Removable singularities and condenser capacities

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1. Introduction

In [3] Ahlfors and Beurling gave a characterization in terms of extremal distances of the removable singularities for the class of analytic functions with finite Dirichlet integral. The purpose of this paper is to generalize and extend this result in several directions.

Let $A(G)$ be a class of analytic or harmonic functions defined for any open set G in the complex plane $\mathbf{C}$ or in d-dimensional Euclidean space $\mathbf{R}^d$. We say that a compact set E is removable for A if for some open G containing E every function in $A(G \setminus E)$ can be extended to a function in $A(G)$.

If $G \subset \mathbf{C}$ we denote by $AD^p(G)$, $p > 0$, the class of analytic functions f in G such that $\int_G |f'(z)|^p dm(z) < \infty$, m being plane Lebesgue measure.

Ahlfors and Beurling [3] proved that a set E is removable for AD^2 if and only if the removal of E does not change extremal distances. (See Theorem 7 below for a more precise statement.) Their proof uses the conformal invariance of the class AD^2, so it does not immediately generalize to $p \neq 2$.

In order to treat this problem for all p, $1 < p < \infty$, we first reformulate it by means of duality as an approximation problem in the Sobolev space W_1^q, $q = p/(p-1)$. (Theorem 1.) It is then quite easy to give a necessary and sufficient condition for a set to be removable for AD^p, $1 < p < \infty$. Our condition (Theorem 4) is that E be a »null set» for a certain condenser capacity, which for $p = 2$ is »conjugate» to the extremal distance considered by Ahlfors and Beurling (Lemmas 3 and 4).

Our main result, however, is that this necessary and sufficient condition is equivalent to a local, apparently much weaker condition (Theorem 6). Thus, condenser capacity has an instability property similar to the instability that a wide class of capacities is known to have.

The above approximation problem in W_1^q can just as well be formulated and solved in d dimensions, and this d-dimensional problem is also equivalent to a problem of removability. In fact, instead of considering functions in AD^p in plane regions it is equivalent to consider their real parts, i.e. real valued harmonic functions u with vanishing periods and such that $\int |\text{grad}\, u|^p dm < \infty$. This class of functions can also be defined in $\mathbf{R}^d$.

Thus, following Rodin and Sario [25; p. 254] we denote for $G \subset \mathbf{R}^d$ by $FD^p(G)$ the class of real valued harmonic functions u in G such that $\int_G |\text{grad}\, u|^p dm_d < \infty$ (m_d is d-dimensional Lebesgue measure), and such that u has no flux, i.e. $\int_c * du = \int_c \partial u/\partial n\, dS = 0$ for all $(d-1)$-cycles c in G. By the de Rham theorem the last condition is equivalent to saying that the $(d-1)$-form $*du$ is exact. Since our results extend to d dimensions at little extra cost, this is the generality in which we shall treat the removability problem.

For comparison we also include characterizations of the removable singularities for the spaces AL^p of analytic functions in plane L^p and (which is equivalent for $d = 2$) HD^p of harmonic functions u with $\int |\text{grad}\, u|^p dm_d < \infty$. (Theorems 1 and 2.)

There are also possibilities of generalizing to solutions of more general elliptic equations, but we leave these aside. Cf. Harvey and Polking [11].

In the last section of the paper we discuss mainly the case of linear sets in the plane. We give conditions for removability (Theorem 13) which for $p = 2$ improve a theorem of Ahlfors and Beurling [3].

For more information about removable singularities we refer in addition to [3] to the book by Sario and Nakai [27], and the bibliography given there. Some results on FD^2 are due to Yamamoto [31].

2. Preliminaries

For any open G in $\mathbf{R}^d$ and $1 \leq q < \infty$ we denote by $W_1^q(G)$ the Sobolev space of locally integrable real valued functions f on G whose derivatives in the distribution sense are functions in $L^q(G)$. We write $W_1^q(\mathbf{R}^d) = W_1^q$. As usual $C_0^\infty(\mathbf{R}^d) = C_0^\infty$ and $C_0^\infty(G)$ denote the infinitely differentiable functions with compact support (in G). When G is bounded $\| |\text{grad}\, f| \|_{L^q(G)}$ is a norm on $C_0^\infty(G)$ by the Poincaré inequality, and the closure of $C_0^\infty(G)$ in this norm is denoted $\overset{\circ}{W}{}_1^q(G)$.

It is well known that for $q > d$ functions in W_1^q are continuous but for $q \leq d$ this is no longer true. The deviation from continuity is measured by a q-capacity which in a natural way is associated with the space W_1^q. For compact sets K this q-capacity is defined by

$$C_q(K) = \inf_\omega \int_{\mathbf{R}^d} |\text{grad}\, \omega|^q dm,$$

where the infimum is taken over all $\omega \in C_0^\infty$ such that $\omega \geq 1$ on K. In the case $q \leq d$ the ω are restricted to $C_0^\infty(B)$ for some fixed large ball B which contains K in its interior.

The definition is extended to arbitrary $E \subset \mathbf{R}^d$ by setting

$$C_q(E) = \sup\{C_q(K); K \subset E, \ K \ \text{compact}\}.$$

For $q > d$ only the empty set has q-capacity zero. C_d is a conformal invariant in $\mathbf{R}^d$, called conformal capacity. C_2 is classical Newtonian (logarithmic for $d = 2$) capacity.

For an account of some of the properties of C_q and W_1^q we refer to T. Bagby [4]. See also Meyers [22], Adams and Meyers [1], Maz'ja and Havin [21], Hedberg [15].

3. The dual problem

We show by means of duality that the problem of characterizing the removable sets for FD^p (AD^p), $1 < p < \infty$, is equivalent to an approximation problem in W_1^q, $q = p/(p-1)$. For comparison we include the corresponding characterization for HD^p (AL^p). For $p = 2$ the result is found in Royden [26].

We denote by $C_E^\infty(\mathbf{R}^d) = C_E^\infty$ or $C_E^\infty(G)$ the subset of C_0^∞ or $C_0^\infty(G)$ which consists of functions ϕ such that $\operatorname{grad}\phi$ belongs to $C_0^\infty(\complement E)$. I.e. C_E^∞ is the subalgebra of C_0^∞ which consists of functions that are constant on each component of some neighborhood of E.

THEOREM 1. a) *E is removable for HD^p or AL^p, $1 < p < \infty$, if and only if $C_0^\infty(G \setminus E)$ is dense in $\mathring{W}_1^q(G)$, $q = p/(p-1)$, for some bounded open $G \supset E$.*

b) *E is removable for FD^p or AD^p, $1 < p < \infty$, if and only if $C_E^\infty(G)$ is dense in $\mathring{W}_1^q(G)$, $q = p/(p-1)$, for some bounded open $G \supset E$.*

Proof. We prove the theorem for HD^p and FD^p. For AL^p and AD^p the proof is simpler and is omitted.

Suppose $C_E^\infty(G)$ is dense in $\mathring{W}_1^q(G)$. Then clearly $m_d E = 0$, so if $u \in HD^p(G \setminus E)$ the partial derivatives u_i are defined almost everywhere in G and belong to $L^p(G)$. We claim that there is a distribution T in G whose partial derivatives $D_i T$ equal u_i. By a theorem of L. Schwartz a necessary and sufficient condition for this is that the distribution partial derivatives $D_j u_i$ satisfy $D_j u_i = D_i u_j$ for all i and j. In other words, we claim that $\int_G u_i D_j \phi dm = \int_G u_j D_i \phi dm$ for all $\phi \in C_0^\infty(G)$. But for all $\phi \in C_E^\infty(G)$ it is easily seen that

$$\int_G u_i D_j \phi dm = \int_{G \setminus E} u_i D_j \phi dm = -\int_{G \setminus E} u D_i D_j \phi dm = -\int_{G \setminus E} u D_j D_i \phi dm = \int_G u_j D_i \phi dm.$$

If $C_E^\infty(G)$ is dense in $C_0^\infty(G)$ the assertion follows from Hölder's inequality. Thus there is a distribution T in G which coincides with u in $G \setminus E$. By another theorem of L. Schwartz T is a function in $W_1^p(G)$, i.e. u has an extension to $W_1^p(G)$, in particular u is locally in L^p.

