Value distributions of entire functions in regions of small growth

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1. Statement of results

Let $f(z)$ be an entire function of finite order ϱ. It is classical (cf. [2, Ch. 4]; [6, Ch. 1] that a *proximate order* $\varrho(r)$ may be associated with $f(z)$ so that the corresponding *indicator* function

$$h(\theta) = \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r^{\varrho(r)}} \qquad (0 \leq \theta \leq 2\pi)$$

is continuous, 2π-periodic, and trigonometrically convex. Let $I = (\alpha, \beta)$ be an open interval with

$$h(\theta) \leq 0 \quad \alpha \leq \theta \leq \beta, \tag{1.1}$$

and choose $\theta_0, \alpha < \theta_0 < \beta$. We say that the complex number a is *maximally assumed* near $\{\arg z = \theta_0\}$ if there is some $\varepsilon > 0$ such that for all $\delta > 0$

$$\limsup_{r \to \infty} \frac{n(r, a, \theta_0, \delta)}{r^{\varrho(r)}} \geq \varepsilon; \tag{1.2}$$

here $n(r, a, \theta_0, \delta)$ denotes the number of roots of $f(z) - a$, including multiplicity, in the region $\{|z| < r\} \cap \{|\arg z - \theta_0| < \delta\}$. The set of all maximally assumed values near $\{\arg z = \theta_0\}$ for a given $\varepsilon > 0$ will be denoted by $\mathfrak{Z}(\theta_0, \varepsilon)$.

More generally, for a closed subinterval $I_1 = [\alpha_1, \beta_1]$ of I, let $n(r, a, I_1)$ denote the number of roots of $f(z) - a$, including multiplicity, in the region

$$\{|z| < r\} \cap \{\alpha_1 < \arg z < \beta_1\},$$

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and set

$$\mathfrak{Z}(I_1, \varepsilon) = \left\{a; \limsup_{r\to\infty} \frac{n(r, a, I_1)}{r^{r(\varrho)}} \geq \varepsilon\right\}. \tag{1.3}$$

Note that $\mathfrak{Z}(I_1, \varepsilon) \supset \bigcup_{\alpha_1<\theta<\beta_1} \mathfrak{Z}(\theta, \varepsilon)$.

THEOREM 1A. *Let $I_1 = [\alpha_1, \beta_1]$ be a closed subinterval of $I = (\alpha, \beta)$ where* (1.1) *is satisfied. Then there exists a positive sequence $\{\sigma_n\}$,*

$$\sigma_{n+1}/\sigma_n \to \infty, \tag{1.4}$$

and a sequence $\{a_n\}$ of complex numbers with the property that if $w \in \mathfrak{Z}(I_1, \varepsilon)$, then

$$|w - a_n| < e^{-\sigma_n} \tag{1.5}$$

for infinitely many n.

THEOREM 1B. *Let a sequence $\{a_n\}$ of complex numbers be given along with a positive sequence $\{\sigma_n\}$ satisfying* (1.4), *and let*

$$\mathfrak{Z} = \bigcap_{m>0} \bigcup_{n\geq m} \{w; |w - a_n| < e^{-\sigma_n}\}. \tag{1.6}$$

Then there exists an entire function of finite order whose indicator vanishes on an interval $I = (\alpha, \beta)$, and such that for some $\theta_0 \in (\alpha, \beta)$ and some $\varepsilon > 0$

$$\mathfrak{Z}(\theta_0, \varepsilon) \supset \mathfrak{Z}. \tag{1.7}$$

2. Remarks

The indicator $h(\theta)$ is non-negative on a set which includes an interval of length π/ϱ, so the hypothesis (1.1) requires $\varrho > \frac{1}{2}$. Since $\varrho(r) \to \varrho$, it is no loss of generality to suppose

$$\tfrac{1}{2} < \varrho(r) \leq 2\varrho \quad (r \geq 0). \tag{2.1}$$

The examples of Theorem 1B have order ϱ for any $\varrho \in (\frac{1}{2}, 1)$, with ε in (1.7) equal to $\pi^{-1}\sin\varrho$. By considering $f(z^n)$ $(n = 2, 3, \ldots)$ we obtain examples for all orders $\varrho > \frac{1}{2}, \varrho \neq 1$, and a more intricate construction, which we do not give here, yields functions of order 1 which satisfy (1.7) for some $\varepsilon > 0$. There is probably a relation between the largest ε allowed in (1.7) and the variables ϱ and $(\beta - \alpha)$.

In [7, p. 55], G. Valiron asserted that $\mathfrak{Z}(\theta_0, \varepsilon)$, for a fixed $\varepsilon > 0$, can never be as large as the complement of a single point with respect to the finite plane (i.e., $\{\arg z = \theta_0\}$ cannot be a Borel direction of $f(z)$); as far as I am aware, he never published a proof. Since $\sum e^{-\sigma_n} < \infty$, it follows from Theorem 1A that $\bigcup_{\varepsilon>0}\mathfrak{Z}(\theta_0, \varepsilon)$ has (planar) measure zero.

The characterizations of $\mathfrak{Z}(\theta_0, \varepsilon)$ and $\mathfrak{Z}(I_1, \varepsilon)$ given here invite comparison with the recent study of A. Hyllengren [4] on Valiron deficiencies of meromorphic functions of finite order. Hyllengren showed that if f is meromorphic and of finite order, and if $\Delta[\varepsilon] = \{a; \Delta(a) \geq \varepsilon\}$, where $\Delta(a)$ is the Valiron deficiency of the complex number a, then $\Delta[\varepsilon]$ is contained in a set of the form (1.6) where the σ_n satisfy $\sigma_{n+1}/\sigma_n = 0(1)$, rather than (1.4). Thus, the considerably smaller sets $\mathfrak{Z}(I_1, \varepsilon)$ are also of capacity zero and have Hausdorff measure zero for all measure functions $h(t)$ such that

$$\int_0 h(t)(-\log t)^{-1}t^{-1}dt < \infty.$$

(I thank Prof. Hyllengren for several discussions on these matters).

The function $e^z, \alpha = \frac{1}{2}\pi, \beta = \frac{1}{2}3\pi$, shows that Theorem 1A is false when I_1 is replaced by I.

