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Continuity and differentiability of Nemytskii operators on the Hardy space $\mathcal{H}^{1,1}(\mathbf{T}^1)$

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Abstract. Let $\mathcal{H}^{1,1}(\mathbf{T}^1)$ denote the Hardy space of real-valued functions on the unit circle with weak derivatives in the usual real Hardy space $\mathcal{H}^1(\mathbf{T}^1)$. It is shown that when the weak derivative of a locally Lipschitz continuous function f has bounded variation on compact sets the Nemytskii operator F, defined by $F(u)=f\circ u$, maps $\mathcal{H}^{1,1}(\mathbf{T}^1)$ continuously into itself. A further condition sufficient for the continuous Fréchet differentiability of F is then added.

Introductory remarks

Let $L^1(\mathbf{T}^1)$ denote the Banach space of real-valued Lebesgue integrable 'functions' on the unit circle $\mathbf{T}^1=\mathbf{R}/2\pi\mathbf{Z}$ and let $L\log^+ L$ be the linear space of functions v for which $|v|\log(1+|v|)\in L^1(\mathbf{T}^1)$. For $v\in L^1(\mathbf{T}^1)$, let $\mathcal{C}v$ denote the Hilbert transform of v, also known as the function conjugate to v, whose value at $x\in\mathbf{T}^1$ is given almost everywhere by the Cauchy principle value integral

$$\mathcal{C}v(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{v(y)}{\tan\left(\frac{1}{2}(x-y)\right)}\,dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{v(x-y)}{\tan\left(\frac{1}{2}y\right)}\,dy.$$

A function $v\in L^1(\mathbf{T}^1)$ is said to be in the real Hardy space $\mathcal{H}^1(\mathbf{T}^1)$ if $\mathcal{C}v\in L^1(\mathbf{T}^1)$ and, for $v\in L^1(\mathbf{T}^1)$, Zygmund's lemma implies that $|v|\in\mathcal{H}^1(\mathbf{T}^1)$ if and only if $v\in L\log^+ L$. (Zygmund's lemma [6, Vol. I, VII, (2.8) and (2.10)] states that if $u\geq\alpha>-\infty$ and $u\in\mathcal{H}^1(\mathbf{T}^1)$ then $u\in L\log^+ L$.) The Hardy space $\mathcal{H}^1(\mathbf{T}^1)$ is a Banach space with the norm $\|u\|_{\mathcal{H}^1(\mathbf{T}^1)}=\|u\|_{L^1(\mathbf{T}^1)}+\|\mathcal{C}u\|_{L^1(\mathbf{T}^1)}$.

Let $f\colon\mathbf{R}\to\mathbf{R}$ be a function and define a Nemytskii operator [4] F on spaces of functions u by $F(u)=f\circ u$. (Nemytskii operators are sometimes called superposition operators [1], [3].) The mapping $v\mapsto|v|$ is a Nemytskii operator which maps $L^1(\mathbf{T}^1)$ to itself but does not map $\mathcal{H}^1(\mathbf{T}^1)$ to itself.

Let $\mathcal{H}^{1,1}(\mathbf{T}^1)$ denote the Banach space of all real-valued absolutely continuous functions u on $\mathbf{T}^1$ for which $u'\in\mathcal{H}^1(\mathbf{T}^1)$, where the norm is $\|u\|_{\mathcal{H}^{1,1}(\mathbf{T}^1)}=$

$\|u\|_{\mathcal{H}^1(\mathbf{T}^1)}+\|u'\|_{\mathcal{H}^1(\mathbf{T}^1)}$. In [2, Remark 1, p. 200] Janson used $IH(\mathbf{T}^1)$ to denote our space $\mathcal{H}^{1,1}(\mathbf{T}^1)$ and observed that a Nemytskii operator F maps $IH(\mathbf{T}^1)$ into itself if and only if f is locally Lipschitz continuous. (In fact Janson's proof yields the stronger result that a Nemytskii operator maps $\mathcal{H}^{1,1}(\mathbf{T}^1)$ into the space $W^{1,1}(\mathbf{T}^1)$ of absolutely continuous functions on $\mathbf{T}^1$ if and only if f is locally Lipschitz continuous, in which case it maps $\mathcal{H}^{1,1}(\mathbf{T}^1)$ into itself.)

Here we are concerned with sufficient conditions for continuity and differentiability of F on $\mathcal{H}^{1,1}(\mathbf{T}^1)$. Marcus and Mizel [3] have shown that any Nemytskii operator from $W^{1,1}(\mathbf{T}^1)$ to itself is continuous. While it is not clear whether such a result holds for $\mathcal{H}^{1,1}(\mathbf{T}^1)$, we will see that f' being locally of bounded variation ensures that F maps $\mathcal{H}^{1,1}(\mathbf{T}^1)$ continuously into itself. (In particular, $u\mapsto|u|$ maps $\mathcal{H}^{1,1}(\mathbf{T}^1)$ continuously into itself.) We also show that if f'' is locally Lipschitz continuous then F is continuously Fréchet differentiable on $\mathcal{H}^{1,1}(\mathbf{T}^1)$.