Suppose now $C_0^\infty(G \setminus E)$ is dense in $\mathring{W}_1^q(G)$. For any $u \in HD^p(G \setminus E)$ and $\phi \in C_0^\infty(G \setminus E)$ we have by Green's formula

$$\int_{G\setminus E} u\Delta\phi dm = - \int_{G\setminus E} \operatorname{grad} u \cdot \operatorname{grad} \phi dm = \int_{G\setminus E} \phi\Delta u dm = 0.$$

It follows that $\int_G u\Delta\phi dm = -\int_G \operatorname{grad} u \cdot \operatorname{grad} \phi dm = 0$ for all $\phi \in C_0^\infty(G)$. By Weyl's lemma u is harmonic in G, i.e. $u \in HD^p(G)$.

Then let $u \in FD^p(G \setminus E)$ and suppose that $C_E^\infty(G)$ is dense in $\mathring{W}_1^q(G)$. Let $\phi \in C_E^\infty(G)$. The support of $\operatorname{grad} \phi$ is a compact set in $G \setminus E$. Only finitely many components of its complement Ω intersect E, and we denote these by Ω_i. On each of these ϕ equals a constant, a_i. Let γ be a $(d-1)$-cycle in $(G \setminus E) \cap \Omega$ which is homologous to zero in $G \setminus E$, and set $\gamma \cap \Omega_i = \gamma_i$. By Green's formula

$$\int_{G\setminus E} u\Delta\phi dm = - \int_{G\setminus E} \operatorname{grad} u \cdot \operatorname{grad} \phi dm = \int_{G\setminus E} \phi\Delta u dm - \int_\gamma \phi(* du) =$$
$$= - \sum_i a_i \int_{\gamma_i} * du = 0.$$

It follows as before that $\int_G u\Delta\phi dm = 0$ for all $\phi \in C_0^\infty(G)$, so again u is harmonic in G, i.e. $u \in FD^p(G)$.

Suppose conversely that $C_0^\infty(G \setminus E)$ is not dense in $\mathring{W}_1^q(G)$. Then there is a non-zero distribution T with support in E and continuous on $\mathring{W}_1^q(G)$. Clearly $S = T * |x|^{2-d}$ is harmonic in $G \setminus E$. We claim that S belongs to $HD^p(G \setminus E)$.

There are functions $u_i \in L^p(G)$ such that for $\phi \in C_0^\infty(G)$ $(T, \phi) = \int \sum_i u_i D_i \phi dm$. Then $(S, \phi) = (T, \phi * |x|^{2-d}) = \int \sum_i u_i D_i(\phi * |x|^{2-d}) dm$. Thus

$$(D_j S, \phi) = -(S, D_j\phi) = - \int \sum_i u_i D_j D_i (\phi * |x|^{2-d}) dm.$$

By the Calderón-Zygmund theory

$$\|D_j D_i(\phi * |x|^{2-d})\|_{L^q} \leq C\|\phi\|_{L^q}, \quad \text{so} \quad |(D_j S, \phi)| \leq C \sum \|u_i\|_{L^p} \|\phi\|_{L^q}.$$

It follows that $D_j S \in L^p$, i.e. $S \in W_1^p(G)$. (See also Theorem 2 below.)

If in addition $C_E^\infty(G)$ is not dense in $\mathring{W}_1^q(G)$ there is a distribution T with the same properties as above and which also annihilates $C_E^\infty(G)$. $S = T * |x|^{2-d}$ again belongs to $W_1^p(G)$ and to $HD^p(G \setminus E)$, and we claim that S now has vanishing periods, i.e. $S \in FD^p(G \setminus E)$.

Let $\phi \in C_E^\infty(G)$ and let γ be a $(d-1)$-cycle as above off the support of grad ϕ. Then $(S, \Delta\phi) = (T, |x|^{2-d} * \Delta\phi) = C(T, \phi) = 0$, so by Green's formula

$$0 = \int_G S\Delta\phi dm = -\int_G \operatorname{grad} S \cdot \operatorname{grad} \phi dm = \int_{G\setminus E} \phi\Delta S dm - \int_\gamma \phi(* \, dS) =$$

$$= - \sum_i a_i \int_{\gamma_i} * \, dS.$$

Since the a_i are arbitrary, $\int_{\gamma_i} * \, dS = 0$ for all i. It follows that $\int_c * \, dS = 0$ for all $(d-1)$-cycles in $G \setminus E$.

The following corollaries are obvious.

COROLLARY 1. *Let A be HD^p, AL^p, FD^p or AD^p for $1 < p < \infty$, and let E be compact. If all functions f in $A(G \setminus E)$ can be extended to $A(G)$ for some bounded open G containing E, then the same is true for all such G.*

COROLLARY 2. *The property of being removable for HD^p, AL^p, FD^p or AD^p, $1 < p < \infty$, is local, i.e. E is removable if and only if every x in E has a compact neighborhood whose intersection with E is removable.*

Theorem 1 says that proving that E is removable for FD^p (or AD^p) is equivalent to proving a »Stone-Weierstrass property» for $C_E^\infty(G)$ in $\overset{\circ}{W}{}_1^q(G)$. For $q > d$ $\overset{\circ}{W}{}_1^q(G)$ is an algebra, but for $q \leq d$ it is not. The subspace $\overset{\circ}{W}{}_1^q(G) \cap L^\infty$, however, is a Banach algebra under the norm $\|f\| = \|f\|_\infty + \{\int |\operatorname{grad} f|^q dm\}^{1/q}$, and the closure of $C_0^\infty(G)$ in the same norm is again an algebra, which we denote by $\mathcal{M}_0^q(G)$. It consists of the functions in $\overset{\circ}{W}{}_1^q(G)$ that are continuous and tend to zero at the boundary. Such »Royden algebras» have been studied by Royden and recently by L. Lewis [18] and J. Lelong-Ferrand [17]. We shall later prove a »Stone-Weierstrass theorem» (Theorem 12) for these algebras also, i.e. we shall characterize the compact sets E such that $C_E^\infty(G)$ is dense in $\mathcal{M}_0^q(G)$, $1 < q < \infty$. A necessary condition is obviously that C_E^∞ is point separating, which is the case if and only if E is completely disconnected.

4. Necessary and sufficient conditions for removability

The removable sets for HD^p and AL^p, $1 < p < \infty$, allow a simple characterization which is undoubtedly known. However, it does not seem to be given explicitly in the literature, except for $p = 2$ which is classical (see e.g. Carleson [6; Th. VII:1]), so for completeness we include it here.

THEOREM 2. *A compact set* $E \subset \mathbf{R}^d$ *is removable for* HD^p (AL^p), $1 < p < \infty$, *if and only if* $C_q(E) = 0$ *(i.e. if and only if* $E = \emptyset$ *in the case* $p < d/(d-1)$*).*

Proof. First assume $C_q(E) > 0$. Then there is a measure $\mu > 0$ supported by E such that the potential $U_1^\mu(x) = \int |x-y|^{1-d} d\mu(y)$ is in L^p_{loc} (see e.g. Meyers [22; Th. 14]. The potential $U_2^\mu(x) = \int |x-y|^{2-d} d\mu(y)$ ($\int \log 1/|x-y| d\mu(y)$ in case $d = 2$) is harmonic in $\complement E$, and its gradient in the distribution sense is $\operatorname{grad} U_2^\mu(x) = \int \operatorname{grad}_x |x-y|^{2-d} d\mu(y)$, which is majorized by $U_1^\mu(x)$. Thus U_2^μ belongs to $HD^p(G \setminus E)$ for all bounded G, but clearly not to $HD^p(G)$. Note however that $U_2^\mu \in W_1^p(G)$.

In the other direction the theorem follows from Theorem 1 and the following lemma.

