Ark. Mat., **39** (2001), 283–309

Reverse hypercontractivity over manifolds

Fernando Galaz-Fontes(1), Leonard Gross and Stephen Bruce Sontz(2)

Abstract. Suppose that X is a vector field on a manifold M whose flow, $\exp tX$, exists for all time. If μ is a measure on M for which the induced measures $\mu_t \equiv (\exp tX)_* \mu$ are absolutely continuous with respect to μ, it is of interest to establish bounds on the $L^p(\mu)$ norm of the Radon–Nikodym derivative $d\mu_t/d\mu$. We establish such bounds in terms of the divergence of the vector field X. We then specialize M to be a complex manifold and derive reverse hypercontractivity bounds and reverse logarithmic Sobolev inequalities in some holomorphic function spaces. We give examples on $\mathbf{C}^m$ and on the Riemann surface for $z^{1/n}$.

1. Introduction

E. Carlen, [C], has shown that the Ornstein–Uhlenbeck semigroup has some surprising reverse hypercontractivity properties when restricted to holomorphic function spaces. Denote by γ the Gauss measure on $\mathbf{C}^m$ with density const. $\exp(-\frac{1}{2}|x|^2)$, and by A the nonnegative self-adjoint Dirichlet form operator on $L^2(\gamma)$ determined by $(Af, g)_{L^2(\gamma)} = \int_{\mathbf{C}^m} \nabla f \cdot \nabla \bar{g}\, d\gamma$. It is well known that if $1 < p_0 < p_1 < \infty$ then the hypercontractivity inequality

$$\|e^{-TA} f\|_{L^{p_1}(\gamma)} \le \|f\|_{L^{p_0}(\gamma)}, \quad f \in L^{p_0}(\gamma), \tag{1.1}$$

holds if T is sufficiently large. But Carlen showed that if f is holomorphic and $T > 0$ is sufficiently *small* then

$$\|e^{-TA} f\|_{L^{p_1}(\gamma)} \ge C \|f\|_{L^{p_0}(\gamma)} \tag{1.2}$$

for some constant C depending on p_0, p_1, T and m. One can even allow $0 < p_0 < p_1 < \infty$.

In a recent paper, [S1], the third author showed that reverse logarithmic Sobolev inequalities hold in the holomorphic category for Gauss measure on $\mathbf{C}^m$.

(1) Research supported in part by CONACyT, Mexico, grant 32725-E.
(2) Research supported in part by CONACyT, Mexico, grant 32146-E.

This was discussed further in the papers [S2] and [GS]. In view of the known connection between hypercontractivity inequalities such as (1.1) and logarithmic Sobolev inequalities, [G1], [G2], it is reasonable to expect that Carlen's reverse hypercontractive inequalities (1.2) are linked to these new reverse logarithmic Sobolev inequalities. It is the purpose of this paper to explore this connection and to do so in a quite general context.

We will actually show that both reverse hypercontractivity and reverse logarithmic Sobolev inequalities over certain complex manifolds are simply consequences of the fact that a Dirichlet form operator reduces to a first order differential operator when applied to holomorphic functions. Our method of proof of reverse hypercontractivity for complex manifolds extends the method first introduced for Gauss measure in [GS]. In order to carry out this extension it is necessary to estimate the L^p norms of the Radon–Nikodym derivative $d\mu_t/d\mu$, where μ is a given probability measure on a (not necessarily complex) manifold M and μ_t is the measure induced from it by a smooth flow, $\exp(tX)$, on M. Estimates of this sort seem to have been first studied in the ground breaking work of Ana Bela Cruzeiro, [Cr1], [Cr2]. Her estimates have been further exploited in the work of B. Driver, [D], Bogachev and Mayer-Wolf, [BM], and Cipriano and Cruzeiro, [CC]. These papers are concerned primarily with the problem of global existence and uniqueness of the flow for a not necessarily smooth vector field X, in both finite and infinite dimensions and in the quasi-invariance of the flow. Our concern here is in obtaining good estimates for $\|d\mu_t/d\mu\|_{L^p(\mu)}$ for $p>1$. We will make a refinement of Cruzeiro's estimates using a variant of the infinitesimal technique that underlies the method of [G1]. To this end we consider a smooth function $r\colon[0,T]\to[1,p]$ and estimate the derivative $d\|f\circ\exp(-tX)\|_{r(t)}/dt$ from below, using a kind of reverse coercivity inequality. The resulting estimate is a functional of the function r. We are able to solve the Euler equation for this functional in the Gaussian case, yielding the exact value of $\|d\mu_t/d\mu\|_{L^p(\mu)}$ in that case. Our estimates are sensitive enough to distinguish between X and $-X$.

We then apply this real manifold theorem to certain Dirichlet form operators over complex manifolds to obtain reverse hypercontractivity in the sense of (1.2).

By way of examples we will give Gaussian and non-Gaussian measures on $\mathbf{C}^m$. (Sections 2 and 5.) We will also show that our method applies to the Riemann surface for $z^{1/n}$ with a natural measure on it. (Section 6.) In Section 7 we will show that reverse hypercontractivity fails for the weighted Bergman spaces. Moreover in the case of the unweighted Bergman space exactly one of our hypotheses breaks down, showing the key role of this hypothesis.

Reverse hypercontractive inequalities of the form $\|e^{-tA}f\|_q\geq\|f\|_p$ for $f\geq 0$ (and therefore non-holomorphic) have been explored by C. Borell and S. Janson, [B1],

[B2], [BJ]. In their work the indices p and q are typically related by $-\infty<q<p<1$. In a sense this range of indices complements ours. Yet the subject matter of their work is quite different from ours in that the generator A is a genuinely second order elliptic operator in their work while it degenerates into a first order operator in our holomorphic context.

2. The L^p norm of the density induced by a smooth flow

Notation 2.1. Denote by M a finite dimensional manifold and let μ be a probability measure on M with a smooth, strictly positive density in each coordinate chart. Denote by X a smooth vector field on M whose flow, $\exp(tX)\equiv e^{tX}$, exists for all real t. Let

$$\mu_t = (\exp(tX))_* \mu. \tag{2.1}$$

Since μ_t also has a strictly positive density in each coordinate chart the Radon–Nikodym derivative

$$J_t = \frac{d\mu_t}{d\mu} \tag{2.2}$$

exists for all real t.

Of course one has $\|J_t\|_{L^1(\mu)}=1$. In this section we will derive estimates for $\|J_t\|_{L^p(\mu)}$ for $p>1$.

Notation 2.2. Let $0\le\varkappa<\infty$ and $0<T<\infty$. Let $e^{\varkappa T}<p<\infty$. A continuously differentiable function $r\colon[0,T]\to[1,p]$ will be called *$\varkappa$-dominant* for T and p if

$$r(0)=1, \quad r(T)=p \tag{2.3}$$

and

$$r'(t)>\varkappa r(t) \quad \text{for } 0\le t\le T. \tag{2.4}$$

If the values of T and p are clear from the context we will simply say that r is $\varkappa$-dominant. Here and in the following r' denotes the derivative of the function r.

Example 2.3. ($\varkappa>0$) Define for $\varkappa>0$,

$$r_\varkappa(t)=ae^{\varkappa t}-b, \quad 0\le t\le T, \tag{2.5}$$

where

$$a=\frac{p-1}{e^{\varkappa T}-1} \tag{2.6}$$

and

$$b=\frac{p-e^{\varkappa T}}{e^{\varkappa T}-1}. \tag{2.7}$$

Then $r_\varkappa'(t)=a\varkappa e^{\varkappa t}=\varkappa r_\varkappa(t)+\varkappa b$. Since $p>e^{\varkappa T}$ we see that $b>0$ and therefore (2.4) holds. One verifies easily that (2.3) also holds.

Example 2.4. ($\varkappa=0$) Define

$$r_0(t)=1+\frac{(p-1)t}{T}. \tag{2.8}$$

Then it is immediate that (2.3) and (2.4) hold for $\varkappa=0$. Writing

$$r_\varkappa(t)=1+(p-1)\frac{e^{\varkappa t}-1}{e^{\varkappa T}-1}$$

one sees that $r_\varkappa$ converges uniformly on $[0,T]$ to r_0 as $\varkappa\downarrow 0$.

Notation 2.5. Let $\varkappa\geq 0$ and let $B\colon(\varkappa,\infty)\to[0,\infty)$ be continuous. For $T>0$ and $e^{\varkappa T}<p<\infty$ define

$$\Lambda(r)=\int_0^T\frac{1}{r(t)}B\left(\frac{r'(t)}{r(t)}\right)dt \tag{2.9}$$

for any $\varkappa$-dominant function r for T and p. Since $r'/r>\varkappa$ on $[0,T]$ and $r(t)\geq 1$ the integrand is continuous in t on $[0,T]$. Hence the integral exists.

For the particular functions in Examples 2.3 and 2.4 we write

$$\lambda_\varkappa(T,p)=\Lambda(r_\varkappa)\quad\text{for } \varkappa\geq 0,\ T>0,\ e^{\varkappa T}<p<\infty. \tag{2.10}$$

Remark 2.6. It will be useful to express $\lambda_\varkappa$ in a more explicit form. For $\varkappa>0$ we may make the change of variables $y(t)=\varkappa b/r(t)$, where $r=r_\varkappa$. Then $y'=-\varkappa br'/r^2$. Also, since $r'=\varkappa r+\varkappa b$ we have $r'/r=\varkappa+y$. But $y(0)=\varkappa b$ and $y(T)=\varkappa b/p$ and $r(t)^{-1}\,dt=(-\varkappa br'/r)^{-1}(-\varkappa br'/r^2)\,dt=(-\varkappa b(\varkappa+y))^{-1}\,dy$. Hence

$$\lambda_\varkappa(T,p)=\frac{1}{\varkappa b}\int_{\varkappa b/p}^{\varkappa b}\frac{B(\varkappa+y)}{\varkappa+y}\,dy,\quad \varkappa>0. \tag{2.11}$$

For $\varkappa=0$ the substitution $y=(p-1)/Tr_0(t)$ gives, similarly,

$$\lambda_0(T,p)=\frac{T}{p-1}\int_{(p-1)/Tp}^{(p-1)/T}\frac{B(y)}{y}\,dy. \tag{2.12}$$

In the following we write $C_c^1(M)^+=\{h\in C_c^1(M):h\geq 0\}$. By p' we denote the conjugate index to p.