The present remarks arose as a natural extension of observations, motivated by questions about functions on the unit disc, in the case $f(t)=\frac{1}{2}t^2$ [5]. Recall that for $v\in\mathcal{H}^1(\mathbf{T}^1)$ the complex-valued function $v+i\mathcal{C}v$ can be interpreted as the boundary values of a holomorphic function V on the unit disc $\mathcal{D}$ in the complex plane. It is well known [6] that the image of $\mathcal{D}$ under V is a connected set, the boundary of which has bounded variation ($v+i\mathcal{C}v$ has bounded variation on $\mathbf{T}^1$) if and only if $v+i\mathcal{C}v$ is absolutely continuous. This in turn is equivalent to the fact that v', the weak derivative of v, is in $\mathcal{H}^1(\mathbf{T}^1)$ in which case $(v+i\mathcal{C}v)'=v'+i\mathcal{C}(v')$.

The treatment here, which is independent of [2] and [3], is self-contained and elementary.

Continuity

Suppose that f is a real-valued function on $\mathbf{R}$ which is locally Lipschitz (Lipschitz continuous on every compact interval) and u is an absolutely continuous function on $\mathbf{T}^1$. It follows from first principles that the composition $f\circ u$ is absolutely continuous on $\mathbf{T}^1$. Therefore, for almost all $x\in\mathbf{T}^1$, the classical derivative of $f\circ u$ at x exists. Note also that f is differentiable at t for almost all $t\in\mathbf{R}$. Suppose now that $t\in\mathbf{R}$ is a point at which f is *not* differentiable and suppose that $u(x)=t$. Then if u is differentiable with *non-zero* derivative at x it is easily verified that $f\circ u$ is *not* differentiable at x. From these observations it follows that, *no matter what finite value is assigned to* $f'(t)$ at points t where f is not differentiable, the formula

$$(f\circ u)'(x)=f'(u(x))u'(x) \tag{1}$$

holds for almost all $x\in\mathbf{T}^1$, where $'$ denotes the classical derivative at points where it exists. This formula also gives the weak derivative of $f\circ u$ almost everywhere on $\mathbf{T}^1$. (The example $f(t)=|t|$ and $u\equiv 0$ illustrates the point discussed in this paragraph.)

Now consider the case when f is convex. At each point $t\in\mathbf{R}$, let $f'_+(t)$ represent the right derivative of f at t. The right derivative always exists and is finite because of convexity, and coincides with the classical derivative almost everywhere. Moreover, at points where the classical derivative f' exists, $t\mapsto f'_+(t)$ is continuous.

If, more generally, f' has bounded variation on every compact interval I, or equivalently if f is the difference of two convex functions on I, the right derivative $f'_+(t)$ is well-defined for all $t\in\mathbf{R}$. In this case we write $f\in DC$ and put $f'=f'_+$ in (1). If u is absolutely continuous and $f\in DC$ we see from the above discussion that, for almost all $x\in\mathbf{T}^1$, the function $G(u)\colon \mathbf{T}^1\times\mathbf{T}^1\to\mathbf{R}$ defined by

$$G(u)(x,y)=f(u(y))-f(u(x))-f'_+(u(x))(u(y)-u(x)) \tag{2}$$

is differentiable with respect to y at $y=x$, and $(\partial/\partial y)G(u)(x,y)|_{y=x}$ is zero for almost all values of x. The following slight variant of the dominated convergence theorem will be useful.

Lemma 1. *Suppose for a sequence $\{(g_n,h_n)\}_{n=1}^{\infty}$ in $L^1(\mathbf{T}^1)\times L^1(\mathbf{T}^1)$, that $|g_n|\le h_n$ almost everywhere. Suppose also that there exists $(g,h)\in L^1(\mathbf{T}^1)\times L^1(\mathbf{T}^1)$ such that every subsequence $\{(g_{n_k},h_{n_k})\}_{k=1}^{\infty}$ of $\{(g_n,h_n)\}_{n=1}^{\infty}$ has a subsequence (also denoted by $\{(g_{n_k},h_{n_k})\}_{k=1}^{\infty}$) with $(g_{n_k},h_{n_k})\to(g,h)$ pointwise almost everywhere and $\int_{-\pi}^{\pi} h_{n_k}\,dx\to\int_{-\pi}^{\pi} h\,dx$, as $k\to\infty$. Then $g_n\to g$ in $L^1(\mathbf{T}^1)$. In particular, if the hypotheses are satisfied with $g_n=h_n$, then $h_n\to h$ in $L^1(\mathbf{T}^1)$.*

