m(r)$$

for $r_0\le r<1$. The corresponding differential equation

$$\widetilde{m}^{(2n)}(r)=\left(\frac{2}{1+r_0}\right)^{2n}\frac{K_n^2}{(1-r)^{2n}}\widetilde{m}(r)$$

has solutions $\widetilde{m}(r)=C(1-r)^{-E(r_0)}$, where $E(r_0)$ is the positive solution of

$$E(E+1)\dots(E+2n-1)=\left(\frac{2}{1+r_0}\right)^{2n}K_n^2.$$

Choosing $C$ large enough, we get

$$m^{(k)}(r_0)<\widetilde{m}^{(k)}(r_0),\quad k=0,1,\dots,2n-1,$$

and so Proposition 8.7 of [13] gives

$$m(r) \le \widetilde{m}(r) \quad \text{for } r_0 \le r < 1. \tag{6.1}$$

*Another proof of* (6.1). The function $\Delta(r) = m(r) - \widetilde{m}(r)$ satisfies

$$\Delta^{(2n)}(r) \le \left(\frac{2}{1+r_0}\right)^{2n} \frac{K_n^2}{(1-r)^n} \Delta(r), \quad r_0 \le r < 1, \tag{6.2}$$

and $\Delta^{(k)}(r_0) < 0$ for $k = 0, 1, \dots, 2n$. Let $r_1 \le 1$ be the largest number such that $\Delta^{(2n)}(r) < 0$ for $r_0 \le r < r_1$. If $r_1 < 1$, then $\Delta^{(2n)}(r_1) = 0$ and $\Delta(r_1) < 0$, which contradicts (6.2). Thus $r_1 = 1$, and (6.1) follows.

We thus have

$$m(r) = \int_0^{2\pi} |f'(re^{i\theta})|^{-n+1} \, d\theta = O((1-r)^{-E(r_0)}).$$

Since $E(r_0) \to E_n$ as $r_0 \to 1$, Theorem 3 is proved.

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