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# Melin–Hörmander inequality in a Wiener type pseudo-differential algebra

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**Abstract.** We prove a Melin–Hörmander inequality for a Banach algebra of pseudo-differential operators whose calculus was developed by Sjöstrand. The main new difficulties in the proof are settled by a stationary phase method tailored to the low-regularity of the symbols.

## Introduction and result

In a series of recent papers ([7], [8]), J. Sjöstrand studied a Wiener algebra of pseudo-differential operators. One of the remarkable features of this class is that the definition does not explicitly involve derivatives of the symbol but only properties of translation invariance in $\mathbf{R}^{2n}$ and Fourier transforms. The definition of this class $\Sigma$ could read as follows.

*Definition* 0.1. (1) We denote by $\Sigma$ the set of $a\in\mathcal{S}'(\mathbf{R}^{2n})$ such that

$$\sup_{Y\in\mathbf{R}^{2n}} |\widehat{\varphi_Y a}(X)| \in L^1(dX), \tag{1}$$

where $\varphi\in\mathcal{S}(\mathbf{R}^{2n})\setminus\{0\}$, $\varphi_Y(\,\cdot\,)=\varphi(\,\cdot\,-Y)$ for $Y\in\mathbf{R}^{2n}$, and $\hat{a}$ stands for the Fourier transform of $a$.

The class $\Sigma$ contains the familiar $S^0_{0,0}$ class, and Sjöstrand develops a calculus of pseudo-differential operators for this class: $L^2$ boundedness, composition, adjoints, sharp Gårding inequality, using a version of the stationary phase method. In [8], the author asks the natural question of Melin inequality for the class $\Sigma$. In the present paper, we give a positive answer to this question and we obtain in fact Hörmander's improvement of Melin's inequality with a gain of $\frac{6}{5}$ derivatives for the class $\Sigma$.

Going back to the proof of the sharp Gårding inequality in [8], one can note that the method of proof relies on the Fourier–Bros–Iagolnitzer transform, which

(1) See also Boulkhemair [2] for a different definition, and also Proposition 1 below.

provides a positive quantization, and also on a careful study of the remainders. It turns out that the remainders provided by the Fourier–Bros–Iagolnitzer method are still relevant for the class $\Sigma$, if one starts with a symbol with two derivatives in $\Sigma$. We shall follow an analogous course for our result. Hörmander's proof involves several remainders, all of them with an explicit expression as oscillatory integrals. We shall prove the stability of the class $\Sigma$ for these integrals. It would be interesting to characterize the class of integrals for which this phenomenon occurs. In particular it seems that the usual proof of the Fefferman–Phong inequality does *not* provide good remainders for the Sjöstrand class $\Sigma$. This fact is essentially due to the induction step and the implicit "bending" of the phase space related to it.

Let us now describe our results. We equip from now on $\mathbf{R}^{2n}$ with its canonical symplectic form $\sigma=\sum_{j=1}^{n} d\xi_j\wedge dx_j$. The dual $g^\sigma$ of a positive definite quadratic form on $\mathbf{R}^{2n}$ with respect to $\sigma$ is defined by $g^\sigma(T)=\sup_{g(Y)=1}\sigma(T,Y)^2$. We choose once and for all some positive quadratic form $\Gamma$ satisfying $\Gamma^\sigma=\Gamma$.

We recall that if $Q$ is a positive semi-definite quadratic form such that its polarized version is given by $Q(X,Y)=\sigma(X,FY)$, then the spectrum of $F/i$ is contained in $\mathbf{R}$, and $\operatorname{Tr}_+(Q)$ is the sum of its positive elements counted with multiplicities. We can then write the following lower-bound result for second-order real-valued polynomials. For such a polynomial $A$, we have (see [6])

$$\operatorname{Re} A(x,D)\geq 0 \iff \inf(A)+\tfrac{1}{2}\operatorname{Tr}_+(A'')\geq 0. \tag{2}$$

The Weyl quantization of a function $a\in\mathcal{S}(\mathbf{R}^{2n})$ is defined by

$$(a^w u)(x)=\frac{1}{(2\pi)^n}\iint_{\mathbf{R}^{2n}} e^{i(x-y,\xi)}a\Big(\frac{x+y}{2},\xi\Big)u(y)\,dy\,d\xi,\quad u\in\mathcal{S}(\mathbf{R}^n). \tag{3}$$

The Weyl quantization of an arbitrary element $a\in\mathcal{S}'(\mathbf{R}^{2n})$ is the continuous operator from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ with distribution kernel given by

$$\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x-y,\xi)}a\Big(\frac{x+y}{2},\xi\Big)\,d\xi, \tag{4}$$

where the integral is interpreted in the distribution sense. We can also define the $\sharp$-product, induced by the composition of symbols. For any $a_1$ and $a_2$ in $\mathcal{S}(\mathbf{R}^{2n})$ we have $(a_1\sharp a_2)^w=a_1^w\circ a_2^w$, where

$$(a_1\sharp a_2)(X)=\frac{1}{\pi^{2n}}\iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X-X_1,X-X_2)}a_1(X_1)a_2(X_2)\,dX_1\,dX_2. \tag{5}$$

We shall prove the following theorem, with a "gain" of $\frac{6}{5}$ derivatives, under a weak regularity assumption on the symbol.

**Theorem 0.2.** *Let $a\in W^{3,\infty}(\mathbf{R}^{2n})$ be real-valued and such that $a^{(3)}\in\Sigma$. Consider then the semi-classical family of symbols $\{a_\Lambda\}_{\Lambda\geq 1}$ where for all $\Lambda\geq 1$ we let $a_\Lambda=\Lambda^2 a(\Lambda^{-1/2}\cdot)$. Suppose that there exists $C_0\geq 1$ such that for all $t\geq 0$ and $\Lambda\geq 1$,*

$$a''_\Lambda+t\Gamma\geq 0 \quad\Longrightarrow\quad a_\Lambda+\tfrac{1}{2}\operatorname{Tr}_+(a''_\Lambda+C_0t\Gamma)\geq 0. \tag{6}$$

*Then there is $C'_0\geq 0$ such that $a^w_\Lambda+C'_0\Lambda^{4/5}\geq 0$ for all $\Lambda\geq 1$.*

The main part of this work is the very precise study of the number of derivatives of the symbol really needed in the study of remainders. It appears that we only need three derivatives of $a$, and that we can assume that $a^{(3)}$ is in $\Sigma$, without any more reference to derivatives. Naturally, here we cannot use the powerful tools of asymptotic quantization, as in the original paper of Hörmander [5, Theorem 6.2]. Nevertheless, the good behavior of the algebra $\Sigma$ under compositions of symbols is sufficient to get a similar result.

As in [5, Theorem 6.2], we shall use the fact that we can reduce the problem to the study of three terms associated to the symbol—a commutator term, an integral remainder term, and an oscillatory term to be treated using (2)—the first two of them being the remainders. During their study, we will also see the necessity of tailoring the phase space into conformal boxes of size $\Lambda^{1/10}$ (in place of $\Lambda^{1/2}$ in the semi-classical metric).

In the first part of this paper we give some properties of the class $\Sigma$. In particular we give alternative definitions of it and we study how its element behave under $\sharp$-product. In the second part, we give the proof of Theorem 0.2. At first we study the integral remainder term, then the commutator term, and eventually we recall briefly the method employed by Hörmander in [5, Theorem 6.2] for the oscillatory term, which can be used here without change.

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## 1. Miscellaneous properties of the class $\Sigma$

We refer to [7] and [8] for a detailed study of the class $\Sigma$ (see also [2] and [3]). We recall here some properties needed later. Here we first notice that $\Sigma$ is a Banach space for the norm

$$\|a\|_\Sigma=\int_{\mathbf{R}^{2n}}\sup_Y|\widehat{\varphi_Y a}(X)|\,dX,$$

depending on $\varphi\in\mathcal{S}(\mathbf{R}^{2n})\setminus\{0\}$. Since the natural duality on the phase space $\mathbf{R}^{2n}$ is induced by $\sigma$, we shall use the following definition of the Fourier transform on $\mathcal{S}'(\mathbf{R}^{2n})$

$$(7)\qquad \hat{a}(X)=(\mathcal{F}a)(X)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X.Y)}a(Y)\,dY,$$

where the integral is to be taken in the distribution sense. This *twisted* Fourier transform has properties similar to the usual one, in particular we have $\mathcal{F}^2=\mathrm{Id}$ and $\mathcal{F}$ is unitary on $L^2(\mathbf{R}^{2n})$.

### 1.1. An alternative description of the class $\Sigma$

The aim of this section is to give other definitions of the class $\Sigma$. In particular we see that its elements can also be thought of as averages of elements in $S^0=S^0_{0.0}$ multiplied by exponential functions.

**Proposition 1.1.** *Let $a\in\mathcal{S}'(\mathbf{R}^{2n})$. Then $a\in\Sigma$ if and only if any of the following five equivalent conditions is satisfied:*

(i) *There is a function $a^*\in L^1(\mathbf{R}^{2n})$ and a function $\varphi\in\mathcal{S}(\mathbf{R}^{2n})\setminus\{0\}$ such that*

$$|\widehat{\varphi_Y a}(X)|\le a^*(X),$$

*when $X,Y\in\mathbf{R}^{2n}$, where $\varphi_Y(X)=\varphi(X-Y)$.*

(ii) *There is for every $\varphi\in\mathcal{S}(\mathbf{R}^{2n})$ a function $a^*_\varphi\in L^1(\mathbf{R}^{2n})$ such that*

$$|\widehat{\varphi_Y a}(X)|\le a^*_\varphi(X),$$

*when $X,Y\in\mathbf{R}^{2n}$.*

(iii) *There is a function $a^*\in L^1(\mathbf{R}^{2n})$ such that*

$$|\widehat{\varphi_Y a}(X)|\le\Big(\sum_{|\alpha|\le 2n+1}\|\varphi^{(\alpha)}\|_{L^1}\Big)a^*(X),$$

*when $X,Y\in\mathbf{R}^{2n}$ and $\varphi\in\mathcal{S}(\mathbf{R}^{2n})$.*

(iv) *One may write*

$$a(X)=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X.Y)}\chi(X,Y)\,dY,$$

*where $\chi$ is a smooth function such that $v\int_{\mathbf{R}^{2n}}\sup_X|\partial_X^\alpha\chi(X,Y)|\,dY<\infty$ for every $\alpha\in\mathbf{N}^{2n}$.*

(v) *There is a measure space* $\Omega$ *together with measurable mappings*

$$\Omega \ni \omega \longmapsto Y_\omega \in \mathbf{R}^{2n} \quad \textit{and} \quad \Omega \ni \omega \longmapsto a_\omega \in S^0$$

*such that* $\int_\Omega p(a_\omega)\, d\omega < \infty$ *for every semi-norm* $p$ *on* $S^0$, *and such that*

$$a(X) = \int_\Omega e^{-2i\sigma(X,Y_\omega)} a_\omega(X)\, d\omega.$$

*Remark* 1.2. Recall that (i) is exactly Definition 0.1. Moreover, the norm in $\Sigma$ is comparable with the norm in $L^1(\mathbf{R}^{2n})$ of the functions $a^*$, $a^*_\varphi$ and $p(a_\omega)$ for suitable $p$.

Since multiplication of a symbol in $S^0$ by the function $X \mapsto e^{-2i\sigma(X,Y)}$, where $Y \in \mathbf{R}^{2n}$, does not change the norm in $L^2(\mathbf{R}^{2n})$ of its Weyl quantization, it is immediate from (iv) that $a^w(x,D)$ is continuous in $L^2(\mathbf{R}^n)$ (with a norm that can be estimated by the norm of $a$ in $\Sigma$) when $a \in \Sigma$.