Notations. A constant which depends only on ε (of (1.2)), $\beta - \alpha, \beta_1 - \alpha_1$, or $\varrho(r)$ (where $\varrho(r)$ is subject to (2.1)) will be given without reference to these quantities. Most inequalities are valid only for sufficiently large $r = |z|$, and such an inequality will be qualified by $r > r_0$ or $r > r_0(K)$; in the latter case, r_0 depends on K as well as $\varrho(r), \beta_1 - \alpha_1, \beta - \alpha$ or ε. Any of these expressions will be freely used to denote different constants in different contexts.

3. Proof of Theorem 1A

We first need a Proposition which allows (1.3) to be replaced by a more convenient condition.

PROPOSITION 1. *For* $\alpha \in \mathfrak{Z}(I_1, \varepsilon)$, *let*

$$R(a) = \left\{r; \frac{n(r, a, I_1)}{r^{\varrho(r)}} < 3\varepsilon/4\right\}. \tag{3.1}$$

Then there exists $M^\infty > 1$ *and* $r_1 = r_1(a)$ *such that*

$$n(r, a, I_1) - n(r', a, I_1) > \tfrac{1}{2}\varepsilon r^{\varrho(r)} \; (r \in R(a), r_1(a) < r' \leq r/M^\infty). \tag{3.2}$$

LEMMA 1. *With* ε *as in* (1.2), *there exist* r_0, M_0 *with*

$$(r/M_0)^{\varrho(r/M_0)} < 4^{-1}\varepsilon r^{\varrho(r)} \quad (r > r_0). \tag{3.3}$$

Proof. Choose M_0 so that for some $\xi > 0$,

$$M_0^{-1/2}e^{\xi} < 4^{-1}\varepsilon; \tag{3.4}$$

there is no harm in supposing ξ so small that

$$\xi \log M_0 \leq 1. \tag{3.5}$$

Now $\varrho'(t)t \log t \to 0$ as $t \to \infty$, so there is $r_1(\xi)$ with

$$|\varrho'(t)t \log t| < \tfrac{1}{2}\xi^2 \quad (t > r_1(\xi)); \tag{3.6}$$

further there is r_0 $(\geq r_1(\xi))$ so that

$$\log\left(1 + \frac{1}{p-1}\right) < 2p \quad \left(p > \frac{\log r_0}{\log M_0}\right). \tag{3.7}$$

Then if $M_0^{-1}r > r_0$, (3.5)—(3.7) yield that

$$\begin{aligned} |\varrho(r) - \varrho(r/M_0)| &\leq \tfrac{1}{2}\xi^2 \log\left\{1 + \frac{\log M_0}{\log r - \log M_0}\right\} \\ &\leq \xi^2 \log M_0 (\log r)^{-1} \leq \xi(\log r)^{-1}, \end{aligned} \tag{3.8}$$

so (2.1), (3.4) and (3.8) lead to

$$(r/M_0)^{\varrho(r/M_0)} r^{-\varrho(r)} = M_0^{-\varrho(r/M_0)} r^{\varrho(r/M_0)-\varrho(r)} \leq M_0^{-1/2} e^{\xi} < 4^{-1}\varepsilon \quad (r > r_0),$$

which is (3.3).

LEMMA 2. *There exists* $r_0(a)$ *with*

$$n(r, a, I_1) \leq 2(2r)^{\varrho(2r)} \quad (r > r_0(a)). \tag{3.9}$$

Proof. This is an immediate consequence of Jensen's theorem [2, p. 9], the defining inequality $\log M(r) \leq \{1 + o(1)\}r^{\varrho(r)}$ and

$$n(r, a, I_1) \log 2 \leq n(r, a) \log 2 \leq \int_r^{2r} n(t, a)t^{-1}dt \leq N(2r, a) \quad (r \geq 1).$$

It is now easy to obtain Proposition 1. Lemma 1 (with $\frac{1}{2}M_0$ in place of M_0) and Lemma 2 imply that there are $M_0, r_0(a)$ with

$$n(r/M_0, a, I_1) \leq 4(\log 2)^{-1}(2r/M_0)^{\varrho(2r/M_0)} \leq \varepsilon r^{\varrho(r)} \quad (r > r_0(a)),$$

and the Proposition, with $M^\infty = M_0$, follows from this and the obvious inequality

$$n(r, a, I_1) - n(r', a, I_1) \geq n(r, a, I_1) - n(r/M_0, a, I_1).$$

It is also useful to have a slight sharpening of (1.1). According to (1.1), there exists $\phi(r) \to 0$ $(r \to \infty)$ with

$$\max_{\alpha \leq \theta \leq \beta} \log |f(re^{i\theta})| \leq \phi(r) r^{\varrho(r)} \quad (r > 0). \tag{3.10}$$

(cf. [6], p. 71). For $K > 1$ consider the closed regions $D(K, r)$ and $D_1(K, r)$ given by

$$D(K, r) = \{te^{i\theta};\, r/2K \leq t \leq 2Kr, \alpha \leq \theta \leq \beta\}, \tag{3.11}$$

$$D_1(K, r) = \{te^{i\theta};\, r/K \leq t \leq Kr, \alpha_1 \leq \theta \leq \beta_1\}. \tag{3.12}$$

Since the function $(r'/r)z$ maps $D(K, r)$ onto $D(K, r')$ and $D_1(K, r)$ onto $D_1(K, r')$ it follows that there is a positive constant $\tau(M)$ with the property that

$$\inf G_{D(K, r)}(z, b) = \tau(K) \quad (z, b \in D_1(K, r), \quad r > 0) \tag{3.13}$$

where $G_{D(K, r)}(z, b)$ is the Green's function for $D(K, r)$ with pole at b.