Theorem 2.7. *Let $\varkappa \ge 0$. The following statements are equivalent:*

(a) $\int_M Xh\,d\mu \le s\int_M h(z)\log(h(z)/\|h\|_1)\,d\mu(z)+B(s)\|h\|_1$ *for all* $h\in C_c^1(M)^+$ *and* $s>\varkappa$;

(b) $\|h\circ e^{-TX}\|_p \ge \|h\|_1 e^{-\Lambda(r)}$ *for all* $T>0$, $p>e^{\varkappa T}$, $h\in C_c^1(M)^+$ *and for all* $\varkappa$-*dominant functions* r;

(c) $\|h\circ e^{-TX}\|_p \ge \|h\|_1 e^{-\lambda_\varkappa(T,p)}$ *for all* $T>0$, $p>e^{\varkappa T}$ *and* $h\in C_c^1(M)^+$;

(d) $\|J_T\|_{p'} \le e^{\Lambda(r)}$ *for all* $T>0$, $p>e^{\varkappa T}$, *and for all* $\varkappa$-*dominant functions* r;

(e) $\|J_T\|_{p'} \le e^{\lambda_\varkappa(T,p)}$ *for all* $T>0$ *and* $p>e^{\varkappa T}$.

We will prove Theorem 2.7 after establishing the following three lemmas.

Lemma 2.8. *Let* $T>0$ *and* $1<p<\infty$. *Let* $0<C<\infty$. *Then*

$$\|J_T\|_{p'} \le C \tag{2.13}$$

if and only if

$$C\|h\circ e^{-TX}\|_p \ge \|h\|_1 \quad \textit{for all } h\in C_c^1(M)^+. \tag{2.14}$$

Proof. Since $h\circ e^{TX}$ is in $C_c^1(M)^+$ if and only if $h\in C_c^1(M)^+$ the inequality in (2.14) may be replaced by the inequality

$$C\|h\|_p \ge \|h\circ e^{TX}\|_1. \tag{2.15}$$

But if (2.13) holds then

$$\|h\circ e^{TX}\|_1 = \int_M h(z)J_T(z)\,d\mu(z) \le \|h\|_p\|J_T\|_{p'} \le C\|h\|_p.$$

Thus (2.15) and hence (2.14) holds. For the converse we just need to know that $C_c^1(M)^+$ is dense in $(L^p)^+$, because the validity of $\int_M fJ_T\,d\mu \le C\|f\|_p$ for all $f\in (L^p)^+$ implies (2.13). Now if $f\in C_c(M)^+$ then also $f^{1/2}\in C_c(M)^+$. By using a partition of unity one sees that there exists a sequence $h_n\in C_c^1(M)$ such that $h_n\to f^{1/2}$ uniformly on M. Then also $h_n^2\to f$ uniformly. Therefore, since $C_c(M)^+$ is dense in $(L^p)^+$, so is $C_c^1(M)^+$. Now assume (2.14) holds. If $f\in(L^p)^+$ and f_n is a sequence in $C_c^1(M)^+$ that converges to f in L^p norm and pointwise a.e. then $f_n\circ e^{TX}$ converges to $f\circ e^{TX}$ a.e. and is Cauchy in L^1 norm by (2.15). Thus (2.15) follows for f. $\square$

Lemma 2.9. *Let $h\in C_c^1(M)^+$ and suppose that $r\colon[0,T]\to[1,p]$ is continuously differentiable. Let*

$$h_t(z)=h(\exp(-tX)z) \quad \text{for } z\in M \text{ and } t\geq 0. \tag{2.16}$$

Then

$$\frac{d}{dt}\|h_t\|_{r(t)}=\|h_t\|_{r(t)}^{1-r(t)}\frac{1}{r(t)}\left(\frac{r'}{r}\int_M h_t^r\log\left(\frac{h_t^r}{\|h_t\|_{r(t)}^{r(t)}}\right)d\mu-\int_M Xh_t^{r(t)}\,d\mu\right) \tag{2.17}$$

for $0\leq t\leq T$.

Proof. If h is supported in a compact set K then h_t is supported in the compact set $\exp(tX)K$. For any real number $a\geq 1$ the derivative $dh_t^a(z)/dt=-Xh_t^a(z)$ and is also continuous in t and z and has compact support on M. The following computation, a variant of that in [G1, Lemma 1.1], is therefore easily justified. Let $v(t)=\int_M h_t(z)^{r(t)}\,d\mu(z)$. Then we have

$$\frac{dv}{dt}=\int_M r'h_t^r\log h_t\,d\mu-\int_M Xh_t^r\,d\mu.$$

Hence

$$\begin{aligned}\frac{d}{dt}\|h_t\|_{r(t)}&=\frac{dv^{1/r(t)}}{dt}=\frac{1}{r}v^{(r^{-1}-1)}\frac{dv}{dt}-\frac{r'}{r^2}v^{1/r}\log v\\&=\frac{1}{r}v^{(r^{-1}-1)}\left(\frac{r'}{r}\left(\int_M h_t^r\log h_t^r\,d\mu-v\log v\right)-\int_M Xh_t^r\,d\mu\right).\quad\square\end{aligned}$$

Lemma 2.10. *Let $s>\varkappa\geq 0$. Then, for $\varkappa>0$,*

$$\lim_{T\downarrow 0}\lambda_\varkappa(T,e^{sT})=0 \tag{2.18}$$

and

$$\frac{\partial}{\partial T}\lambda_\varkappa(T,e^{sT})\bigg|_{T=0^+}=B(s). \tag{2.19}$$

For $\varkappa=0$ we have

$$\lim_{T\downarrow 0}\lambda_0(T,1+sT)=0 \tag{2.20}$$

and

$$\frac{\partial}{\partial T}\lambda_0(T,1+sT)\Big|_{T=0^+}=B(s). \tag{2.21}$$

Proof. For $\varkappa>0$ we have, upon expanding the following exponentials, $b(T)\equiv b=(e^{sT}-e^{\varkappa T})/(e^{\varkappa T}-1)=((s-\varkappa)+O(T))/(\varkappa+O(T))$. So $b(T)\to(s-\varkappa)/\varkappa$, as $T\downarrow 0$, and $\varkappa b\to s-\varkappa\equiv y_0>0$. Moreover $b(T)$ has a real analytic extension to a neighborhood of $T=0$. Therefore, writing $p=p(T)=e^{sT}$, we have

$$\frac{1}{\varkappa b}\left(\frac{\partial(\varkappa b)}{\partial T}-\frac{\partial(\varkappa b/p)}{\partial T}\right)=\frac{1}{b}\left(\frac{\partial b}{\partial T}-\frac{1}{p}\frac{\partial b}{\partial T}+\frac{bp'}{p^2}\right)\to s,\quad \text{as } T\downarrow 0.$$

The derivative of the factor $(\varkappa b)^{-1}$ in front of the integral in (2.11) does not contribute in the limit as $T\downarrow 0$, because the upper and lower limits both converge to y_0. Since $B(\varkappa+y)$ is continuous at y_0 the lemma follows. The proof for $\varkappa=0$ is similar and a little bit simpler. □

Remark 2.11. Any C^2 function p: $[0,\delta)\to[1,\infty)$ such that $p(0)=1$, $p(T)>e^{\varkappa T}$ and $p'(0)=s$ would work in this lemma in place of e^{sT}.

Proof of Theorem 2.7. By Lemma 2.8 the statements (b) and (d) are equivalent and the statements (c) and (e) are equivalent. So it suffices to show that (a), (b) and (c) are equivalent. We will show that (a) ⇒ (b) ⇒ (c) ⇒ (a).

Assume that (a) holds. Let $h\in C_c^1(M)^+$ and define h_t by (2.16). Let r be a $\varkappa$-dominant function for T and p. Fix $t\in[0,T]$ and put $s=r'(t)/r(t)$ in (a) with h replaced by $h_t^{r(t)}$. Inserting the resulting inequality into the identity (2.17) yields

$$\frac{d}{dt}\|h_t\|_{r(t)}\geq-\|h_t\|_r^{1-r}\frac{1}{r}\|h_t\|_r^r B\left(\frac{r'}{r}\right).$$

Hence

$$\frac{d}{dt}\log\|h_t\|_{r(t)}\geq-\frac{1}{r(t)}B\left(\frac{r'}{r}\right).$$

Integration of this inequality from 0 to T gives

$$\|h_T\|_p\geq\|h\|_1 e^{-\Lambda(r)}\quad\text{for all } h\in C_c^1(M)^+, \tag{2.22}$$

which is (b). Now (c) is just a special case of (b), with r chosen as in Example 2.3 or 2.4. It remains to show that (c) implies (a).