*Proof.* We shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Let $\varphi \in \mathcal{S}(\mathbf{R}^{2n})$, and consider $\varphi_0 \in \mathcal{S}(\mathbf{R}^{2n}) \setminus \{0\}$ and $a_0^* \in L^1(\mathbf{R}^{2n})$ such that $|\widehat{\varphi_{0,Y}a}| \le a_0^*$ for all $Y \in \mathbf{R}^{2n}$. Consider further $\psi_0 \in \mathcal{S}(\mathbf{R}^{2n})$ such that $\int_{\mathbf{R}^{2n}} \psi_0\, dX = 1$ and such that $\operatorname{supp} \psi_0$ is contained in the interior of $\operatorname{supp} \varphi_0$. Let us first suppose that $a$ belongs to the Schwartz space $\mathcal{S}(\mathbf{R}^{2n})$. We can write, for $X \in \mathbf{R}^{2n}$,

$$\begin{aligned}\widehat{\varphi_Y a}(X) &= \frac{1}{\pi^n} \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Z)} \varphi(Z-Y) a(Z)\, dZ \\ &= \frac{1}{\pi^n} \iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X,Z)} \varphi(Z-Y) \psi_0(Z-T) a(Z)\, dT\, dZ.\end{aligned}$$

Using the condition on the supports, we get

$$\begin{aligned}\widehat{\varphi_Y a}(X) &= \frac{1}{\pi^n} \iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X,Z)} \varphi(Z-Y) \gamma_0(Z-T) \varphi_0(Z-T) a(Z)\, dT\, dZ \\ &= \frac{1}{\pi^{2n}} \iiint_{\mathbf{R}^{6n}} e^{-2i\sigma(X-U,Z)} \varphi(Z-Y) \gamma_0(Z-T) \widehat{\varphi_{0,T}a}(U)\, dU\, dT\, dZ,\end{aligned}$$

where $\gamma_0 = \psi_0/\varphi_0$ and $\varphi_{0,T} = \varphi_0(\cdot - T)$. A simple argument of approximation yields the same formula for arbitrary $a$ in $\mathcal{S}'(\mathbf{R}^{2n})$ (see the beginning of Subsection 1.3 below). Now choose a symplectic basis of $\mathbf{R}^{2n}$ in which $\mathbf{R}^{2n} \ni X = (X_1, \dots, X_{2n})$ and $\Gamma = \sum_{j=1}^{2n} |dX_j|^2$. Let us consider for $j \in \mathbf{N}_n = \{1, \dots, n\}$ the differential operators

$$\mathcal{P}_j = i + \frac{1}{2i} \partial_{Z_{j+n}}, \quad \mathcal{P}_{j+n} = i - \frac{1}{2i} \partial_{Z_j}, \tag{8}$$

for which we have,

$$\mathcal{P}_j e^{-2i\sigma(X-U,Z)} = (i+X_j-U_j)e^{-2i\sigma(X-U,Z)}, \quad j=1,\dots,2n,$$

and let us denote by ${}^t\mathcal{P}_j$ the transpose of $\mathcal{P}_j$. Consider now a finite partition of unity

$$(9) \qquad 1=\sum_{j=1}^{2n}\psi_j(X-U) \quad \text{for which} \quad (1+X_j^2)\geq\frac{\langle X\rangle^2}{2n} \text{ on } \operatorname{supp}\psi_j,$$

where $\langle X\rangle=(1+\Gamma(X))^{1/2}$. Using this and making $N=2n+1$ integrations by parts with the operators defined in (8), we can write

$$\widehat{\varphi_Y a}(X)=\frac{1}{\pi^{2n}} \times\sum_{j=1}^{2n}\iiint_{\mathbf{R}^{6n}}\frac{e^{-2i\sigma(X-U,Z)}}{(i+X_j-U_j)^N}\,{}^t\mathcal{P}_j^N\varphi(Z-Y)\gamma_0(Z-T)\psi_j(X-U)\widehat{\varphi_{0,T}a}(U)\,dU\,dT\,dZ.$$

Since $|\widehat{\varphi_{0,T}a}|\leq a_0^*$ for all $T\in\mathbf{R}^{2n}$, we get that there is a constant $C_{n,N}>0$ such that

$$|\widehat{\varphi_Y a}(X)|\leq C_{n,N}\sum_{j=1}^{2n}\iiint_{\mathbf{R}^{6n}}\frac{|{}^t\mathcal{P}_j^N\varphi(Z-Y)\gamma_0(Z-T)\psi_j(X-U)|a_0^*(U)}{\langle X-U\rangle^N}\,dU\,dT\,dZ.$$

If we integrate over $T$ and then over $Z$, and since $N=2n+1$, we get that there is a constant $C_n>0$ independent of $\varphi$ such that

$$(10) \qquad |\widehat{\varphi_Y a}(X)|\leq C_n\Big(\sum_{|\alpha|\leq 2n+1}\|\partial^\alpha\varphi\|_{L^1}\Big)\int_{\mathbf{R}^{2n}}\frac{a_0^*(U)}{\langle X-U\rangle^{2n+1}}\,dU.$$

The right-hand side $a_\varphi^*$ of (10) satisfies the conditions of (ii).

(ii) $\Rightarrow$ (iii) The proof is immediate from formula (10).

(iii) $\Rightarrow$ (iv) Let us choose $\varphi\in C_0^\infty(\mathbf{R}^{2n})$ supported in the ball $B$ of radius 1 for $\Gamma$ and such that $\int_{\mathbf{R}^{2n}}\varphi\,dX=1$. From (iii) we get that there is $a_\varphi^*\in L^1(\mathbf{R}^{2n})$ such that $|\widehat{\varphi_Y a}|\leq a_\varphi^*$ for all $Y\in\mathbf{R}^{2n}$. Now consider $\widetilde{\varphi}\in C_0^\infty(\mathbf{R}^{2n})$ equal to 1 on $B$. Then we can write, for $X\in\mathbf{R}^{2n}$,

$$\begin{aligned} a(X)&=\int_{\mathbf{R}^{2n}}\varphi(X-Z)a(X)\widetilde{\varphi}(X-Z)\,dZ \\ &=\frac{1}{\pi^n}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X,Y)}\widehat{\varphi_Z a}(Y)\widetilde{\varphi}(X-Z)\,dY\,dZ. \end{aligned}$$

Let us define

$$\chi(X,Y)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}}\widehat{\varphi_Z a}(Y)\tilde{\varphi}(X-Z)\,dZ.$$

We notice that $\chi(\cdot,Y)\in S^0$ for fixed $Y$, and that for all $\alpha\in\mathbf{N}^{2n}$, there exists $C_\alpha$ such that $|\partial_X^\alpha\chi(\cdot,Y)|\le C_\alpha a_\varphi^*(Y)$ for all $Y$, which implies (iv).

(iv) $\Rightarrow$ (v) This is immediate if we take $\omega\in\Omega=\mathbf{R}^{2n}\ni Y$ and $a_\omega=\chi(\cdot,\omega)\in S^0$.

(v) $\Rightarrow$ (i) Let $\varphi\in\mathcal{S}(\mathbf{R}^{2n})\setminus\{0\}$. From (v) we can write that

$$\widehat{\varphi_Y a}(X)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}}\int_\Omega e^{-2i\sigma(Z,Y_\omega)-2i\sigma(X,Z)}a_\omega(Z)\varphi(Z-Y_\omega)\,d\omega\,dZ.$$

Let us consider the following differential operator acting on $Z$

$$\mathcal{P}=1-\frac{1}{4}\sum_{j=1}^{2n}\partial_{Z_j}^2, \tag{11}$$

for which we have ${}^t\mathcal{P}=\mathcal{P}$ and, for all $X\in\mathbf{R}^{2n}$ and $\omega\in\Omega$,

$$\mathcal{P}e^{-2i\sigma(X-Y_\omega,Z)}=\langle X-Y_\omega\rangle^2 e^{-2i\sigma(X-Y_\omega,Z)}.$$

If we make $2n+2$ integrations by parts, we get

$$\widehat{\varphi_Y a}(X)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}}\int_\Omega \frac{e^{-2i\sigma(X-Y_\omega,Z)}\mathcal{P}^{n+1}\varphi(Z-Y_\omega)a_\omega(Z)}{\langle X-Y_\omega\rangle^{2n+2}}\,d\omega\,dZ.$$

If we take the modulus, we get that there is a semi-norm $p$ on $S^0$ such that

$$\begin{aligned}|\widehat{\varphi_Y a}(X)|&\le\int_{\mathbf{R}^{2n}}\int_\Omega\frac{p(a_\omega)}{\langle X-Y_\omega\rangle^{2n+2}}\Bigl(\sum_{|\alpha|\le 2n+2}|\partial_Z^\alpha\varphi(Z-Y_\omega)|\Bigr)\,d\omega\,dZ\\&\le C_\varphi\int_\Omega\frac{p(a_\omega)}{\langle X-Y_\omega\rangle^{2n+2}}\,d\omega\overset{\text{def}}{=}a^*(X).\end{aligned}$$

Obviously $a^*\in L^1(\mathbf{R}^{2n})$, so we have proved (i) and the proposition. □

### 1.2. Functional properties and the class $\Sigma$

At first we want to study the behavior of the class $\Sigma$ under dilation. This property, already noticed in [3, Proposition 3.2], will be used as such but is also a good preamble to the proofs coming later.

**Proposition 1.3.** *Let us consider $a\in\Sigma$. Then $a(\theta\,\cdot)\in\Sigma$ for all $\theta\in]0,1]$, and $a\mapsto a(\theta\,\cdot)$ is uniformly continuous with respect to $\theta\in]0,1]$ in $\Sigma$. In a similar way $a(\cdot-X_0)\in\Sigma$ for all $X_0\in\mathbf{R}^{2n}$ and $a\mapsto a(\cdot-X_0)$ is uniformly continuous with respect to $X_0$ in $\Sigma$.*

*Proof.* Let $\theta\in]0,1]$. The proof is immediate from Proposition 1.1 if one makes use of (iv) to write, for $X\in\mathbf{R}^{2n}$,

$$a(\theta X)=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(\theta X,Y)}\chi(\theta X,Y)\,dY=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y')}\chi_\theta(X,Y')\,dY', \tag{12}$$

where $\chi_\theta(X,Y')=\theta^{-2n}\chi(\theta X,Y'/\theta)$ satisfies

$$\sup_{\theta\in]0,1]}\int_{\mathbf{R}^{2n}} p(\chi_\theta(\cdot,Y'))\,dY'<\infty$$

for every semi-norm $p$ on $S^0$. Therefore $a(\theta\,\cdot)$ satisfies (iv) of Proposition 1.1 uniformly with respect to $\theta$. As for the second part of the proposition, it is enough to write, for all $X_0,X\in\mathbf{R}^{2n}$,

$$a(X-X_0)=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)}\chi_0(X,Y)\,dY, \tag{13}$$

where $\chi_0(X,Y)=e^{2i\sigma(X_0,Y)}\chi(X-X_0,Y)$ satisfies the uniform property

$$\sup_{X_0\in\mathbf{R}^{2n}} p(\chi_0(\cdot,Y))\le a_p^*(Y),\quad a_p^*\in L^1(\mathbf{R}^{2n}),$$

for every semi-norm $p$ on $S^0$. This is actually a manifestation of the translation invariance of $\Sigma$. It implies in particular the second part of the lemma, and the proof is complete. □

The following lemma enlightens the behavior of the class $\Sigma$ under derivation, and also clarifies the conditions on $a$ in the main theorem, Theorem 0.2.