LEMMA 1. *Suppose* E *is compact and* $C_q(E) = 0$, $1 < q < \infty$. *Then* $C_0^\infty(G \setminus E)$ *is dense in* $\mathring{W}_1^q(G)$ *for all bounded open* G.

Proof. It is enough to show that any ϕ in $C_0^\infty(G)$ can be approximated. Let $\varepsilon > 0$ be arbitrary. There exists an $\omega \in C_0^\infty$ such that $\omega = 1$ in a neighborhood of $\operatorname{supp} \phi \cap E$ and $\int |\operatorname{grad} \omega|^q dm < \varepsilon$. Then $\phi(1-\omega) \in C_0^\infty(G \setminus E)$, and

$$\int |\operatorname{grad} \phi\omega|^q dm = \int |\phi \operatorname{grad} \omega + \omega \operatorname{grad} \phi|^q dm \leq$$

$$\leq 2^{q-1} \max |\phi|^q \varepsilon + 2^{q-1} \max |\operatorname{grad} \phi|^q \int_G |\omega|^q dm \leq \text{const.} \cdot \varepsilon$$

by the Poincaré inequality.

The removable sets for FD^p (and AD^p) cannot be characterized in such simple terms. However the following simple theorem may be worth recording.

THEOREM 3. *Let* $E \subset \mathbf{C}$ *be compact and let* $G \supset E$ *be open. Then every function* f *in* $W_1^p(G) \cap AD^p(G \setminus E)$, $1 < p < \infty$, *belongs to* $AD^p(G)$ *if and only if* $m_2 E = 0$.

Proof. Let $f \in W_1^p(G) \cap AD^p(G \setminus E)$, $G \supset E$. Then $\partial f / \partial \bar{z} \in L^p(G)$ and vanishes on $G \setminus E$. If $m_2 E = 0$ it follows by Weyl's lemma that f is analytic in G.

Conversely, suppose $m_2 E > 0$. Then $f(z) = \int_E (\zeta - z)^{-1} dm_2(\zeta)$ is analytic off E, and f is not identically zero, because $\lim_{z \to \infty} |zf(z)| = m_2 E$. By the Calderón-Zygmund theorem $f \in W_1^p(G)$ for all $p > 1$.

In order to characterize the removable sets for FD^p and AD^p we introduce a kind of condenser q-capacity, $q = p/(p-1)$.

Definition. Let R be a d-dimensional, open rectangle with, say, sides parallel to the coordinate planes. Let E be compact. The d condenser q-capacities of R with respect to E are

$$\Gamma_q^{(i)}(R/E) = \inf_{\omega} \int_{R} |\text{grad}\,\omega|^q dm, \quad i = 1, 2, \ldots, d,$$

where the infimum is taken over all $\omega \in C_E^\infty$ such that $\omega(x) = 0$ on one of the sides of R parallel to the coordinate plane $x_i = 0$, and $\omega(x) = 1$ on the opposite side. If $E = \emptyset$ we write $\Gamma_q^{(i)}(R)$, which is the ordinary condenser q-capacity of R.

By the usual strict convexity and variation argument one sees that there is a unique extremal function u in $W_1^q(R)$ which satisfies the equation

$$\text{div}\,(|\text{grad}\,u|^{q-2}\,\text{grad}\,u) = 0 \quad \text{in} \quad R \setminus E \; (\Delta u = 0 \quad \text{for} \quad q = 2).$$

When $E = \emptyset$ the solution is linear. As is well known it follows that if R has edges of length a_i perpendicular to the plane $x_i = 0$, then $\Gamma_q^{(i)}(R) = a_i^{-q} m_d R$.

THEOREM 4. *Let E be a compact set in $\mathbf{R}^d$. For E to be removable for FD^p (AD^p), $1 < p < \infty$, it is necessary and sufficient that*

$$\Gamma_q^{(i)}(R/E) = \Gamma_q^{(i)}(R), \quad i = 1, 2, \ldots, d, \quad q = p/(p-1),$$

for some open rectangle (or all open rectangles) R containing E.

Proof. Suppose first E is removable. Then by Theorem 1 C_E^∞ is dense in C_0^∞ in the W_1^q sense. It follows immediately that the capacities $\Gamma_q^{(i)}(R/E)$ and $\Gamma_q^{(i)}(R)$ are the same for all R containing E.

Suppose conversely that there is a rectangle R containing E such that $\Gamma_q^{(i)}(R/E) = \Gamma_q^{(i)}(R)$, $i = 1, 2, \ldots, d$. By Theorem 1 it is clearly enough to show that the restriction of C_E^∞ to R is dense in $W_1^q(R)$. For each i, $i = 1, 2, \ldots, d$, there is a sequence $\{\phi_n\}_1^\infty$ in C_E^∞, such that the ϕ_n take the prescribed boundary values, and such that $\int_R |\text{grad}\,\phi_n|^q dm \to \Gamma_q^{(i)}(R/E) = \Gamma_q^{(i)}(R)$. By strict convexity $\{\phi_n\}$ tends strongly in W_1^q to the extremal function for $\Gamma_q^{(i)}(R)$, which is of the form $ax_i + b$. It follows that all polynomials of the first degree can be approximated in $W_1^q(R)$ by functions in C_E^∞. Since W_1^q is closed under truncation (see e.g. Deny and Lions [7; Th. 3.2]) we can also assume that the approximating functions are bounded, and the theorem then follows from the following lemma and the density of polynomials in $W_1^q(R)$.

LEMMA 2. *Suppose ϕ and ψ belong to $W_1^q(R)$, and that $\|\phi\|_\infty \leq M$, and $\|\psi\|_\infty \leq M$. Let $\{\phi_n\}_1^\infty$ and $\{\psi_n\}_1^\infty$ be sequences such that*

$$\lim_{n\to\infty}\int_R |\text{grad}\,(\phi-\phi_n)|^q dm = 0, \quad \lim_{n\to\infty}\int_R |\text{grad}\,(\psi-\psi_n)|^q dm = 0,$$

$\|\phi_n\|_\infty \le M$, *and* $\|\psi_n\|_\infty \le M$. *Then there is a subsequence* $\{\phi_{n_i}\psi_{n_i}\}$ *such that* $\lim_{i\to\infty}\int_R |\text{grad}\,(\phi\psi-\phi_{n_i}\psi_{n_i})|^q dm = 0$.

Proof. $\text{grad}\,(\phi\psi-\phi_n\psi_n) = \phi\,\text{grad}\,\psi + \psi\,\text{grad}\,\phi - \phi_n\,\text{grad}\,\psi_n - \psi_n\,\text{grad}\,\phi_n = (\phi-\phi_n)\,\text{grad}\,\psi + \phi_n\,\text{grad}\,(\psi-\psi_n) + (\psi-\psi_n)\,\text{grad}\,\phi + \psi_n\,\text{grad}\,(\phi-\phi_n)$.

$\int |\phi_n\,\text{grad}\,(\psi-\psi_n)|^q dm \le M^q \int |\text{grad}\,(\psi-\psi_n)|^q dm \to 0$, and similarly for the fourth term. We can choose a sequence $\{n_i\}$ such that both ϕ_{n_i} and ψ_{n_i} converge pointwise almost everywhere to ϕ and ψ. The convergence to zero of

$$\int |(\phi-\phi_{n_i})\,\text{grad}\,\psi|^q dm \quad \text{and} \quad \int |(\psi-\psi_{n_i})\,\text{grad}\,\phi|^q dm$$

follows from the Lebesgue convergence theorem.

The following corollaries to Theorem 4 are due to Ahlfors and Beurling [3] for $p = d = 2$.

COROLLARY 1. *E is removable for FD^p (AD^p), $1 < p < \infty$, if the projection of E on each of the coordinate axes has one-dimensional measure zero.*

COROLLARY 2. *Let E be a Cartesian product of a compact one-dimensional set with itself, e.g. a d-dimensional Cantor set. Then E is removable for FD^p (AD^p), $1 < p < \infty$, if and only if $m_dE = 0$.*

Corollary 1 follows easily from the theorem. The sufficiency in Corollary 2 then follows. The necessity was already observed in the course of the proof of Theorem 1.