LEMMA 3. *There exists an increasing unbounded function* $K(r)$ $(r > 0)$ *such that, if* ε *is the constant of* (1.2) *and* τ *is given by* (3.13)

$$\max_{\zeta \in D(K(r), r)} \log |f(\zeta)| < \tfrac{1}{8}\varepsilon\tau(K(r))r^{\varrho(r)} \quad r > r_0(\phi) \tag{3.14}$$

and, further,

$$\tau(K(r))r^{\varrho(r)} = o\{\tau(K(s))s^{\varrho(s)}\} \quad (r, s \to \infty, s/r \to \infty). \tag{3.15}$$

Proof. Let $K_1 = 4$ and for $j = 2, 3, \ldots$ determine K_j as the largest solution of

$$\tau(K_j) \geq 2^{-1/4}\tau(K_{j-1}), \tag{3.16}$$

$$K_{j-1} < K_j \leq 2K_{j-1}. \tag{3.17}$$

Since $\tau(K)$ is a continuous function of K, it follows that K_j exists and $K_j \to \infty$ as $j \to \infty$. If $r_1(j)$ is chosen so large that

$$|\varrho(t) - \varrho(r)| \leq \log 2(\log r)^{-1} \quad (r_1(j) \leq r/K_j \leq t \leq K_j r) \tag{3.18}$$

(this is possible, as can be seen from the proof of (3.8) in Lemma 1), then (2.1) and simple manipulations give

$$\phi(t)t^{\varrho(t)}r^{-\varrho(r)} \leq \phi(t)K_j^{2\varrho}r^{\varrho(t)-\varrho(r)} \leq 2\phi(t)K_j^{2\varrho} \quad (r_1(j) \leq r/K_j \leq t \leq K_j r)$$

Since $\phi(t) \to 0$, we now have an $r_0(j, \phi)$ $(\geq r_1(j))$ with the property that

$$\phi(t)t^{\varrho(t)} \leq 2^{-13/4}\varepsilon\tau(K_j)r^{\varrho(r)} \; (r_0(j, p) \leq r/K_j \leq t \leq K_j r). \tag{3.19}$$

Let us further require that $r_0(j + 1, \phi) \geq K_j^2 r_0(j, \phi)$, and let $K(r) = K_j$ when $K_j r_0(j, \phi) \leq r \leq K_{j+1}r_0(j + 1, \phi)$. It is easy to see from (3.10), (3.16) and (3.19) that (3.14) holds as well as

$$K(r)^{-1}r \to \infty \quad (r \to \infty). \tag{3.20}$$

To complete the proof of Lemma 3, we show (3.15). Suppose $16r \leq t \leq 32r$, with r so large that (3.18) is satisfied with $K_j > 32$. Then (3.16) and (3.17) imply that $\tau(K(t)) > 2^{-1/4}\tau(K(r))$ and this, (2.1) and (3.18) lead to

$$\frac{\tau(K(t))t^{\varrho(t)}}{\tau(K(r))r^{\varrho(r)}} \geq 2^{-1/4}16^{1/2}r^{\varrho(t)-\varrho(r)} \geq 2^{3/4} \quad (r_1(K_j) \leq t/32 \leq r \leq t/16), \tag{3.21}$$

and iteration of (3.21) easily gives (3.15).

Finally, we can prove Theorem 1A. Let $a \in \mathcal{Z}(I_1, \varepsilon)$ and let $R(a)$ be as in (3.1). Let $r^*(a)$ be so large that, with M^∞ as in (3.2), $K(r) > M^\infty$ if $r > r^*(a)$ and

$$\log^+ |a| \leq \tfrac{1}{8}\varepsilon\tau(K(r))r^{\varrho(r)} \quad (r^*(a) \leq K(r)^{-1}r)\,; \tag{3.22}$$

(3.15) and (3.20) show that $r^*(a)$ exists. We write $R^*(a)$ for $R(a) \cap (r^*(a), \infty)$. Then if $r \in R^*(a)$, $z \in D_1(r)$ and $\{b_n\}$ are the roots of $f - a$ in $D_1(K(r), r)$, we have from Poisson's formula ([2], p. 7)

$$\begin{gathered}\log |f(z) - a| \leq \int\limits_{\zeta \in \partial D} \log |f(\zeta) - a| K(\zeta, z)d\zeta - \sum G(z, b_n) \\ (z \in D_1(K(r), r), \quad r \in R^*(a)).\end{gathered} \tag{3.23}$$

Here $K > 0$, $\int K(\zeta, z)d\zeta = 1$. Then (3.14) and (3.22) show

$$\log |f(\zeta) - a| \leq \tfrac{1}{4}\varepsilon\tau(K(r))r^{\varrho(r)} \quad (\zeta \in \partial D_1(K(r), r), \quad r \in R^*(a)),$$

and since $\{b_n\}$ are in $D_1(K(r), r)$, (3.2) and (3.13) imply that

$$\sum G(z, b_n) \geq \tfrac{1}{2}\varepsilon\tau(K(r))r^{\varrho(r)}.$$

Thus

$$\log |f(z) - a| \leq \tfrac{1}{4}\varepsilon\tau(K(r))r^{\varrho(r)} = -\sigma(r, a) \quad (z \in D(K(r), r), r \in R^*(a)). \tag{3.24}$$

Hence if $a' \in \mathcal{Z}(I_1, \varepsilon)$, and $|a' - a| > \frac{1}{4}\varepsilon\tau(K(r))r^{\varrho(r)}$, it follows that

$$R(a') \cap (K(r)^{-1}r, K(r)r) = \emptyset \text{ if } r \in R(a).$$

Thus let $\{t_m\} \to \infty$ so slowly that

$$t_{m+1}/t_m \leq \inf K(t) \quad (K(t_{m-1}))^{-1}t_{m-1} \leq t \leq K(t_{m+1})t_{m+1};\ m = 1, 2, \ldots)$$

and let $J_m = [t_m, t_{m+1}]$. First let m_1 be the least positive integer with

$$J_{m_1} \cap \{\cup R^*(a);\, a \in \mathcal{Z}(I_1, \varepsilon)\} \neq \emptyset,$$

and choose $r_{m_1} \in J_{m_1}, a_{m_1} \in \mathcal{Z}(I_1, \varepsilon)$ with $r_{m_1} \in R^*(a_{m_1})$. Then let m_2 be the least positive integer $> m_1$ with

$$J_{m_2} \cap \{(K(r_{m_1})r_{m_1}, \infty)\} \cap \{\cup R^*(a);\, a \in \mathcal{Z}(I_1, \varepsilon)\} \neq \emptyset,$$

and choose $r_{m_2} \in J_{m_2}, a_{m_2} \in \mathcal{Z}(I_1, \varepsilon)$ with $r_{m_2} \in R^*(a_{m_2}) \ldots$ This gives sequences $\{r_{m_n}\}, \{a_{m_n}\}$ which we label simply as $\{r_n\}, \{a_n\}$, and let

$$\sigma_n = \tfrac{1}{4}\varepsilon\tau(K(r_n))r^{\varrho(r_n)}. \tag{3.25}$$

Since the $\{r_n\}$ increase, and $r_{n+1}/r_{n-1} > K(r_{n-1})$, we have that $r_{n+1}/r_n \to \infty$ and so, from (3.15), $\sigma_{n+1}/\sigma_n \to \infty$ which is (1.7). Finally, let $a \in \mathfrak{Z}(I_1, \varepsilon)$, and $s \in R^*(a)$. Then s belongs to some interval J_m and the construction given guarantees that there is an r_p $(p = m - 1$ or $m)$ which belongs to the sequence $\{r_n\}$ with either $1 < s/r_p < K(r_p)$ or $1 > s/r_p > K(r_p)^{-1}$. Then (3.24) and (3.25) with $a = a_p$, $r = r_p$ ensure that $|a - a_p| < e^{-\sigma_p}$, and Theorem 1A is established.