**Lemma 1.4.** *Let $K\in\mathbf{N}$ and assume that $a\in W^{K,\infty}(\mathbf{R}^{2n})$ and that $a^{(K)}\in\Sigma$. Then $a^{(k)}\in\Sigma$ when $k\le K$ and for any $\varphi\in\mathcal{S}(\mathbf{R}^{2n})$, there is an $a^*_{k,\varphi}\in L^1(\mathbf{R}^{2n})$ such that*

$$|\widehat{\varphi_Y a^{(k)}}(X)|\le\langle X\rangle^{k-K}a^*_{k,\varphi}(X),\quad X,Y\in\mathbf{R}^{2n}. \tag{14}$$

*Proof.* We shall first prove this lemma in the case $K=1$ (i.e. $a'\in\Sigma$). We can write, for $\varphi\in\mathcal{S}(\mathbf{R}^{2n})$,

$$\widehat{\varphi_Y a}(X)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Z)}\varphi(Z-Y)a(Z)\,dZ.$$

If we use again the partition of unity $\{\psi_j\}_{j=1}^{2n}$ introduced in (9), we can write

$$\widehat{\varphi_Y a}(X)=\sum_{j=1}^{2n}\psi_j(X)\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Z)}\varphi(Z-Y)a(Z)\,dZ.$$

If we also use the differential operators $\{\mathcal{P}_j\}_{j=1}^{2n}$ defined in (8) and acting on the variable $Z$, for which we have in the related coordinates

$$\mathcal{P}_j e^{-2i\sigma(X,Z)}=(i+X_j)e^{-2i\sigma(X,Z)},\quad j=1,\dots,2n,$$

we get

$$\widehat{\varphi_Y a}(X)=\sum_{j=1}^{2n}\frac{\psi_j(X)}{i+X_j}\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Z)}\,{}^t\mathcal{P}_j(\varphi_Y a)(Z)\,dZ.$$

We first notice that for all $j\in\mathbf{N}_{2n}$ the function defined for all $X\in\mathbf{R}^{2n}$ by

$$\tilde{\psi}_j(X)\stackrel{\text{def}}{=}\psi_j(X)\frac{\langle X\rangle}{i+X_j}$$

belongs to $L^\infty(\mathbf{R}^{2n})\cap\mathcal{C}^\infty(\mathbf{R}^{2n})$. We can also notice that for all $j\in\mathbf{N}_{2n}$, there is a function $\widetilde{\varphi}_j\in\mathcal{S}(\mathbf{R}^{2n})$ such that, for all $Y\in\mathbf{R}^{2n}$,

$${}^t\mathcal{P}_j\varphi_Y a=\widetilde{\varphi}_{j,Y}a+\varphi_Y\tilde{\partial}_j a,$$

where $\tilde{\partial}_j=-(1/2i)\partial_{j+n}$ for $j\in\{1,\dots,n\}$ and $\tilde{\partial}_j=(1/2i)\partial_{j-n}$ for $j\in\{n+1,\dots,2n\}$. Therefore we can write

$$\begin{aligned}\widehat{\varphi_Y a}(X)&=\frac{1}{\pi^n\langle X\rangle}\sum_{j=1}^{2n}\tilde{\psi}_j(X)\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Z)}(\widetilde{\varphi}_{j,Y}a(Z)+\varphi_Y\tilde{\partial}_j a(Z))\,dZ\\ &=\frac{1}{\pi^n\langle X\rangle}\sum_{j=1}^{2n}\tilde{\psi}_j(X)\big(\widehat{\widetilde{\varphi}_{j,Y}a}(X)+\widehat{\varphi_Y\tilde{\partial}_j a}(X)\big).\end{aligned}\tag{15}$$

By hypothesis we have $\sup_{j,Y}|\widehat{\varphi_Y\tilde{\partial}_j a}|\in L^1(\mathbf{R}^{2n})$ so we only have to prove that the first term in (15) also belongs uniformly to $L^1(\mathbf{R}^{2n})$. For this it is sufficient to prove that $a\in\Sigma$. We shall prove this result in Lemma 1.5 below, and therefore the proof of Lemma 1.4 in the case $K=1$ is complete.

Let us now suppose that the lemma is proved for $K\in\mathbf{N}$, and consider $a\in W^{K+1,\infty}(\mathbf{R}^{2n})$ such that $a^{(K+1)}\in\Sigma$. As in formula (15), we can write, for $k\in\{0,\dots,K\}$,

$$\widehat{\varphi_Y a^{(k)}}(X)=\frac{1}{\pi^n\langle X\rangle}\sum_{j=1}^{2n}\tilde{\psi}_j(X)\big(\widehat{\widetilde{\varphi}_{j,Y}a^{(k)}}(X)+\widehat{\varphi_Y\tilde{\partial}_j a^{(k)}}(X)\big).\tag{16}$$

Now for $j\in\{1,\dots,2n\}$, $(\tilde\partial_j a)^{(K)}\in\Sigma$ and $\tilde\partial_j a\in W^{K,\infty}(\mathbf{R}^{2n})$, therefore we can apply the lemma at rank $K$ to $\tilde\partial_j a$ and we get that for every $k\in\{0,\dots,K\}$ there is a function $b^*_{j,k,\varphi}\in L^1(\mathbf{R}^{2n})$ such that

$$|\widehat{\varphi_Y\tilde\partial_j a^{(k)}}(X)|\le\langle X\rangle^{k-K}b^*_{j,k,\varphi}(X),\quad X,Y\in\mathbf{R}^{2n}. \tag{17}$$

Moreover, Lemma 1.5 implies that $a^{(K)}\in\Sigma$ and $a\in W^{K,\infty}(\mathbf{R}^{2n})$, therefore we also get that for every $k\in\{0,\dots,K\}$ there is a function $c^*_{j,k,\varphi}\in L^1(\mathbf{R}^{2n})$ such that

$$|\widehat{\tilde\varphi_{j,Y}a^{(k)}}(X)|\le\langle X\rangle^{k-K}c^*_{j,k,\varphi}(X),\quad X,Y\in\mathbf{R}^{2n}. \tag{18}$$

Formulas (16)–(18) complete the proof of the lemma. □

**Lemma 1.5.** *Assume that $a\in W^{1,\infty}(\mathbf{R}^{2n})$ and $a'\in\Sigma$. Then $a\in\Sigma$.*

*Proof.* We can write, for $\varphi\in\mathcal{S}(\mathbf{R}^{2n})$,

$$\widehat{\varphi_Y a}(X)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}}e^{-2i\sigma(X,Z)}\varphi(Z-Y)a(Z)\,dZ. \tag{19}$$

For $Y,Z\in\mathbf{R}^{2n}$ we can write

$$a(Z)=a(Y)+\left(\int_0^1 a'(Y+\theta(Z-Y))\,d\theta\right)\cdot(Z-Y).$$

Therefore we can split (19) into two terms $\widehat{\varphi_Y a}(X)=I_1(X,Y)+I_2(X,Y)$ with

$$I_1(X,Y)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}}e^{-2i\sigma(X,Z)}\varphi(Z-Y)a(Y)\,dZ=e^{-2i\sigma(X,Y)}\widehat\varphi(X)a(Y), \tag{20}$$

$$I_2(X,Y)=\frac{1}{\pi^n}\int_{\mathbf{R}^{2n}}e^{-2i\sigma(X,Z)}\varphi(Z-Y)\left(\int_0^1 a'(Y+\theta(Z-Y))\,d\theta\right)\cdot(Z-Y)\,dZ. \tag{21}$$

Since $a\in L^\infty(\mathbf{R}^{2n})$ and $\varphi\in\mathcal{S}(\mathbf{R}^{2n})$ we get $\sup_Y|I_1(\cdot,Y)|\le\|a\|_\infty|\widehat\varphi|\in L^1(\mathbf{R}^{2n})$.

As for $I_2$, Proposition 1.3 and the Banach space structure of $\Sigma$ imply that $Z\mapsto\int_0^1 a'(Y+\theta(Z-Y))\,d\theta\in\Sigma$ uniformly in $Y\in\mathbf{R}^{2n}$. Using (iv) in Proposition 1.1 we get that, for all $Y\in\mathbf{R}^{2n}$, there is a function $(Z,T)\mapsto\chi_Y(Z,T)$ such that

$$\int_0^1 a'(Y+\theta(Z-Y))\,d\theta=\int_{\mathbf{R}^{2n}}e^{-2i\sigma(Z,T)}\chi_Y(Z,T)\,dT,$$

where $Z\mapsto\chi_Y(Z,T)$ is smooth and satisfies the following strong estimate: for every semi-norm $p$ on $S^0$, there is an $a_p^*\in L^1(\mathbf{R}^{2n})$ such that

$$p(\chi_Y(\cdot,T))\le a_p^*(T)\quad\text{for all } Y,T\in\mathbf{R}^{2n}. \tag{22}$$

Let us denote by $\widetilde{\varphi}$ the function $\mathbf{R}^{2n} \ni X \mapsto \varphi(X)X \in \mathbf{R}^{2n}$. We can write, for all $X, Y \in \mathbf{R}^{2n}$,

$$I_2(X,Y) = \frac{1}{\pi^n} \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Z)} \left( \int_0^1 a'(Y+\theta(Z-Y))\, d\theta \right) \cdot \widetilde{\varphi}(Z-Y)\, dZ \tag{23}$$

$$= \frac{1}{\pi^n} \iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X-T,Z)} \chi_Y(Z,T) \cdot \widetilde{\varphi}(Z-Y)\, dZ\, dT. \tag{24}$$

Let us now again $n+1$ times use the differential operator $\mathcal{P}$ acting on the variable $Z$ defined in (11), for which we have

$$\mathcal{P} e^{-2i\sigma(X-T,Z)} = \langle X-T \rangle^2 e^{-2i\sigma(X-T,Z)}.$$

We get, for all $X, Y \in \mathbf{R}^{2n}$,

$$I_2(X,Y) = \frac{1}{\pi^n} \iint_{\mathbf{R}^{4n}} \frac{e^{-2i\sigma(X-Y_\omega,Z)} \mathcal{P}^{n+1} \chi_Y(Z,T) \cdot \widetilde{\varphi}(Z-Y)}{\langle X-T \rangle^{2n+2}}\, dT\, dZ.$$

If we take the modulus, we get that there is a semi-norm $p$ on $S^0$ such that

$$\begin{aligned} |I_2(X,Y)| &\le \iint_{\mathbf{R}^{4n}} \frac{p(\chi_Y(\,\cdot\,,T))}{\langle X-T \rangle^{2n+2}} \left( \sum_{|\alpha| \le 2n+2} |\partial_Z^\alpha \widetilde{\varphi}(Z-Y)| \right) dT\, dZ \\ &\le C_{\widetilde{\varphi}} \int_{\mathbf{R}^{2n}} \frac{a_p^*(T)}{\langle X-T \rangle^{2n+2}}\, dT \stackrel{\text{def}}{=} a_{\widetilde{\varphi}}^*(X). \end{aligned}$$

Since $a_{\widetilde{\varphi}}^* \in L^1(\mathbf{R}^{2n})$ we have proved the lemma. □

We summarize, for further reference, some important properties and embeddings of the class $\Sigma$.

**Lemma 1.6.** *The following embeddings are continuous:*

$$\mathcal{S}(\mathbf{R}^{2n}) \hookrightarrow S_{0,0}^0 \hookrightarrow \Sigma \hookrightarrow C^0(\mathbf{R}^{2n}) \cap L^\infty(\mathbf{R}^{2n}) \quad \text{and} \quad \mathcal{F}(L^1(\mathbf{R}^{2n})) \hookrightarrow \Sigma \hookrightarrow \mathcal{L}(L^2(\mathbf{R}^n)). \tag{25}$$

All these properties are immediate except for the last one: it means that the Weyl quantization of an element $a$ in $\Sigma$ is a bounded operator $a^w$ on $L^2(\mathbf{R}^n)$. Moreover this quantization is continuous from $\Sigma$ to $\mathcal{L}(L^2(\mathbf{R}^n))$. We refer to Section 3 of [7], and also Proposition 1.10 below or Remark 1.2 for details. We shall use these results in the study of the remainders introduced in (46). We now want to study the behavior of the class $\Sigma$ under the $\sharp$-product.

## 1.3. Symbolic calculus and the class $\Sigma$

The aim of this section is to prove Proposition 1.10 below. Let us first recall some results: In [7], Sjöstrand notices that $\mathcal{S}(\mathbf{R}^{2n})$ is not dense in $\Sigma$ for the norm, and introduces the notion of *narrow convergence* in $\Sigma$. We say that a sequence $a_\nu \in \Sigma$ converges narrowly to $a\in\Sigma$ if and only if $a_\nu \to a$ in $\mathcal{S}'(\mathbf{R}^{2n})$, and if there is a function $a^*\in L^1(\mathbf{R}^{2n})$ and a function $\varphi\in\mathcal{S}(\mathbf{R}^{2n})\setminus\{0\}$ such that

$$|\widehat{\varphi_Y a_\nu}(X)| \le a^*(X) \quad \text{for all } \nu,$$

when $X, Y\in\mathbf{R}^{2n}$, where $\varphi_Y(X)=\varphi(X-Y)$. It appears that $\mathcal{S}(\mathbf{R}^{2n})$ is dense in $\Sigma$ for the narrow convergence. Let us write

$$a(X) = \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)}\chi(X,Y)\,dY, \quad X\in\mathbf{R}^{2n},$$

where $\chi$ satisfies the hypothesis of (iv) in Proposition 1.1. As in [7], let us choose $\mathcal{S}(\mathbf{R}^{2n})\ni a_\nu\colon X\mapsto\psi(X/\nu)(\Phi_\nu * a)(X)$ for $\nu\in\mathbf{N}$, where $\psi,\Phi\in\mathcal{S}(\mathbf{R}^{2n})$, $\int_{\mathbf{R}^{2n}}\Phi\,dX=1$, $\psi(0)=1$ and $\Phi_\nu=\nu^{2n}\Phi(\nu\,\cdot)$. We can then write, for all $\nu$,

$$a_\nu(X) = \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)}\chi_\nu(X,Y)\,dY, \tag{26}$$

where one possible choice for $\chi_\nu$ is

$$\chi_\nu(X,Y) = \int_{\mathbf{R}^{2n}} e^{2i\sigma(Z,Y)}\psi\left(\frac{X}{\nu}\right)\Phi_\nu(Z)\chi(X-Z,Y)\,dZ. \tag{27}$$

It is then easy to check that

(i) $a_\nu$ converges narrowly to $a\in\Sigma$,
(ii) $\chi_\nu$ belongs to $L^1(\mathbf{R}^{4n})$ and $\chi_\nu(\,\cdot\,,Y)\in S^0$ uniformly with respect to $Y$,
(iii) for every semi-norm $p$ on $S^0$, there exists a semi-norm $p'$ such that

$$\int_{\mathbf{R}^{2n}} p(\chi_\nu(\,\cdot\,,Y))\,dY \le \int_{\mathbf{R}^{2n}} p'(\chi(\,\cdot\,,Y))\,dY, \quad \nu\in\mathbf{N}. \tag{28}$$

We have the following result.