That the vanishing of m_dE is by no means sufficient for removability in general is well known (see Ahlfors-Beurling [3; Th. 14]).

Since removability is a local property it is desirable to have a local condition for removability in terms of condenser capacity. For this reason we need to apply the definition of $\Gamma_q^{(i)}(R/E)$ in a situation where E is allowed to intersect the boundary of R.

Now, one complication arises from the fact that when $q \le d-1$ there are continua with zero q-capacity. (For $q > d-1$ this cannot happen, because for a continuum E with diameter δ one then has $C_q(E) \ge \text{const.}\,\delta$. (See e.g. Maz'ja [20; Lemma 5].)) This means that a set E can be removable for FD^p although there are rectangles R such that no function in C_E^∞ can be 1 on one side of R and 0 on the opposite side. On the other hand, no continuum E with $C_q(E) > 0$

can be removable for FD^p. In order to see this one only has to set $\mu = \mu_1 - \mu_2$ where μ_1 and μ_2 are positive measures with support in different parts of E such that $\mu_1(E) = \mu_2(E)$ and such that $U_1^{\mu_i}(x) = \int |x - y|^{1-d} d\mu_i(y)$, $i = 1, 2$, are in L^p. This is possible by Meyers [22; Th. 14]. Then $U_2^{\mu}(x) = \int |x - y|^{2-d} d\mu(y)$ is in FD^p for some neighborhood of E and does not vanish identically.

Therefore, if a set E is removable for FD^p it has to be completely disconnected after a set of q-capacity zero has been removed. We can thus without loss of generality assume that E is completely disconnected.

We prefer now to work with capacities of ring domains instead of rectangles. By a *ring* we mean a bounded domain G such that the complement $\complement G$ has exactly two components.

Definition. Let G be a ring in $\mathbf{R}^d$, and let E be closed. The condenser q-capacity of G with respect to E is

$$\Gamma_q(G/E) = \inf_{\omega} \int_G |\operatorname{grad} \omega|^q dm_d,$$

where the infimum is taken over all $\omega \in C_E^\infty$ such that $\omega = 1$ on the bounded component of $\complement G$ and $\omega = 0$ on the unbounded component.

The condition for removability then takes the following form.

THEOREM 5. *Let $E \subset \mathbf{R}^d$ be compact and completely disconnected. Then E is removable for FD^p (AD^p), $1 < p < \infty$, if and only if $\Gamma_q(G/E) = \Gamma_q(G)$, $q = p/(p-1)$, for all rings G.*

The necessity is proved as in Theorem 4. A proof of the sufficiency of the condition can easily be given, but since this also follows from Theorem 6 below, we do not give it here.

The main result of this paper is that the condition in Theorem 5 is equivalent to an apparently much weaker condition. The situation is comparable to the »instability» of analytic capacity and potential theoretic capacities (see e.g. Vituškin [30; Ch. VI: 1], Gončar [10], Hedberg [14; p. 162] and [15; Th. 9]).

We denote the annulus $\{y \in \mathbf{R}^d; r < |y - x| < R\}$ by $A(x, r, R)$.

THEOREM 6. *Let $E \subset \mathbf{R}^d$ be compact with $m_d E = 0$, and let $1 < q < \infty$. Suppose that for all $x \in E$, with the possible exception of a compact set $E_0 \subset E$ with $C_q(E_0) = 0$, there exist a number $K(x) < \infty$, and sequences $\{r_n(x)\}_1^\infty$ and $\{R_n(x)\}_1^\infty$ decreasing to zero in such a way that*

$$1 + \frac{1}{K(x)} < \frac{R_n(x)}{r_n(x)} < K(x), \tag{1}$$

and so that

$$\Gamma_q(A(x, r_n(x), R_n(x))/E) < K(x) \cdot \Gamma_q(A(x, r_n(x), R_n(x))) \tag{2}$$

for all n.

Then C_E^∞ *is dense in* $\mathring{W}_1^q$, *i.e.* E *is removable for* FD^p (AD^p), $p = q/(q-1)$, *and* $\Gamma_q(G/E) = \Gamma_q(G)$ *for all rings* G.

The proof of this theorem is somewhat technical. We shall therefore first discuss how Theorems 4 and 5 compare with the characterization of removable sets for AD^2 given by Ahlfors and Beurling, and then give the proof of Theorem 6 and some related results.

5. Comparison with the Ahlfors-Beurling theorems

In [3] Ahlfors and Beurling gave a characterization of the removable sets for AD^2 in terms of extremal lengths. Let the plane compact set E be contained in a rectangle R with, say, sides parallel to the coordinate axes. We denote the extremal distance between the vertical sides of R with respect to $R \setminus E$ by $\lambda^{(1)}(R \setminus E)$, and between the horizontal sides by $\lambda^{(2)}(R \setminus E)$. Then the theorem of Ahlfors and Beurling is the following.

THEOREM 7. *E is removable for AD^2 if and only if for some rectangle (or all rectangles) R containing E in its interior*

$$\lambda^{(i)}(R \setminus E) = \lambda^{(i)}(R), \quad i = 1, 2.$$

The sufficiency of this condition was obtained as a consequence of the following theorem, also proved in [3].

THEOREM 8. *E is removable for AD^2 if and only if every region which is conformally equivalent with $\complement E$ has a complement of zero area.*

We shall show directly that Theorem 4, when specialized, is equivalent to Theorem 7, and we thus obtain new proofs of Theorems 7 and 8.

We first define, for a d-dimensional rectangle R, another condenser p-capacity with respect to E.

Definition. $\Gamma_p^{(i)}(R \setminus E) = \inf_\omega \int_{R \setminus E} |\text{grad}\, \omega|^p dx$, $i = 1, 2, \ldots, d$, where the infimum is taken over all $\omega \in C^\infty(\complement E)$ such that $\omega = 1$ on one of the sides of R parallel to the hyperplane $x_i = 0$, and $\omega = 0$ on the other one.

Note that always

$$\Gamma_p^{(i)}(R \setminus E) \leq \Gamma_p^{(i)}(R) \leq \Gamma_p^{(i)}(R/E).$$

The bridge between Theorems 4 and 7 is provided by Lemmas 3 and 4 below. They are of course not new (see the remarks below), but for the reader's convenience we give some details. We again assume that E is a plane compact set contained in the interior of a rectangle R.

LEMMA 3. $\Gamma_2^{(i)}(R/E) = 1/\Gamma_2^{(j)}(R \setminus E)$, $i = 1$, $j = 2$ *or* $i = 2$, $j = 1$.

Proof. Let R have vertical sides F_1 and F_1' of length a and horizontal sides F_2 and F_2' of length b. First assume that E is bounded by finitely many analytic curves.

The sets of functions ω competing in the extremal problems defining $\Gamma_2^{(1)}(R/E)$ and $\Gamma_2^{(2)}(R \setminus E)$ are convex in $W_1^2(R)$ and $W_1^2(R \setminus E)$ respectively. Therefore there are unique extremal functions u in $W_1^2(R)$ and v in $W_1^2(R \setminus E)$. Both are harmonic in $R \setminus E$, $u = 0$ on F_1, $u = 1$ on F_1', and $v = 0$ on F_2, $v = 1$ on F_2'. u is clearly constant on each component of the interior of E. Both u and v can be continued harmonically by reflection over the boundaries, so they have continuous boundary values and boundary derivatives.

The usual variation shows that $\int_R \operatorname{grad} u \cdot \operatorname{grad} \phi dm = 0$ for all ϕ in C_E^∞ with support off $F_1 \cup F_1'$. By Green's formula

$$\int_{\partial R \cup \partial E} \phi(* du) = \int_{\partial R \cup \partial E} \phi \frac{\partial u}{\partial n} ds = 0,$$

for all such ϕ, and it follows that $\partial u/\partial n = 0$ on $F_2 \cup F_2'$, and that $\int (\partial u/\partial n) ds = 0$ for every component c of ∂E, i.e. u has no periods in $R \setminus E$.