Assume that (c) holds and that $h\in C_c^1(M)^+$. Suppose first that $\varkappa>0$. Let $s>\varkappa$ and put $p=p(T)=e^{sT}$ into (c). By (2.18) the right-hand side of (c) converges to $\|h\|_1$, as $T\downarrow 0$. Clearly the left-hand side converges to $\|h\|_1$ also. So we may differentiate (c) at $T=0$. By (2.19) one obtains for the derivative on the right-hand side $-\|h\|_1 B(s)$. To compute the derivative of the left-hand side we may apply (2.17) with $r(t)=e^{st}$. We find, at $T=0$,

$$\frac{d}{dT}\|h\circ e^{-TX}\|_{e^{sT}}\Big|_{T=0^+}=s\int_M h\log\Big(\frac{h}{\|h\|_1}\Big)\,d\mu-\int_M Xh\,d\mu,$$

which is therefore $\geq -\|h\|_1 B(s)$. The resulting inequality is exactly (a). The proof for $\varkappa=0$ is similar. One puts $p=1+sT$ into (c). $\square$

Definition 2.12. The *μ-divergence* of X is the function W defined by

$$\int_M X\varphi\,d\mu=\int_M \varphi W\,d\mu \quad \text{for all } \varphi\in C_c^1(M). \tag{2.23}$$

If one writes equation (2.23) in a local coordinate chart (U,x) with support $\varphi\subset U$ one sees immediately that W exists and is a C^∞ function on U. By using a smooth partition of unity it follows that there is a unique function W on M satisfying (2.23). Clearly W is real and in $L^1_{\mathrm{loc}}(M,\mu)$. All of our results on reverse hypercontractivity and reverse logarithmic Sobolev inequalities will depend on W. In regard to the terminology in Definition 2.12 note that if X is a smooth vector field on $\mathbf{R}^n$ and $d\mu=\varrho(x)\,dx$ then $W=-\operatorname{div}X-X\log\varrho$.

Notation 2.13. Let

$$B(s)=\log\Big(\int_M e^{W(x)/s}\,d\mu(x)\Big)^s, \quad 0<s<\infty. \tag{2.24}$$

Define

$$\varkappa=\inf\{s>0:B(s)<\infty\}. \tag{2.25}$$

We will assume throughout that $B(s)<\infty$ for some $s\in(0,\infty)$ and therefore

$$\varkappa<\infty. \tag{2.26}$$

This imposes a strong restriction on the positive part of W but no restriction on the negative part. We will assume throughout that

$$W\in L^1(\mu). \tag{2.27}$$

In this case (2.23) also holds for $\varphi=1$, as will be shown in Lemma 4.4. Therefore $\int_M W\,d\mu=0$. Jensen's inequality now shows that $\int e^{W/s}\,d\mu\geq\exp\big(\int W\,d\mu/s\big)=1$. Thus $B(s)\geq 0$, which is consistent with our assumption in Notation 2.5.

Theorem 2.14. (L^p bound) *Let $\varkappa \ge 0$ and $T>0$ and assume*

$$e^{\varkappa T} < p < \infty. \tag{2.28}$$

Let r be a $\varkappa$-dominant function for T and p. Define B by (2.24) and Λ by (2.9). Then

$$\|J_T\|_{p'} \le e^{\Lambda(r)}. \tag{2.29}$$

In particular

$$\|J_T\|_{p'} \le e^{\lambda_\varkappa(T,p)}. \tag{2.30}$$

Note that, in view of (2.11) and (2.12), the inequality (2.30) can be written

$$\|J_T\|_{p'} \le \exp\biggl(\frac{1}{\varkappa b}\int_{\varkappa b/p}^{\varkappa b}\biggl(\log \int_M e^{W/(\varkappa+y)}\,d\mu\biggr)\,dy\biggr) \quad \text{for } \varkappa>0 \tag{2.31}$$

and

$$\|J_T\|_{p'} \le \exp\biggl(\frac{T}{p-1}\int_{(p-1)/Tp}^{(p-1)/T}\biggl(\log \int_M e^{W/y}\,d\mu\biggr)\,dy\biggr) \quad \text{for } \varkappa=0. \tag{2.32}$$

The proof of Theorem 2.14 depends on the following lemma.

Lemma 2.15. *Suppose that W is a real valued function in $L^p(\mu)$ for some $p\in[1,\infty)$. Suppose that h is a nonnegative function in $L^{p'}(\mu)$. Define $\varkappa$ by (2.25). If $s>\varkappa$ then*

$$\int_M hW\,d\mu \le s\biggl(\int_M h\log h\,d\mu - \|h\|_1 \log\|h\|_1\biggr) + \|h\|_1 B(s). \tag{2.33}$$

Proof. This proof is a slight variation of that given in [G1, Theorem 7]. Let $a>0$ and apply Young's inequality, $xy\le x\log x - x + e^y$, which is valid for $x\ge 0$ and $y\in\mathbf{R}$, to the numbers $x=sh(z)/a$ and $y=W(z)/s$ to find

$$\begin{aligned}\int_M \frac{h(z)}{a}W(z)\,d\mu(z) &= \int_M \frac{sh(z)}{a}\frac{W(z)}{s}\,d\mu(z)\\ &\le \int_M \frac{sh(z)}{a}\log\biggl(\frac{sh(z)}{a}\biggr)\,d\mu(z) - \biggl\|\frac{sh}{a}\biggr\|_1 + \int_M e^{W(z)/s}\,d\mu(z).\end{aligned}$$

Note that all terms are well defined since $h\log h$ is integrable. Multiplying by a we get

$$\int_M h(z)W(z)\,d\mu(z) \le \int_M sh(z)\log\left(\frac{sh(z)}{a}\right)d\mu(z) - s\|h\|_1 + ae^{B(s)/s}. \tag{2.34}$$

Put $a = s\|h\|_1 e^{-B(s)/s}$ in (2.34) to conclude that

$$\begin{aligned}\int_M h(z)W(z)\,d\mu(z) &\le s\int_M h(z)\log\left(\frac{h(z)e^{B(s)/s}}{\|h\|_1}\right)d\mu(z) - s\|h\|_1 + s\|h\|_1\\ &= s\left(\int_M h(z)\log h(z)\,d\mu(z) - \|h\|_1\log\|h\|_1\right) + \|h\|_1 B(s). \quad\square\end{aligned}$$

Proof of Theorem 2.14. For any function $h\in C_c^1(M)^+$ we have $\int_M Xh\,d\mu = \int_M hW\,d\mu$ by (2.23). Lemma 2.15 now shows that condition (a) of Theorem 2.7 holds. But (2.29) and (2.30) are just restatements of conditions (d) and (e) of Theorem 2.7. $\square$

Example 2.16. (Gauss measure on $\mathbf{R}^n$.) Let

$$d\gamma_c(x) = \frac{e^{-|x|^2/2c}}{(2\pi c)^{n/2}}\,dx \tag{2.35}$$

denote the Gauss measure on $\mathbf{R}^n$. We will be especially interested in the dilation vector field

$$X = \frac{1}{c}x\cdot\nabla. \tag{2.36}$$

This vector field arises naturally in the context of Dirichlet forms in holomorphic function spaces, [G3, Section 5]. We may compute W as follows. Let $\varphi\in C_c^1(\mathbf{R}^n)$. Then

$$\begin{aligned}\int_{\mathbf{R}^n}\varphi(x)W\,d\gamma_c(x) &= \int_{\mathbf{R}^n}(X\varphi)\,d\gamma_c(x) = \frac{1}{c}\int_{\mathbf{R}^n}\sum_{j=1}^n\left(x_j\frac{\partial\varphi}{\partial x_j}\right)\frac{e^{-|x|^2/2c}}{(2\pi c)^{n/2}}\,dx\\ &= \int_{\mathbf{R}^n}\varphi(x)\frac{1}{c}\left(\frac{|x|^2}{c}-n\right)d\gamma_c(x).\end{aligned}$$

Hence

$$W(x) = \frac{1}{c}\left(\frac{|x|^2}{c}-n\right). \tag{2.37}$$

A straightforward Gaussian integration now shows that

$$\int_{\mathbf{R}^n} e^{W(x)/s}\, d\gamma_c(x) = e^{-n/sc}\left(1-\frac{2}{sc}\right)^{-n/2}, \quad sc>2. \tag{2.38}$$

The integral is infinite if $sc\le 2$. Hence

$$\varkappa = \frac{2}{c} \tag{2.39}$$

and

$$B(s) = -\frac{n}{2}\left(\varkappa + s\log\left(1-\frac{\varkappa}{s}\right)\right), \quad s>\varkappa. \tag{2.40}$$

Thus (2.26) holds. Since W is quadratic (2.27) also holds. Hence Theorem 2.14 is applicable.

For fixed $T>0$ and $p>e^{\varkappa T}$ we may seek the optimal $\varkappa$-dominant function r for the inequality $\|J_T\|_{p'}\le e^{\Lambda(r)}$, cf. Theorem 2.7(d). This is the function r which minimizes $\Lambda(r)$ when B is defined by (2.24). A straightforward but lengthy computation of the Euler equation for this minimization problem gives $r''-\varkappa r'=0$. The general solution is $r(t)=ae^{\varkappa t}-b$ and the solution that matches the boundary conditions (2.3) is exactly that given by (2.5)–(2.7). Rather than show directly that this solution, $r_\varkappa$, gives a minimum of $\Lambda(r)$ we will compute $e^{\Lambda(r_\varkappa)}$ and $\|J_T\|_{p'}$ and show that they are equal. In view of Theorem 2.7(d) this will show that $r_\varkappa$ minimizes $\Lambda(r)$. To this end we will evaluate the integral (2.11). Equation (2.40) gives $B(s)/s=-\frac{1}{2}n(\varkappa/s+\log(1-\varkappa/s))$. Hence

$$\frac{B(\varkappa+y)}{\varkappa+y} = \frac{n}{2}\left(\log\frac{\varkappa+y}{y} - \frac{\varkappa}{\varkappa+y}\right). \tag{2.41}$$