**Theorem 1.7.** (Theorem 1.1 in [8]) *Assume $\Phi$ is a non-degenerate quadratic form on $\mathbf{R}^{2n}$. Then the convolution operator $a\mapsto e^{i\Phi}*a$ is bounded from $\Sigma$ to $\Sigma$ and is continuous in the sense of narrow convergence.*

*Remark* 1.8. Following [8] we can notice that the tensor product of two functions in $\Sigma=\Sigma(\mathbf{R}^{2n})$ is in the corresponding Sjöstrand class $\Sigma(\mathbf{R}^{4n})$, and that the restriction to an (even-dimensional) subspace $F \subset \mathbf{R}^{2n}$ of a function in $\Sigma$ is also in the corresponding Sjöstrand class. Using this and Theorem 1.7 we get that $ab=(a\otimes b)|_{\mathrm{Diag}}\in\Sigma$ and $a\sharp b=\pi^{-2n}(e^{-2i\sigma}*(a\otimes b))|_{\mathrm{Diag}}\in\Sigma$, where $\sigma$ is considered as a function on $\mathbf{R}^{2n}\times\mathbf{R}^{2n}$. We therefore get that $\Sigma$ is an algebra for the usual product *and* for the $\sharp$-product defined in (5).

We now want to state some results about the asymptotic expansion of (5). Let us first define $\widetilde{\mathcal{F}}=\mathcal{F}\otimes\mathcal{F}$, the twisted Fourier transform on $\mathcal{S}'(\mathbf{R}^{2n}\times\mathbf{R}^{2n})$. We have the immediate properties

$$\widetilde{\mathcal{F}}(u*v)=\pi^{2n}\widetilde{\mathcal{F}}u\widetilde{\mathcal{F}}v \quad\text{and}\quad \widetilde{\mathcal{F}}(e^{-2i\lambda\sigma})=\frac{e^{2i\sigma/\lambda}}{|\lambda|^{2n}}, \tag{29}$$

when $\lambda\in\mathbf{R}\setminus\{0\}$, where $*$ is the usual convolution on $\mathbf{R}^{4n}$, and $\sigma$ is considered as a function on $\mathbf{R}^{2n}\times\mathbf{R}^{2n}$. Using this we can write, for $a_1,a_2\in\mathcal{S}(\mathbf{R}^{2n})$ and $X\in\mathbf{R}^{2n}$,

$$(a_1\sharp a_2)(X)=\frac{1}{\pi^{2n}}(e^{-2i\sigma}*(a_1\otimes a_2))(X,X)=(\widetilde{\mathcal{F}}(e^{2i\sigma}(\hat a_1\otimes\hat a_2)))(X,X).$$

If we observe that $(1/2i)\sigma(\partial_X,\partial_Y)\circ\widetilde{\mathcal{F}}=\widetilde{\mathcal{F}}\circ(2i\sigma)$, where $\sigma$ to the right is the multiplication operator, and perform a Taylor expansion of the exponential at order $m-1$, we get

$$(a_1\sharp a_2)(X)=\sum_{p<m}\frac{1}{p!}\left(\frac{1}{2i}\sigma(\partial_{X_1},\partial_{X_2})\right)^p a_1(X_1)a_2(X_2)\bigg|_{X_1=X_2=X}+R_m(a_1,a_2)(X), \tag{30}$$

where

$$R_m(a_1,a_2)(X)=\int_0^1\frac{(1-\theta)^{m-1}}{(m-1)!}(\widetilde{\mathcal{F}}(e^{2i\theta\sigma}(2i\sigma)^m(\hat a_1\otimes\hat a_2)))(X,X)\,d\theta.$$

Using (29) we can write

$$\begin{aligned}R_m(a_1,a_2)(X)=\int_0^1&\frac{(1-\theta)^{m-1}}{(m-1)!(\theta\pi)^{2n}}\\ &\times\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X-X_1,X-X_2)/\theta}\left(\frac{1}{2i}\sigma(\partial_{X_1},\partial_{X_2})\right)^m a_1(X_1)a_2(X_2)\,dX_1\,dX_2\,d\theta.\end{aligned}$$

It is natural to introduce the following expression, for $\theta\in]0,1]$ and $X\in\mathbf{R}^{2n}$,

$$(a_1\sharp_\theta a_2)(X)=\frac{1}{(\theta\pi)^{2n}}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X-X_1,X-X_2)/\theta}a_1(X_1)a_2(X_2)\,dX_1\,dX_2. \tag{31}$$

Using the change of variables $Y_1=X+(X_1-X)/\theta$, $Y_2=X_2$, we also get

$$(a_1\sharp_\theta a_2)(X)=\frac{1}{\pi^{2n}}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X-Y_1,X-Y_2)}a_1(X+\theta(Y_1-X))a_2(Y_2)\,dY_1\,dY_2. \tag{32}$$

For further reference we now state some properties of the symbolic calculus in $S^0=S^0_{0,0}$.

**Lemma 1.9.** *Assume $a_1, a_2 \in S^0$. Then we have the following estimates:*

(i) *For any semi-norm $p$ on $S^0$, there is another semi-norm $p'$ on $S^0$ such that $p(a_1 \sharp_\theta a_2) \le p'(a_1) p'(a_2)$ for any $\theta \in ]0,1]$.*

(ii) *If $a_2 \in \mathcal{S}(\mathbf{R}^{2n})$, then $a_1 \sharp_\theta a_2 \in \mathcal{S}(\mathbf{R}^{2n})$ and for any semi-norm $q$ on $\mathcal{S}(\mathbf{R}^{2n})$, there is another semi-norm $q'$ on $\mathcal{S}$ and a semi-norm $p'$ on $S^0$ such that $q(a_1 \sharp_\theta a_2) \le p'(a_1) q'(a_2)$ for any $\theta \in ]0,1]$.*

(iii) *If we set $\tau_Y a(X) = e^{-2i\sigma(X,Y)} a(X)$ when $X, Y \in \mathbf{R}^{2n}$, then*

$$(\tau_{Y_1} a_1) \sharp_\theta (\tau_{Y_2} a_2) = e^{2i\theta\sigma(Y_1,Y_2)} \tau_{Y_1+Y_2} (a_1(\cdot + \theta Y_2) \sharp_\theta a_2(\cdot - \theta Y_1)), \tag{33}$$

*when $Y_1, Y_2 \in \mathbf{R}^{2n}$ and $\theta \in ]0,1]$.*

*Proof.* Let us first prove the first estimate. We shall use formula (32): Consider $a_1$ and $a_2$ in $\mathcal{S}(\mathbf{R}^{2n})$, and $\alpha \in \mathbf{N}^{2n}$. For all $X \in \mathbf{R}^{2n}$ and $\theta \in ]0,1]$ we can write

$$\begin{aligned}\partial_X^\alpha (a_1 \sharp_\theta a_2)(X) = \sum_{|\beta| \le |\alpha|} \frac{C_{\alpha,\beta}}{\pi^{2n}} \iint_{\mathbf{R}^{4n}} & (-2i\sigma(Y_1 - Y_2))^{\alpha-\beta} e^{-2i\sigma(X-Y_1, X-Y_2)} \\ & \times (\partial_X^\beta a_1)(X + \theta(Y_1 - X))(1-\theta)^{|\beta|} a_2(Y_2)\, dY_1\, dY_2,\end{aligned} \tag{34}$$

where $X^\alpha = \prod_{j=1}^{2n} X_j^{\alpha_j}$ and $\sigma(Y) = (-\eta, y)$ when $Y = (y, \eta)$. We can integrate by parts both in $Y_1$ and in $Y_2$ using the differential operators

$$\mathcal{P}_1 = \frac{1}{\langle X - Y_2 \rangle^2} (1 - \tfrac{1}{4} \Delta_{Y_1}), \quad \mathcal{P}_2 = \frac{1}{\langle X - Y_1 \rangle^2} (1 - \tfrac{1}{4} \Delta_{Y_2}). \tag{35}$$

If we denote the phase $-2\sigma(X - Y_1, X - Y_2)$ by $\Phi_X$, then we get

$$\mathcal{P}_1 e^{i\Phi_X} = e^{i\Phi_X} \quad \text{and} \quad \mathcal{P}_2 e^{i\Phi_X} = e^{i\Phi_X}. \tag{36}$$

Using each operator $N$ times in (34), we get

$$\begin{aligned}\partial_X^\alpha (a_1 \sharp_\theta a_2)(X) = \sum_{|\beta| \le |\alpha|} & \frac{C_{\alpha,\beta}(1-\theta)^{|\beta|}}{\pi^{2n}} \iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X-Y_1, X-Y_2)} \\ & \times {}^t\mathcal{P}_1^N\, {}^t\mathcal{P}_2^N ((-2i\sigma(Y_1 - Y_2))^{\alpha-\beta} (\partial_X^\beta a_1)(X + \theta(Y_1 - X)) a_2(Y_2))\, dY_1\, dY_2.\end{aligned} \tag{37}$$

Since $\langle Y_1 - Y_2 \rangle \le \sqrt{3} \langle X - Y_1 \rangle \langle X - Y_2 \rangle$, we get that for any $|\gamma| \le |\alpha|$, there is a constant $C_\gamma$ such that

$$|(-2i\sigma(Y_1 - Y_2))^\gamma| \le C_\gamma \langle X - Y_1 \rangle^{|\gamma|} \langle X - Y_2 \rangle^{|\gamma|}. \tag{38}$$

Using this in (37) and choosing $N \geq n+1+\frac{1}{2}|\alpha|$, we get that for all $\alpha$, there is a semi-norm $p'$ (independent of $\theta$) on $S^0$ such that

$$(39) \qquad |\partial_X^\alpha (a_1 \sharp_\theta a_2)(X)| \leq \iint_{\mathbf{R}^{4n}} \frac{p'(a_1)p'(a_2)}{\langle X-Y_2\rangle^{2n+2}\langle X-Y_1\rangle^{2n+2}} \, dY_1 \, dY_2,$$

and therefore the wished estimate.

The second estimate (ii) is also immediate from formula (37). Indeed, for all $N' \in \mathbf{N}$ and $\beta \in \mathbf{N}^{2n}$, there is a semi-norm $q'$ on $\mathcal{S}(\mathbf{R}^{2n})$ such that

$$\frac{|\partial_{Y_2}^\beta a_2(Y_2)|}{\langle X-Y_2\rangle^{N'}} \leq \frac{q'(a_2)}{\langle X\rangle^{N'}}.$$

Using this result and (38), we get from (37) that for all $\alpha \in \mathbf{N}^{2n}$ and $N' \in \mathbf{N}$, there is a semi-norm $p'$ on $S^0$ and a semi-norm $q'$ on $\mathcal{S}(\mathbf{R}^{2n})$ (independent of $\theta$) such that

$$(40) \qquad \left| \frac{\partial_X^\alpha (a_1 \sharp_\theta a_2)(X)}{\langle X\rangle^{N'}} \right| \leq \iint_{\mathbf{R}^{4n}} \frac{p'(a_1)q'(a_2)}{\langle X-Y_2\rangle^{2n+2}\langle X-Y_1\rangle^{2n+2}} \, dY_1 \, dY_2.$$

As for the third result (iii), the proof is due to a simple change of variables. Thus the proof of Lemma 1.9 is complete. □

A version of the following proposition is established in [8]. We here give a proof adapted to our methods of calculus.