Proof. Suppose that $g_n\not\to g$ in $L^1(\mathbf{T}^1)$, as $n\to\infty$. Then there is a number α and a subsequence with $\|g_{n_k}-g\|_{L^1(\mathbf{T}^1)}\ge\alpha>0$ for all k. From the hypothesis we may assume that $(g_{n_k},h_{n_k})\to(g,h)$ pointwise almost everywhere. Hence, by Fatou's lemma,

$$\begin{aligned}\int_{-\pi}^{\pi} 2h\,dx &\le \liminf_{k\to\infty}\int_{-\pi}^{\pi}(h+h_{n_k}-|g_{n_k}-g|)\,dx\\ &=\int_{-\pi}^{\pi}2h\,dx+\liminf_{k\to\infty}-\int_{-\pi}^{\pi}|g_{n_k}-g|\,dx.\end{aligned}$$

It follows that $0\le-\limsup_{k\to\infty}\|g_{n_k}-g\|_{L^1(\mathbf{T}^1)}\le-\alpha<0$, which contradiction proves the claim. □

Recall the properties of C and of integrability-B, which is defined in Zygmund [6].

(i) That $v_n\to v$ in $L^1(\mathbf{T}^1)$ implies that a subsequence $Cv_{n_k}\to Cv$ pointwise almost everywhere.

(ii) For $v\in L^1(\mathbf{T}^1)$, $|\mathcal{C}v|^{[p]}\in L^1(\mathbf{T}^1)$ for all $p\in(0,1)$, where $t^{[p]}=\min\{t,t^p\}$ for $t\geq 0$.

(iii) If $u\in L^1(\mathbf{T}^1)$ then u is integrable-B and the two integrals coincide. (We write this as $\int_{-\pi}^{\pi} u\,dx=(B)\int_{-\pi}^{\pi} u\,dx$.)

(iv) If $u\in L^1(\mathbf{T}^1)$ then $\mathcal{C}u$ is integrable-B and $(B)\int_{-\pi}^{\pi}\mathcal{C}u\,dx=0$.

(v) If u and v are integrable-B, then $u+v$ is integrable-B and

$$(B)\int_{-\pi}^{\pi} u\,dx+(B)\int_{-\pi}^{\pi} v\,dx=(B)\int_{-\pi}^{\pi}(u+v)\,dx.$$

The key is the following observation.

Proposition 2. *For $v\in L^1(\mathbf{T}^1)$, $v\in\mathcal{H}^1(\mathbf{T}^1)$ if and only if the positive part of $\mathcal{C}v$ is in $L^1(\mathbf{T}^1)$.*

Proof. The 'only if' part is clear from the definition of $\mathcal{H}^1(\mathbf{T}^1)$. Suppose that $v\in L^1(\mathbf{T}^1)$ and that $u=(\mathcal{C}v)^+\in L^1(\mathbf{T}^1)$, where $w^+(x)\equiv\max\{w(x),0\}$ for any function w. Then $u-\mathcal{C}v\geq 0$ almost everywhere. Therefore, for all $p\in(0,1)$, (ii)–(v) give that

$$\begin{aligned}\int_{-\pi}^{\pi}(u-\mathcal{C}v)^{[p]}\,dx&=(B)\int_{-\pi}^{\pi}(u-\mathcal{C}v)^{[p]}\,dx\\&\leq(B)\int_{-\pi}^{\pi}(u-\mathcal{C}v)\,dx=(B)\int_{-\pi}^{\pi}u\,dx=\int_{-\pi}^{\pi}u\,dx.\end{aligned}$$

When $p\nearrow 1$ we learn from Fatou's lemma that $u-\mathcal{C}v\in L^1(\mathbf{T}^1)$. Since $u\in L^1(\mathbf{T}^1)$, the result follows. □

Remark. A trivial consequence of this observation and Zygmund's lemma is that if $u\in L^1(\mathbf{T}^1)$ and $\mathcal{C}u\geq\alpha$ for some $\alpha\in\mathbf{R}$, then $\mathcal{C}u\in L\log^+L$. □

For any absolutely continuous function u and $f\in DC$, let $\mathcal{F}(u)$ be defined for almost all $x\in\mathbf{T}^1$ by

$$\mathcal{F}(u)(x)\equiv f'_+(u(x))\mathcal{C}u'(x)-\mathcal{C}(f'_+(u)u')(x). \tag{3}$$

Proposition 3. *Suppose that f is convex on $\mathbf{R}$ and u is absolutely continuous on $\mathbf{T}^1$. Then $\mathcal{F}(u)(x)\geq 0$ for almost all $x\in\mathbf{T}^1$.*