An indefinite integral of (2.41) is $\frac{1}{2}ny\log((\varkappa+y)/y)$, as one can verify by differentiation. Substituting the limits from (2.11) and simplifying we find

$$\lambda_\varkappa(T,p) = \frac{n}{2}\left(\log\frac{b+1}{b} - \frac{1}{p}\log\frac{b+p}{b}\right),$$

where b is given by (2.7). To simplify this further put

$$\alpha = \frac{p-1}{p-e^{\varkappa T}},$$

which is greater than one. From (2.7) we see that $b+1=(p-1)/(e^{\varkappa T}-1)$ and $b+p=e^{\varkappa T}(p-1)/(e^{\varkappa T}-1)$. So $(b+1)/b=\alpha$ and $(b+p)/b=\alpha e^{\varkappa T}$. Hence

$$\lambda_\varkappa(T,p)=\frac{1}{2}n\left(\left(1-\frac{1}{p}\right)\log\alpha-\frac{\varkappa T}{p}\right). \tag{2.42}$$

Thus

$$e^{\lambda_\varkappa(T,p)}=(\alpha^{(1-p^{-1})}e^{-\varkappa T/p})^{n/2}. \tag{2.43}$$

By Theorem 2.7 we therefore have

$$\left\|\frac{d(e^{TX})_*\gamma_c}{d\gamma_c}\right\|_{L^{p'}(\gamma_c)}\le(\alpha^{(1-p^{-1})}e^{-\varkappa T/p})^{n/2}, \tag{2.44}$$

if $p>e^{\varkappa T}$. Now the measure $\mu_T\equiv(e^{TX})_*\gamma_c$ is also Gaussian and the left-hand side of (2.44) is computable. We have, from (2.36),

$$e^{TX}x=e^{T/c}x \quad \text{for all } T\in\mathbf{R}. \tag{2.45}$$

Therefore, substituting $x=e^{-T/c}y$ in the following equations, we have

$$\int_{\mathbf{R}^n}f(x)\,d\mu_T(x)=\int_{\mathbf{R}^n}f(e^{T/c}x)\,d\gamma_c(x)=\int_{\mathbf{R}^n}f(y)J_T(y)\,d\gamma_c(y),$$

where

$$J_T(y)=e^{(1-e^{-\varkappa T})|y|^2/(2c)}e^{-(n/2)\varkappa T} \tag{2.46}$$

with $\varkappa=2/c$ as in (2.39). A straightforward Gaussian computation now gives

$$\|J_T\|_{L^{p'}(\gamma_c)}=\left(\left(\frac{p-1}{p-e^{\varkappa T}}\right)^{1-p^{-1}}e^{-\varkappa T/p}\right)^{n/2}, \quad 1\le e^{\varkappa T}<p<\infty. \tag{2.47}$$

Comparing with (2.43) we see that

$$\|J_T\|_{p'}=e^{\lambda_\varkappa(T,p)}. \tag{2.48}$$

Hence the function $r_\varkappa$ minimizes $\Lambda(r)$ in the Gaussian case.

Notice that if we change X to its negative then W changes to $-W$. But $-W$ is bounded above. Hence $\varkappa=0$ for $-X$ instead of $2/c$. Thus our bounds are sensitive to a change in the sign of the vector field X.

3. Reverse hypercontractivity over complex manifolds

Notation 3.1. Let M be a finite dimensional manifold with Riemannian metric g. Denote by μ a probability measure on M. We will always assume that μ has a strictly positive smooth density in each coordinate patch. Associated to the triple (M, g, μ) is the Dirichlet form operator $\nabla^*\nabla$ on $C^\infty(M)$, which is defined by

$$(3.1)\quad (\nabla^*\nabla f, \varphi)_{L^2(\mu)} = \int_M g(\nabla f(z), \nabla\bar{\varphi}(z))\, d\mu(z), \quad f \in C^\infty(M),\ \varphi \in C_c^\infty(M).$$

We wish to allow f and φ to be complex valued. So in (3.1), g should be extended complex bilinearly to the complexified tangent spaces $T_z(M)\otimes\mathbf{C}$. In addition to the differential operator $\nabla^*\nabla$ we want to make use of the following self-adjoint version. Let Q be the closed quadratic form in $L^2(M, \mu)$ with core $C_c^\infty(M)$, which is given by

$$(3.2)\qquad Q(f) = \int_M g(\nabla f, \nabla\bar{f})\, d\mu \quad \text{for } f \in C_c^\infty(M).$$

There is a unique nonnegative self-adjoint operator A in $L^2(\mu)$ such that $\mathcal{D}(Q) \equiv$ domain $Q=$ domain $A^{1/2} \equiv \mathcal{D}(A^{1/2})$ and

$$(3.3)\qquad Q(f) = \|A^{1/2} f\|^2, \quad f \in \mathcal{D}(A^{1/2}).$$

We want to study the action of the semigroup e^{-tA} in subspaces of $L^p(\mu)$ consisting of holomorphic functions. To this end we will take M henceforth to be a complex manifold of complex dimension m and the Riemannian metric g to be Hermitian. Let us recall briefly that this means that M can be coordinatized in local coordinate patches by complex valued functions $z_1, \dots, z_m$ and the transition functions from one coordinate system to another one are holomorphic on the overlap. Moreover if $z_j = x_j + iy_j$ then g is Hermitian if and only if

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}\right) = g\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_j}\right), \quad j = 1, \dots, m,$$

just as in the complex plane.

Definition 3.2. The operator $\nabla^*\nabla$ is *holomorphic* if, for any function $f \in C^\infty(M)$, $\nabla^*\nabla f$ is holomorphic in any open set in which f is holomorphic. We also call the triple (M, g, μ) holomorphic in this case.

In particular if f is holomorphic then so is $\nabla^*\nabla f$.

Let $\mathcal{H} = \mathcal{H}(M)$ be the space of all holomorphic functions on M. Then $\mathcal{H}$ is invariant under $\nabla^*\nabla$ when $\nabla^*\nabla$ is holomorphic. This will reflect itself in similar invariance properties for the self-adjoint version A in holomorphic function spaces such as $\mathcal{H}\cap L^2(\mu)$.

Now if $f\in C^\infty(M)$ and is holomorphic in some coordinate patch U then the Cauchy–Riemann equations, $\partial f/\partial\bar{z}_j=0$, can easily be used to show that the second order terms in $\nabla^*\nabla f$ are zero in U and that one simply has $\nabla^*\nabla f(z)=\sum_{j=1}^m \varphi_j(z)\partial f/\partial z_j$ for some functions $\varphi_j\in C^\infty(U)$. Moreover the coefficients φ_j are holomorphic in U if and only if $\nabla^*\nabla$ is holomorphic (over U). In more invariant terminology this may be stated as: there exists a unique complex vector field Z of type $(1,0)$ such that for any function $f\in C^\infty(M)$

$$\nabla^*\nabla f = Zf \tag{3.4}$$

in any open set in which f is holomorphic. It is not hard to see that $\nabla^*\nabla$ is holomorphic if and only if Z is holomorphic.

Any complex vector field Z on M can be written in the form

$$Z=\tfrac{1}{2}(X-iY) \tag{3.5}$$

for some unique real vector fields X and Y. The vector fields Z, X and Y are clearly determined by the triple (M,g,μ). The properties of X and Y will play a fundamental role.

Standing Assumptions 3.3. We will assume throughout that $\nabla^*\nabla$ is holomorphic and that the flow of the vector field Y exists for all time, i.e., that Y is complete. We will assume further that Y is Killing. That is, the flow of Y preserves the metric g.

Under these assumptions there are some holomorphic function spaces that are invariant under the semigroup e^{-tA}. These are the spaces we are interested in. Let

$$\begin{aligned}
&\mathcal{H}^2 = L^2(\mu) \text{ closure of } \mathcal{H}\cap\mathcal{D}(Q), && \\
&\mathcal{H}^p = \mathcal{H}^2\cap L^p(\mu) && \text{for } 2<p<\infty, \\
&\mathcal{H}^p = \text{closure of } \mathcal{H}^2 \text{ in } L^p(\mu) && \text{for } 0<p<2.
\end{aligned}$$

The previous discussion is a summary of the structures introduced in [G3], where the invariance of these holomorphic function spaces under the contraction semigroup e^{-tA} is shown.

When (M,g) is complete then one simply has $\mathcal{H}^p=\mathcal{H}\cap L^p(\mu)$ for $p\geq 2$. See [G3, Theorem 2.14] for a proof. If (M,g) is not complete then $\mathcal{H}^2$ could be a proper subspace of $\mathcal{H}\cap L^2(\mu)$. This occurs in the interesting case of the Riemann surface for $z^{1/n}$. Our theorems are applicable to this case, which will be discussed in Section 6.

It is shown in [G3, Corollary 2.12] that the semigroup e^{-tA} relates to the vector field X by the identity

$$e^{-tA}f = f\circ\exp(-tX) \quad \text{for } t\geq 0 \text{ and } f\in\mathcal{H}^p \tag{3.6}$$

for any $p>0$ when the one-sided flow $\exp(-tX)$ exists for all $t\geq 0$. This is the key identity that we will use to relate the present section to Section 2.

Unlike the L^p spaces it can happen that $\mathcal{H}^q$ is not dense in $\mathcal{H}^p$ for some $q>p$. An example is given in [G3, Section 5]. But we will rule out these uninteresting cases by assuming henceforth that

$$\mathcal{H}^q \text{ is dense in } \mathcal{H}^p \quad \text{if } 0<p<q<\infty. \tag{3.7}$$

For the vector field X we may define its μ divergence as in Definition 2.12.