**Proposition 1.10.** *The operation $\sharp_\theta$ is uniformly continuous with respect to $\theta \in ]0,1]$ from $\Sigma \times \Sigma$ to $\Sigma$.*

*Proof.* In the spirit of Remark 1.8, we have, for $a_1, a_2 \in \Sigma$ and $\theta \in ]0,1]$, $a_1 \sharp_\theta a_2 = (\theta\pi)^{-2n}(e^{-2i\sigma/\theta} * (a_1 \otimes a_2))|_{\mathrm{Diag}} \in \Sigma$, where $\sigma$ is considered as a function on $\mathbf{R}^{2n} \times \mathbf{R}^{2n}$. Theorem 1.7 implies that $\sharp_\theta$ is continuous for the narrow convergence from $\Sigma \times \Sigma$ to $\Sigma$. We want to prove that it is uniformly bounded with respect to $\theta \in ]0,1]$. For this let us first choose $a_1, a_2 \in \mathcal{S}(\mathbf{R}^{2n})$ and $\chi_1, \chi_2 \in L^1(\mathbf{R}^{4n})$ such that

$$a_j(X) = \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)} \chi_j(X,Y) \, dY, \quad j=1,2,$$

and satisfying $\int_{\mathbf{R}^{2n}} p(\chi_j(\cdot, Y)) \, dY \leq C_p(a_j)$ for any semi-norm $p$ on $S^0$. We can write, for $X \in \mathbf{R}^{2n}$ and $\theta \in ]0,1]$,

$$(41) \quad (a_1 \sharp_\theta a_2)(X) = \frac{1}{\pi^{2n}} \iiiint_{\mathbf{R}^{8n}} e^{-2i\sigma(X-X_1, X-X_2)/\theta} \tau_{Y_1}\chi_1(X_1, Y_1) \tau_{Y_2}\chi_2(X_2, Y_2) \, dY_1 \, dY_2 \, dX_1 \, dX_2.$$

Fubini's theorem implies then that

$$(a_1\sharp_\theta a_2)(X)=\iint_{\mathbf{R}^{4n}} \tau_{Y_1}\chi_1(\cdot\,.Y_1)\sharp_\theta\tau_{Y_2}\chi_2(\cdot\,,Y_2)\,dY_1\,dY_2,$$

where $\tau_Y$ is defined in (iii) of Lemma 1.9. The same lemma gives

$$a_1\sharp_\theta a_2(X)=\iint_{\mathbf{R}^{4n}} e^{-2i\sigma(Y_1+Y_2,X)}\chi_\theta(X,Y_1,Y_2)\,dY_1\,dY_2, \tag{42}$$

where $\chi_\theta(X,Y_1,Y_2)=e^{2i\theta\sigma(Y_1,Y_2)}\chi_1(\cdot+\theta Y_2,Y_1)\sharp_\theta\chi_2(\cdot-\theta Y_1,Y_2)$. From part (i) of Lemma 1.9, we get that for any semi-norm $p$ on $S^0$, there is a semi-norm $p'$ on $S^0$ such that

$$p(\chi_\theta(\cdot\,,Y_1,Y_2))\le p'(\chi_1(\cdot\,,Y_1))p'(\chi_2(\cdot\,,Y_2)) \quad \text{for all } Y_1,Y_2,\theta. \tag{43}$$

In (42) we recognize the form (v) of Proposition 1.1 with $\Omega=\mathbf{R}^{4n}$, $\omega=(Y_1,Y_2)$ and $Y_\omega=Y_1+Y_2$. Since the estimate in (43) is uniform in $\theta$, we have proved that the mappings

$$\mathcal{S}(\mathbf{R}^{2n})\times\mathcal{S}(\mathbf{R}^{2n})\ni(a_1,a_2)\longmapsto a_1\sharp_\theta a_2\in\Sigma$$

is uniformly continuous (in $\theta$) when $\mathcal{S}(\mathbf{R}^{2n})\times\mathcal{S}(\mathbf{R}^{2n})$ is equipped with the topology from $\Sigma\times\Sigma$.

Now let us choose sequences $a_{1,\nu},a_{2,\nu}\in\mathcal{S}(\mathbf{R}^{2n})$ converging narrowly to $a_1,a_2\in\Sigma$, respectively, and $\chi_{1,\nu},\chi_{2,\nu}\in L^1(\mathbf{R}^{4n})$ related to $a_{1,\nu}$ and $a_{2,\nu}$, respectively, as in (26) and (27). From (42) and (43) applied to $a_{1,\nu}$ and $a_{2,\nu}$, and (28) in the beginning of Section 1.3 we get that $a_{1,\nu}\sharp_\theta a_{2,\nu}$ converges narrowly to $a_1\sharp_\theta a_2\in\Sigma$, and

$$\|a_1\sharp_\theta a_2\|_\Sigma\le C\|a_1\|_\Sigma\|a_2\|_\Sigma$$

with $C>0$ independent of $\theta$. The proof of Proposition 1.10 is now complete. □

## 2. Proof of the main theorem

### 2.1. Beginning of the proof

Let $a\in W^{3,\infty}(\mathbf{R}^{2n})$ such that $a^{(3)}\in\Sigma$. Consider the family of symbols $\{a_\Lambda\}_{\Lambda\ge1}$, where we write $a_\Lambda=\Lambda^2a(\Lambda^{-1/2}\cdot)$ for $\Lambda\ge1$. Now, for $\varepsilon\in]0,1]$ to be chosen later, let us introduce a continuous partition of unity of the phase space $\mathbf{R}^{2n}$ for the semi-classical family of metrics depending on $\Lambda\ge1$,

$$g_\varepsilon=\Lambda^{-2\varepsilon}\Gamma. \tag{44}$$

More precisely, as $\Gamma$ we choose a non-negative even function $\varphi$ supported in the ball of radius 1 such that $\int_{\mathbf{R}^{2n}} \varphi^2(s)\,ds=1$, and we write

$$\varphi_{\varepsilon Z}(X)=\varphi(\Lambda^{-\varepsilon}(X-Z)). \tag{45}$$

We shall use the following important property established in [5, Lemma 6.6], for which we give a direct proof.

**Lemma 2.1.** *Let $\tilde{\varphi},\tilde{\psi}\in\mathcal{S}(\mathbf{R}^{2n})$ with $\tilde{\varphi}_{\varepsilon Z}$ and $\tilde{\psi}_{\varepsilon Z}$ associated in the same way as in* (45). *Then we have $\int_{\mathbf{R}^{2n}}(\overline{\tilde{\varphi}_{\varepsilon Z}}\sharp\tilde{\psi}_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ=(\tilde{\psi},\tilde{\varphi})_{L^2(\mathbf{R}^{2n})}$ for $X\in\mathbf{R}^{2n}$.*

*Proof.* Let us first make the following simple observations:

(i) $\int_{\mathbf{R}^{2n}}(a_1\sharp a_2)(X)\,dX=\int_{\mathbf{R}^{2n}}a_1(X)a_2(X)\,dX$ for all $a_1,a_2\in\mathcal{S}(\mathbf{R}^{2n})$;

(ii) the mapping $(a_1,a_2)\mapsto a_1\sharp a_2$ commutes with translations.

The second result is due to a simple change of variables. As for the first one, we can write

$$\begin{aligned}\int_{\mathbf{R}^{2n}}(a_1\sharp a_2)(X)\,dX&=\frac{1}{\pi^{2n}}\iiint_{\mathbf{R}^{6n}}e^{-2i\sigma(X-X_1,X-X_2)}a_1(X_1)a_2(X_2)\,dX_1\,dX_2\,dX\\&=\frac{1}{\pi^{n}}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X,X-X_2)}\hat{a}_1(X-X_2)a_2(X_2)\,dX_2\,dX.\end{aligned}$$

If we use the change of variable $Y=X-X_2$ we get

$$\begin{aligned}\int_{\mathbf{R}^{2n}}(a_1\sharp a_2)(X)\,dX&=\frac{1}{\pi^{n}}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X,Y)}\hat{a}_1(Y)a_2(X_2)\,dX_2\,dY\\&=\int_{\mathbf{R}^{2n}}a_1(X)a_2(X)\,dX.\end{aligned}$$

Now assume $\tilde{\varphi},\tilde{\psi}\in\mathcal{S}(\mathbf{R}^{2n})$. We can write, for all $X\in\mathbf{R}^{2n}$,

$$\begin{aligned}\int_{\mathbf{R}^{2n}}(\overline{\tilde{\varphi}_{\varepsilon Z}}\sharp\tilde{\psi}_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ&=\int_{\mathbf{R}^{2n}}(\overline{\tilde{\varphi}}(\Lambda^{-\varepsilon}(\,\cdot\,-Z))\sharp\tilde{\psi}(\Lambda^{-\varepsilon}(\,\cdot\,-Z)))(X)\Lambda^{-2n\varepsilon}\,dZ\\&=\int_{\mathbf{R}^{2n}}(\overline{\tilde{\varphi}}(\Lambda^{-\varepsilon}\cdot)\sharp\tilde{\psi}(\Lambda^{-\varepsilon}\cdot))(X-Z)\Lambda^{-2n\varepsilon}\,dZ\end{aligned}$$

from (ii), and if we let $Y=X-Z$ we get

$$\begin{aligned}\int_{\mathbf{R}^{2n}}(\overline{\tilde{\varphi}_{\varepsilon Z}}\sharp\tilde{\psi}_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ&=\int_{\mathbf{R}^{2n}}(\overline{\tilde{\varphi}}(\Lambda^{-\varepsilon}\cdot)\sharp\tilde{\psi}(\Lambda^{-\varepsilon}\cdot))(Y)\Lambda^{-2n\varepsilon}\,dY\\&=\int_{\mathbf{R}^{2n}}\overline{\tilde{\varphi}}(\Lambda^{-\varepsilon}Y)\tilde{\psi}(\Lambda^{-\varepsilon}Y)\Lambda^{-2n\varepsilon}\,dY\\&=\int_{\mathbf{R}^{2n}}\overline{\tilde{\varphi}}(Y)\tilde{\psi}(Y)\,dY\\&=(\tilde{\psi},\tilde{\varphi})_{L^2(\mathbf{R}^{2n})},\end{aligned}$$

where the second equality is due to (i) above. The proof is complete. $\square$

We can then apply this lemma to the function $\varphi$, and we obtain that, for all $X$,

$$a_\Lambda(X)=\int_{\mathbf{R}^{2n}}(a_\Lambda\sharp\varphi_{\varepsilon Z}\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ=R_\Lambda(X)+S_\Lambda(X)+A_\Lambda(X). \tag{46}$$

The term $S_\Lambda$ is a commutator,

$$S_\Lambda(X)=\int_{\mathbf{R}^{2n}}((a_\Lambda\sharp\varphi_{\varepsilon Z}-\varphi_{\varepsilon Z}\sharp a_\Lambda)\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ. \tag{47}$$

As for the other ones, the sum $A_\Lambda(X)+R_\Lambda(X)=\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon Z}\sharp a_\Lambda\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ$ is a splitting according to a Taylor expansion of $a_\Lambda$ at order 2: let us write, for all $X,Z\in\mathbf{R}^{2n}$,

$$\begin{aligned} a_{\Lambda,Z}(X)&=a_\Lambda(Z)+a'_\Lambda(Z)(X-Z)+\tfrac{1}{2}a''_\Lambda(Z)(X-Z)^2,\\ R_{\Lambda,Z}(X)&=\frac{1}{2}\int_0^1 a^{(3)}(Z+\theta(X-Z))(X-Z)^3(1-\theta)^2\,d\theta.\end{aligned} \tag{48}$$

We define

$$A_\Lambda(X)=\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon Z}\sharp a_{\Lambda,Z}\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ\quad\text{(oscillatory term)}, \tag{49}$$

$$R_\Lambda(X)=\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon Z}\sharp R_{\Lambda,Z}\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ\quad\text{(integral remainder term)}. \tag{50}$$

Expressions (47), (49) and (50) lead to formula (46). We shall begin with studying the remainders $S_\Lambda$ and $R_\Lambda$.

## 2.2. Study of the integral remainder

We shall establish the following result.

**Lemma 2.2.** *The remainder $R_\Lambda$ defined in* (50) *belongs to the class $\Sigma$. Moreover there is $C_1$ such that we have $\|R_\Lambda\|_\Sigma\le C_1\Lambda^{1/2+3\varepsilon}$ for all $\varepsilon\in]0,1]$ and $\Lambda\ge1$.*

*Proof.* By $f_{\Lambda,Z}$ we denote, from now on, the function

$$X\longmapsto f_{\Lambda,Z}(X)=\Lambda^{-1/2}\frac{1}{2}\int_0^1(1-\theta)^2a_\Lambda^{(3)}(Z+\theta(X-Z))\,d\theta \tag{51}$$

which is uniformly in $\Sigma$ with respect to $\Lambda\ge1$ and $Z\in\mathbf{R}^{2n}$, by Proposition 1.3 and the fact that $\Sigma$ is a Banach space. We can therefore write

$$R_\Lambda(X)=\Lambda^{1/2+3\varepsilon}\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon Z}\sharp(f_{\Lambda,Z}(\cdot)(\Lambda^{-\varepsilon}(\cdot-Z))^3)\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ. \tag{52}$$

To obtain the result it is sufficient to study the term inside the preceding integral. Let us state a first lemma.

