On holomorphy and compactness in Banach spaces

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If E and F are complex Banach spaces and U is a non-void open subset of E, $\mathcal{H}(U; F)$ denotes the complex vector space of all holomorphic mappings from U into F. There are several locally convex topologies which may be considered on $\mathcal{H}(U; F)$. Among them some are natural in the sense that they coincide with the usual topology on $\mathcal{H}(U; F)$ for E finite dimensional. This paper presents results about the characterization of the relatively compact subsets of $\mathcal{H}(U; F)$ for one of these topologies. Theorems of Nachbin and Aron appear as special cases of these results.

Let $\mathcal{T}$ be the family of all continuous mappings τ from U into $\mathbf{R}$ such that $0 < \tau(x) \leq d(x, \partial U) =$ distance of x to the boundary ∂U of U. For each $\tau \in \mathcal{T}$ let $\mathcal{B}_\tau(U)$ be the collection of all the finite unions of closed balls $\bar{B}_\varrho(x)$ with center $x \in U$ and radius $\varrho < \tau(x)$. $\mathcal{H}_\tau(U; F)$ denotes the complex vector space of all the holomorphic mappings from U into F which are bounded over the elements of $\mathcal{B}_\tau(U)$. In $\mathcal{H}_\tau(U; F)$ we consider the topology of uniform convergence over the elements of $\mathcal{B}_\tau(U)$.

PROPOSITION 1. *Let p be a seminorm defined in $\mathcal{H}_\tau(U; F)$. The following statements are equivalent:*

(i) *There are $c > 0$ and $B \in \mathcal{B}_\tau(U)$ such that*

$$p(f) \leq c \sup \{\|f(t)\|;\ t \in B\} \text{ for every } f \text{ in } \mathcal{H}_\tau(U; F).$$

(ii) *There are $c > 0$ and $B \in \mathcal{B}_\tau(U)$, $B = \bigcup_{i=1}^k \bar{B}_{\varrho_i}(x_i)$, $\varrho_i < \tau(x_i)$, $i = 1, 2, \ldots, k$, such that*

$$p(f) \leq c \sup \{\sum_{n=0}^{\infty} \varrho_i^n \|(n!)^{-1} \widehat{d^n} f(x_i)\|; \quad i = 1, 2, \ldots, k\}$$

for each f *in* $\mathcal{H}_\tau(U; F)$.

Proof. Let B of (i) be of the form $B = \bigcup_{i=1}^k \bar{B}_{\varrho_i}(x_i)$, $\varrho_i < \tau(x_i)$, $i = 1, 2, \ldots, k$. If $t \in \bar{B}_{\varrho_i}(x_i)$ and $f \in \mathcal{H}_\tau(U; F)$, we have

$$\|f(t)\| \leq \sum_{n=0}^{\infty} \|(n!)^{-1} \widehat{d^n} f(x_i)(t - x_i)\| \leq \sum_{n=0}^{\infty} \varrho_i^n \|(n!)^{-1} \widehat{d^n} f(x_i)\|.$$

From this it is easy to show that (i) implies (ii).

Let B be as in (ii). If $r > 0$ is such that $r < \tau(x_i) - \varrho_i$, $i = 1, 2, \ldots, k$, we have $B' = \bigcup_{i=1}^k \bar{B}_{\varrho_i + r}(x_i) \in \mathcal{B}_\tau(U)$ and, for $f \in \mathcal{H}_\tau(U; F)$,

$$\|(n!)^{-1} \widehat{d^n} f(x_i)\| \leq (\varrho_i + r)^{-n} \sup \{\|f(t)\|; t \in B'\}, \quad n = 0, 1, \ldots,$$

via the Cauchy inequalities. Thus

$$\sum_{n=0}^{\infty} (n!)^{-1} \varrho_i^n \|\widehat{d^n} f(x_i)\| \leq (1 - \varrho_i(\varrho_i + r)^{-1})^{-1} \sup \{\|f(t)\|; t \in B'\}$$

for every f in $\mathcal{H}_\tau(U; F)$. From this we get that (ii) implies (i).

It is not difficult to see that

$$\mathcal{H}(U; F) = \bigcup_{\tau \in \mathcal{T}} \mathcal{H}_\tau(U; F).$$

On the other hand $\mathcal{H}_\tau(U; F)$ is a Fréchet space if E is separable. For further details see [1], [2], [3]. Let τ_l be the locally convex topology in $\mathcal{H}(U; F)$ which is the inductive limit of the topologies of the $\mathcal{H}_\tau(U; F)$, $\tau \in \mathcal{T}$. It is easy to see that τ_l is finer than the Nachbin topology τ_ω. (See [4]).

Let τ_{0f} denote the topology on $\mathcal{H}(U; F)$ of uniform convergence over the finite dimensional compact subsets of U. A subset of $\mathcal{H}(U; F)$ is τ_{0f}-bounded if and only if it is locally bounded on U. See [5].

PROPOSITION 2. *If* $\mathcal{X}$ *is a subset of* $\mathcal{H}(U; F)$ *which is* τ_{0f}*-bounded, then there is* $\tau \in \mathcal{T}$ *such that* $\mathcal{X}$ *is contained and bounded in* $\mathcal{H}_\tau(U; F)$.