4. Proof of Theorem 1B

Let $\{\sigma_n\}$ be a sequence for which $\sigma_{n+1}/\sigma_n \to \infty$, and, for a fixed $\varrho \in (\tfrac{1}{2}, 1)$, let

$$\sigma_n = -9(\cos \pi\varrho)r_n^{\varrho} - \log 2 \quad (n = 1, 2, \ldots). \tag{4.1}$$

Then

$$r_{n+1}/r_n \to \infty, \tag{4.2}$$

and (4.1) yields a relation between r_n and σ_n which we keep for the remainder of this paper. Given σ_n or r_n, which satisfy (1.4) or (4.2), there is no loss of generality in decreasing the ratios σ_{n+1}/σ_n or r_{n+1}/r_n so that also

$$\frac{(\log \sigma_{n-1})^6}{(\log \sigma_{n+1})^2} \to \infty. \tag{4.3}$$

We may then state Theorem 1B more precisely as

THEOREM 1B'. *For* $\tfrac{1}{2} < \varrho < 1$, *let* $\{\sigma_n\}$ *be a sequence which satisfies* (1.4) *and* (4.3), *and define* $\{r_n\}$ *by* (4.1); *finally let* $\{a_n\}$ *be a sequence with*

$$|a_n| < \min\left\{\frac{(\log r_{n-1})^6}{(\log r_{n+1})^2}, \tfrac{1}{2}(\log r_{n-1})^6\right\}. \tag{4.4}$$

Then there exists an entire function $f(z)$ *with*

$$\log M(r, f) \sim r^{\varrho} \quad (r \to \infty) \tag{4.5}$$

and, if $h(\theta)$ *is the indicator of* $f(z)$ *with respect to* $\varrho(r) = \varrho$,

$$h(\theta) \le 0 \quad (|\arg z - \pi| < \tfrac{1}{2}(\pi - \pi/2\varrho)). \tag{4.6}$$

Further, we have for all $\delta > 0$, *in the notation of* (1.2), *that*

$$\liminf_{n\to\infty} \frac{n(r_n, w, \pi, \delta)}{r_n^{\varrho(r_n)}} \ge \pi^{-1} \sin \pi\varrho \tag{4.7}$$

for all $w \in \bigcap_m \bigcup_{m>n} C_n$, *where*

$$C_n = \{w; |w - a_n| < e^{-\varrho_n}\}. \tag{4.8}$$

The function $f(z)$ is obtained by Riemann surface methods, and depends on the existence of an auxiliary entire function $g(z)$ which satisfies Theorem 1B' with all a_n identically zero. We list the requisite properties of $g(z)$ below in Proposition 2, and then show how to modify g to obtain f. In § 5 is a proof of Proposition 2.

PROPOSITION 2. *There exists an entire function* $g(z)$ *which satisfies* (4.5) *and* (4.6). *Further, if* $\{r_n\}$ *is the sequence which appears in Theorem 1B', there exist sequences* $\{R_n\}$ *with* $R_n/r_n \to \infty$ *and* $r_{n+1}/R_n \to \infty$, *and* $\{\eta_n\} \to 0$ *such that*

$$\inf_{r_n/2 \leq r \leq 2r_n} \frac{n(r, w, \pi, \delta)}{r^\varrho} \geq (1 - \eta_n)\pi^{-1} \sin \pi\varrho \tag{4.9}$$

for all w *satisfying*

$$|w| < 2e^{-\sigma_n}. \tag{4.10}$$

Finally, we can choose $\varepsilon_n \to 0$ *so slowly that*

$$\varepsilon_n R_n^\varrho > (\log R_n)^7 \tag{4.11}$$

with that property that if

$$D_n = \{R_{n-1} < |z| < R_n\} \cap \{\pi \geq |\arg z| > \pi/4\}, \tag{4.12}$$

and $E_n = \partial D_n$, *then*

$$\log |g(z)| > \varepsilon_{n-1}(R_{n-1})^\varrho \quad (n > n_0; z \in E_n). \tag{4.13}$$

We accept this Proposition for now, and produce $f(z)$ using an indirect approach. Using $g(z)$, we shall construct a continuous function $F(z)$ which is regular in the complement of certain simply-connected resgions

$$\{\Delta_{m,n}\} \quad (n = 1, 2, \ldots; m = 1, \ldots, k(n))$$

with $\Delta_{m,n} \subset D_n$ for all m and n, where D_n is defined in (4.12). Inside the $\{\Delta_{m,n}\}$, F will not be holomorphic, but will be nearly so in the following sense: each $\Delta_{m,n}$ can be divided into three subregions in each of which $F(z) = F(x, y) = u(x, y) + iv(x, y)$ has continuous partial derivatives, and

$$|F_{\bar{z}}/F_z| \leq A(\log |z|)^{-2} \quad (\text{a.e.} \quad z \in \Delta_{m,n}) \tag{4.14}$$

for some positive constant A, where, as usual

$$\begin{aligned} F_z &= \frac{1}{2}(u_x + v_y) + \frac{i}{2}(v_x - u_y), \\ F_{\bar{z}} &= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y). \end{aligned} \tag{4.15}$$

This will imply that the dilatation $p(z)$ of F (cf. [3, p. 439], [5, p. 18]) satisfies

$$(0 \leq)\, p(z) - 1 \leq A(\log |z|)^{-2}, \quad \text{(a.e.)} \tag{4.16}$$

and so

$$\iint_{|z|>1} \{p(z) - 1\} \frac{dxdy}{|z|^2} < \infty. \tag{4.17}$$

Finally, we will show that

$$F \text{ maps the plane topologically onto a Riemann surface } \mathcal{F}. \tag{4.18}$$