Theorem 3.4. (Reverse hypercontractivity.) *Suppose that the Standing Assumptions* 3.3 *hold, that the flow* $\exp tX$ *exists for all time and that* (2.26) *and* (2.27) *hold. Let* $T>0$. *Suppose that* $p_0>0$ *and* $p_1>p_0e^{\varkappa T}$. *Let* $p=p_1/p_0$. *Let* r *be a* $\varkappa$*-dominant function for* T *and* p. *Then*

$$\|e^{-TA}f\|_{p_1}\geq \|f\|_{p_0}\exp\biggl(-\frac{\Lambda(r)}{p_0}\biggr) \quad \textit{for } f\in\mathcal{H}^{p_1}. \tag{3.8}$$

Proof. Let $f\in\mathcal{H}^{p_1}$ and write $h=e^{-TA}f$. By (3.6) we have $h=f\circ e^{-TX}$. Hence $f=h\circ e^{TX}$. Thus

$$\begin{aligned}\|f\|_{p_0}^{p_0} &= \int_M |h\circ e^{TX}|^{p_0}\,d\mu = \int_M |h|^{p_0}\,d\mu_T = \int_M |h|^{p_0}J_T\,d\mu\\ &\leq \bigl\||h|^{p_0}\bigr\|_p\|J_T\|_{p'} = \|h\|_{p_1}^{p_0}\|J_T\|_{p'}.\end{aligned}$$

The inequality (2.29) now yields $\|f\|_{p_0}\leq\|h\|_{p_1}e^{\Lambda(r)/p_0}$, which is (3.8). □

The following corollary was first derived in [GS].

Corollary 3.5. (Gauss measure.) *Take* $\mu=\gamma_c$ (cf. (2.35)) *on* $\mathbf{C}^m$ *and let* g *be the standard metric on* $\mathbf{C}^m$. *Suppose that* $T>0$, $p_0>0$ *and* $p_1>p_0e^{2T/c}$. *Then*

$$\|e^{-TA}f\|_{p_1}\geq\|f\|_{p_0}\biggl(e^{-\varkappa T/p_1}\biggl(\frac{p_1-p_0}{p_1-e^{\varkappa T}p_0}\biggr)^{1/p_0-1/p_1}\biggr)^{-m}, \quad f\in\mathcal{H}^{p_1}, \tag{3.9}$$

where $\varkappa=2/c$.

Proof. Choose $r=r_\varkappa$ in (3.8). Then $e^{\Lambda(r)}=e^{\lambda_\varkappa(T,p)}$, which is given, e.g., by the right-hand side of (2.47). Put $p=p_1/p_0$ and $n=2m$ to get (3.9). □

Remark 3.6. For Gauss measure one has in the limiting case $p_1=p_0e^{2T/c}$ a hypercontractive inequality which goes the other way from (3.9), namely, ([C], [G3], [J1], [J2], [Z]),

$$\|e^{-TA}f\|_{p_1}\le\|f\|_{p_0}. \tag{3.10}$$

Remark 3.7. E. Carlen was the first to derive a reverse hypercontractive inequality in the holomorphic category, [C, Theorem 4]. He obtained an inequality for Gauss measure similar to (3.9) but with a smaller coefficient of $\|f\|_{p_0}$. His coefficient, in the present notation, is $((p_1-e^{2T/c}p_0)/p_1)^{m/p_0}$. The comparison of his methods with ours is especially interesting in that he used a logarithmic Sobolev inequality in a key step in his proof. Such an inequality is usually used to prove forward hypercontractive inequalities such as (3.10). Our method does not use a logarithmic Sobolev inequality, but rather, is based on a simple use of Hölder's inequality, in the manner first introduced for Gauss measure in [GS]. In fact in the next section we will prove a *reverse* logarithmic Sobolev inequality.

It should be noted that although the proof of (3.9) is based on the exact value (2.47), nevertheless equality in (3.9) does not hold for any nonzero holomorphic function. This is shown in [GS]. The best constant in (3.9) is not at present known.

4. Carlen's identity and reverse logarithmic Sobolev inequalities

In this section we will continue the assumptions of Sections 2 and 3, specifically the assumptions on M, g, μ and X stated in Theorem 3.4. In addition we will assume that

$$W\in\bigcap_{1\le p<\infty}L^p(\mu). \tag{4.1}$$

The following identity reduces, in the Gaussian case, to an integral identity first discovered by E. Carlen, [C, equation (1.7)]. Note that the semigroup e^{-tA} is a contraction semigroup in $L^p(\mu)$ for $1\le p\le\infty$ because A is a Dirichlet form operator. By the L^p domain of A we mean the domain of the infinitesimal generator, A_p, of this semigroup as a semigroup in $L^p(\mu)$.

Theorem 4.1. (Carlen's identity.) *Suppose that $f\in\mathcal{H}(M)$. If $p\ge2$ and $f\in L^q(\mu)$ for some $q>p$ and f is in the L^p domain of A then*

$$4\int_M|\nabla|f|^{p/2}|^2\,d\mu=\int_M|f|^pW\,d\mu. \tag{4.2}$$

If $0<p<2$ *and* f *is in the* L^2 *domain of* A *then*

$$4\int_M |\nabla |f|^{p/2}|^2\,d\mu \le \int_M |f|^p W\,d\mu. \tag{4.3}$$

Remark 4.2. It seems likely that (4.3) is also an equality. But we have run into technical problems concerning the behavior of f near its zeros. In the simple case in which $M=\mathbf{C}$ and f is a polynomial it is not hard to show that equality holds in (4.3). For a version of (4.2) which holds for all $p\in(0,\infty)$ and which avoids the singularities at $f=0$ see Equation (4.13).

Corollary 4.3. (Reverse logarithmic Sobolev inequality.) *Let* $p>0$ *and assume* f *is holomorphic on* M. *If* $p\ge 2$ *assume that* $f\in L^q(\mu)\cap(L^p$ *domain of* $A)$ *for some* $q>p$. *If* $0<p<2$ *assume that* f *is in the* L^2 *domain of* A. *Then*

$$4\int_M |\nabla |f|^{p/2}|^2\,d\mu \le s\left(\int_M |f|^p\log|f|^p\,d\mu - \|f\|_p^p\log\|f\|_p^p\right)+\|f\|_p^p B(s) \tag{4.4}$$

if $s>\varkappa$.

The theorem and corollary will be proved in the following lemmas.

Lemma 4.4. *If* $h\in C^1(M)\cap L^q(\mu)$ *for some* $q\in(1,\infty]$ *and* $Xh\in L^1(\mu)$ *then*

$$\int_M Xh\,d\mu = \int_M hW\,d\mu. \tag{4.5}$$

In order for (4.5) *to hold it suffices that* $W\in L^{q'}(\mu)$ *in place of* (4.1).

Proof. Choose a sequence g_n in $C_c^1(M)$ such that $0\le g_n\le 1$ and such that g_n converges to one on M uniformly on compact sets. Let $0\le u\in C_c^\infty(\mathbf{R})$ satisfy $\int_{-\infty}^{\infty} u(t)\,dt=1$ and $u(t)=0$ if $|t|\ge 1$. For $z\in M$ define

$$f_n(z)=\int_{-\infty}^{\infty} g_n(e^{tX}z)u(t)\,dt.$$

Then f_n is in $C^1(M)$ and $0\le f_n\le 1$. Moreover f_n has compact support because the map $(t,z)\to e^{-tX}z$ is jointly continuous, so that if g_n is supported in a compact set K and $H=\{e^{-tX}z:|t|\le 1,\ z\in K\}$ then f_n is supported in the compact set H. Now $f_n(z)\to 1$ for each $z\in M$ because $g_n(e^{tX}z)\to 1$ uniformly on the compact set $\{e^{tX}z:|t|\le 1\}$. Using $Xf_n(z)=df_n(e^{sX}z)/ds|_{s=0}$ one sees that

$$Xf_n(z)=-\int_{-\infty}^{\infty} g_n(e^{tX}z)u'(t)\,dt.$$

Hence $|Xf_n(z)| \le \int_{-\infty}^{\infty} |u'(t)|\, dt$. Thus the functions Xf_n are uniformly bounded. Moreover $\lim_{n\to\infty} Xf_n(z) = -\int_{-\infty}^{\infty} u'(t)\, dt = 0$. If h satisfies the hypotheses of the lemma then $f_n(z)h(z)$ is in $C_c^1(M)$. Hence by (2.23),

$$\int_M f_n hW\, d\mu = \int_M X(f_n h)\, d\mu = \int_M ((Xf_n)h + f_n Xh)\, d\mu.$$

Since $Xf_n \to 0$ boundedly, the dominated convergence theorem applies to all the terms in the last equality and yields (4.5) in the limit, as $n\to\infty$. □

Lemma 4.5. *Let $f \in \mathcal{H}(M)$. Suppose that $0<p<2$ and that f is in the L^2 domain of A or that $2 \le p < \infty$ and f is in the L^p domain of A. In the latter case assume also that $f \in L^q(\mu)$ for some $q>p$. Let $\varepsilon>0$ and put*

$$k(z) = |f(z)|^2 + \varepsilon. \tag{4.6}$$

Then

$$p\int_M (Af)\bar{f} k^{p/2-1}\, d\mu = \int_M k^{p/2} W\, d\mu \tag{4.7}$$

and both integrands are in $L^1(\mu)$.