Similarly $\int_{R \setminus E} \operatorname{grad} v \cdot \operatorname{grad} \phi dm = 0$ for all ϕ in C^∞ with support off $F_2 \cup F_2'$. Thus $\int_{\partial R \cup \partial E} \phi(\partial v/\partial n) ds = 0$ for all such ϕ, and it follows that $\partial v/\partial n = 0$ on ∂E and on $F_1 \cup F_1'$.

Moreover, Green's formula

$$\int_G |\operatorname{grad} (u_1 - u_2)|^2 dm = \int_{\partial G} (u_1 - u_2) \frac{\partial (u_1 - u_2)}{\partial n} ds$$

shows that both u and v are uniquely determined by these boundary conditions.

Since u does not have any periods it has a single-valued conjugate harmonic function u^* in $R \setminus E$. The flow lines for u are level lines for u^*, and conversely. Therefore $\partial u^*/\partial n = 0$ on ∂E and on $F_1 \cup F_2'$, and $u^* = \text{const.}$ on F_2 and on F_2'. We can assume that $u^* = 0$ on F_2, and then it follows from the uniqueness of v that $u^* = cv$ for some constant c.

Again applying Green's formula we find

$$\Gamma_2^{(1)}(R/E) = \int_{R \setminus E} |\operatorname{grad} u|^2 dm = \int_{\partial R \cup \partial E} u \frac{\partial u}{\partial n} ds = \int_{F_1'} \frac{\partial u}{\partial n} ds = u^*(F_2') = c.$$

On the other hand

$$\int_{R\setminus E} |\text{grad } u|^2 dm = \int_{R\setminus E} |\text{grad } u^*|^2 dm = c^2 \int_{R\setminus E} |\text{grad } v|^2 dm = c^2 \Gamma_2^{(2)}(R \setminus E).$$

Thus $\Gamma_2^{(1)}(R/E) = c = 1/\Gamma_2^{(2)}(R \setminus E)$.

Moreover the analytic function $f = u + iu^*$ maps $R \setminus E$ univalently onto a region bounded by a rectangle S and finitely many vertical slits. The length of the horisontal sides of S is 1 and of the vertical sides c. Clearly $\Gamma_2^{(1)}(R) = 1/\Gamma_2^{(2)}(R) = a/b$, so $c \geq a/b$.

In order to finish the proof we have to prove that if $E = \bigcap_1^\infty E_n$, $E_{n+1} \subset E_n$, then $\lim_{n\to\infty} \Gamma_2^{(i)}(R \setminus E_n) = \Gamma_2^{(i)}(R \setminus E)$, and $\lim_{n\to\infty} \Gamma_2^{(i)}(R/E_n) = \Gamma_2^{(i)}(R/E)$. This is easily done by arguments similar to those used by Ziemer [32; Lemma 3.9].

LEMMA 4. $\Gamma_2^{(j)}(R \setminus E) = 1/\lambda^{(j)}(R \setminus E)$, $j = 1, 2$.

Proof. Assume first, as before, that E is bounded by finitely many analytic curves. Let v be the extremal function defined above (for $j = 2$), i.e. v is harmonic in $R \setminus E$, $v = 0$ on F_2, $v = 1$ on F_2', and $\partial v/\partial n = 0$ on $\partial E \cup F_1 \cup F_1'$, and $\int_{R\setminus E} |\text{grad } v|^2 dm = \Gamma_2^{(2)}(R \setminus E)$. Then $\dfrac{1}{\lambda^{(2)}(R \setminus E)} = \int_{R\setminus E} |\text{grad } v|^2 dm$.

This is Theorem 4—5 in Ahlfors [2; p. 65], and »almost certainly due to Beurling». (See [2; p. 81].)

The extension to general E is again easy. Details are found in Ziemer [32; Th. 2.5.1] or [33; Lemma 2.3].

In a similar way one obtains a counterpart to Theorem 5. If G is a ring domain in $\mathbf{R}^d$ we denote by $M_p(G \setminus E)$ the p-modulus of the family of hypersurfaces (curves for $d = 2$) in $G \setminus E$ that separate the components of $\complement G$. I.e. for $p = d = 2$ $M_2(G \setminus E) = \lambda(G \setminus E)^{-1}$, where $\lambda(G \setminus E)$ is the extremal length of this family of curves.

THEOREM 9. *If $E \subset \mathbf{C}$ and G is a ring then $\Gamma_2(G/E) = M_2(G \setminus E)^{-1}$. Thus E is removable for AD^2 if and only if*

$$M_2(G \setminus E) = M_2(G)$$

for all rings G.

Remarks. Capacities of the types $\Gamma_p^{(i)}(R \setminus E)$ and $\Gamma_p(G \setminus E)$ have been studied extensively. Thus Lemma 4 is a special case of a theorem of J. Hesse [16] who extended earlier work of Beurling, Fuglede [8], Gehring [9], Ziemer [33] and others. See also Ohtsuka [23]. Nullsets for these capacities have been investigated by Väisälä [29] and Ziemer [32; Th. 3.14]. Capacities of the type $\Gamma_p^{(i)}(R/E)$ and

$\Gamma_p(G/E)$ on the other hand do not seem to have been studied before. A theorem similar to Lemma 3 in a general situation on open Riemann surfaces, but in terms of conjugate extremal lengths, was given by Marden and Rodin [19]. See also the monograph by Rodin and Sario [25; p. 124]. Generalizations of Lemma 3 and Theorem 9 to $p \neq 2$ and to higher dimensions do not seem to be known. A »conjugate» problem was solved, at least for $p \geq 2$, by Ziemer [32], and Bardet and Lelong-Ferrand [5], who extended earlier results of Fuglede [8], Gehring [9], and others.

In view of these results the following conjecture seems reasonable: For any compact E and any ring $G \subset \mathbf{R}^d$ $\Gamma_q(G/E)^{1/q} = M_p(G \setminus E)^{-1/p}$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$.

6. Proof of Theorem 6

In order to prove Theorem 6 it is by Theorem 1 enough to show that every function in $C_0^\infty(\complement E_0)$ can be approximated in the $\overset{\circ}{W}{}_1^q$ norm by functions in C_E^∞. The proof is by a direct construction of the approximation functions. The construction is similar to one used previously by the author in [13].

We denote the closure of C_E^∞ in $\overset{\circ}{W}{}_1^q$ by $\overline{C_E^\infty}$. Various constants independent of x are denoted by C.

LEMMA 5. *Under the assumptions of Theorem* 6

$$\Gamma_q(A(x, r_n(x), R_n(x))) \leq CK(x)^{q-1}R_n(x)^{d-q}, \quad 1 < q < \infty.$$

Proof. It is well known that

$$\Gamma_q(A(x, r, R)) = C\left\{\int_r^R t^{(d-1)/(1-q)}dt\right\}^{1-q},$$

corresponding to the extremal function $u(y) = C|y - x|^{(d-q)/(1-q)}$. The assumption $R_n(x)/r_n(x) > 1 + 1/K(x)$ gives the result after a short computation.

We shall start by covering the set $E \setminus E_0$ with balls in a special way by means of the following well-known lemma. See e.g. Stein [28; I. 1.7] for a proof.

LEMMA 6. *Let E be a measurable subset of $\mathbf{R}^d$ which is covered by the union of a family of balls $\{B(x, r)\}$ with bounded diameter. Then from this family a subsequence $\{B(x_i, r_i)\}_{i=1}^\infty$ can be selected so that $E \subset \bigcup_1^\infty B(x_i, r_i)$ but the balls $B(x_i, r_i/5)$ are all disjoint.*

For every integer $i \geq 1$ we set $E_i = \{x \in E; i \leq K(x) < i + 1\}$, so $E = \bigcup_{i=0}^\infty E_i$, where by assumption $C_q(E_0) = 0$.