**Lemma 2.3.** *Let $b\in\mathcal{S}'(\mathbf{R}^{2n})$, $a,c\in\mathcal{S}(\mathbf{R}^{2n})$. Then, for all $X\in\mathbf{R}^{2n}$,*

$$(a\sharp b\sharp c)(X)=\int_{\mathbf{R}^{2n}}\Psi_{a,c}\big(X-\tfrac{1}{2}Y,Y\big)b(X-Y)\,dY,$$

*where the integral is interpreted in the distribution sense, and $\Psi_{a,c}\in\mathcal{S}(\mathbf{R}^{4n})$ is defined by*

$$\Psi_{a,c}(U,T)=\frac{1}{\pi^{2n}}\int_{\mathbf{R}^{2n}}e^{2i\sigma(T,Y')}a(U-Y')c(U+Y')\,dY'. \tag{53}$$

*Proof.* It is sufficient to prove the preceding decomposition for $b\in\mathcal{S}(\mathbf{R}^{2n})$. We can write, for all $X_4\in\mathbf{R}^{2n}$,

$$\begin{aligned}(a\sharp b)(X_4)&=\frac{1}{\pi^{2n}}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X_4-X_1,X_4-X_2)}a(X_1)b(X_2)\,dX_1\,dX_2\\&=\frac{1}{\pi^{n}}\int_{\mathbf{R}^{2n}}e^{-2i\sigma(X_2,X_4)}\hat{a}(X_4-X_2)b(X_2)\,dX_2,\end{aligned}$$

therefore, for all $X\in\mathbf{R}^{2n}$, letting $Y_1=X_4-X_2$,

$$\begin{aligned}&(a\sharp b\sharp c)(X)=\frac{1}{\pi^{2n}}\iint_{\mathbf{R}^{4n}}e^{-2i\sigma(X-X_4,X-X_3)}a\sharp b(X_4)c(X_3)\,dX_3\,dX_4\\&=\frac{1}{\pi^{3n}}\iiint_{\mathbf{R}^{6n}}e^{-2i\sigma(X-X_4,X-X_3)-2i\sigma(X_2,X_4)}\hat{a}(X_4-X_2)b(X_2)c(X_3)\,dX_2\,dX_3\,dX_4\\&=\frac{1}{\pi^{3n}}\iiint_{\mathbf{R}^{6n}}e^{-2i\sigma(X_2+X-X_3,Y_1)}\hat{a}(Y_1)e^{-2i\sigma(X-X_2,X-X_3)}b(X_2)c(X_3)\,dY_1\,dX_2\,dX_3\\&=\frac{1}{\pi^{2n}}\iint_{\mathbf{R}^{4n}}a(X_2+X-X_3)e^{-2i\sigma(X-X_2,X-X_3)}b(X_2)c(X_3)\,dX_2\,dX_3.\end{aligned}$$

If we set $Y=X-X_2$ and $Y'=X_3-\frac{1}{2}(X+X_2)$, we get

$$a\sharp b\sharp c(X)=\int_{\mathbf{R}^{2n}}\Psi_{a,c}\big(X-\tfrac{1}{2}Y,Y\big)b(X-Y)\,dY,$$

where $\Psi_{a,c}$ has the desired form (53). $\square$

Now we can use this lemma to transform the integrand in (52):

$$\begin{aligned}&(\varphi_{\varepsilon Z}\sharp(f_{\Lambda,Z}(\cdot)(\Lambda^{-\varepsilon}(\cdot-Z))^3)\sharp\varphi_{\varepsilon Z})(X)\\&\qquad=\int_{\mathbf{R}^{2n}}\Psi_{\varphi_{\varepsilon Z},\varphi_{\varepsilon Z}}\big(X-\tfrac{1}{2}Y,Y\big)f_{\Lambda,Z}(X-Y)(\Lambda^{-\varepsilon}(X-Y-Z))^3\,dY\end{aligned} \tag{54}$$

in the terminology of Lemma 2.3 for $\Psi_{\varphi_{\varepsilon Z},\varphi_{\varepsilon Z}}$. If we set $\Psi=\Psi_{\varphi,\varphi}$, a simple computation gives, for all $U,T,Z\in\mathbf{R}^{2n}$,

$$\Psi_{\varphi_{\varepsilon Z},\varphi_{\varepsilon Z}}(U,T)=\Lambda^{2n\varepsilon}\Psi(\Lambda^{-\varepsilon}(U-Z),\Lambda^{\varepsilon}T). \tag{55}$$

We know that $f_{\Lambda,Z}$ is uniformly in $\Sigma$ with respect to $Z\in\mathbf{R}^{2n}$ and $\Lambda\geq 1$, but in fact we have a much stronger result: following (iv) in Proposition 1.1, we can write

$$f_{\Lambda,Z}(X)=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y')}\chi_{\Lambda,Z}(X,Y')\,dY', \tag{56}$$

where $\chi_{\Lambda,Z}\in S^0$ and where, for every $p$ semi-norm in $S^0$ and $\Lambda\geq 1$, there is a function $a^*_{p,\Lambda}\in L^1(\mathbf{R}^{2n})$ such that $p(\chi_{\Lambda,Z}(\,\cdot\,,Y'))\leq a^*_{p,\Lambda}(Y')$ and $\sup_{\Lambda\geq 1}\int_{\mathbf{R}^{2n}} a^*_{p,\Lambda}(Y')\,dY'<\infty$. We can then write from (54), (55) and (56),

$$\begin{aligned}(\varphi_{\varepsilon Z}\sharp(f_{\Lambda,Z}(\,\cdot\,)(\Lambda^{-\varepsilon}(\cdot-Z))^3)\sharp\varphi_{\varepsilon Z})(X)=&\iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X-Y,Y')}\chi_{\Lambda,Z}(X-Y,Y')\\ &\times(\Lambda^{-\varepsilon}(X-Y-Z))^3\Psi\big(\Lambda^{-\varepsilon}\big(X-\tfrac12 Y-Z\big),\Lambda^{\varepsilon}Y\big)\Lambda^{2n\varepsilon}\,dY\,dY'.\end{aligned} \tag{57}$$

Since $\Psi\in\mathcal{S}(\mathbf{R}^{4n})$, we get that for all $N\in\mathbf{N}$ there are two constants $C_N$ and $C'_N$ such that

$$\begin{aligned}\Psi\big(\Lambda^{-\varepsilon}\big(X-\tfrac12 Y-Z\big),\Lambda^{\varepsilon}Y\big)&\leq\frac{C_N}{\big\langle\Lambda^{-\varepsilon}\big(X-\frac12 Y-Z\big)\big\rangle^{N+3}\langle\Lambda^{\varepsilon}Y\rangle^{2N+3}}\\ &\leq\frac{C'_N}{\langle\Lambda^{-\varepsilon}(X-Y-Z)\rangle^3\langle\Lambda^{\varepsilon}Y\rangle^N\langle\Lambda^{-\varepsilon}(X-Z)\rangle^N},\end{aligned} \tag{58}$$

where the second inequality comes from the immediate inequalities

$$\begin{aligned}&\big\langle\Lambda^{-\varepsilon}\big(X-\tfrac12 Y-Z\big)\big\rangle\langle\Lambda^{\varepsilon}Y\rangle\geq C\langle\Lambda^{-\varepsilon}(X-Z)\rangle,\\ &\big\langle\Lambda^{-\varepsilon}\big(X-\tfrac12 Y-Z\big)\big\rangle\langle\Lambda^{\varepsilon}Y\rangle\geq C\langle\Lambda^{-\varepsilon}(X-Y-Z)\rangle\end{aligned}$$

with $C>0$. Hence

$$\begin{aligned}\big|e^{2i\sigma(Y,Y')}\chi_{\Lambda,Z}(X-Y,Y')(\Lambda^{-\varepsilon}(X-Y-Z))^3\Psi\big(\Lambda^{-\varepsilon}\big(X-\tfrac12 Y-Z\big),\Lambda^{\varepsilon}Y\big)\big|&\\ \leq\frac{a^*_{0,\Lambda}(Y')}{\langle\Lambda^{\varepsilon}Y\rangle^N\langle\Lambda^{-\varepsilon}(X-Z)\rangle^N}&.\end{aligned} \tag{59}$$

We can therefore write

$$\begin{aligned}&\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon Z}\sharp(f_{\Lambda,Z}(\,\cdot\,)(\Lambda^{-\varepsilon}(\,\cdot\,-Z))^3)\sharp\varphi_{\varepsilon Z})(X)\Lambda^{-2n\varepsilon}\,dZ\\ &\qquad=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y')}\widetilde{\chi}_{\Lambda}(X,Y')\,dY',\end{aligned} \tag{60}$$

with, for all $X, Y' \in \mathbf{R}^{2n}$ and $\Lambda \geq 1$,

$$\tag{61} \begin{aligned} \widetilde{\chi}_\Lambda(X, Y') = \iint_{\mathbf{R}^{4n}} & e^{2i\sigma(Y, Y')} \chi_{\Lambda, Z}(X - Y, Y') (\Lambda^{-\varepsilon}(X - Y - Z))^3 \\ & \times \Psi\big(\Lambda^{-\varepsilon}(X - \tfrac{1}{2}Y - Z), \Lambda^{\varepsilon} Y\big)\, d(\Lambda^{-\varepsilon} Z)\, d(\Lambda^{\varepsilon} Y). \end{aligned}$$

Now (59) with $N = 2n+1$ shows that $Y' \mapsto \sup_X |\widetilde{\chi}_\Lambda(X, Y')|$ belongs to $L^1(dY')$ uniformly with respect to $\Lambda \geq 1$. An analogous study gives the same result for any derivative with respect to $X$ of $\widetilde{\chi}_\Lambda$. We recognize the fourth characterization in Proposition 1.1. This completes the proof. □

### 2.3. Study of the commutator term

Let us now study the commutator term $S_\Lambda$ in (46). We shall establish the following result.

**Lemma 2.4.** *The remainder $S_\Lambda$ belongs to the class $\Sigma$. Moreover there exists $C_2$ such that we have $\|S_\Lambda\|_\Sigma \leq C_2 \Lambda^{1-2\varepsilon}$ for all $\varepsilon \in ]0, 1]$ and $\Lambda \geq 1$.*

*Proof.* Let us recall that the symbol of $S_\Lambda$ is defined in (47) by the formula

$$\tag{62} S_\Lambda(X) = \int_{\mathbf{R}^{2n}} ((a_\Lambda \sharp \varphi_{\varepsilon Z} - \varphi_{\varepsilon Z} \sharp a_\Lambda) \sharp \varphi_{\varepsilon Z})(X) \Lambda^{-2n\varepsilon}\, dZ.$$

We start using the asymptotic formula (30) at order 2 and get

$$\tag{63} a_\Lambda \sharp \varphi_{\varepsilon Z} - \varphi_{\varepsilon Z} \sharp a_\Lambda = \frac{1}{i}\{a_\Lambda, \varphi_{\varepsilon Z}\} + U_{\Lambda, Z},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket, and where, for all $Z, X \in \mathbf{R}^{2n}$ and $\Lambda \geq 1$,

$$\tag{64} \begin{aligned} U_{\Lambda, Z}(X) &= 2\,\mathrm{Im} \int_0^1 \frac{1-\theta}{(\pi\theta)^{2n}} \iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X - X_1, X - X_2)/\theta} \\ &\qquad \times \left(\frac{1}{2i}\sigma(\partial_{X_1}, \partial_{X_2})\right)^2 a_\Lambda(X_1) \varphi_{\varepsilon Z}(X_2)\, dX_1\, dX_2\, d\theta \\ &= 2\,\mathrm{Im} \sum_{|\alpha| + |\beta| = 2} C_{\alpha, \beta} \int_0^1 (1-\theta)(\partial_x^\alpha \partial_\xi^\beta a_\Lambda \sharp_\theta \partial_\xi^\alpha \partial_x^\beta \varphi_{\varepsilon Z})(X)\, d\theta \end{aligned}$$

for some constants $C_{\alpha, \beta}$. We can therefore split $S_\Lambda$ into two parts

$$\tag{65} S_\Lambda(X) = \frac{1}{i} \int_{\mathbf{R}^{2n}} (\{a_\Lambda, \varphi_{\varepsilon Z}\} \sharp \varphi_{\varepsilon Z})(X) \Lambda^{-2n\varepsilon}\, dZ + \int_{\mathbf{R}^{2n}} (U_{\Lambda, Z} \sharp \varphi_{\varepsilon Z})(X) \Lambda^{-2n\varepsilon}\, dZ.$$

For the second term of the sum, we shall prove the following lemma.