Proof. Note first that $k \in C^\infty(M)$ and is bounded away from zero. For $p \ge 2$ we have $f \in \mathcal{H} \cap \mathcal{D}(A_p) \subset \mathcal{H} \cap \mathcal{D}(A_2) \subset \mathcal{H}^2$. For $0<p<2$ we clearly also have $f \in \mathcal{H}^2$. Hence $Af = Zf = (Z+\bar{Z})f = Xf$. Since X is a real vector field we have

$$Xk^{p/2} = \tfrac{1}{2} p k^{p/2-1} Xk = \tfrac{1}{2} p k^{p/2-1}((Xf)\bar{f} + f\overline{Xf}) = \tfrac{1}{2} p k^{p/2-1}((Af)\bar{f} + f\overline{Af}).$$

Hence

$$Xk^{p/2}(z) = p \operatorname{Re}((Af)(z)\overline{f(z)} k^{p/2-1}(z)). \tag{4.8}$$

Now the left-hand side of (4.7) is real by [G3, Proposition 4.2]. So

$$p\int_M (Af)\bar{f} k^{p/2-1}\, d\mu = \int_M Xk^{p/2}\, d\mu. \tag{4.9}$$

If $p \ge 2$ and $r = q/p$ then $(k^{p/2})^r = k^{q/2} = (|f|^2+\varepsilon)^{q/2}$ which is in $L^1(\mu)$. Moreover the right-hand side of (4.8) is in $L^1(\mu)$ by [G3, Lemma 4.1]. Hence $Xk^{p/2} \in L^1(\mu)$. Thus by (4.5) we have

$$\int_M Xk^{p/2}\, d\mu = \int_M k^{p/2} W\, d\mu. \tag{4.10}$$

If $p<2$ then $(k^{p/2})^{2/p} = k \in L^1(\mu)$. So the same argument again yields (4.10). Equation (4.7) now follows from (4.9) and (4.10). □

Corollary 4.6. *Under the hypotheses of Lemma* 4.5 *we have*

(4.11)
$$p(Af, fk^{p/2-1}) \le s\left(\int_M k^{p/2}\log k^{p/2}\,d\mu - \|k^{1/2}\|_p^p \log\|k^{1/2}\|_p^p\right) + \|k^{1/2}\|_p^p B(s)$$

for $s > \varkappa$ *and* $p > 0$. *Moreover, if* $p \ge 1$ *then*

$$p(Af, f|f|^{p-2}) \le s\int_M |f|^p \log\left(\frac{|f|}{\|f\|_p}\right)^p d\mu + \|f\|_p^p B(s) \quad \text{for } s > \varkappa. \tag{4.12}$$

Here $(\,\cdot\,,\cdot\,)$ *refers to the* $L^2(\mu)$ *inner product.*

Proof. The inequality (4.11) follows from (4.7) by applying Lemma 2.15 to the function $h = k^{p/2} = (|f|^2+\varepsilon)^{p/2}$. Now if $p \ge 2$ then $fk^{p/2-1}$ converges to $f|f|^{p-2}$ in $L^{p'}(\mu)$ by dominated convergence as $\varepsilon \downarrow 0$ because $\big\|f|f|^{p-2}\big\|_{p'} = \|f\|_p < \infty$. Since $Af \in L^p(\mu)$ the left-hand side of (4.11) converges to the left-hand side of (4.12), as $\varepsilon \downarrow 0$. Also by dominated convergence, and for all $p > 0$, the right-hand side of (4.11) converges to the right-hand side of (4.12). This proves (4.12) for $p \ge 2$. If $1 \le p < 2$ then f and Af are in $L^2(\mu)$, by assumption. In this case one verifies that $fk^{p/2-1} \to f|f|^{p-2}$ in $L^2(\mu)$, which proves (4.12) in this interval also.

Note. The inequalities (4.11) and (4.12) are reverse coercivity inequalities because A is a second order differential operator. We will see that the reverse logarithmic Sobolev inequality (4.4) is, informally, just another form of these inequalities, given the integration by parts identity (4.14) for holomorphic functions. Although (4.12) is more perspicuous than (4.11) it seems to be less useful because of the technical problems associated with the zeros of f.

The inequality (4.12) was conjectured in [S1] for the case of Gauss measure on $\mathbf{C}^n$ for $p \ne 2$. For $p = 2$ and Gauss measure a variant of (4.12) was proved in [S1] with a different coefficient in the norm term.

Lemma 4.7. *Under the hypotheses of Lemma* 4.5 *we have*

$$\int_M |\nabla k^{p/4}|^2\,d\mu + \frac{p\varepsilon}{4}\int_M k^{p/2-2}|\nabla f|^2\,d\mu = \frac{1}{4}\int_M k^{p/2} W\,d\mu \tag{4.13}$$

and all integrands are integrable.

Proof. Combining equations (4.27) and (4.6) of [G3] we get

$$\int_M |\nabla k^{p/4}|^2\,d\mu + \frac{p\varepsilon}{4}\int_M k^{p/2-2}|\nabla f|^2\,d\mu = \frac{p}{4}(Af, fk^{p/2-1}). \tag{4.14}$$

Apply (4.7) to the last term to find (4.13). □

Remark 4.8. Integration by parts identities such as (4.14) would be difficult to verify for $|f(z)|^{p/2}$ rather than $(|f(z)|^2+\varepsilon)^{p/4}$ because of the singularity of $|f(z)|^{p/2}$ at the zeros of f. Although Equation (4.2) is exactly (4.13) with $\varepsilon=0$ it was necessary to prove (4.13) for $\varepsilon>0$ because of difficulty at the zeros of f (cf. [G3, Section 4]). In order to prove Theorem 4.1 it will be necessary now to show that the second term on the left-hand side of (4.13) goes to zero, as $\varepsilon\downarrow 0$. We have only been able to do this for $p\geq 2$. But this seems likely to be correct for all $p>0$. The next two lemmas and corollary are devoted to showing that one may let $\varepsilon\downarrow 0$ in the terms on the left-hand side of (4.13).

Lemma 4.9. *Let $0<p<\infty$ and let f be in $\mathcal{H}(M)$. If $0<p<2$ assume that f is in the L^2 domain of A. If $p\geq 2$ assume that f is in the L^p domain of A. Let $\delta>0$. Then*

$$\lim_{\varepsilon\downarrow 0}\varepsilon\int_{|f|^2\geq\delta}(|f|^2+\varepsilon)^{p/2-2}|\nabla f|^2\,d\mu=0,\quad 0<p<\infty. \tag{4.15}$$

Proof. Let $k(z)=|f(z)|^2+\varepsilon$. For any real number $x\geq\delta$ we have $(x+\varepsilon)/x=1+\varepsilon/x\leq 1+\varepsilon/\delta$. So $|f|^2\leq k\leq|f|^2(1+\varepsilon/\delta)$ if $|f|^2\geq\delta$. Suppose first that $0<p\leq 4$. Then $\frac{1}{2}p-2\leq 0$. Hence

$$(|f|^2)^{p/2-2}\geq k^{p/2-2}\geq(|f|^2)^{p/2-2}\Big(1+\frac{\varepsilon}{\delta}\Big)^{p/2-2}$$

wherever $|f|^2\geq\delta$. By [G3, Lemma 4.1 and Proposition 4.2] the right-hand side of (4.14) is finite under the hypotheses of the present lemma. Therefore both terms on the left-hand side of (4.14) are also finite, and in particular the second term on the left-hand side. Therefore

$$\Big(1+\frac{\varepsilon}{\delta}\Big)^{p/2-2}\int_{|f|^2\geq\delta}|f|^{p-4}|\nabla f|^2\,d\mu\leq\int_{|f|^2\geq\delta}k^{p/2-2}|\nabla f|^2\,d\mu<\infty.$$

So

$$\varepsilon\int_{|f|^2\geq\delta}k^{p/2-2}|\nabla f|^2\,d\mu\leq\varepsilon\int_{|f|^2\geq\delta}|f|^{p-4}|\nabla f|^2\,d\mu$$

which goes to zero, as $\varepsilon\downarrow 0$.

Suppose now that $p>4$. Then $\frac{1}{2}p-2>0$. So $(|f|^2)^{p/2-2}|\nabla f|^2\leq k^{p/2-2}|\nabla f|^2$ which is integrable over M. Hence

$$\varepsilon\int_{|f|^2\geq\delta}k^{p/2-2}|\nabla f|^2\,d\mu\leq\varepsilon\Big(1+\frac{\varepsilon}{\delta}\Big)^{p/2-2}\int_{|f|^2\geq\delta}(|f|^2)^{p/2-2}|\nabla f|^2\,d\mu$$

which goes to zero, as $\varepsilon\downarrow 0$. □

Corollary 4.10. *Let $p \geq 2$ and let $f \in \mathcal{H}(M)$. If f is in the L^p domain of A then*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_M (|f|^2+\varepsilon)^{p/2-2} |\nabla f|^2 \, d\mu = 0. \tag{4.16}$$

Proof. By Lemma 4.9 it suffices to prove that, for fixed $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \int_{|f|^2<\delta} (|f|^2+\varepsilon)^{p/2-2} |\nabla f|^2 \, d\mu = 0. \tag{4.17}$$

We may assume that f is not identically zero. Now $\mu(\{z \in M : f(z)=0\})=0$ because f is holomorphic. Moreover $\varepsilon(|f(z)|^2+\varepsilon)^{p/2-2}|\nabla f(z)|^2$ converges to zero at any point where $f(z) \neq 0$. Furthermore, for any fixed $\varepsilon > 0$, $(|f(z)|^2+\varepsilon)^{p/2-2}$ is bounded away from zero on $\{z : |f(z)|^2 < \delta\}$. By the argument given in the proof of Lemma 4.9 we have

$$\int_{|f|^2<\delta} (|f|^2+\varepsilon)^{p/2-2} |\nabla f|^2 \, d\mu < \infty.$$

Hence $\int_{|f|^2<\delta} |\nabla f|^2 \, d\mu < \infty$. But $\varepsilon(|f|^2+\varepsilon)^{p/2-2} \leq (|f|^2+\varepsilon)^{p/2-1} \leq (\delta+\varepsilon)^{p/2-1}$ if $|f(z)|^2 \leq \delta$ because $\frac{1}{2}p-1 \geq 0$. We may therefore apply the dominated convergence theorem to conclude the validity of (4.17). □

Lemma 4.11. *Let $0<p<\infty$. Suppose that $f \in \mathcal{H}(M)$ and that*

$$\int_M \left|\nabla(|f|^2+\varepsilon)^{p/4}\right|^2 d\mu < \infty \quad \textit{for some } \varepsilon > 0.$$