For $\delta > 0$ we set $G_\delta = \{x; \text{dist}(x, E) < \delta\}$. Since $m_d E = 0$ by assumption, we have $\lim_{\delta\to 0} m_d G_\delta = 0$. We choose an arbitrary $\varepsilon > 0$ which will be kept fixed, and then we choose a sequence $\{\delta_i\}_{i=1}^\infty$ such that

$$\sum_{i=1}^{\infty} i^{d+q} m_d G_{\delta_i} < \varepsilon.$$

For $x \in E_i$, $i \geq 1$, we can also assume that $R_n(x) < \delta_i$ for all n.

It follows from Lemma 6 that each E_i, $i = 1, 2, \ldots$, can be covered by a (possibly infinite) sequence of balls $B(x_j, r_j)$, $x_j \in E_i$, $r_j = r_{n_j}(x_j)$, so that $B(x_j, R_j) \subset G_{\delta_i}$, $R_j = R_{n_j}(x_j)$, and so that

$$\sum_j R_j^d \leq (i+1)^d \sum_j (r_j/5)^d \leq C i^d m_d G_{\delta_j}.$$

Taking the union of these coverings and reenumerating we obtain a covering of $E \setminus E_0$ by balls $B(x_j, r_j)$ so that $R_j < \delta_1$ and

$$\sum_{j=1}^{\infty} K(x_j)^q R_j^d \leq C \sum_{i=1}^{\infty} i^{d+q} m_d G_{\delta_i} < C\varepsilon.$$

For each of these balls there is by assumption and Lemma 5 a function $\phi_j \in C_E^\infty$ so that $\phi_j = 1$ on $B(x_j, r_j)$, $\phi_j = 0$ off $B(x_j, R_j)$, and

$$\int |\text{grad}\, \phi_j|^q dm < CK(x_j)^q R_j^{d-q}.$$

Thus

$$\sum_j R_j^q \int |\text{grad}\, \phi_j|^q dm < C\varepsilon. \tag{3}$$

Let g be the function to be approximated. We can assume that $0 \leq g \leq 1$ and that $|\text{grad}\, g| \leq 1$. We shall construct an approximation to g in an inductive way by using the functions ϕ_j and the following lemma.

LEMMA 7. *If* ϕ_1 *and* ϕ_2 *belong to* $\overline{C_E^\infty}$, *then the functions* $\max(\phi_1, \phi_2)$ *and* $\min(\phi_1, \phi_2)$ *belong to* $\overline{C_E^\infty}$.

A similar lemma was proved in [13; Lemma 2] and we omit the proof here.

Denote the support of g by S. Then $E \cap S$ is compact, so we can select a finite subsequence $\{B(x_j, r_j)\}_{j=1}^J$ that covers $E \cap S$. We choose η so $0 < \eta \leq \min\{R_j; 1 \leq j \leq J\}$, and so that $1/\eta$ is an integer.

Then we set

$$L_k = \{x; g(x) \geq k\eta\}, \quad k = 0, 1, 2, \ldots, 1/\eta,$$

and define $g_k(x)$ for each k by

$$g_k(x) = \begin{cases} 0, & x \notin L_k \\ g(x) - k\eta, & x \in L_k \setminus L_{k+1} \\ \eta, & x \in L_{k+1}. \end{cases}$$

For each $j = 1, 2, \ldots, J$ there is an integer $\mu(j)$ such that $B(x_j, R_j) \subset L_{\mu(j)}$, but $B(x_j, R_j) \not\subset L_{\mu(j)+1}$, and there is an integer $\nu(j) \geq \mu(j)$ such that $B(x_j, r_j) \cap L_{\nu(j)} \neq \emptyset$, but $B(x_j, r_j) \cap L_{\nu(j)+1} = \emptyset$.

Set $\psi_j = (\nu(j) - \mu(j))\eta\phi_j$ if $\nu(j) > \mu(j)$, and $\psi_j = \eta\phi_j$ otherwise. Since $|\text{grad } g| \leq 1$, $\text{dist}(\complement L_i, L_{i+1}) \geq \eta$. Thus

$$(\nu(j) - \mu(j) - 1)\eta \leq \text{dist}(\complement L_{\mu(j)+1}, L_{\nu(j)}) \leq R_j + r_j.$$

Since $\eta \leq R_j$ we have $\psi_j \leq 3R_j\phi_j$, so by (3)

$$\sum \int |\text{grad } \psi_j|^q dm \leq C\varepsilon. \tag{4}$$

Set $\Omega = \bigcup_1^J B(x_j, r_j)$, and set $\varrho = \text{dist}(\complement \Omega, E \cap S)$. Define a function $\chi(x)$ by

$$\chi(x) = \begin{cases} 0, & \text{dist}(x, E \cap S) \leq \varrho/2 \\ 2/\varrho \, \text{dist}(x, E \cap S) - 1, & \varrho/2 \leq \text{dist}(x, E \cap S) \leq \varrho \\ 1, & \text{dist}(x, E \cap S) \geq \varrho \end{cases}$$

Then $g_k\chi = 0$ in a neighborhood of E, so $g_k\chi \in \overline{C_E^\infty}$, and $g_k\chi = g_k$ outside Ω. Now, set $\Psi_0(x) = \max\{\psi_j(x); B(x_j, R_j) \subset L_0\}$, and set $h_0(x) = \max\{g_0(x)\chi(x), \Psi_0(x)\}$. Then $h_0 \in \overline{C_E^\infty}$ by Lemma 7, $h_0(x) \geq \eta$ on L_1 and $h_0(x) = g(x)$ on $\complement L_1 \setminus (\text{supp } \Psi_0)$.

Set $l_0(x) = \min\{h_0(x), \eta\}$. Then $l_0 \in \overline{C_E^\infty}$, by Lemma 7 and $l_0(x) = \eta$ on L_1.

Next, set $\Psi_1(x) = \max\{\psi_j(x); B(x_j, R_j) \subset L_1\}$, and set $h_1(x) = \max\{g_1(x)\chi(x), \Psi_1(x)\}$. Then $h_1 \in \overline{C_E^\infty}$, and $h_1(x) = 0$ outside L_1. Set $k_1(x) = \max\{h_0(x), l_0(x) + h_1(x)\}$. Then also $k_1 \in \overline{C_E^\infty}$, and we claim that $k_1(x) \geq 2\eta$ on L_2. In fact, if $x \in L_2$, and x belongs to some $B(x_j, r_j)$ such that $B(x_j, R_j)$ intersects $\complement L_1$, then $\psi_j(x) \geq 2\eta$, so $h_0(x) \geq 2\eta$. If $x \in L_2$ and x belongs to some $B(x_j, r_j)$ such that $B(x_j, R_j) \subset L_1$, then $\psi_j(x) \geq \eta$ so $l_0(x) + h_1(x) \geq 2\eta$. If $x \in L_2 \setminus \Omega$, then $h_1(x) \geq g_1(x)\chi(x) = g_1(x) = \eta$, so again $k_1(x) \geq 2\eta$. On $\complement L_1$ we have $h_1(x) = 0$, so $k_1(x) = k_0(x) = h_0(x)$.

Moreover, if $x \in L_i$, $i > 2$, and x belongs to some $B(x_j, r_j)$ such that $B(x_j, R_j)$ intersects $\complement L_2$, then $k_1(x) \geq i\eta$. In fact, either $B(x_j, R_j)$ intersects $\complement L_1$, and then $\psi_j(x) \geq i\eta$, so $h_0(x) \geq i\eta$, or else $B(x_j, R_j) \subset L_1$, and then $\Psi_1(x) \geq \psi_j(x) \geq (i-1)\eta$, and $k_1(x) \geq l_0(x) + \Psi_1(x) \geq i\eta$.

Set $l_1(x) = \min\{k_1(x), 2\eta\}$, and continue the construction in the same way.

Assume that $k_{n-1} \in \overline{C_E^\infty}$ has been constructed, so that $k_{n-1}(x) = k_{n-2}(x)$ on $\complement L_{n-1}$, $k_{n-1}(x) \geq n\eta$ on L_n, and $k_{n-1}(x) \geq i\eta$ if $x \in L_i$, $i > n$, and if x belongs to some $B(x_j, r_j)$ such that $B(x_j, R_j)$ intersects $\complement L_n$.