The utility of (4.17) and (4.18) arises from results of O. Teichmüller and P. Belinskii ([5, Ch. 5, § 6]). For these conditions imply that $\mathcal{F}$ is parabolic and, if $f_1(\zeta)$ maps the ζ-plane conformally onto $\mathcal{F}$, then for a suitable choice of A, the induced transformation $\zeta(z) = A^{-1}f_1^{-1}(F(z))$ satisfies

$$\zeta(z) \sim z \quad (z \to \infty). \tag{4.19}$$

Although F is not regular, we have $\max_{|z|=r} |F(z)| \sim r^{\varrho}$, and this and (4.18) allow the expressions $h(\theta)$ and $n(r, a, \theta_0, \delta)$ to be defined for $F(z)$ as if F were entire. Our explicit construction of F will guarantee that (4.9) is satisfied for those w which belong to infinitely many of the discs (4.8) so that (4.19) yields that $f(z) = f_1(Az)$ meets all conditions of Theorem 1B'.

Thus we start with $g(z)$, as in Proposition 2, and for $z \in D_n$ describe how to achieve the $F(z)$ which will satisfy (4.17) and (4.18). Let

$$\tau_n = (\log R_{n-1})^6 \tag{4.20}$$

and consider the closed subsets $\Delta_{m,n}$ of D_n in which

$$|g(z)| \leq \tau_n \quad (z \in \Delta_{m,n}) \tag{4.21}$$

Note that (4.11), (4.12), (4.13) and (4.21) imply that $\Delta_{m,n} \subset D_n$ for all m. Thus we may consider n fixed in this construction. If a_n satisfies (4.4), consider the Mobius transformation

$$Lw = e^{ip_n}\tau_n^2 \frac{w - a_n}{\tau_n^2 - \bar{a}_n w} \tag{4.22}$$

which maps the disc $\{|w| \leq \tau_n\}$ to itself, with p_n chosen so that $L(\tau_n) = \tau_n$. Then L induces a map s so that

$$L(\tau_n e^{is(\theta)}) = \tau_n e^{i\theta} \quad s(0) = 0. \tag{4.23}$$

for all θ, and we can now define the mapping H from $\{|w| \leq \tau_n\}$ to itself as

$$H(ue^{iv}) = \begin{cases} ue^{iv} & 0 \leq u \leq \frac{1}{2}\tau_n \\ u \exp\left[i\left\{v + (s(v) - v)\ \dfrac{\log u - \log \tau_n/2}{\log 2}\right\}\right] & \frac{1}{2}\tau_n \leq u \leq \tau_n \end{cases} \tag{4.24}$$

and define $F(z)$ for $z \in \Delta_{m,n}$ by

$$F(z) = H \circ L \circ g(z) \quad (z \in \Delta_{m,n}) \tag{4.25}$$

where L is specified in (4.21). For $z \notin \bigcup_m \Delta_{m,n}$ we set $F(z) = g(z)$; it then follows from (4.22)—(4.25) that F is continuous in the full plane.

The next task is to show that F satisfies (4.7) and (4.8). Consider the disc $\{|w - a_n| < e^{-\sigma_n}\}$. Since $R_n > r_n$, (4.4) and (4.20) imply that this disc is inside $\{|w| \leq \frac{1}{2}\tau_n\}$. It thus follows from (4.22), (4.3), (4.4), and (4.20) and the interlacing of the $\{R_n\}$, $\{r_n\}$ that there is a constant A (independent of n) with

$$|1 - |L'(w)|| \leq A \left|\frac{a_n}{\tau_n}\right| \leq A\ \frac{(\log r_{n-1})^6}{(\log R_{n-1})^6}\ \frac{1}{(\log r_{n+1})^2} \leq A(\log r_{n+1})^{-2} \tag{4.26}$$

which tends to zero as $n \to \infty$. Now $L(0) = a_n$, so (4.26) implies that if n is sufficiently large, the inverse of $\{|w - a_n| < e^{-\sigma_n}\}$ under L is contained in $\{|z| < 2e^{-\sigma_n}\}$, and (4.7) follows from (4.9), (4.10) and (4.19).

It remains but to verify (4.17) (or (4.16)) and (4.18). Evidently $F_{\bar z} = 0$ if $z \notin \bigcup_{m,n} \Delta_{m,n}$, and the representation (4.25) shows that it suffices to show

$$(H(w))_{\bar w}/H(w))_w \leq A(\log|z|)^{-2} \quad (w = g(z), z \in \Delta_{m,n}); \tag{4.27}$$

further since g is a regular, the explicit formula (4.24) shows we need only consider these z for which $\frac{1}{2}\tau_n \leq |g(z)| \leq \tau_n$. We cut $A = \{w; \frac{1}{2}\tau_n \leq |w| \leq \tau_n\}$ along the axis $\{\arg w = 0\}$ and write $ue^{iv} = \exp(U + iV)$. Then (4.4) may be written $H(ue^{iv}) = \exp\{k(U + iV)\} = \exp\{K(U + iV) + iK^*(U + iV)\}$ with

$$\begin{aligned} K(U + iV) &= U \\ K^*(U + iV) &= V + (s(V) - V)\left(\frac{U - \log \tau_n/2}{\log 2}\right) \end{aligned} \tag{4.28}$$

for

$$\log \tau_n - \log 2 \leq U \leq \log \tau_n, \quad 0 \leq V \leq 2\pi.$$

Since $\exp\{\}$ is conformal, we have that

$$H(w)_{\bar w}/H(w)_w = k(W)_{\bar W}/k(W)_W, \tag{4.29}$$

and we can compute the left side of (4.29) using (4.25), (4.23), (4.26) and (4.28). Thus (4.22), (4.23) and (4.26) show that $|s'(V) - 1| \leq A|a_n/\tau_n|$, and $|s(V) - V| \leq 2\pi A|a_n/\tau_n|$. It is then easy to show that

$$K_U = 1, \quad |K_V^* - 1| \leq |s'(V) - 1| \leq A|a_n/\tau_n|$$
$$K_V = 0, \quad |K_U^*| \leq |s(V) - V| \leq 2\pi A|a_n/\tau_n|,$$

so that, for perhaps a different constant A

$$|F_{\bar{z}}/F_z| \leq A|a_n/\tau_n| \quad (z \in \Delta_{m,n})$$

and thus (cf. (4.26))

$$|F_{\bar{z}}/F_z| \leq A(\log r_{n+1})^{-2} \leq A(\log |z|)^{-2} \quad (z \in \Delta_{m,n});$$

since $D_n \subset \{|z| < r_{n+1}\}$ and this proves (4.16). To obtain (4.18), we observe that the image of $\Delta_{m,n}$ by g is a bordered Riemann surface, and hence so is the image of $\Delta_{m,n}$ under F. F is also regular in the complement of the $\Delta_{m,n}$, and since F is uniquely defined on $\partial\Delta_{m,n}$, (4.18) follows from standard gluing arguments (cf. [1, pp. 117—119]).