Then

$$\lim_{\varepsilon \downarrow 0} \int_M \left|\nabla(|f|^2+\varepsilon)^{p/4}\right|^2 d\mu = \int_M \left|\nabla |f|^{p/2}\right|^2 d\mu, \quad 0<p<\infty. \tag{4.18}$$

Proof. We may assume that f is not identically zero. The integrand on the right-hand side of (4.18) should be interpreted as undefined on $\{z \in M : f(z)=0\}$. Since this is a set of μ measure zero the integral is well defined. At a point z such that $f(z) \neq 0$ we have $\nabla(|f|^2+\varepsilon)^{p/4} = \frac{1}{4}p(|f|^2+\varepsilon)^{p/4-1}\nabla|f|^2$. So

$$\left|\nabla(|f|^2+\varepsilon)^{p/4}\right|^2 = \left(\tfrac{1}{4}p\right)^2 (|f|^2+\varepsilon)^{p/2-2} \left|\nabla|f|^2\right|^2, \quad \varepsilon \geq 0. \tag{4.19}$$

If $\frac{1}{2}p-2 \geq 0$ the right-hand side decreases to $\left|\nabla|f|^{p/2}\right|^2$. By the dominated convergence theorem, (4.18) holds because $\mu(\{z : f(z)=0\})=0$. If $\frac{1}{2}p-2<0$ the right-hand side of (4.19) increases to $\left|\nabla|f|^{p/2}\right|^2$. So the monotone convergence theorem applies. □

Proof of Theorem 4.1. Consider first the case $p \geq 2$. The identity (4.13) holds under the hypothesis of Theorem 4.1. By Lemma 4.11 the first term on the left-hand side of (4.13) converges to $\int_M \left|\nabla |f|^{p/2}\right|^2 d\mu$, as $\varepsilon \downarrow 0$. By Corollary 4.10 the second term on the left-hand side converges to zero. The right-hand side of (4.13) converges to $\frac{1}{4}\int_M |f|^p W \, d\mu$ by dominated convergence since $(|f|^2+\varepsilon)^{p/2}W$ is integrable for some $\varepsilon > 0$. This proves (4.2). The proof of the inequality (4.3) is similar. One need only drop the positive second term in (4.13), obtaining an inequality which persists in the limit $\varepsilon \downarrow 0$. $\square$

Proof of Corollary 4.3. This follows from Theorem 4.1 by applying Lemma 2.15 to (4.2) for $p \geq 2$ and to (4.3) for $0 < p < 2$. $\square$

Remark 4.12. The reverse hypercontractivity and reverse logarithmic Sobolev inequalities that we have proven are valid specifically for holomorphic functions. But other surprising "reversals" of long established inequalities relating to logarithmic Sobolev inequalities have recently been found in nonholomorphic categories. See, for example, the exposition of M. Ledoux, [L, Section 7], concerning reversed Herbst inequalities and the paper of F. Y. Wang, [W].

5. Other measures on $\mathbf{C}^m$

It is essential that the Dirichlet form operator $\nabla^*\nabla$ associated to a triple (M, g, μ) leaves invariant the space of holomorphic functions on M. Otherwise the semigroup e^{-tA} does not even leave any reasonable holomorphic function spaces invariant. In order for $\nabla^*\nabla$ to leave $\mathcal{H}(M)$ invariant the metric g and measure μ must be properly related. We will describe a class of measures μ on $\mathbf{C}^m$ and corresponding metrics g for which $\nabla^*\nabla$ leaves $\mathcal{H}(\mathbf{C}^m)$ invariant. For these we will compute the function W defined in (2.23) and thereby show the existence of a large class of measures on $\mathbf{C}^m$ for which reverse hypercontractivity holds. This class of measures has already been studied in [G3, Example 5.1]. We will use the notation from that example.

Let φ be a strictly positive function in $C^\infty([0,\infty))$. Assume that its derivative $\varphi'(s) < 0$ for $0 \leq s < \infty$. Define $w(z) = \varphi(|z|^2)$ for $z \in \mathbf{C}^m$ and $\varrho(z) = -b\varphi'(|z|^2)$ for some constant $b > 0$. Let $\sigma(z) = w(z)/\varrho(z)$. Choose b so that the measure

$$d\mu(x) = \varrho(x)\, dx$$

is normalized. Let g be the metric on $\mathbf{C}^m \equiv \mathbf{R}^{2m}$ given by $g_{ij}(x) = \delta_{ij}/\sigma(x)$. Then $(\mathbf{C}^m, g, \mu)$ is a holomorphic triple satisfying our Standing Assumptions 3.3. (See [G3, Section 5].)

It is shown in [G3, Section 5] that

$$X=\frac{2}{b}\sum_{k=1}^{m}\left(x_k\frac{\partial}{\partial x_k}+y_k\frac{\partial}{\partial y_k}\right)$$

which we will write as $X=(2/b)x\cdot\nabla$. If $\psi\in C_c^\infty(\mathbf{C}^m)$ then

$$\begin{aligned}\int_{\mathbf{C}^m}(X\psi)\,d\mu&=\int_{\mathbf{C}^m}(X\psi)\varrho(x)\,dx=\frac{2}{b}\int_{\mathbf{C}^m}(x\cdot\nabla\psi)\varrho(x)\,dx\\&=-\frac{2}{b}\int_{\mathbf{C}^m}\psi(x)\nabla\cdot(x\varrho(x))\,dx=-\frac{2}{b}\int_{\mathbf{C}^m}\psi(x)(2m\varrho(x)+x\cdot\nabla\varrho)\,dx\\&=-\frac{2}{b}\int_{\mathbf{C}^m}\psi(x)\left(2m+\frac{1}{\varrho(x)}x\cdot\nabla\varrho\right)\varrho(x)\,dx.\end{aligned}$$

So $W(x)=-(2/b)(2m+\varrho(x)^{-1}x\cdot\nabla\varrho)$. Now $\nabla\varrho=-b\varphi''(|x|^2)\nabla|x|^2=-2b\varphi''(|x|^2)x$. Therefore $x\cdot\nabla\varrho=-2b|x|^2\varphi''(|x|^2)$. Thus

$$\frac{1}{\varrho(x)}x\cdot\nabla\varrho=\frac{2|x|^2\varphi''(|x|^2)}{\varphi'(|x|^2)}.$$

Hence

$$W(x)=-\frac{4}{b}\left(m+\frac{|x|^2\varphi''(|x|^2)}{\varphi'(|x|^2)}\right). \tag{5.1}$$

For example, if we take $\varphi(s)=(2\pi c)^{-m}e^{-s/2c}$ and $b=2c$ then $\varrho(x)$ is Gaussian and $\varrho\,dx=d\gamma_c$. Moreover $\varphi''(s)=-(2c)^{-1}\varphi'(s)$. The identity (5.1) therefore reduces to $W(x)=c^{-2}|x|^2-2m/c$, in agreement with (2.37).

Since

$$\exp\left(\frac{B(a)}{a}\right)=\int_{\mathbf{C}^m}e^{W(x)/a}(-b\varphi'(|x|^2))\,dx$$

it is clear from (5.1) and the Gaussian case that, if $\varphi(s)=e^{-v(s)}$ then $B(a)$ will be finite for some a if $v(s)$ deviates just a little from linear (e.g. on a compact set, provided $v'(s)>0$ on $[0,\infty)$). Clearly (2.27) and (4.1) also hold if $v(s)$ differs slightly from linear. The vector field X is two-sided complete because it is the same as in the Gaussian case.

Thus we have a large class of non-Gaussian measures on $\mathbf{C}^m$ for which reverse hypercontractivity, (3.8), holds, Carlen's identity, (4.2), holds and the reverse logarithmic Sobolev inequalities, (4.4), hold.

6. The Riemann surface for $z^{1/n}$

Choose an integer $n\geq 2$. Denote by M_n the n sheeted Riemann surface associated to $z^{1/n}$. Let $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$. Then M_n is a covering space of $\mathbf{C}^*$ with n leaves.

Let $\alpha: M_n \to \mathbf{C}^*$ be the covering map and let g be the standard Riemannian metric on M_n. This is the metric that makes α_* an isometry at each point. That is, $g = dx^2 + dy^2$ in the obvious local coordinates x and y lifted from $\mathbf{C}^*$. We take μ to be the measure on M_n whose density with respect to the Riemann area element $dx\,dy$ is $(1/n)p_c(\alpha(z))$ where $p_c(w) = (2\pi c)^{-1} e^{-|w|^2/2c}$ for $w \in \mathbf{C}^*$. In other words we divide the Gaussian density p_c equally among the n sheets. Then μ is a probability measure on M_n. This example was extensively discussed in [G4].

It was shown in [G4, Section 6] that the triple (M_n, g, μ) is holomorphic and that the Standing Assumptions 3.3 hold. This example differs from those in the preceding section, not only because of the different topology of the underlying manifold, but also because $\mathcal{H}^2$ is of codimension $n-1$ in $\mathcal{H} \cap L^2(\mu)$. From [G4, Theorem 6.1] we will use the form of the vector field X. It is

$$X = \frac{1}{c}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \tag{6.1}$$

in the obvious local coordinates x and y.

All of our results are applicable to this example.