**Lemma 2.5.** *The term* $U_\Lambda = \int_{\mathbf{R}^{2n}} U_{\Lambda,Z} \sharp \varphi_{\varepsilon Z} \Lambda^{-2n\varepsilon}\, dZ$ *belongs to the class* $\Sigma$. *Moreover there is* $C_2'$ *such that* $\|U_\Lambda\|_\Sigma \le C_2' \Lambda^{1-2\varepsilon}$ *for all* $\varepsilon \in ]0,1]$ *and* $\Lambda \ge 1$.

*Proof.* Using Lemma 1.5 and the definition of the rescaling, we have, for all $\alpha$ and $\beta$ with $|\alpha|+|\beta|=2$, and uniformly with respect to $\Lambda$, $\varepsilon$ and $Z$,

$$\Lambda^{-1}\partial_x^\alpha \partial_\xi^\beta a_\Lambda \in \Sigma \quad \text{and} \quad \Lambda^{2\varepsilon}\partial_x^\alpha \partial_\xi^\beta \varphi_{\varepsilon Z} \in \Sigma. \tag{66}$$

Proposition 1.10 then implies that the function $f_{\Lambda,Z} = \Lambda^{-1+2\varepsilon} U_{\Lambda,Z}$ belongs to $\Sigma$ uniformly with respect to $\Lambda$ and $Z$. From (iv) in Proposition 1.1, we can write, for all $Z$ and $X$,

$$f_{\Lambda,Z}(X) = \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)} \chi_{\Lambda,Z}(X,Y)\, dY, \tag{67}$$

where $\chi_{\Lambda,Z}$ satisfies the following uniform properties induced by (64) and Proposition 1.10: for every $p$ semi-norm on $S^0$, and $\Lambda \ge 1$, there exists $a_{p,\Lambda}^*$ uniformly in $L^1(\mathbf{R}^{2n})$ such that $p(\chi_{\Lambda,Z}(\cdot, Y)) < a_{p,\Lambda}^*(Y)$ for all $Z$ and $Y$. We can then write, for all $Z$, $X$ and $\Lambda$,

$$\begin{aligned}(f_{\Lambda,Z} \sharp \varphi_{\varepsilon,Z})(X) &= \int_{\mathbf{R}^{2n}} ((e^{-2i\sigma(\cdot,Y)} \chi_{\Lambda,Z}(\cdot, Y)) \sharp \varphi_{\varepsilon,Z})(X)\, dY \\ &= \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)} (\chi_{\Lambda,Z}(\cdot, Y) \sharp \varphi_{\varepsilon,Z}(\cdot - Y))(X)\, dY,\end{aligned} \tag{68}$$

according to an easy computation yielding $(\tau_Y a) \sharp b = \tau_Y(a \sharp b(\cdot - Y))$ when $a, b \in \Sigma$, where $\tau_Y a(X) = e^{-2i\sigma(X,Y)} a(X)$. From arguments similar to those in the proof of Lemma 1.9, and the fact that $\sharp$ commutes with the translations, we easily get that for all $N \in \mathbf{N}$ and $\alpha \in \mathbf{N}^{2n}$, there exists $p'$ and $q'$ semi-norms on $S^0$ and $\mathcal{S}(\mathbf{R}^{2n})$, respectively, such that, for all $X$, $Y$, $Z$ and $\Lambda$,

$$|\partial_X^\alpha (\chi_{\Lambda,Z}(\cdot, Y) \sharp \varphi_{\varepsilon,Z}(\cdot - Y))(X)| \le \frac{p'(\chi_{\Lambda,Z}(\cdot, Y)) q'(\varphi)}{\langle \Lambda^{-\varepsilon}(X-Y-Z)\rangle^N} \le \frac{a_{p',\Lambda}^*(Y) q'(\varphi)}{\langle \Lambda^{-\varepsilon}(X-Y-Z)\rangle^N}. \tag{69}$$

Now we can apply the Fubini theorem and we get from (69) that if $N = 2n+1$, the function defined for all $Y$ by

$$X \longmapsto \tilde{\chi}_\Lambda(X,Y) = \int_{\mathbf{R}^{2n}} (\chi_{\Lambda,Z}(\cdot, Y) \sharp \varphi_{\varepsilon,Z}(\cdot - Y))(X) \Lambda^{-2n\varepsilon}\, dZ \tag{70}$$

is in $S^0$ and satisfies $\sup_{\Lambda \ge 1} \int_{\mathbf{R}^{2n}} p(\chi_\Lambda(\cdot, Y))\, dY < \infty$ for any semi-norm $p$ on $S^0$. Therefore $X \mapsto \Lambda^{-1+2\varepsilon} U_\Lambda(X) = \int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)} \tilde{\chi}_\Lambda(X,Y)\, dY$ belongs to the class $\Sigma$ uniformly in $\Lambda$. The proof of Lemma 2.5 is complete. $\square$

In order to conclude the study of the commutator term $S_\Lambda$ (and Lemma 2.4), let us have a look at the first term in the sum (65).

**Lemma 2.6.** *The term $V_\Lambda=(1/i)\int_{\mathbf{R}^{2n}}\{a_\Lambda,\varphi_{\varepsilon Z}\}\sharp\varphi_{\varepsilon Z}\Lambda^{-2n\varepsilon}\,dZ$ belongs to the class $\Sigma$. Moreover there is $C_2''$ such that $\|V_\Lambda\|_\Sigma\le C_2''\Lambda^{1-2\varepsilon}$ for all $\Lambda\ge 1$ and $\varepsilon\in]0,1]$.*

*Proof.* Let us write $\sigma(T)=(-\tau,t)$ when $T=(t,\tau)$, and define, for all $Z,X\in\mathbf{R}^{2n}$ and $\Lambda\ge 1$,

$$
\begin{aligned}
V_{\Lambda,Z}(X)&=\Big(\frac{1}{i}\{a_\Lambda,\varphi_{\varepsilon Z}\}\sharp\varphi_{\varepsilon Z}\Big)(X)=\Big(\frac{1}{i}(a'_\Lambda\sigma(\varphi'_{\varepsilon Z}))\sharp\varphi_{\varepsilon Z}\Big)(X)\\
&=\frac{1}{i\pi^{2n}}\iint_{\mathbf{R}^{4n}}a'_\Lambda(Y_1)\sigma(\varphi'_{\varepsilon Z}(Y_1))\varphi_{\varepsilon Z}(Y_2)e^{-2i\sigma(X-Y_1,X-Y_2)}\,dY_1\,dY_2.
\end{aligned}
\tag{71}
$$

In order to separately consider the derivatives of order 1 an the one of order 2 of $a_\Lambda$, we perform a Taylor expansion of $a'_\Lambda$ with respect to $X$,

$$
a'_\Lambda(Y_1)=a'_\Lambda(X)+\int_0^1 a''_\Lambda(X+\theta(Y_1-X))(Y_1-X)\,d\theta.
$$

We can then write

$$
\begin{aligned}
V_{\Lambda,Z}(X)=\frac{1}{i}a'_\Lambda(X)(\sigma(\varphi'_{\varepsilon Z})\sharp\varphi_{\varepsilon Z})(X)&+\frac{1}{i\pi^{2n}}\int_0^1\iint_{\mathbf{R}^{4n}}a''_\Lambda(X+\theta(Y_1-X))(Y_1-X)\\
&\times\sigma(\varphi'_{\varepsilon Z}(Y_1))\varphi_{\varepsilon Z}(Y_2)e^{-2i\sigma(X-Y_1,X-Y_2)}\,dY_1\,dY_2\,d\theta.
\end{aligned}
\tag{72}
$$

As already noticed in the original paper of Hörmander [5], Lemma 2.1 implies that the first term disappears when we integrate over $Z$, since $\varphi$ is even. Therefore we only have to deal with the second part of (72). Let us denote it by $W_{\Lambda,Z}(X)$. We first notice that

$$
(Y_1-X)e^{-2i\sigma(X-Y_1,X-Y_2)}=-\tfrac{1}{2}\sigma(D_{Y_2})e^{-2i\sigma(X-Y_1,X-Y_2)}.
\tag{73}
$$

It allows us to write

$$
\begin{aligned}
W_{\Lambda,Z}(X)=\frac{1}{2\pi^{2n}}\int_0^1\iint_{\mathbf{R}^{4n}}&a''_\Lambda(X+\theta(Y_1-X))\sigma(\varphi'_{\varepsilon Z}(Y_1))\sigma(\varphi'_{\varepsilon Z}(Y_2))\\
&\times e^{-2i\sigma(X-Y_1,X-Y_2)}\,dY_1\,dY_2\,d\theta.
\end{aligned}
\tag{74}
$$

We shall again use the characterization of the class $\Sigma$ given in (iv) of Proposition 1.1. We first notice that the functions

$$
f_\Lambda=\Lambda^{-1}a''_\Lambda\quad\text{and}\quad\Psi_{\varepsilon Z}=\Lambda^{\varepsilon}\sigma(\varphi'_{\varepsilon Z}(\,\cdot\,))
\tag{75}
$$

are in $\Sigma$ and $\mathcal{S}(\mathbf{R}^{2n})$, respectively, uniformly with respect to $\Lambda$. We can write

$$f_\Lambda(X)=\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)}\chi_\Lambda(X,Y)\,dY,$$

where $X\mapsto\chi_\Lambda(X,Y)$ is in $S^0$ and is such that for all semi-norms $p$ on $S^0$, and $\Lambda\geq 1$, there is a function $a^*_{p,\Lambda}$ uniformly in $L^1(\mathbf{R}^{2n})$ such that $p(\chi_\Lambda(\cdot,Y))\leq a^*_{p,\Lambda}(Y)$ for all $Y\in\mathbf{R}^{2n}$. We can then write

$$\begin{aligned} W_{\Lambda,Z}(X) &= \Lambda^{1-2\varepsilon}\frac{1}{2\pi^{2n}}\int_0^1\iint_{\mathbf{R}^{4n}} e^{-2i\sigma(X-Y_1,X-Y_2)}f_\Lambda(X+\theta(Y_1-X)) \\ &\quad\times\Psi_{\varepsilon Z}(Y_1)\Psi_{\varepsilon Z}(Y_2)\,dY_1\,dY_2\,d\theta \\ &= \Lambda^{1-2\varepsilon}\frac{1}{2\pi^{2n}}\int_0^1\iiint_{\mathbf{R}^{6n}} e^{-2i\sigma(X-Y_1,X-Y_2)-2i\sigma(X+\theta(Y_1-X),Y)} \\ &\quad\times\chi_\Lambda(X+\theta(Y_1-X),Y)\Psi_{\varepsilon Z}(Y_1)\Psi_{\varepsilon Z}(Y_2)\,dY_1\,dY_2\,dY\,d\theta. \end{aligned}$$

If we choose $X_1=X-Y_1$, $X_2=X-Y_2$, then we have

$$\begin{aligned} W_{\Lambda,Z}(X) &= \Lambda^{1-2\varepsilon}\frac{1}{2\pi^{2n}}\int_0^1\iiint_{\mathbf{R}^{6n}} e^{-2i\sigma(X_1,X_2)-2i\sigma(X-\theta X_1,Y)} \\ &\quad\times\chi_\Lambda(X-\theta X_1,Y)\Psi_{\varepsilon Z}(X-X_1)\Psi_{\varepsilon Z}(X-X_2)\,dX_1\,dX_2\,dY\,d\theta. \end{aligned}$$