5. Proof of Proposition 2

The methods used here rely heavily on Chapters 1 and 2 of [6].

Suppose $g_0(z)$ is a canonical product of order $\varrho, \frac{1}{2} < \varrho < 1$ with $g_0(0) \neq 0$, and let $\{b_n\}$ be the roots of g_0. Many functions can play the role of g_0 below, but all will have, for some absolute constant K,

$$n(r, 0) < Kr^{\varrho} \tag{5.1}$$

(K may be taken as 6, for example). Let $r_0 > 0, A > 0$ be given, and define products $\pi_1(z)$ and $\pi_2(z)$ by

$$\pi_1(z) = \prod_{|b_n| < A^{-2}r_0} (1 - z/b_n); \quad \pi_2(z) = \prod_{|b_n| > A^2 r_0} (1 - z/b_n). \tag{5.2}$$

The discussion of [6, pp. 62—3] and (5.1) imply that, given $\varepsilon_1 > 0$, there exists $A_0(\varepsilon_1)$ (which also depends on the absolute constant K of (5.1)) such that if $A \geq A_0(\varepsilon_1)$

$$|\log|\pi_1(z)|| + |\log|\pi_2(z)|| < \varepsilon_1 r^{\varrho} \tag{5.3}$$

if

$$r_0 A^{-1} < |z| < r_0 A. \tag{5.4}$$

One further element of flexibility will be needed. Let M be a (large) positive integer and let $\{h_m(\theta)\}$ ($m = 0, \pm 1, \ldots \pm M$) be a family of 2π-periodic trigonometrically convex functions of order $\varrho, \frac{1}{2} < \varrho < 1$. Thus each h_m is continuous, has right and left-hand derivatives which agree off an at most countable set of θ, and

$$s_m(\theta) = h'_m(\theta) - \varrho^2 \int_0^\theta h_m(\phi)d\phi \quad (0 \le \theta \le 2\pi) \tag{5.5}$$

increases (in (5.5), h'_m denotes either the right or left-hand derivative of h_m). In our situation, $s_m(\theta)$ will increase only by simple jumps at one or three values of θ, and there exists a set $E(M) = \{\theta_0, \theta_1, \theta_{-1}, \theta_2, \theta_{-2}\}$ outside of which all functions $h_m(\theta)$ are continuously differentiable. To measure the denseness of the family $\{h_m\}$ let

$$q(M) = \max_{-M \le m \le M-1} \max_{\theta \notin E(M)} |h'_{m+1}(\theta) - h'_m(\theta)|. \tag{5.6}$$

Then for each m, Chapter 2 of [6] yields an entire function f_m whose indicator is $h_m(\theta)$. This f_m has several properties which are useful here and so we indicate the salient features of the construction. For $0 \le \theta \le 2\pi$, let

$$\Delta_m(\theta) = (2\pi\varrho)^{-1} \lim_{\delta \downarrow 0} \{h'_m(\theta + \delta) - h'_m(\theta - \delta)\} \tag{5.7}$$

measure the jump of the derivative of h_m at θ, and observe from our convention that $\Delta_m(\theta) = 0$ for all $\theta \notin E(M)$. Then for $j = -2, \ldots, 2$ we place $n_{j,m}(r)$ zeros of $f(z)$ on $\{\arg z = \theta_j\}$ to satisfy

$$|n_{j,m}(r) - \Delta_m(\theta_j)r^\varrho| < 1; \tag{5.8}$$

$f_m(z)$ is the canonical product whose zeros are so distributed. Then, to each $\varepsilon_1 > 0$ is a p'_M with the property that if $|z| = r > p'_M$

$$r^\varrho h_m(\theta) - \varepsilon_1 r^\varrho < \log|f_m(z)| < r^\varrho h_m(\theta) + \varepsilon_1 r^\varrho \; (-M \le m \le M) \tag{5.9}$$

save for points z contained in circles $C_{m,k}$ whose radii $r_{m,k}$ $(k = 1, 2, \ldots)$ satisfy

$$r^{-1} \sum{}^r r_{m,k} < \varepsilon_1 \quad (-M \le m \le M, r \ge p'_M) \tag{5.10}$$

(the symbol $\sum^r$ means summation over those k such that $C_{m,k}$ intersects $\{|z| \le r\}$). Also, we obtain from (5.6), (5.7) and (5.8) that given $\varepsilon_2 > 0$, there exists a $q_1 > 0$ and p''_M such that if $q(M) \le q_1$, then

$$|n_{j,m}(r) - n_{j,m+1}(r)| < \varepsilon_2 r^\varrho \quad r > p''_M. \tag{5.11}$$

Finally, we let $p_M = \max(p'_M, p''_M)$.

For $N = 1, 2, \ldots$, let $\varepsilon_1(N) = N^{-2}$ and then consider a family of $2M + 1$ trigonometrically convex functions $h_m(\theta)$ where the specific choice of M will be made later. Easiest to define is

$$h_0(\theta) = \cos \varrho\theta \quad (|\theta| \le \pi);$$

the remaining functions are divided into two classes, each of M functions. Those in Class I will be labelled $h_1, \ldots, h_M$ and we first describe these. Choose θ_1, $0 < \theta_1 < \pi - \pi/2\varrho$ with

$$\cos \varrho\theta_1 = N^{-1} \tag{5.12}$$

and, in the interval $0 \leq \theta \leq \theta_1$, let $h_m(\theta) = h_0(\theta)$ for $1 \leq m \leq M$. Next, we define