Proof. Let x be a point at which the partial derivative of $G(u)(x,y)$ with respect to y at $y=x$ exists and is zero. From (2) and the convexity of f, $G(u)(x,y)\geq 0$ for all $y\in\mathbf{R}$. Therefore, by definition,

$$\begin{aligned} f'_+(u(x))\mathcal{C}u'(x)-\mathcal{C}(f'_+(u)u')(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(f'_+(u(x))-f'_+(u(y)))u'(y)}{\tan\left(\frac{1}{2}(x-y)\right)}\,dy \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial/\partial y)G(u)(x,y)}{\tan\left(\frac{1}{2}(x-y)\right)}\,dy \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{G(u)(x,y)}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy \geq 0. \quad \square \end{aligned}$$

Proposition 4. *Suppose $f\in DC$. Then $f\circ u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$ for all $u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$.*

Proof. Suppose that $u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$. Then there is a compact interval I such that $u(x)\in I$ for all $x\in\mathbf{T}^1$. Since $f\in DC$ it suffices to restrict attention to the case when f is convex on $\mathbf{R}$. Since $u'\in\mathcal{H}^1(\mathbf{T}^1)$ and

$$\mathcal{C}(f'_+(u)u') = f'_+(u)\mathcal{C}u' - \mathcal{F}(u),$$

we find, from Proposition 3, that $(\mathcal{C}(f'_+(u)u'))^+\in L^1(\mathbf{T}^1)$. Hence $f'_+(u)u'\in\mathcal{H}^1(\mathbf{T}^1)$, by Proposition 2. However $f'_+(u)u'$ is the weak derivative of $f(u)$. Hence $f(u)\in\mathcal{H}^{1,1}(\mathbf{T}^1)$. $\square$

Remark. Suppose that $u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$. Then it follows from elementary calculus that

$$\left|u(x)-\frac{1}{2\pi}\int_{-\pi}^{\pi}u(x)\,dx\right| \leq \frac{1}{2}\int_{-\pi}^{\pi}|u'(x)|\,dx, \tag{4}$$

and therefore

$$\int_{-\pi}^{\pi}u\mathcal{C}u'\,dx \leq \frac{1}{2}\|u\|^2_{\mathcal{H}^{1,1}(\mathbf{T}^1)}, \quad u\in\mathcal{H}^{1,1}(\mathbf{T}^1). \quad \square$$

Corollary 5. *For $u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$*

$$0\leq\frac{1}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\frac{u(x)-u(y)}{\sin\left(\frac{1}{2}(x-y)\right)}\right)^2 dy\,dx = \int_{-\pi}^{\pi}u\mathcal{C}u'\,dx \leq \frac{1}{2}\|u\|^2_{\mathcal{H}^{1,1}(\mathbf{T}^1)}. \tag{5}$$

Proof. This follows from taking $f(t)=\frac{1}{2}t^2$ in the proof of Proposition 3 and integrating over $[-\pi,\pi]$, using Proposition 4 and the preceding remark. $\square$

Next we have the following result.

Corollary 6. *$\mathcal{H}^{1,1}(\mathbf{T}^1)$ is an algebra in which multiplication is continuous.*

Proof. Let $f(t)=\frac{1}{2}t^2$, $t\in\mathbf{R}$, in Proposition 3. For $u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$ let

$$0\le Qu(x)=u(x)\mathcal{C}u'(x)-\mathcal{C}(uu')(x),\quad x\in\mathbf{T}^1.$$

Hence $(\mathcal{C}(uu'))^+\le|u\mathcal{C}u'|$ and that $\|(\mathcal{C}(uu'))^+\|_{L^1(\mathbf{T}^1)}\le\frac{1}{2}\|u\|^2_{\mathcal{H}^{1,1}(\mathbf{T}^1)}$ follows. Since $\int_{-\pi}^{\pi}\mathcal{C}(uu')\,dx=0$ it follows that $\|\mathcal{C}((u^2)')\|_{L^1(\mathbf{T}^1)}\le2\|u\|^2_{\mathcal{H}^{1,1}(\mathbf{T}^1)}$. The result follows. □

Let $W^{1,1}(\mathbf{T}^1)$ denote the Banach space of real-valued absolutely continuous functions on $\mathbf{T}^1$ with norm $\|u\|_{W^{1,1}(\mathbf{T}^1)}=\|u\|_{L^1(\mathbf{T}^1)}+\|u'\|_{L^1(\mathbf{T}^1)}$.

Remark. When f is locally Lipschitz, the Nemytskii operator F maps $W^{1,1}(\mathbf{T}^1)$ continuously into itself [3], but a simpler result is sufficient here; for completeness we include the proof.