Theorem 6.1. *The μ divergence, W, of X satisfies* (2.26) *and* (4.1). *The reverse hypercontractive inequality* (3.9) *holds.*

Proof. If $\psi \in C_c^\infty(M_n)$ and is supported in one leaf over a disk in $\mathbf{C}^*$ then, in local coordinates x and y, we may compute

$$\begin{aligned}
\int_{M_n} (X\psi)\, d\mu &= \frac{1}{c}\int_{\mathbf{C}^*} ((x\partial_x + y\partial_y)\psi(x,y))\frac{1}{n}p_c(x,y)\, dx\, dy \\
&= -\frac{1}{cn}\int_{\mathbf{C}^*} \psi(x,y)(2p_c + (x\partial_x + y\partial_y)p_c(x,y))\, dx\, dy \\
&= \frac{1}{c}\int_{\mathbf{C}^*} \psi(x,y)\left(\frac{|z|^2}{c} - 2\right)\frac{1}{n}p_c(x,y)\, dx\, dy.
\end{aligned}$$

So

$$W(z) = \frac{1}{c}\left(\frac{|z|^2}{c} - 2\right). \tag{6.2}$$

where z is a local coordinate on one sheet. Clearly W takes the same value at all points in M_n which project to the same point in $\mathbf{C}^*$. The function W has the same appearance as in the Gaussian case on $\mathbf{C}$, (2.37), but is defined on M_n rather than on $\mathbf{C}$. Since W is invariant under change of leaf the integral $\int_{M_n} e^{W(z)/s}\, d\mu$ can be evaluated by integrating over $\mathbf{C}^*$. The function $B(s)$ is therefore the same as in the

Gaussian case and is given by (2.40). So (2.26) holds. Since $\|W\|_{L^p}$ is the same as in the Gaussian case, (4.1) also holds. Since $B(s)$ is the same as in the Gaussian case the computations leading to (3.9) are the same as in Example 2.16. The flow of X is again given by (2.45), which should now be interpreted on M_n. Thus all the results of Sections 2, 3 and 4 are applicable to this example.

7. The weighted Bergman spaces

The weighted Bergman spaces give another example of spaces for which our Standing Assumptions 3.3 hold. But the vector field X is not two-sided complete. We will show that reverse hypercontractivity fails.

Let $M=\{z\in\mathbf{C}:|z|<1\}$ and take the metric g to be $g=(1-|z|^2)^{-1}(dx^2+dy^2)$. For any $\lambda>-1$ define $d\mu_\lambda(z)=a_\lambda(1-|z|^2)^\lambda\,dx\,dy$, where a_λ is a normalization constant. The weighted Bergman spaces are the holomorphic function spaces $\mathcal{H}\cap L^2(M,\mu_\lambda)$. It was shown in [G4, Section 5] that $\nabla^*\nabla$ is holomorphic, that our Standing Assumptions 3.3 hold and that

$$X=2(\lambda+1)r\frac{\partial}{\partial r} \tag{7.1}$$

in polar coordinates. Moreover, for all $p\in(0,\infty)$, $\mathcal{H}^p=\mathcal{H}\cap L^p(\mu_\lambda)$ if $\lambda\geq 0$. (For $\lambda<0$ one must use non-Dirichlet boundary conditions to obtain this equality. See [G4, Section 5].) If $\varphi\in C_c^\infty(M)$ then a straightforward computation in polar coordinates shows that

$$\int_M (X\varphi)\,d\mu_\lambda=\int_M \varphi(z)W_\lambda(z)\,d\mu_\lambda(z),$$

where

$$W_\lambda(z)=4(\lambda+1)\left(\frac{\lambda}{1-|z|^2}-(\lambda+1)\right). \tag{7.2}$$

It follows from (7.2) that (2.26) fails if $\lambda>0$, while (2.27) fails if $\lambda<0$. The assumption (4.1) fails whenever $\lambda\neq 0$. If $\lambda=0$ then (2.26) and (4.1) hold. But in all cases X is not two-sided complete because $\exp(tX)$ is a dilation if $t>0$, as one sees from (7.1). The case $\lambda=0$ is of principal interest to us here because all of the hypotheses of Theorem 3.4 hold in this case except that X is not two-sided complete.

If $T>0$ and $p_1>p_0>0$ then a reverse hypercontractive inequality

$$\|e^{-TA}f\|_{p_1}\geq C\|f\|_{p_0},\quad C>0,\ f\in\mathcal{H}\cap L^{p_1}(\mu_\lambda) \tag{7.3}$$

cannot hold for any $\lambda>-1$ and in particular for $\lambda=0$. To see this pick an integer $k\geq 1$ such that

$$\int_M |1-z|^{-kp_0}\,d\mu_\lambda(z)=\infty.$$

For each number $b>1$ let $f_b(z)=(b-z)^{-k}$. Then $f_b\in\mathcal{H}(M)\cap L^\infty(\mu_\lambda)$ while $\|f_b\|_{p_0}\to\infty$, as $b\downarrow 1$. But, writing $\alpha=2(\lambda+1)$, equation (7.1) suggests that $(e^{-TA}f_b)(z)=f_b(e^{-T\alpha}z)$ for $z\in M$. A complete proof of this identity is given in [G4, Section 5] for all λ. (For $\lambda<0$ one must choose appropriate boundary conditions for A.) So $|f_b(e^{-T\alpha}z)|\le(b-e^{-T\alpha})^{-k}\le(1-e^{-T\alpha})^{-k}$ for all $b>1$ and for all $z\in M$. Hence $\|e^{-TA}f_b\|_{p_1}\le(1-e^{-T\alpha})^{-k}<\infty$ for all $b>1$. Therefore the inequality (7.3) cannot hold.

References

[BM] BOGACHEV, V. and MAYER-WOLF, E., Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions, *J. Funct. Anal.* **167** (1999), 1–68.

[B1] BORELL, C., Convexity in Gauss space, in *Aspects statistiques et aspects physiques des processus gaussiens* (*Saint-Flour, 1980*), Colloq. Internat. CNRS 307, pp. 27–37, CNRS, Paris, 1981.

[B2] BORELL, C., Positivity improving operators and hypercontractivity, *Math. Z.* **180** (1982), 225–234.

[BJ] BORELL, C. and JANSON, S., Converse hypercontractivity, in *Séminaire d'Initiation à l'analyse, 21e annee, 1981/1982* (Choquet, G., Rogalski, M., and Saint-Raymond, J., eds.), Exp. **4**, Univ. Paris VI, Paris, 1982.

[C] CARLEN, E. A., Some integral identities and inequalities for entire functions and their applications to the coherent state transform, *J. Funct. Anal.* **97** (1991), 231–249.

[CC] CIPRIANO, F. and CRUZEIRO, A. B., Flows associated to tangent processes on the Wiener space, *J. Funct. Anal.* **166** (1999), 310–331.

[Cr1] CRUZEIRO, A. B., Équations différentielles ordinaires: Non explosion et mesures quasi-invariantes, *J. Funct. Anal.* **54** (1983), 193–205.

[Cr2] CRUZEIRO, A. B., Équations différentielles sur l'espace de Wiener et formules de Cameron–Martin non-linéaires, *J. Funct. Anal.* **54** (1983), 206–227.

[D] DRIVER, B. K., Integration by parts and quasi-invariance for heat kernel measures on loop groups, *J. Funct. Anal.* **149** (1997), 470–547.

[GS] GALAZ-FONTES, F. and SONTZ, S. B., On two reverse inequalities in the Segal–Bargmann space, in *Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory* (*Berkeley, Calif., 1999*), (Hislop, P. and Warchall, H., eds.), Electron. J. Differ. Equ. Conf. **4**, pp. 103–111, Southwest Texas State Univ., San Marcos, Tex., 2000.

[G1] GROSS, L., Logarithmic Sobolev inequalities, *Amer. J. Math.* **97** (1975), 1061–1083.

[G2] GROSS, L., Logarithmic Sobolev inequalities and contractivity properties of semi-groups, in *Dirichlet Forms* (*Varenna, 1992*), (Dell'Antonio, G. and Mosco, U., eds.), Lecture Notes in Math. **1563**, pp. 54–88, Springer-Verlag, Berlin–Heidelberg, 1993.

[G3] GROSS, L., Hypercontractivity over complex manifolds, *Acta Math.* **182** (1999), 159–206.

[G4] GROSS, L., Strong hypercontractivity and relative subharmonicity, to appear in *J. Funct. Anal.*

[J1] JANSON, S., On hypercontractivity for multipliers on orthogonal polynomials, *Ark. Mat.* **21** (1983), 97–110.

[J2] JANSON, S., On complex hypercontractivity, *J. Funct. Anal.* **151** (1997), 270–280.

[L] LEDOUX, M., Concentration of measure and logarithmic Sobolev inequalities, in *Séminaire de Probabilités XXXIII* (Azèma, J., Èmery, M., Ledoux, M. and Yor, M., eds.), Lecture Notes in Math. **1709**, pp. 120–216, Springer-Verlag, Berlin–Heidelberg, 1999.

[S1] SONTZ, S. B., A reverse log-Sobolev inequality in the Segal–Bargmann space, *J. Math. Phys.* **40** (1999), 1677–1695.

[S2] SONTZ, S. B., On some reverse inequalities in the Segal–Bargmann space, in *Differential Equations and Mathematical Physics* (*Birmingham, Al., 1999*), (Weikard, R. and Weinstein, G., eds.), AMS/IP Stud. Adv. Math. **16**, pp. 361–373, Amer. Math. Soc., Providence, R. I., 2000.

[W] WANG, F.-Y., Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, *Probab. Theory Relat. Fields* **109** (1997), 417–424.

[Z] ZHOU, Z., The contractivity of the free Hamiltonian semigroup in the L^p space of entire functions, *J. Funct. Anal.* **96** (1991), 407–425.

Received April 25, 2000

Fernando Galaz-Fontes
Centro de Investigación en Matemáticas
Guanajuato, Gto. 36240
Mexico
email: galaz@fractal.cimat.mx

Leonard Gross
Department of Mathematics
Cornell University
Ithaca, NY 14853
U.S.A.
email: gross@math.cornell.edu

Stephen Bruce Sontz
Universidad Autónoma Metropolitana-Iztapalapa
Mexico DF 09340
Mexico
Current address:
Centro de Investigación en Matemáticas
Guanajuato, Gto. 36240
Mexico
email: sontz@fractal.cimat.mx