$$h_M(\pi) = (2N)^{-1} \tag{5.13}$$

and then, for $1 \leq m < M$,

$$h_m(\pi) = h_0(\pi) + \frac{m}{M}(h_M(\pi) - h_0(\pi)). \tag{5.14}$$

For $\theta_1 < \theta < \tau, h_m$ is the unique portion of a sinusoid of period $2\pi/\varrho$ which at π and θ_1 interpolates the values $h_m(\pi)$ and $h_0(\theta_1)$ (to see how this sinusoid is constructed, cf. [6], p. 52; uniqueness follows since $\pi - \theta_1 < \pi\varrho^{-1}$). Next, for $\pi \leq \theta < 2\pi$ let $h_m(\theta) = h_m(2\pi - \theta)$. Thus, in the enumeration of $E(M), \theta_1$ the solution of (5.12), $\theta_{-1} = -\theta_1$ and $\theta_0 = \pi$. The functions in Class II are written $h_{-1}, \ldots, h_{-M}$, and are constructed as in (5.12), (5.13) and (5.14) save that m is replaced by $-m, N$ by $N+1$ and θ_1 by θ_2, where θ_2 is defined by the equation $\cos \varrho\theta_2 = (N+1)^{-1}$. Note that the functions $h_m(\theta)$ are 2π-periodic and trigonometrically convex. The easiest way to establish this convexity is to verify that each $s_m(\theta)$ (defined in (5.5)) increases. To see that s_m increases, we observe that $h_m(\theta)$ is a continuous function and is sinusoidal at all points of continuity of h_m'; at the remaining points of the domain h_m' has a positive jump discontinuity.

We can now relate the choice of M to N and the sequence $\{r_n\}$ which is specified in the statement of Proposition 2. Choose $\{t_n\}$ with

$$r_n/t_{n-1} = t_n/r_n \tag{5.15}$$

so that both sides of (5.15) tend to infinity as $n \to \infty$. With $\varepsilon_1(N) = N^{-2}$ as mentioned above, in (5.3), (5.9) and (5.10), choose $A = A_N$ so large that (5.3) holds with $\varepsilon_1 = \varepsilon_1(N)$ and then choose p_M, M $(M = M(N))$ and ε_2 $(= \varepsilon_2(N))$ so that if the $\{f_m\}$ are chosen as in (5.8), then (5.11) may be sharpened to

$$|n_{j,m}(A_N^2 r) - n_{j,m-1}(A_N^2 r)| < r^\varrho (N \log A_N)^{-2} \quad (r > p_M). \tag{5.16}$$

According to (5.7) and (5.8), (5.16) can be achieved by making $|\Delta_m(\theta) - \Delta_{m-1}(\theta)|$ small for all θ, and these differences will be diminished if $q(M)$ is small, i.e. if M is large.

We next choose $n(N)$ so large that $n(N) > n(N-1)$,

$$\log (t_{n+1}/t_n) > 4(2M(N)+1) \log A_{M(N)} \quad (n > n(N)) \tag{5.17}$$

and in addition, with $p_{M(N)}$ selected so that (5.9) and (5.10) hold with our choice of ε_1, we also have

$$r_n > p_{M(N)} \quad (n > n(N)). \tag{5.18}$$

For each $n, n(N) \leq n < n(N+1)$, the interval (t_n, t_{n+1}) is divided into

$(2M(N)+1)$ intervals $(\alpha_j(n), \beta_j(n))$ with $\alpha_i(n)/\beta_i(n) = \alpha_j(n)/\beta_j(n)$ for all i and j $(-M(N) \le i, j \le M(N))$. When the value of n is clear from the context, we abbreviate $\alpha_j(n)$ and $\beta_j(n)$ by α_j and β_j. We set, for each n, T_j $(= T_j(n)) = \{z; \alpha_j \le |z| < \beta_j\}$, and for the moment suppose $n \ne n(N+1) - 1$. Then in T_j, g is assigned the same zeros as the corresponding f_j if $j \ge 0$, and as f_{-j} if $j < 0$; if $n = n(N+1) - 1$, then in T_j g has the same zeros as f_{-j} for all j. (This special definition, when $n = n(N+1) - 1$, allows a smooth connection near $\{|z| = t_{n(N+1)}\}$). Finally in $\{|z| < t_1\}$, g is assigned the same zeros as $f_{M(1)}$. With $\{b_n\}$ these zeros, we set $g(z) = \prod(1 - z/b_n)$.

The point of this construction is that if $z \in T_j$ and $f_{j(z)}$ is the proper choice of f_j or f_{-j}, as explained above, then

$$\log |g(z)| = \log |f_{j(z)}(z)| + \mu_j(z) \quad (z \in T_j) \tag{5.19}$$

where, for large n,

$$|\mu_j(z)| < 2N^{-3/2}|z|^\varrho \tag{5.20}$$

outside circles C_k of radius r_k such that

$$r^{-1}\sum^r r_k \le \varepsilon_1(N) = o(N^{-1}) \quad (n(N) < n \le n(N+1)) \tag{5.21}$$

(cf. (5.10)). Granting this for the moment, it is easy to complete the proof of Proposition 2. Indeed, (5.21) implies there exist $\{R_n\} \to \infty$ with $R_n/t_n \to 1$ $(n \to \infty)$ such that (5.19) and (5.20) hold on all of $\{|z| = R_n\}$. In particular, this, (5.9) and the fact that $|h_{\pm M}(\theta)| \ge (2N+2)^{-1}$ $(0 \le \theta \le 2\pi)$ imply for large n that

$$\log |g(R_n e^{i\theta})| \ge R_n^\varrho\{(2N+2)^{-1} - N^{-2} - 2N^{-3/2}\} \ge (3N)^{-1}R_n^\varrho \\ (0 \le \theta \le 2\pi, n > n_0) \tag{5.22}$$

On the rays $\{\arg z = \pm\pi/4\}$ we have

$$\log |g(re^{\pm i\pi/4})| > \tfrac{1}{2}(\cos \tfrac{1}{4}\pi)r^\varrho \quad (r > r_0) \tag{5.23}$$

Since $h_m(\theta) \ge \cos(\frac{1}{4}\pi)$ for all θ with $0 \le \theta \le \frac{1}{4}\pi$, (5.23) is clear from (5.9) and (5.19) if z does not belong to the circles estimated in (5.10); if z is interior to one of these circles, then it follows from (5.8), (5.19) and (5.20) that $f(z)$ does not vanish in the circle, and so (5.23) follows from (5.9), (5.19), (5.20) and the minimum principle. Thus (5.22) and (5.23) imply that (4.13) holds with any $\varepsilon_n \ge (4N)^{-1}$ $(n > n_0, n(N) < n \le n(N+1))$, so (4.11) can be achieved as well, by increasing the numbers $n(N)$ if necessary.