Lemma 7. *Suppose that $f\in DC$. Then $F:W^{1,1}(\mathbf{T}^1)\to W^{1,1}(\mathbf{T}^1)$ is continuous.*

Proof. Since $f\in DC$ it suffices to consider the case when f is convex on $\mathbf{R}$. In this case, by our earlier discussion, $(F(u))'=f'_+(u)u'$ almost everywhere. Let $u_n\to u$ in $W^{1,1}(\mathbf{T}^1)$. It suffices to show that $f'_+(u_n)u'_n\to f'_+(u)u'$ in $L^1(\mathbf{T}^1)$. Since $u'_n\to u'$ in $L^1(\mathbf{T}^1)$ and f'_+ is bounded on bounded sets, it is enough, using Lemma 1, to show that every subsequence $\{f'_+(u_{n_k})u'_{n_k}\}_{k=1}^{\infty}$ of $\{f'_+(u_n)u'_n\}_{n=1}^{\infty}$ has a subsequence which converges pointwise almost everywhere to $f'_+(u)u'$.

Every subsequence of $\{u'_{n_k}\}_{k=1}^{\infty}$ has a subsequence (also denoted by $\{u'_{n_k}\}_{k=1}^{\infty}$) which converges to u' on a set U of full measure. Let $E\subset U$ denote the set on which u' exists, let $E_0=\{x\in E:u'(x)=0\}$ and let $E_1=E\setminus E_0$. Clearly $f'_+(u_{n_k}(x))u'_{n_k}(x)\to 0=f'_+(u(x))u'(x)$ for $x\in E_0$. Moreover, the earlier discussion ensures that $f'(u(x))$ exists for almost all $x\in E_1$. Therefore, for almost all $x\in E_1$, the function $t\mapsto f'_+(t)$ is continuous at $t=u(x)$. Hence $f'_+(u_{n_k}(x))u'_{n_k}(x)\to f'_+(u(x))u'(x)$ at all such points. We have shown that $f'_+(u_{n_k}(x))u'_{n_k}(x)\to f'_+(u(x))u'(x)$ for almost all $x\in E$. Since E has full measure this completes the proof. □

Proposition 8. *For $f\in DC$, the Nemytskii operator $F:\mathcal{H}^{1,1}(\mathbf{T}^1)\to\mathcal{H}^{1,1}(\mathbf{T}^1)$ is continuous.*

Proof. As with the proof of Proposition 4 and Lemma 7, it suffices to prove the result for convex f. Also, by Lemma 7, it is now enough to show that $\mathcal{C}(f'_+(u_n)u'_n)\to\mathcal{C}(f'_+(u)u')$ in $L^1(\mathbf{T}^1)$, as $n\to\infty$, where $\{u_n\}_{n=1}^{\infty}$ is a subsequence of a sequence converging to u in $\mathcal{H}^{1,1}(\mathbf{T}^1)$. Let $g_n=(\mathcal{C}(f'_+(u_n)u'_n))^+$ and let $0\le M=\sup\{f'_+(u_n(x)):x\in\mathbf{T}^1,\ n\in\mathbf{N}\}<\infty$. Then by Proposition 3,

$$0\le g_n=(\mathcal{C}(f'_+(u_n)u'_n))^+\le|f'_+(u_n)\mathcal{C}u'_n|\le M|\mathcal{C}u'_n|$$

almost everywhere, for all n. By Lemma 7 and (i), every subsequence of $\{g_n\}_{n=1}^{\infty}$ has a subsequence which converges almost everywhere to $g=(\mathcal{C}(f'_+(u)u'))^+$. Let subsequences be indexed by n and let $h_n=M|\mathcal{C}u'_n|$. Then $h_n\to h$ in $L^1(\mathbf{T}^1)$, where $h=M|\mathcal{C}u'|$. Now an application of Lemma 1 shows that $g_n\to g$ in $L^1(\mathbf{T}^1)$. By Proposition 4, $\mathcal{C}(f'_+(u_n)u'_n)\in L^1(\mathbf{T}^1)$ and has zero integral (by (iii) and (iv)). Therefore for a subsequence of the negative parts, $(\mathcal{C}(f'_+(u_n)u'_n)^-\to(\mathcal{C}(f'_+(u)u')^-$ almost everywhere and $\int_{-\pi}^{\pi}(\mathcal{C}(f'_+(u_n)u'_n)^-\,dx\to\int_{-\pi}^{\pi}(\mathcal{C}(f'_+(u)u')^-\,dx$, as $n\to\infty$. The result now follows from the last statement in Lemma 1. □

Remark. From the preceding proof it follows that if f is convex,

$$\int_{-\pi}^{\pi}|\mathcal{C}(f'_+(u)u')(x)|\,dx\le 2\int_{-\pi}^{\pi}|f'_+(u(x))\mathcal{C}u'(x)|\,dx,$$

and therefore that F maps bounded sets into bounded sets in $\mathcal{H}^{1,1}(\mathbf{T}^1)$ when f' is locally of bounded variation. □

