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A short proof of a theorem of Bertilsson by direct use of Löwner's method

Karl-Joachim Wirths

Abstract. Let S denote the class of schlicht functions. D. Bertilsson proved recently that for $f \in S$, $p<0$ and $1 \le N \le 2|p|+1$ the modulus of the Nth Taylor coefficient of $(f')^p$ takes its maximal value if f is the Koebe function. Here a short proof of a generalisation of this result is presented.

Let S denote the class of functions f holomorphic and univalent in the unit disk $\mathbf{D}$ and normalized as usual by $f(0)=f'(0)-1=0$. Many extremal problems in the class S have as an extremal function the Koebe function

$$k(z)=\frac{z}{(1-z)^2}$$

or one of its rotations.

In his doctoral thesis [2] and in [1] D. Bertilsson proved among other things the following result.

Theorem 1. *Let $f \in S$, $p<0$ and*

$$(f'(z))^p = 1+\sum_{N=1}^{\infty} c_N((f')^p) z^N.$$

Then for $1 \le N \le 2|p|+1$ the assertion

$$|c_N((f')^p)| \le |c_N((k')^p)|$$

is valid.

The best impression of the nice but involved proof of this theorem is given in the review of [1] in the Mathematical Reviews of the American Mathematical Society (MR 99i:30030) by K. Pearce which we cite for the convenience of the reader:

The proof of Theorem 1 is similar to the proof of the Bieberbach conjecture. The author uses Löwner's equation to construct a system of linear differential (coefficient) equations, which he reformulates as a control problem. He studies the control problem, following de Branges's lead, by examining a related quadratic expression, $H(t)$, and shows that $dH(t)/dt$ is positive-semidefinite, which implies that the control problem is solvable. The proof that $dH(t)/dt$ is positive-semidefinite is quite elaborate and exploits both recursive and inductive relationships among the components of $dH(t)/dt$.

The aim of the present paper is to give a short proof of Theorem 1 which is based on Löwner's method as well. In the following Theorem 2 which is a variant of the Schur–Jabotinski theorem (compare [3], Theorem 1.9.a) and in the proof of Theorem 3 below we show that it is possible to use Löwner's method for the inverses of functions in S to prove Theorem 1 (see [5]).

Theorem 2. *Let $p<0$, $N\in\mathbf{N}$, $f\in S$ and let $F\colon f(\mathbf{D})\to\mathbf{C}$ be f's inverse function. For the Taylor coefficient $b_N(N,f,p)$ defined by*

$$\frac{F'(w)^{|p|+1}}{F(w)^{N+1}}=\frac{1}{w^{N+1}}\sum_{n=0}^{\infty}b_n(N,f,p)w^n$$

the assertion

$$c_N((f')^p)=b_N(N,f,p)$$

is valid.

Proof. Let Γ be a Jordan curve surrounding the origin counterclockwise in $f(\mathbf{D})$. Then

$$b_N(N,f,p)=\frac{1}{2\pi i}\int_{\Gamma}\frac{F'(w)^{|p|+1}}{F(w)^{N+1}}\,dw=\frac{1}{2\pi i}\int_{F(\Gamma)}\frac{f'(z)^p}{z^{N+1}}\,dz=c_N((f')^p).$$

Theorem 2 indicates that Theorem 1 is an immediate consequence of the following theorem which we prove analogously to the proofs in [4].

Theorem 3. *Let N, f, p and $b_n(N,f,p)$ be defined as in Theorem 2. Then for $n\in\mathbf{N}$ and $1\le N\le 2|p|+1$ the assertion*

$$|b_n(N,f,p)|\le|b_n(N,k,p)|$$

is valid.

Proof. If $f\in S$, f can be embedded into a subordination chain. It follows that F has a representation

$$F(w)=\lim_{t\to\infty}\Phi(e^{-t}w,t),\qquad \frac{\partial\Phi(w,t)}{\partial t}=w\left(\frac{\partial\Phi(w,t)}{\partial w}\right)p(w,t),$$

where the partial differential equation holds in some neighbourhood of $w=0$ (depending on t),

$$p(w,t)=\sum_{n=0}^{\infty} p_n(t)w^n,\ p_0(t)=1,\ \operatorname{Re} p(w,t)>0 \quad \text{for } w\in\mathbf{D},\ t\geq 0,$$

and

$$\Phi(w,0)=w.$$

For details see [5] and [6].

Using the above formulae and setting

$$K(w,t)=\frac{(\partial\Phi(w,t)/\partial w)^{|p|+1}}{\Phi(w,t)^{N+1}}=\frac{1}{w^{N+1}}\sum_{n=0}^{\infty} B_n(t)w^n$$

we get

$$\frac{\partial K(w,t)}{\partial t}=(|p|+1)\frac{\partial(wp(w,t))}{\partial w}K(w,t)+\frac{\partial K(w,t)}{\partial w}wp(w,t)$$

and

$$\sum_{n=0}^{\infty}\frac{dB_n(t)}{dt}w^n=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}((|p|+1)(n-j+1)+j-N-1)B_j(t)p_{n-j}(t)\right)w^n.$$

This yields the initial value problems

$$\frac{dB_0(t)}{dt}=(|p|-N)B_0(t),\quad B_0(0)=1,$$

and for $n\in\mathbf{N}$,

$$\frac{dB_n(t)}{dt}-(n-N+|p|)B_n(t)=\sum_{j=0}^{n-1}(|p|(n-j+1)+n-N)B_j(t)p_{n-j}(t),$$
$$B_n(0)=0.$$

The crucial fact for the following arguments is the chain of inequalities

$$|p|(n-j+1)+n-N\geq 2|p|+n-N\geq 2|p|+1-N\geq 0,$$

which is valid for $0\leq j\leq n-1$ and $1\leq N\leq 2|p|+1$.

It should be mentioned here that the only case in which equality is attained at every place in this chain is the case $n=1$, $j=0$, $N=2|p|+1$. This is of importance if one wants to discuss the question of the possible extremal functions.

The above initial value problems have the solutions

$$B_0(t)=e^{(|p|-N)t}$$

and for $n\in\mathbf{N}$

$$B_n(t)=\int_0^t e^{(n-N+|p|)(t-\tau)}\sum_{j=0}^{n-1}(|p|(n-j+1)+n-N)B_j(\tau)p_{n-j}(\tau)\,d\tau,$$

in particular, $B_1(t)\equiv 0$ if $N=2|p|+1$.

This shows that $\operatorname{Re}B_n(t)$ is maximal for fixed t if we choose $B_j(\tau)$, $\tau\in[0,t]$, $j=1,\dots,n-1$, real and maximal, and any p_n, $n\in\mathbf{N}$, equal to the constant 2. As a consequence of this and of

$$b_n(N,f,p)=\lim_{t\to\infty}e^{-t(n-N+|p|)}B_n(t)$$

we get that $\max\{\operatorname{Re}b_n(N,f,p)\,|\,f\in S\}$ is attained if

$$p(w,t)=\frac{1+w}{1-w}.$$

Now the assertion of Theorem 3 follows immediately from the fact that the problems of finding the maximum of the real part and the maximum of the modulus for the coefficients in question are equivalent up to a rotation.

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Karl-Joachim Wirths
Institut für Analysis
Technische Universität Braunschweig
DE-38106 Braunschweig
Germany