Similar reasoning gives (4.5) and (4.6). Indeed, when (5.20) is valid, these conclusions follow from the construction of the $h_m(\theta)$ since $\max_\theta h_m(\theta) = 1$, and the inequality $h_m(\theta) \le 2N^{-1}$ when $|\theta - \pi| < \pi/2\varrho$ and $-M(N) \le m \le M(N)$.

Finally, we consider (4.9) and (4.10), and let $\{s_n\}$ be a sequence with $\frac{1}{2}r_n \le s_n \le 2r_n$. Then s_n is well-contained in $T_0(n)$ in the sense that $s_n/\alpha_0(n) \to \infty$ as $n \to \infty$. Let $\{\delta_n\} \downarrow 0$ and $\{A_n\} \to \infty, \{S_n\} \to \infty$ be sequences (with

$\alpha_0(n) < A_n < S_n < s_n$, $A_n \sim \alpha_0(n)$ and $S_n \sim s_n$ as $n \to \infty$) so that (5.19), with $m = 0$, holds on all of $\{|z| = A_n\}, \{|z| = S_n\}$ and the segments

$$\{\arg z = \pi \pm \delta_n, A_n \leq |z| \leq S_n\}.$$

Then if D_n^* denotes that region bounded by these curves which contains a segment of the negative axis, our construction implies that g has at least

$$k(S_n) = \pi^{-1} \sin \pi\varrho\, (S_n^\varrho - A_n^\varrho) - 2 = \pi^{-1} \sin \pi\varrho\, S_n^\varrho(1 + o(1))$$

zeros in D_n^* where the rate at which $o(1)$ tends to zero depends on n but not the choice of $s_n \in [\frac{1}{2}r_n, 2r_n]$; further, (5.9), with $m = 0$, and (5.19) yield that

$$\log |g(\zeta)| \geq \log |f_0(\zeta)| - |\mu_0(\zeta)| \geq 2 \cos \pi\varrho\, |\zeta|^\varrho - o(1)|\zeta|^\varrho$$
$$\geq 3 \cos \pi\varrho\, |\zeta|^\varrho \geq 3 \cos \pi\varrho\, (3r_n)^\varrho \geq 9 \cos \pi\varrho\, r_n^\varrho \quad (\zeta \in \partial D_n^*, n > n_0).$$

Hence, by Rouché's theorem $g(z)$ assumes every value w with

$$|w| < \exp (9 \cos \pi\varrho\, r_n^\varrho) \tag{5.24}$$

at least $k(S_n)$ times for $z \in D_n^*$.

For a fixed $\delta > 0$ and all large n, if w satisfies (5.24)

$$\begin{aligned} n(s_n, w, \pi, \delta) - n(R_n, w, \pi, \delta) &\geq n(S_n, w, \pi, \delta_n) - n(A_n, w, \pi, \delta_n) \\ &\geq \pi^{-1} \sin \pi\varrho\, s_n^\varrho(1 + o(1)), \end{aligned} \tag{5.23}$$

and so (4.9) and (4.10) are consequences of (5.22), (5.23) and the definition (4.1).

We conclude by sketching a proof of (5.19) and (5.20) provided z avoids the circles estimated by (5.21). Let $z_0 \in T_j$, $|z| = r_0$ and, for convenience of notation, suppose $-M(N) + 1 \leq j = j(z_0) \leq M(N) - 1$. Then the interval $(A_N^{-2}r_0, A_N^2 r_0)$ meets at most one T_k $(k \neq j)$ (cf. (5.17)). Let $\{b_n\}$ and $\{b_{n,j}\}$ denote respectively the zeros of $g(z)$ and $f_j(z)$, and given a sequence $\{a_n\}$, define $\pi^*(1 - z/a_n)$ to be the product over those n with $rA_N^{-2} \leq |a_n| < r_0 A_N^2$.

Since $\varepsilon_1(N) = N^{-2}$, (5.2) gives

$$|\log |g(z)| - \log |f_j(z)|| \leq |\log|\pi^*(1 - z/b_{n,j})| - \log |\pi^*(1 - z/b_n)|| + 2N^{-2}r^\varrho. \tag{5.26}$$

However, the $\{b_n\}$ and $\{b_{n,j}\}$ agree in T_j, and thus (5.16) implies that

$$|\log |\pi^*(1 - z/b_{n,j})| - \log |\pi^*(1 - z/b_n)|| = |\log \prod_{n=1}^{p(z_0)} |(1 - z/a_n)^{\varepsilon_n}||, \tag{5.27}$$

where $\varepsilon_n = \pm 1$, and $p(z_0) \leq (N \log A_N)^{-2} r^\varrho$. Let

$$Q_{z_0}(z) = \prod_{n=1}^{p(z_0)} (1 - z/a_n),$$

It is clear that if Q_1 is any partial product of Q_{z_0}, then

$$\log |Q_1(z)| \leq \log \{(1 + A_N^2)^{r^\varrho (N \log A_N)^{-2}}\} \leq 3N^{-2}(\log A_N)^{-1} r^\varrho \quad (N > N_0) \tag{5.28}$$

and Cartan's estimate ([6], p. 21 with $2eR = 12r$) and the manipulations leading to (5.27) ensure that outside circles C_k whose radii satisfy

$$\sum' r_k \leq 6r(NA_N^2)^{-1} \tag{5.29}$$

that

$$\begin{aligned} \log |Q_1(z)| &\geq -[2 + \log (12eNA_N^2)] \max_{|\zeta|=12r} \log |Q_1(\zeta)| \\ &\geq -3[2 + \log (12eNA_N^2)]N^{-2}(\log A_N)^{-1}r^{\varrho}. \end{aligned} \tag{5.30}$$

From (5.28) and (5.30), it is clear that (5.27) is estimated by

$$\left|\log |\pi^*(1 - z/b_{n,j})| - \log |\pi^*(1 - z/b_n)|\right| \leq N^{-3/2}r^{\varrho} \quad (r > r_0), \tag{5.31}$$

and (5.26) and (5.31) give (5.20). Finally, the bound (5.21) is a direct consequence of (5.29).

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