By contrast with the mapping $u\mapsto f'_+(u)u'$, which is continuous from $W^{1,1}(\mathbf{T}^1)$ to $L^1(\mathbf{T}^1)$, we now show that $u\mapsto f'_+(u)\mathcal{C}u'$ need not be continuous from $\mathcal{H}^{1,1}(\mathbf{T}^1)$ to $L^1(\mathbf{T}^1)$. (As a consequence of this remark and Proposition 8, $\mathcal{F}:\mathcal{H}^{1,1}(\mathbf{T}^1)\to L^1(\mathbf{T}^1)$ is well defined but not necessarily continuous when $f\in DC$. If, in addition, f' is continuous then it follows from Proposition 8 and the dominated convergence theorem that $\mathcal{F}:\mathcal{H}^{1,1}(\mathbf{T}^1)\to L^1(\mathbf{T}^1)$ is continuous.) First a simple observation.

Lemma 9. *Let $u:\mathbf{T}^1\to\mathbf{R}$ be a non-negative smooth function which is zero on an open interval I, but not identically zero. Then for all $x,y\in I$ with $x>y$, $\mathcal{C}u(x)-\mathcal{C}u(y)<0$. In particular, $\mathcal{C}u'\not\equiv 0$ on I.*

Proof. Let $x>y$, $x,y\in I$. Then, since $u\equiv 0$ on I,

$$\mathcal{C}u(x)-\mathcal{C}u(y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{u(x-z)-u(y-z)}{\tan(\frac12 z)}\,dz=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial/\partial z)\int_{y-z}^{x-z}u(t)\,dt}{\tan(\frac12 z)}\,dz$$
$$=-\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\int_{y-z}^{x-z}u(t)\,dt}{\sin^2(\frac12 z)}\,dz<0.\quad\square$$

Proposition 10. *Suppose that $f(t)=|t|$, $t\in\mathbf{R}$. Then $\mathcal{F}:\mathcal{H}^{1,1}(\mathbf{T}^1)\to L^1(\mathbf{T}^1)$ is not continuous.*

Proof. Let $u\in\mathcal{H}^{1,1}(\mathbf{T}^1)$ be as described in Lemma 9 and let $v\in\mathcal{H}^{1,1}(\mathbf{T}^1)$ be a non-negative, smooth function which is non-zero and has compact support in I. Now for x in the support of v and $\varepsilon>0$,

$$[f'_+(u+\varepsilon v)\mathcal{C}(u+\varepsilon v)']_x=\operatorname{sgn}(\varepsilon)(\mathcal{C}u'(x)+\varepsilon\mathcal{C}v'(x)).$$

The result now follows since $\mathcal{C}u' \not\equiv 0$ on the support of v, by Lemma 9. This shows that $w \mapsto f'_+(w)\mathcal{C}w'$ is not continuous from $\mathcal{H}^{1,1}(\mathbf{T}^1)$ into $L^1(\mathbf{T}^1)$ which, by Proposition 8, is equivalent to the required result. □

Remark. We finish this section with a useful inequality. Suppose that f is convex and that there exists $0 \le \alpha \le \beta$ such that $\alpha(a-b)^2 \le (a-b)(f'_+(a)-f'_+(b)) \le \beta(a-b)^2$ for all $a,b \in I = \{u(x) : x \in \mathbf{T}^1\}$, where $u \in \mathcal{H}^{1,1}(\mathbf{T}^1)$. Then

$$0 \le \alpha \int_{-\pi}^{\pi} u\mathcal{C}u'\,dx \le \int_{-\pi}^{\pi} f'_+(u)\mathcal{C}u'\,dx \le \beta \int_{-\pi}^{\pi} u\mathcal{C}u'\,dx.$$

To see this simply note by symmetry and the proof of Proposition 3 that, for all $x \in \mathbf{T}^1$,

$$\begin{aligned}\int_{-\pi}^{\pi} f'_+(u(x))\mathcal{C}u'(x)\,dx &= \int_{-\pi}^{\pi} [f'_+(u(x))\mathcal{C}u(x) - \mathcal{C}(f'_+(u)u')(x)]\,dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{G(u)(x,y)}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy\,dx \\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{G(u)(x,y)+G(u)(y,x)}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy\,dx \\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(f'_+(u(x))-f'_+(u(y)))(u(x)-u(y))}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy\,dx.\end{aligned}$$

This identity in the special case when $f(u) = \frac{1}{2}u^2$ (see (5)), and the general case when f'_+ satisfy the hypotheses of this remark, combine to give the required result. □