Now we can integrate by parts with respect to $X_1$ and $X_2$ in the preceding equality, using the following differential operators

$$\mathcal{P}_1=\frac{1}{\langle X_2-\theta Y\rangle^2}\left(1-\tfrac{1}{4}\Delta_{X_1}\right),\quad \mathcal{P}_2=\frac{1}{\langle X_1\rangle^2}\left(1-\tfrac{1}{4}\Delta_{X_2}\right), \tag{76}$$

where $\Delta_{X_j}$, $j=1,2$, is the Laplacian in the $X_j$ variables. If we by $\Phi_X$ denote the phase function

$$\Phi_X=-2\sigma(X_1,X_2)-2\sigma(X-\theta X_1,Y),$$

we have $\mathcal{P}_j e^{i\Phi_X}=e^{i\Phi_X}$ for $j=1,2$. We can therefore write

$$\begin{aligned} W_{\Lambda,Z}(X) &= \Lambda^{1-2\varepsilon}\frac{1}{2\pi^{2n}}\int_0^1\iiint_{\mathbf{R}^{6n}} e^{i\Phi_X} \\ &\quad\times\mathcal{P}_1^{n+1}\mathcal{P}_2^{n+1}\chi_\Lambda(X-\theta X_1,Y)\Psi_{\varepsilon Z}(X-X_1)\Psi_{\varepsilon Z}(X-X_2)\,dX_1\,dX_2\,dY\,d\theta. \end{aligned}$$

It is easy to check that for all $\alpha\in\mathbf{N}^{2n}$ and $\Lambda\geq 1$, there is a function $a^*_{\alpha,\Lambda}$ in $L^1(\mathbf{R}^{2n})$ uniformly with respect to $\Lambda$ such that, for all $X,Y_1,Y_2,Y,Z\in\mathbf{R}^{2n}$ and $\theta\in]0,1]$,

$$\begin{aligned} &|\partial_X^\alpha\mathcal{P}_1^{n+1}\mathcal{P}_2^{n+1}\chi_\Lambda(X-\theta X_1,Y)\Psi_{\varepsilon Z}(X-X_1)\Psi_{\varepsilon Z}(X-X_2)| \\ &\qquad\leq\frac{a^*_{\alpha,\Lambda}(Y)}{\langle X_2-\theta Y\rangle^{2n+2}\langle X_1\rangle^{2n+2}\langle\Lambda^{-\varepsilon}(X-X_1-Z)\rangle^{2n+2}}. \end{aligned} \tag{77}$$

Let us first recall that $V_\Lambda(X)=\int_{\mathbf{R}^{2n}} W_{\Lambda,Z}(X)\Lambda^{-2n\varepsilon}\,dZ$ and then set

$$\widetilde{\chi}_\Lambda(X,Y)=\frac{1}{2\pi^{2n}}\int_0^1\iiint_{\mathbf{R}^{6n}} e^{-2i\sigma(X_1,X_2)+2i\sigma(\theta X_1,Y)}\mathcal{P}_1^{n+1}\mathcal{P}_2^{n+1}\chi_\Lambda(X-\theta X_1,Y)$$
$$\times\Psi_{\varepsilon Z}(X-X_1)\Psi_{\varepsilon Z}(X-X_2)\,dX_1\,dX_2\Lambda^{-2n\varepsilon}\,dZ\,d\theta.$$

We notice that $V_\Lambda(X)=\Lambda^{1-2\varepsilon}\int_{\mathbf{R}^{2n}} e^{-2i\sigma(X,Y)}\widetilde{\chi}_\Lambda(X,Y)\,dY$. Moreover, (77) implies that $\widetilde{\chi}_\Lambda(\,\cdot\,,Y)$ is in $S^0$ for all $Y$ and $\Lambda$, and for all semi-norms $p$ on $S^0$,

$$\sup_{\Lambda\geq 1}\int_{\mathbf{R}^{2n}} p(\widetilde{\chi}_\Lambda(\,\cdot\,,Y))\,dY<\infty.$$

This implies that $V_\Lambda\in\Sigma$ with the estimate of Lemma 2.6. The proof is complete. □

### 2.4. Study of the oscillatory term and the end of the proof

The study of the two remainders $R_\Lambda$ and $S_\Lambda$ and the continuous embedding $\Sigma\hookrightarrow\mathcal{L}(L^2(\mathbf{R}^n))$ (see Lemma 1.6) allow us to write the following lemma.

**Lemma 2.7.** *For $\varepsilon=\frac{1}{10}$, there is $C_3>0$ such that we have $\|R_\Lambda^w+S_\Lambda^w\|_{\mathcal{L}(L^2)}\leq C_3\Lambda^{4/5}$ for all $\Lambda\geq 1$.*

*Proof.* The result is immediate if we optimize the bounds inherited from Lemmas 2.2 and 2.4. We write

$$1-2\varepsilon=\tfrac{1}{2}+3\varepsilon,$$

and get $\varepsilon=\frac{1}{10}$ and $1-2\varepsilon=\frac{1}{2}+3\varepsilon=\frac{4}{5}$. The study of remainders is complete. □

We have now to deal with the oscillatory part of the symbol in (46),

$$A_\Lambda(X)=\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon,Z}\sharp a_{\Lambda,Z}\sharp\varphi_{\varepsilon,Z})(X)\Lambda^{-2n\varepsilon}\,dZ,\quad X\in\mathbf{R}^{2n},\ \Lambda\geq 1,\tag{78}$$

where $a_{\Lambda,Z}$ is the polynomial in the Taylor expansion of order 2 at $Z$ of $a_\Lambda$, already defined in (48). Hörmander's study in [5] and, in particular, Lemma 6.4 can be applied without change. It is an analytic lemma about the lower bound of a function in $C^3(\mathbf{R}^{2n})$. Since $\Sigma\hookrightarrow C^0(\mathbf{R}^{2n})$ (see Lemma 1.6) we can write the following lemma, assuming $\|a\|_{3,\infty}\leq 1$ and $|a^{(3)}(X;Y_1,Y_2,Y_3)|\leq 6$ for $\Gamma(X)\leq 1$ and $\Gamma(Y_j)\leq 1$, $j=1,2,3$.

**Lemma 2.8.** ([5, Lemma 6.4]) *Consider the semi-classical family $a_\Lambda$, defined in Theorem 0.2. Then for all $Z\in\mathbf{R}^{2n}$, there exists $s\in[0,\Lambda^{1/2}]$ and $Y\in\mathbf{R}^{2n}$ with $\Gamma(Y-Z)=s^2$ such that*

$$a_\Lambda(Y)\leq a_{\Lambda,Z}(X)+(3s\Gamma(X-Z)-2s^3)\Lambda^{1/2},\quad X\in\mathbf{R}^{2n},\tag{79}$$

$$a''_{\Lambda,Z} - 6s\Lambda^{1/2}\Gamma \leq a''_\Lambda(Y) \leq a''_{\Lambda,Z} + 6s\Lambda^{1/2}\Gamma. \tag{80}$$

This lemma implies that the quadratic form $a''_\Lambda(Y) + 12s\Lambda^{1/2}\Gamma$ is non-negative. Therefore, using (6), we obtain that the second-order polynomial

$$B(X) = a_{\Lambda,Z}(X) + (6sC_2\Gamma(X-Z) - 2s^3)\Lambda^{1/2} \tag{81}$$

satisfies $\inf(B) + \frac{1}{2}\operatorname{Tr}_+(B'') \geq 0$, and $B^w \geq 0$ follows from (2). This gives

$$0 \leq \varphi^w_{\varepsilon,Z} a^w_{\Lambda,Z} \varphi^w_{\varepsilon,Z} + (\varphi_{\varepsilon,Z} \sharp (6sC_2\Lambda^{1/2}\Gamma(X-Z) - 2s^3\Lambda^{1/2}) \sharp \varphi_{\varepsilon,Z})^w. \tag{82}$$

When $u \in \mathcal{S}(\mathbf{R}^n)$ we therefore have

(83)

$$(a^w_{\Lambda,Z}\varphi^w_{\varepsilon,Z}u, \varphi^w_{\varepsilon,Z}u) \geq 2s^3\Lambda^{1/2}\|\varphi^w_{\varepsilon,Z}u\|^2 - 6sC_2\Lambda^{1/2+2\varepsilon}\Gamma(\Lambda^{-\varepsilon}(X-Z))^w\varphi^w_{\varepsilon,Z}u, \varphi^w_{\varepsilon,Z}u.$$

Using (30) at order 2, we get that there is a function $\gamma_\Lambda$ in $\mathcal{S}(\mathbf{R}^{2n})$ uniformly with respect to $\Lambda$, such that

$$(\Gamma(\Lambda^{-\varepsilon}(X-Z))) \sharp \varphi_{\varepsilon,Z} = \gamma_\Lambda(\Lambda^{-\varepsilon}(X-Z)) \stackrel{\text{def}}{=} \gamma_{\varepsilon,Z}(X). \tag{84}$$

Using (83) we get

$$(a^w_{\Lambda,Z}\varphi^w_{\varepsilon,Z}u, \varphi^w_{\varepsilon,Z}u) \geq -3sC_2\Lambda^{1/2+2\varepsilon}(\|\varphi^w_{\varepsilon,Z}u\|^2 + \|\gamma^w_{\varepsilon,Z}u\|^2). \tag{85}$$

Now we can notice that for $s > 0$,

$$\begin{aligned} 2s^3\Lambda^{1/2}\|\varphi^w_{\varepsilon,Z}u\|^2 - 6sC_2\Lambda^{1/2+2\varepsilon}\, \|\varphi^w_{\varepsilon,Z}u\|\, \|\gamma^w_{\varepsilon,Z}u\| + \frac{9C_2^2}{2s}\Lambda^{1/2+4\varepsilon}\|\gamma^w_{\varepsilon,Z}u\|^2 \\ = 2s\Lambda^{1/2}\left(s\|\varphi^w_{\varepsilon,Z}u\| - \Lambda^{2\varepsilon}\frac{3C_2}{2s}\|\gamma^w_{\varepsilon,Z}u\|\right)^2 \geq 0. \end{aligned} \tag{86}$$

Using (83) again we therefore get

$$\begin{aligned} (a^w_{\Lambda,Z}\varphi^w_{\varepsilon,Z}u, \varphi^w_{\varepsilon,Z}u) &\geq -\frac{9C_2^2}{2s}\Lambda^{1/2+4\varepsilon}(\|\gamma^w_{\varepsilon,Z}u\|^2) \\ &\geq -\frac{9C_2^2}{2s}\Lambda^{1/2+4\varepsilon}(\|\gamma^w_{\varepsilon,Z}u\|^2 + \|\varphi^w_{\varepsilon,Z}u\|^2). \end{aligned} \tag{87}$$

Moreover, we can notice that there exists $C_3 \geq 0$ such that

$$\min_{s \in ]0,\Lambda^{1/2}]}\left(\frac{9C_2^2}{2s}\Lambda^{1/2+4\varepsilon}, 3sC_2\Lambda^{1/2+2\varepsilon}\right) = C_3\Lambda^{1/2+3\varepsilon}. \tag{88}$$

From (85) and (87) we get

$$(a^w_{\Lambda,Z}\varphi^w_{\varepsilon,Z}u,\varphi^w_{\varepsilon,Z}u) \geq -C_3\Lambda^{1/2+3\varepsilon}(\|\gamma^w_{\varepsilon,Z}u\|^2+\|\varphi^w_{\varepsilon,Z}u\|^2), \tag{89}$$

hence

$$\varphi_{\varepsilon,Z}\sharp a_{\Lambda,Z}\sharp\varphi_{\varepsilon,Z} \geq -C_3\Lambda^{1/2+3\varepsilon}(\gamma_{\varepsilon,Z}\sharp\gamma_{\varepsilon,Z}+\varphi_{\varepsilon,Z}\sharp\varphi_{\varepsilon,Z}).$$

Using $\frac{1}{2}+3\varepsilon=\frac{4}{5}$ and integrating over $Z$ yields

$$\int_{\mathbf{R}^{2n}} \varphi_{\varepsilon,Z}\sharp a_{\Lambda,Z}\sharp\varphi_{\varepsilon,Z}\Lambda^{-2n\varepsilon}\,dZ \geq -C_3\Lambda^{4/5}\int_{\mathbf{R}^{2n}}(\varphi_{\varepsilon,Z}\sharp\varphi_{\varepsilon,Z}+\gamma_{\varepsilon,Z}\sharp\gamma_{\varepsilon,Z})\Lambda^{-2n\varepsilon}\,dZ. \tag{90}$$

According to Lemma 2.1 we have

$$\begin{aligned}\left(\int_{\mathbf{R}^{2n}} \varphi_{\varepsilon,Z}\sharp a_{\Lambda,Z}\sharp\varphi_{\varepsilon,Z}\Lambda^{-2n\varepsilon}\,dZ\right)^w &\geq -C_3\Lambda^{4/5}(\|\varphi\|^2_{L^2(\mathbf{R}^{2n})}+\|\gamma_\Lambda\|^2_{L^2(\mathbf{R}^{2n})})\\ &\geq -C_3'\Lambda^{4/5}.\end{aligned} \tag{91}$$

The proof of Theorem 0.2 is complete. □

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