Then set
$$l_{n-1}(x) = \min\{k_{n-1}(x), n\eta\},$$
$$\Psi_n(x) = \max\{\psi_j(x); B(x_j, R_j) \subset L_n\},$$
$$h_n(x) = \max\{g_n(x)\chi(x), \Psi_n(x)\},$$
and
$$k_n(x) = \max\{k_{n-1}(x), l_{n-1}(x) + h_n(x)\}.$$

Clearly $k_n(x) \in \overline{C_E^\infty}$, and $k_n(x) = k_{n-1}(x)$ on $\complement L_n$.

If $x \in L_{n+1}$, we claim that $k_n(x) \geq (n+1)\eta$. In fact, if $x \in L_{n+1}$, and x belongs to some $B(x_j, r_j)$ such that $B(x_j, R_j)$ intersects $\complement L_n$, then by the induction hypothesis $k_{n-1}(x) \geq (n+1)\eta$, so $k_n(x) \geq (n+1)\eta$. If $x \in B(x_j, r_j)$, and $B(x_j, R_j) \subset L_n$, then $\Psi_n(x) \geq \eta$, so $k_n(x) \geq l_{n-1}(x) + \eta = (n+1)\eta$. Finally, if $x \in L_{n+1} \setminus \Omega$, then $g_n(x)\chi(x) = g_n(x) = \eta$, so again $k_n(x) \geq (n+1)\eta$.

If $x \in L_i$, $i > n+1$, and if x belongs to some $B(x_j, r_j)$ such that $B(x_j, R_j)$ intersects $\complement L_{n+1}$, we claim that $k_n(x) \geq i\eta$. In fact, either $B(x_j, R_j)$ intersects $\complement L_n$, and then $k_{n-1}(x) \geq i\eta$, by the induction hypothesis, so $k_n(x) \geq i\eta$, or else $B(x_j, R_j) \subset L_n$, in which case $\psi_j(x) \geq (i-n)\eta$, so $k_n(x) \geq l_{n-1}(x) + (i-n)\eta = i\eta$.

If $n = 1/\eta$ we find that $k_{n+1} = k_n$. We finally set $k = k_{1/\eta}$, and claim that k approximates the given function g.

It is clear that $k(x) = g(x)$ outside $G_{\delta_1} = G_0$. Moreover, for almost all $x \in L_n \setminus L_{n+1}$, $n = 0, 1, \ldots$, we have either $\operatorname{grad} k(x) = \operatorname{grad} \psi_j(x)$ for some j, or $\operatorname{grad} k(x) = \operatorname{grad}(g_n(x)\chi(x))$ (see e.g. Deny-Lions [7; Th. 3.2], and
$$|\operatorname{grad}(g_n(x)\chi(x))| = |g_n(x) \operatorname{grad} \chi(x) + \chi(x) \operatorname{grad} g(x)| \leq 3.$$

Thus
$$\int |\operatorname{grad}(g-k)|^q dm \leq \int_{G_0} (|\operatorname{grad} g| + |\operatorname{grad} k|)^q dm \leq C m_d G_0 +$$
$$+ C \sum_j \int |\operatorname{grad} \psi_j|^q dm \leq C m_d G_0 + C\varepsilon$$
by (4). Since ε and $m_d G_0$ are arbitrarily small, the theorem follows.

We also observe that $|g(x) - k(x)| \leq 3 \max_j R_j \leq 3\delta_1$, so k also approximates g uniformly.

Variants of the proof of Theorem 6 also give the following results.

THEOREM 10. a) *Suppose E is contained in a C^1-submanifold $M \subset \mathbf{R}^d$ of dimension α, and that $m_\alpha(E) = 0$. Then the conclusion of Theorem 6 is still true if* (2) *is replaced by*
$$\Gamma_q(A(x, r_n(x), R_n(x))/E) < K(x) R_n(x)^{\alpha-q} \tag{5}$$
for all n.

b) *Suppose that the* α*-dimensional Hausdorff measure* $\Lambda_\alpha(E) = 0$ *for some* $\alpha \leq d$. *Then the conclusion of Theorem* 6 *is true if there exists a compact* E_0 *with* $C_q(E_0) = 0$ *so that for all* $x \in E \setminus E_0$ *there is a* $K = K(x)$, $0 < K(x) < 1$, *so that*

$$\overline{\lim_{R\to 0}} \frac{\Gamma_q(A(x, KR, R)/E)}{R^{\alpha-q}} < \infty.$$

Proof. To prove the theorem one only has to modify the covering argument in the first part of the proof of Theorem 6. We keep the notation from that proof.

In case a) we know that $\lim_{\delta\to 0} m_\alpha(G_\delta \cap M) = 0$. We now choose $\{\delta_i\}$ so that

$$\sum_{i=1}^{\infty} i^{\alpha+q} m_\alpha(G_{\delta_i} \cap M) < \varepsilon,$$

and then (3) follows as before.

In case b) we write $E \setminus E_0 = \bigcup_1^\infty E_i$, where E_i is the set where $i^{-1} < K(x) < 1 - i^{-1}$, and where $\Gamma_q(A(x, K(x)R, R)/E) < iR^{\alpha-q}$ for all $R < i^{-1}$. We choose a sequence $\{\delta_i\}$ so that

$$\sum_1^\infty i^{\alpha+1}\delta_i < \varepsilon$$

and cover each E_i with balls $B(x_j, r_j)$ so that $r_j < i^{-2}$ and $\sum_j r_j^\alpha < \delta_i$. We can assume, by doubling the r_j if necessary, that $x_j \in E_i$. It follows, if we set $R_j = r_j/K(x_j)$, that

$$\sum_j R_j^q \Gamma_q(A(x_j, r_j, R_j)/E) < i^{\alpha+1} \sum_j r_j^\alpha < i^{\alpha+1}\delta_i.$$

(3) follows, and then the theorem follows as before.

THEOREM 11. a) *If in Theorems* 6 *and* 10 a) *the function* $K(x)$ *is uniformly bounded, the conclusion is still true if the hypotheses* $m_d(E) = 0$ *or* $m_\alpha(E) = 0$ *are removed.*

b) *Suppose that* $\Lambda_\alpha(E) < \infty$ *for some* $\alpha \leq d$. *Then the conclusion of Theorem* 6 *is true if for every* $\varepsilon > 0$ *there exists a compact* E_0 *with* $C_q(E_0) < \varepsilon$ *and a number* $M < \infty$ *so that for all* $x \in E \setminus E_0$ *there is a* $K = K(x)$, $1/M < K(x) < 1 - 1/M$, *so that*

$$\overline{\lim_{R\to 0}} \frac{\Gamma_q(A(x, KR, R)/E)}{R^{\alpha-q}} < M.$$

Proof. We can no longer claim that $\int |\text{grad}\,(g - k)|^q dm$ in the proof of Theorem 6 is small, but in all cases $\int |\text{grad}\, k|^q dm$ is bounded independently of ε, so there is a weakly convergent sequence of functions k. By the Banach-Saks theorem there is a subsequence $k^{(i)}$ such that $K_n = 1/n \sum_1^n k^{(i)}$ converge strongly. The

$k^{(i)}$, and therefore the K_n, also converge uniformly to g, so g is also the strong limit of the K_n.

THEOREM 12. *Under the additional assumption that E is completely disconnected the equivalent conditions in Theorems 5 and 6 are necessary and sufficient, and the conditions in Theorems 10 and 11 are sufficient for $C_E^\infty(G)$ to be dense in the Royden q-algebra $\mathcal{M}_0^q(G)$ for any bounded domain $G \supset E$.*

In fact, a modification of the proof of Lemma 1 shows that $C_E^\infty(G)$ is dense in $\mathcal{M}_0^q(G)$ if E is completely disconnected and $C_q(E) = 0$. Theorem 12 then follows easily, since, as we noted above, the approximating functions in the proof of Theorem 6 actually converge uniformly.

We finally remark that a Baire category argument (see Sario and Nakai [27; Th. VI. 1. L, p. 371]) shows that if E is compact, and $E = \bigcup_1^\infty E_i$, where the E_i are compact and removable for FD^p, then E is also removable.