Fréchet differentiability

Suppose now that f'' is locally Lipschitz. We will show that the Nemytskii operator F is continuously Fréchet differentiable on $\mathcal{H}^{1,1}(\mathbf{T}^1)$. For $u \in \mathcal{H}^{1,1}(\mathbf{T}^1)$, the obvious candidate for the Fréchet derivative of F at u is the linear operator L_u defined by the product

$$L_u v = v f'(u), \quad v \in \mathcal{H}^{1,1}(\mathbf{T}^1).$$

Proposition 11. *When f'' is locally Lipschitz the operator F on $\mathcal{H}^{1,1}(\mathbf{T}^1)$ is continuously Fréchet differentiable and L_u is the derivative of F at u.*

Proof. For $u \in \mathcal{H}^{1,1}(\mathbf{T}^1)$, $f' \circ u \in \mathcal{H}^{1,1}(\mathbf{T}^1)$ depends continuously on u, by Proposition 8. Hence, for $v \in \mathcal{H}^{1,1}(\mathbf{T}^1)$, the product $vf'(u) \in \mathcal{H}^{1,1}(\mathbf{T}^1)$ depends continuously on $u, v \in \mathcal{H}^{1,1}(\mathbf{T}^1)$, by Corollary 6. It remains only to show that L_u is the Fréchet derivative of F at u. In other words we have to show that when $\|v_n\|_{\mathcal{H}^{1,1}(\mathbf{T}^1)}\to 0$,

$$\lim_{n\to\infty}\frac{\|F(u+v_n)-F(u)-L_u v_n\|_{\mathcal{H}^{1,1}(\mathbf{T}^1)}}{\|v_n\|_{\mathcal{H}^{1,1}(\mathbf{T}^1)}}=0.$$

It is easy to see, from the intermediate value theorem and the hypothesis on f, that the mappings

$$u\longmapsto f(u),\quad u\longmapsto f'(u)u',\quad u\longmapsto f'(u)\mathcal{C}u'$$

are Fréchet differentiable from $\mathcal{H}^{1,1}(\mathbf{T}^1)$ into $L^1(\mathbf{T}^1)$ with derivatives

$$(6)\qquad v\longmapsto f'(u)v,\quad v\longmapsto f''(u)u'v+f'(u)v',\quad v\longmapsto (f''(u)\mathcal{C}u')v+f'(u)\mathcal{C}v'.$$

Therefore it suffices to show that the mapping $u\mapsto\mathcal{C}(f'(u)u')$ is Fréchet differentiable from $\mathcal{H}^{1,1}(\mathbf{T}^1)$ to $L^1(\mathbf{T}^1)$ with derivative

$$v\longmapsto \mathcal{C}((f''(u)u'v+f'(u)v').$$

However, because of the definition of $\mathcal{F}(u)$, given in (3), it suffices to show that $\mathcal{F}:\mathcal{H}^{1,1}(\mathbf{T}^1)\to L^1(\mathbf{T}^1)$ is Fréchet differentiable at u where, as in the proof of Proposition 3,

$$(7)\qquad \mathcal{F}(u)(x)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{G(u)(x,y)}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy.$$

Note first that $G(u)(x,y)=H(u(x),u(y))$ where, by Taylor's theorem,

$$H(a,b)=f(b)-f(a)-f'(a)(b-a)=\tfrac{1}{2}f''(\xi)(b-a)^2$$

for some ξ between a and b. Let

$$h(a,b)=\begin{cases}\dfrac{H(a,b)}{(b-a)^2}, & \text{if } a\neq b,\\[2mm] \tfrac{1}{2}f''(a), & \text{if } a=b.\end{cases}$$

Then h is continuous on $\mathbf{R}^2$, and continuously differentiable on the open set where $a\neq b$. At such points

$$\left.\frac{\partial h}{\partial b}\right|_{(a,b)}=\frac{H(a,b)-H(b,a)}{(a-b)^3}=\frac{1}{2}\frac{f''(\chi)-f''(\zeta)}{a-b},\qquad \chi,\zeta\in[a,b],$$

$$\left.\frac{\partial h}{\partial a}\right|_{(a,b)}=2\frac{H(a,b)-\frac{1}{2}f''(a)(b-a)^2}{(b-a)^3}=\frac{f''(\xi)-f''(a)}{b-a},\qquad \xi\in[a,b].$$

(Here $[a,b]$ denotes the closed interval with end-points a, b, whether $a \le b$ or not.) Since f'' is locally Lipschitz, it follows that ∇h is uniformly bounded on bounded sets of points (a,b) with $a \ne b$. Note that for $a \ne b$,

$$\frac{\partial h}{\partial b} = \frac{h(a,b)-h(b,a)}{a-b} \quad \text{and} \quad \frac{\partial h}{\partial a} = \frac{2}{b-a}(h(a,b)-h(a,a)). \tag{8}$$