Proof. For each u in U let $\tau(u)$ be the supremum of the $\varrho > 0$ such that $\mathcal{X}$ is uniformly bounded in $\bar{B}_\varrho(u) \subset U$. Since $\mathcal{X}$ is locally bounded in U, $0 < \tau(u) \leq d(u, \partial U)$. We prove that

$$|\tau(u) - \tau(v)| \leq \|u - v\| \tag{*}$$

for all u and v in U. We have two cases: (a) $\|u - v\| < \tau(u)$ and (b) $\|u - v\| \geq \tau(u)$. In the first case, if $\|u - v\| < \varrho < \tau(u)$, we have $\bar{B}_{\varrho - \|u - v\|}(v) \subset \bar{B}_\varrho(u)$ and $\varrho - \|u - v\| \leq \tau(v)$. It follows that $\tau(u) - \tau(v) \leq \|u - v\|$. In the second case this inequality holds trivially. If we interchange the roles of u and v in the above reasoning we get (*). Thus $\tau \in \mathcal{T}$ and it is easy to see that this is the τ we need for the proof.

For E separable τ_l is the bornological topology on $\mathcal{H}(U; F)$ associated to τ_{0f}.

Let τ_∞ denote the topology on $\mathcal{H}(U; F)$ of uniform convergence over the compact subsets of U of f and all of its differentials. See [4].

THEOREM 3. *If $\mathcal{X}$ is a τ_{0f}-bounded subset of $\mathcal{H}(U; F)$, then τ_l and τ_∞ determine the same uniform structure over $\mathcal{X}$. In particular τ_l and τ_∞ induce the same topology on $\mathcal{X}$.*

Proof. Let p be a τ_l-continuous seminorm on $\mathcal{H}(U; F)$. Let $\tau \in \mathcal{T}$ be such that $\mathcal{X}$ is contained and bounded in $\mathcal{H}_\tau(U; F)$. Thus there are $c_\tau > 0$ and $B = \bigcup_{i=1}^k \bar{B}_{\varrho_i}(x_i)$, $\varrho_i < \tau(x_i)$, $i = 1, 2, \ldots, k$, such that

$$p(f) \leq c_\tau \sup \{\textstyle\sum_{n=0}^\infty \varrho_i^n \|(n!)^{-1}\widehat{d^n}f(x_i)\|;\ \ i = 1, 2, \ldots, k\}$$

for each f in $\mathcal{H}_\tau(U; F)$. If $0 < r < \tau(x_i)$, $i = 1, 2, \ldots, k$, then the set $B' = \bigcup_{i=1}^k \bar{B}_{\varrho_i + r}(x_i)$ belongs to $\mathcal{B}_\tau(U)$ and there is $C > 0$ such that

$$\sup \{\|f(t)\|;\ t \in B'\} \leq C$$

for every f in $\mathcal{X}$. It follows that

$$\|(n!)^{-1}\widehat{d^n}f(x_i)\| \leq C(\varrho_i + r)^{-n}$$

for $n = 0, 1, \ldots$ and f in $\mathcal{X}$. Let $\mu > 0$ be such that

$$c_\tau \sup \{\textstyle\sum_{n=\mu}^\infty C\varrho_i^n(\varrho_i + r)^{-n};\ \ i = 1, 2, \ldots, k\} \leq 2^{-1}.$$

If the seminorm q is defined on $\mathcal{H}(U; F)$ by

$$q(f) = c_\tau \sup \{\textstyle\sum_{j=0}^\mu \varrho_i^j \|(j!)^{-1}\widehat{d^j}f(x_i)\|;\ \ i = 1, 2, \ldots, k\},$$

then q is τ_∞-continuous and $f \in \mathcal{X}, q(f) \leq 2^{-1}$ imply $p(f) \leq 1$. It follows that if $0 \in \mathcal{X}$, $\mathcal{V}$ is a neighborhood of 0 in the topology of $\mathcal{X}$ induced by τ_∞ if and only if $\mathcal{V}$ is a neighborhood of 0 in the topology of $\mathcal{X}$ induced by τ_l. Given $\mathcal{X} \subset \mathcal{H}(U; F)$ τ_l-bounded, the set $\mathcal{X} - \mathcal{X}$, of the differences between two elements of $\mathcal{X}$, is τ_l-bounded and contains 0. By the preceding remark the neighborhoods of 0 in the topologies of $\mathcal{X} - \mathcal{X}$ induced by τ_∞ and τ_l are the same. It follows that the uniform structures induced over $\mathcal{X}$ by τ_l and τ_∞ are also the same. Denoting by τ_0 the topology of uniform convergence on all compact subsets of U we have:

COROLLARY 4 (Nachbin [4]). *If* $\mathcal{X}$ *is a* τ_0*-bounded subset of* $\mathcal{H}(U; F)$, *then* τ_ω *and* τ_∞ *determine the same uniform structure over* $\mathcal{X}$. *In particular,* τ_ω *and* τ_∞ *induce the same topology on* $\mathcal{X}$.

PROPOSITION 5. *A subset* $\mathcal{X}$ *of* $\mathcal{H}(U; F)$ *is* τ_l*-relatively compact if, and only if,* $\mathcal{X}$ *is* τ_∞*-relatively compact.*

Proof. If $\mathcal{X}$ is τ_{0f}-bounded, it follows that the closures of $\mathcal{X}$ relative to τ_l and τ_∞ coincide (Theorem 3). On the other hand these closures are τ_{0f}-bounded and τ_l and τ_∞ induce the same topology on them. The result now follows trivially.

COROLLARY 6 (Nachbin [4]). *Let* $\mathcal{X}$ *be a subset of* $\mathcal{H}(U; F)$. $\mathcal{X}$ *is* τ_ω*-relatively compact if, and only if, it is* τ_∞*-relatively compact.*

Remark. Aron (see [6]) proved Theorem 3 and Proposition 5 for E separable.

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