7. Sets on hyperplanes

In this section we give some further results in the case when the set E is contained in a hyperplane, especially when $d = 2$ and E is a linear set, in which case we improve a theorem of Ahlfors and Beurling. The discussion also serves to illuminate the difference between the capacities $\Gamma_q(R/E)$ and $\Gamma_q(R \setminus E)$.

We denote the hyperplane $\{x \in \mathbf{R}^d; x_d = 0\}$ by $\mathbf{R}^{d-1}$, and the halfspaces $\{x \in \mathbf{R}^d; x_d > 0\ (x_d < 0)\}$ by $\mathbf{R}_+^d$ $(\mathbf{R}_-^d)$. Suppose the compact set E belongs to $\mathbf{R}^{d-1}$. It is well known that the restriction of W_1^q to $\mathbf{R}^{d-1}$ can be identified with the Besov or Lipschitz space $\Lambda_{1/p}^{q,q}(\mathbf{R}^{d-1})$, i.e. the space of boundary values of harmonic functions $u \in W_1^q(\mathbf{R}_+^d)$. See e.g. Stein [28; VI. 4.4] and references given there. It is easy to see that $C_E^\infty(\mathbf{R}^d)$ is dense in $W_1^q(\mathbf{R}^d)$ if and only if $C_E^\infty(\mathbf{R}^{d-1})$ is dense in $\Lambda_{1/p}^{q,q}(\mathbf{R}^{d-1})$. I.e. the problem can be reduced to the same problem in a Lipschitz space, and for $d > 2$ a solution requires a study of condenser capacities in these spaces. For $d = 1$ a linear functional on $\Lambda_{1/p}^{q,q}(\mathbf{R})$ that annihilates $C_E^\infty(\mathbf{R})$ can be identified with a function $f \in \Lambda_{1/q}^{p,p}(\mathbf{R})$ such that $\int_{\mathbf{R}} f\phi' dm = 0$ for all $\phi \in C_E^\infty(\mathbf{R})$. It follows that $f = 0$ on $\complement E$, i.e. C_E^∞ is dense in $\Lambda_{1/p}^{q,q}$ if and only if every function $f \in \Lambda_{1/q}^{p,p}(\mathbf{R})$ with support on E has to be identically zero. We say that E is a set of uniqueness for $\Lambda_{1/q}^{p,p}(\mathbf{R})$. A good description of these sets of uniqueness can be given in terms of »Bessel capacities» $C_{\alpha,p}$ (see e.g. Meyers [22] for definitions and properties), which are equivalent to the classical Riesz capacities for $p = 2$. We denote the space of Bessel potentials of order α (or equivalently Riesz potentials with respect to the kernel $|x|^{1-\alpha}$) of functions in $L^p(\mathbf{R})$ by $\mathcal{L}_\alpha^p(\mathbf{R})$, and note the following inclusion relations (see Stein [28; V. 5.3], where there are misprints in the statement of the theorem, however).

$$\Lambda_\alpha^{p,p} \subset \mathscr{L}_\alpha^p \subset \Lambda_\alpha^{p,2}, \quad 1 < p \leq 2.$$

$$\Lambda_\alpha^{p,2} \subset \mathscr{L}_\alpha^p \subset \Lambda_\alpha^{p,p}, \quad 2 \leq p < \infty.$$

Moreover $\Lambda_\alpha^{p,2} \subset \Lambda_\alpha^{p',p'}, p' < p$, and $\Lambda_\alpha^{p',p'} \subset \Lambda_\alpha^{p,2}$, $p < p'$, at least locally. Necessary and sufficient conditions for a set to be a uniqueness set for $\mathscr{L}_\alpha^p$ have been given by Polking [24] and the author [15; Th. 9]. For $p\alpha > d$ functions in $\mathscr{L}_\alpha^p$ are continuous, and then the condition is that E has no interior.

We summarize the result. We denote the interval $[x - \delta, x + \delta]$ by $I_x(\delta)$.

THEOREM 13. *Let $E \subset \mathbf{C}$ be compact, and suppose $E \subset \mathbf{R}$, the real axis.*

For E to be removable for AD^p, $1 < p < 2$, it is sufficient that E is a set of uniqueness for $\mathscr{L}_{1/q}^p(\mathbf{R})$. It is necessary that E is a set of uniqueness for $\mathscr{L}_{1/q}^{p'}(\mathbf{R})$ for all $p' < p$.

For E to be removable for AD^2 it is necessary and sufficient that E is a set of uniqueness for $\mathscr{L}_{1/2}^2(\mathbf{R})$.

For E to be removable for AD^p, $p > 2$, it is necessary and sufficient that E is totally disconnected.

E is a set of uniqueness for $\mathscr{L}_{1/q}^p$, $1 < p \leq 2$, if and only if one of the following equivalent conditions is satisfied:

(a) $C_{1/q,p}(I \setminus E) = C_{1/q,p}(I)$ *for some interval I containing E.*

(b) $C_{1/q,p}(I \setminus E) = C_{1/q,p}(I)$ *for every interval I.*

(c) $\displaystyle\overline{\lim_{\delta\to 0}} \frac{C_{1/q,p}(I_x(\delta) \setminus E)}{\delta} > 0$ *for almost all $x \in E$.*

The capacity $C_{1/2,2}(E)$ is equivalent to the plane logarithmic capacity $C_2(E)$, and the result is that E is removable for AD^2 if and only if either

(a) $C_2(I \setminus E) = C_2(I)$ for some interval I containing E,

(b) $C_2(I \setminus E) = C_2(I)$ for every interval I,

or

(c) $\displaystyle\overline{\lim_{\delta\to 0}} \frac{C_2(I_x(\delta) \setminus E)}{\delta} > 0$ for almost all $x \in E$.

Note that $C_2(I_x(\delta)) = 1/(\log 4/\delta)$. Similar results can be given for sets on circles.

The characterization (a) is due to Ahlfors and Beurling [3]. Their result was extended to sets on C^2 curves by Carleson [6; VI. 3]. See also [12; Th. 1] where a related theorem of Carleson was published.

Sharp comparison theorems between the $C_{\alpha,p}$ and Hausdorff measures, and between the $C_{\alpha,p}$ for different α and p have been given by Maz'ja and Havin [21], and by Adams and Meyers [1].

Suppose again that $E \subset \mathbf{R}^{d-1} \subset \mathbf{R}^d$, and consider $\Gamma_p^{(d)}(R \setminus E)$ for a rectangle R that is symmetrically situated with respect to $\mathbf{R}^{d-1}$. Let u be the corresponding extremal function. Then $1 - u(\cdot, -x_d)$ is also an extremal function and since the extremal is unique it follows that $u = \frac{1}{2}$ on $\mathbf{R}^{d-1} \setminus E$.

The restriction of u to $\mathbf{R}^d_+$ belongs to $W^p_1(R^d_+)$, and it can be extended to $W^p_1(\mathbf{R}^d)$ by setting $\tilde{u}(\cdot, -x_d) = u(\cdot, x_d)$. The restriction of $W^p_1(\mathbf{R}^d)$ to $\mathbf{R}^{d-1}$ is again $\Lambda^{p,p}_{1/q}$, and it follows that $\Gamma^{(d)}_p(R \setminus E) = \Gamma^{(d)}_p(R)$ if E is a set of uniqueness for $\Lambda^{p,p}_{1/q}(\mathbf{R}^{d-1})$. By the inclusion relations above and [15; Th. 9] this is the case if $\overline{\lim}_{\delta\to 0} C_{1/q,p}(B(x, \delta) \setminus E)\delta^{1-d} > 0$ for (m_{d-1}) a.e. x in E, in particular if $m_{d-1}(E) = 0$. For $p = 2$ $C_{1/2,2}(E)$ is again equivalent to $C_2(E)$, i.e. to classical capacity with respect to the Newton kernel $|x|^{2-d}$ in $\mathbf{R}^d$. Cf. Väisälä [29], and Ziemer [32; Th. 3.14].

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