For definiteness in formulae later we use the convention that $\nabla h(a,a)=(0,0)$. Now

$$\begin{aligned} &H(a+a',b+b')-H(a,b)-2(a-b)(a'-b')h(a,b)-(a-b)^2\nabla h(a,b)\cdot(a',b') \\ &\quad =(a-b)^2[h(a+a',b+b')-h(a,b)-\nabla h(a,b)\cdot(a',b')] \\ &\qquad +2(a-b)(a'-b')[h(a+a',b+b')-h(a,b)]+(a'-b')^2 h(a+a',b+b'). \end{aligned} \tag{9}$$

When $a=b$ and $(a',b')\in\mathbf{R}^2$, then

$$H(a+a',b+b')-H(a,b)=(a'-b')^2 h(a+a',b+b').$$

Now for $a \ne b$ and $(a',b')\in\mathbf{R}^2$ let

$$k(t)=h(a+ta',b+tb')-h(a,b)-t\nabla h(a,b)\cdot(a',b'), \quad t\in[0,1].$$

Then k is Lipschitz on $[0,1]$ and is continuously differentiable except possibly at one point $t\in[0,1]$. Therefore for $(a',b')\in\mathbf{R}^2$ and $a\ne b$

$$\begin{aligned} K_1(a,b,a',b') &:\equiv h(a+a',b+b')-h(a,b)-\nabla h(a,b)\cdot(a',b')=k(1)-k(0) \\ &= (a',b')\cdot\int_0^1 (\nabla h(a+ta',b+tb')-\nabla h(a,b))\,dt, \end{aligned} \tag{10}$$

where

$$\left|\int_0^1 (\nabla h(a+ta',b+tb')-\nabla h(a,b))\,dt\right|$$

is bounded for (a,b,a',b') in bounded sets in $\mathbf{R}^4$, and, by the dominated convergence theorem, converges to 0, as $(a',b')\to(0,0)$, for fixed $a\ne b$. Let $K_1(a,b,a',b')=0$ when $a=b$ and let

$$K_2(a,b,a',b')\equiv h(a+a',b+b')-h(a,b)\to 0, \quad \text{as } (a',b'),\to(0,0), \tag{11}$$

uniformly for (a,b) in bounded sets.

Therefore, by (9), for all $u, v \in \mathcal{H}^{1,1}(\mathbf{T}^1)$ and $x, y \in \mathbf{T}^1$,

$$\begin{aligned} H(u(x)+v(x),u(y)+v(y)) - H(u(x),u(y)) - 2(u(x)-u(y))(v(x)-v(y))h(u(x),u(y)) \\ -(u(x)-u(y))^2 \nabla h(u(x),u(y))\cdot(v(x),v(y)) \\ = (u(x)-u(y))^2 K_1(u(x),u(y),v(x),v(y)) \\ +2(u(x)-u(y))(v(x)-v(y))K_2(u(x),u(y),v(x),v(y)) \\ +(v(x)-v(y))^2 h(u(x)+v(x),u(y)+v(y)). \end{aligned}$$

It now follows, from Corollary 5, with (7), (8), (10), (11) and the dominated convergence theorem, followed by an integration by parts, that $\mathcal{F}\colon \mathcal{H}^{1,1}(\mathbf{T}^1) \to L^1(\mathbf{T}^1)$ is Fréchet differentiable at u with derivative

$$\begin{aligned} v \mapsto & \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{2(u(x)-u(y))(v(x)-v(y))h(u(x),u(y))}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy \\ & + \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{(u(x)-u(y))(h(u(x),u(y))-h(u(y),u(x)))}{\sin^2\left(\frac{1}{2}(x-y)\right)} v(y)\,dy \\ & - \frac{1}{2\pi} v(x) \int_{-\pi}^{\pi} \frac{(u(x)-u(y))(h(u(x),u(y))-h(u(x),u(x)))}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy \\ & = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{(u(x)-u(y))v(x)f''(u(x))+v(y)(f'(u(y))-f'(u(x)))}{\sin^2\left(\frac{1}{2}(x-y)\right)}\,dy \\ & = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(f''(u(x))v(x)-f''(u(y))v(y))u'(y)+(f'(u(x))-f'(u(y)))v'(y)}{\tan\left(\frac{1}{2}(x-y)\right)}\,dy \\ & = [f''(u)v\mathcal{C}u' + f'(u)\mathcal{C}v'](x) - [\mathcal{C}(f''(u)vu') + \mathcal{C}(f'(u)v')](x). \end{aligned}$$

In the light of (6), this is what is needed to conclude that $F\colon \mathcal{H}^{1,1}(\mathbf{T}^1) \to \mathcal{H}^{1,1}(\mathbf{T}^1)$ is Fréchet differentiable at u with derivative L_u. □

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